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# Integral Geometry of pairs of planes 

Julià Cufí, Eduardo Gallego and Agustí Reventós


#### Abstract

We deal with integrals of invariant measures of pairs of planes in euclidean space $\mathbb{E}^{3}$ as considered by Hug and Schneider. In this paper we express some of these integrals in terms of functions of the visual angle of a convex set. As a consequence of our results we evaluate the deficit in a Crofton-type inequality due to Blaschke.


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## 1. Introduction

The main goal of this paper is to study integrals of invariant measures with respect to euclidean motions in the euclidean space $\mathbb{E}^{3}$, extended to the set of pairs of planes meeting a compact convex set. To carry out this objective we express these integrals in terms of functions of the dihedral visual angle of the convex set from a line and integrate them with respect to an invariant measure in the space of lines.

The first known formula involving the visual angle of a convex set in the euclidean plane $\mathbb{E}^{2}$ is Crofton's formula given in [2]. Other results in this direction were obtained by Hurwitz ([8]), Masotti ([10]) and others, in which the use of Fourier series is the main tool. Recently the authors ([3], [4]) have dealt with a general type of integral formulas, involving the visual angle, from the point of view of Integral Geometry.

When trying to generalize these results to higher dimensions the role played by Fourier series in the case of the plane has to be replaced by the use of spherical harmonics. In this sense Theorem 4.1 plays an important role. After stating and proving this result in dimension 3 we realized that Hug and Schneider ([7]) proved a more general result in any dimension. In fact

[^0]the present paper can be considered in some sense as a complement to [7], the novelty being the introduction of the dihedral visual angle.

In Proposition 3.1 we give a characterization of invariant measures in the space of pairs of planes. These will be the kind of measures considered along the paper.

In section 4, using Hug-Schneider's Theorem [7, p. 349], we give an expression for the integral of the sine of the dihedral visual angle of pairs of planes meeting a given compact convex set $K$ in terms of geometrical properties of $K$, see formula (4.2); also we characterize the compact convex sets of constant width in terms of invariant measures given by Legendre polynomials by means of the following Proposition that completes a result in [7].

Proposition 4.3. Let $K$ be a compact convex set of constant width $\mathcal{W}$ and let $f:[-1,1] \longrightarrow \mathbb{R}$ an even bounded measurable function. Then

$$
\begin{equation*}
\int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\lambda_{0} \pi \mathcal{W}^{2} \tag{1.1}
\end{equation*}
$$

where $u_{i}$ are normal unit vectors to the planes $E_{i}$ and $\lambda_{0}=2 \pi \int_{-1}^{1} f(t) d t$. Moreover if the above equality holds when $f(t)=P_{2 n}(t)$ where $P_{2 n}$ is any Legendre polynomial of degree $2 n, n \neq 0$ then $K$ is of constant width.

In section 5 we assign to any invariant measure on the space of pairs of planes an appropriate function of the dihedral visual angle of a given convex set. The integral of this function with respect to the measure on the space of lines gives the integral of the above measure extended to those planes meeting the convex set. This result is given in the following Theorem.

Theorem 5.2. Let $K$ be a compact convex set and let $f:[-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let $H$ be the $\mathcal{C}^{2}$ function on $[-\pi, \pi]$ satisfying

$$
H^{\prime \prime}(x)=f(\cos (x)) \sin ^{2}(x), \quad-\pi \leq x \leq \pi, \quad H(0)=H^{\prime}(0)=0
$$

Then

$$
\begin{equation*}
\int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\pi H(\pi) F+2 \int_{G \cap K=\emptyset} H(\omega) d G \tag{1.2}
\end{equation*}
$$

where $u_{i}$ are normal unit vectors to the planes $E_{i}, \omega=\omega(G)$ is the visual angle from the line $G$ and $F$ is the area of the boundary of $K$.

Then we relate this result to Blaschke's work [1]. If $K$ is a convex set of mean curvature $M$ and area of its boundary $F$, it is known the following Crofton-Herglotz formula

$$
\int_{G \cap K=\emptyset}\left(\omega^{2}-\sin ^{2} \omega\right) d G=2 M^{2}-\frac{\pi^{3} F}{2}
$$

where $\omega=\omega(G)$ is the dihedral visual angle of $K$ from the line $G$. This equality reveals the significance of the function of the visual angle $\omega^{2}-\sin ^{2} \omega$. One can ask what role does it play the function $\omega-\sin \omega$; this function,
interpreting $\omega$ as the visual angle in the plane is significant thanks to Crofton's formula. In dimension 3 the inequality

$$
\int_{G \cap K=\emptyset}(\omega-\sin \omega) d G \geq \frac{\pi}{4}\left(M^{2}-2 \pi F\right)
$$

was stablished in [1]. Here we provide a simple formulation of the deficit in this inequality by means of the following result.

Theorem 5.3. Let $K$ be a compact convex set with support function $p$, area of its boundary $F$ and mean curvature $M$. Let $L_{u}$ be the length of the boundary of the projection of $K$ on $\operatorname{span}\{u\}^{\perp}$ and let $\omega=\omega(G)$ be the visual angle of $K$ from the line $G$. Then

$$
\begin{aligned}
& \text { i) } \int_{u \in S^{2}} L_{u}^{2} d u=\pi M^{2}+4 \pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1 / 2)^{2}}{\Gamma(n+1)^{2}}\left\|\pi_{2 n}(p)\right\|^{2} \\
& \text { ii) } \int_{G \cap K=\emptyset}(\omega-\sin \omega) d G=\frac{\pi}{4}\left(M^{2}-2 \pi F\right)+\pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1 / 2)^{2}}{\Gamma(n+1)^{2}}\left\|\pi_{2 n}(p)\right\|^{2}
\end{aligned}
$$

whith $\pi_{2 n}(p)$ the projection of the support function $p$ of $K$ on the vector space of spherical harmonics of degree $2 n$.

Moreover equality holds both in (5.5) and (5.6) if and only if $K$ is of constant width.

In section 6 we give a formulation of Theorem 5.2 in terms of Fourier series of the function of the visual angle assigned to an invariant measure. As a consequence one obtains that the integral of any invariant measure in the space of pairs of planes extended to those meeting a compact convex set $K$ is an infinite linear combination of integrals of even powers of the sine of the visual angle of $K$. From this we exhibit in Proposition 6.3 a simple family of polynomial functions that are in some sense a basis for the integrals considered in Theorem 4.1. In fact every invariant integral can be written as an infinite linear combination of integrals with respect to the invariant measures given by those polynomial functions.

## 2. Preliminaries

## Support function

The support function of a compact convex set $K$ in the euclidean space $\mathbb{E}^{3}$ is defined as $p_{K}(u)=\sup \{\langle x, u\rangle: x \in K\}$ for $u$ belonging to the unit sphere $S^{2}$. If the origin $O$ of $\mathbb{E}^{3}$ is an interior point of $K$ then the number $p_{K}(u)$ is the distance from the origin to the support plane of $K$ in the direction given by $u$. The width $w$ of $K$ in a direction $u \in S^{2}$ is $w(u)=p_{K}(u)+p_{K}(-u)$.

From now on we will write $p(u)=p_{K}(u)$ and will assume that $p(u)$ is of class $C^{2}$; in this case we shall say that the boundary of $K, \partial K$, is of class $\mathcal{C}^{2}$.

## Spherical harmonics

Let us recall that a spherical harmonic of order $n$ on the unit sphere $S^{2}$ is the restriction to $S^{2}$ of an harmonic homogeneous polynomial of degree $n$. It is known that every continuous function on $S^{2}$ can be uniformly approximated by finite sums of spherical harmonics (see for instance [6]).

More precisely, the function $p(u)$ can be written in terms of spherical harmonics as

$$
\begin{equation*}
p(u)=\sum_{n=0}^{\infty} \pi_{n}(p)(u) \tag{2.1}
\end{equation*}
$$

where $\pi_{n}(p)$ is the projection of the support function $p$ on the vector space of spherical harmonics of degree $n$. An orthogonal basis of this space is given in terms of the longitude $\theta$ and the colatitude $\varphi$ in $S^{2}$ by

$$
\left\{\cos (j \theta)(\sin \varphi)^{j} P_{n}^{(j)}(\cos \varphi), \quad \sin (j \theta)(\sin \varphi)^{j} P_{n}^{(j)}(\cos \varphi): \quad 0 \leq j \leq n\right\}
$$

where $P_{n}^{(j)}$ denotes the $j$ th derivative of the $n$th Legendre polynomial $P_{n}$ (cf. [6]).

It can be seen that $\pi_{0}(p)=\mathcal{W} / 2=M / 4 \pi$ where $\mathcal{W}=1 / 4 \pi \int_{S^{2}} w(u) d u$ is the mean width of $K$, and $M$ is the mean curvature of $K$. Moreover $\pi_{1}(p)=$ $\langle s(K), \cdot\rangle$ where $s(K)$ denotes the Steiner point of $K(c f .[6$, p. 182]). It is clear that $\pi_{0}(p)$ is invariant under euclidean motions and that $\pi_{1}(p)$ is not. It is known that $\pi_{n}(p)$ is invariant under translations for every $n \neq 1$ (cf. [12, p. 5]).

One can easily check that $K$ has constant width if and only if $\pi_{n}(p)=0$ for $n \neq 0$ even.

## Measures in the space of planes

The space of affine planes $\mathcal{A}_{3,2}$ in $\mathbb{E}^{3}$ is a homogeneous space of the group of isometries of $\mathbb{E}^{3}$. It can also be considered as a line bundle $\pi: \mathcal{A}_{3,2} \longrightarrow \operatorname{Gr}(3,2)$ where $\operatorname{Gr}(3,2)$ is the Grassmannian of planes through the origin in $\mathbb{E}^{3}$ and $\pi(E)$ is the plane parallel to $E$ through the origin. The fiber on $E_{0} \in \operatorname{Gr}(3,2)$ is identified with $E_{0}^{\perp}$. Each plane $E \in \mathcal{A}_{3,2}$ is then uniquely determined by the pair $\left(\pi(E), E \cap \pi(E)^{\perp}\right)$. Every pair $\left(E_{0}, p\right) \in \operatorname{Gr}(3,2) \times \mathbb{R}^{3}$ determines an element $E_{0}+p \in \mathcal{A}_{3,2}$.

We shall also consider the space of affine lines $\mathcal{A}_{3,1}$ in $\mathbb{E}^{3}$; it is a vector bundle $\pi: \mathcal{A}_{3,1} \longrightarrow \operatorname{Gr}(3,1)$ where $\operatorname{Gr}(3,1)$ is the Grassmannian of lines through the origin and every affine line $G \subset \mathbb{E}^{3}$ can be identified with $\left(\pi(G), G \cap \pi(G)^{\perp}\right)$.

Both the isometry group of $\mathbb{E}^{3}$ and the isotropy group of a fixed plane $E \in \mathcal{A}_{3,2}$ are unimodular groups; so the Haar measure of the group of isometries is projected into a isometry-invariant measure $m$ on $\mathcal{A}_{3,2}$.

For a measurable set $B \subset \mathcal{A}_{3,2}$ we consider

$$
m(B)=\int_{\mathcal{A}_{3,2}} \chi_{B}(E) d E:=\int_{\operatorname{Gr}(3,2)}\left(\int_{E_{0}^{\perp}} \chi_{B}\left(E_{0}+p\right) d p\right) d \nu
$$

where $\chi_{B}$ is the characteristic function of $B, d p$ denotes the ordinary Lebesgue measure on $E_{0}^{\perp}$ and $d \nu$ a normalized isometry-invariant measure on $\operatorname{Gr}(3,2)$ such that $\nu(\operatorname{Gr}(3,2))=2 \pi$.

More generally, if $f: \mathcal{A}_{3,2} \rightarrow \mathbb{R}$ and $\bar{f}: \operatorname{Gr}(3,2) \times \mathbb{R}^{3} \rightarrow \mathbb{R}$ are related by $\bar{f}\left(E_{0}, p\right)=f\left(E_{0}+p\right)$ we have

$$
\int_{\mathcal{A}_{3,2}} f(E) d E:=\int_{\operatorname{Gr}(3,2)}\left(\int_{E_{0}^{\perp}} \bar{f}\left(E_{0}, p\right) d p\right) d \nu .
$$

Notice that the only measures on $\mathcal{A}_{3,2}$ invariant under isometries are those of the form $f(E) d E$ with $f$ a constant function.

In a similar way one has a normalized isometry-invariant measure on $\mathcal{A}_{3,1}$ that will be denoted by $d G$. For more details see [9].

## 3. Invariant measures in the space of ordered pairs of planes

We consider measures in the space $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^{3}$ of the form $m_{\tilde{f}}:=\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$. We want to study which functions $\tilde{f}$ give an isometry-invariant measure, that is a measure $m_{\tilde{f}}$ satisfying $m_{\tilde{f}}(B)=$ $m_{\tilde{f}}(g B)$ for every euclidean motion $g$. For instance, it is known that for a given compact convex set $K$ one has $\int_{E \cap K \neq \emptyset} d E=M$. So when $\tilde{f}\left(E_{1}, E_{2}\right)=1$ we have

$$
\begin{equation*}
\int_{K \cap E_{i} \neq \emptyset} d E_{1} d E_{2}=M^{2}=4 \pi^{2} \mathcal{W}^{2} \tag{3.1}
\end{equation*}
$$

where $M$ and $\mathcal{W}$ are the mean curvature and the mean width of $K$, respectively.

Proposition 3.1. The measure given by $\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$ in $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ is invariant under isometries of $\mathbb{E}^{3}$ if and only if $\tilde{f}\left(E_{1}, E_{2}\right)=f\left(\left\langle u_{1}, u_{2}\right\rangle\right)$ where $\pi\left(E_{i}\right)^{\perp}=\operatorname{span}\left\{u_{i}\right\}, i=1,2$ and $f:[-1,1] \rightarrow \mathbb{R}$ is an even measurable function.
Proof. Suppose that $\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$ is invariant. Using the representation of an element $E \in \mathcal{A}_{3,2}$ as a pair $(\pi(E), p)$ where $p=E \cap \pi(E)^{\perp}$ we can write

$$
\tilde{f}\left(E_{1}, E_{2}\right)=F\left(\pi\left(E_{1}\right), p_{1} ; \pi\left(E_{2}\right), p_{2}\right)
$$

for some $F:\left(\operatorname{Gr}(3,2) \times \mathbb{E}^{3}\right)^{2} \rightarrow \mathbb{R}$. For any translation $\tau$ it is

$$
\begin{aligned}
\tilde{f}\left(E_{1}+\tau, E_{2}+\tau\right) & =F\left(\pi\left(E_{1}\right), p_{1}+\left\langle\tau, u_{1}\right\rangle u_{1} ; \pi\left(E_{2}\right), p_{2}+\left\langle\tau, u_{2}\right\rangle u_{2}\right) \\
& =F\left(\pi\left(E_{1}\right), p_{1} ; \pi\left(E_{2}\right), p_{2}\right)
\end{aligned}
$$

the last equality due to the invariance of $\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$. Now, as it can be easily checked, for each pair $p_{i}^{\prime} \in \pi\left(E_{i}\right)^{\perp}, i=1,2$ there is a translation $\tau$ such that $p_{i}^{\prime}=p_{i}+\left\langle\tau, u_{i}\right\rangle u_{i}$, and so $F$ is independent of $p_{1}$ and $p_{2}$ and we can write $\tilde{f}\left(E_{1}, E_{2}\right)=H\left(\pi\left(E_{1}\right), \pi\left(E_{2}\right)\right)$ for some function $H$ on $\operatorname{Gr}(3,2) \times \operatorname{Gr}(3,2)$.

Given $t \in[-1,1]$ consider $(V, W) \in \operatorname{Gr}(3,2)^{2}$ such that $V=\operatorname{span}\{v\}^{\perp}$, $W=\operatorname{span}\{w\}^{\perp}$ and $t=\langle v, w\rangle$ with $v, w$ unit vectors. The function $f(t)=$ $H(V, W)$ is well defined since for any rotation $\theta$ we have that $H(\theta V, \theta W)=$ $H(V, W)$ and it is even. So it is proved that there exists a measurable and even function $f:[-1,1] \rightarrow \mathbb{R}$ such that

$$
\tilde{f}\left(E_{1}, E_{2}\right)=f\left(\left\langle u_{1}, u_{2}\right\rangle\right)
$$

If $\tilde{f}$ is as above it is clear that $\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$ gives rise to an isometryinvariant measure.

## 4. Integral of functions of pairs of planes meeting a convex set

Let $K$ be a compact convex set in the euclidean space $\mathbb{E}^{3}$. According to equality (3.1) it is a natural question to evaluate

$$
\int_{E_{i} \cap K \neq \emptyset} \tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}
$$

where $\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$ is an isometry-invariant measure on $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$. This can be done in terms of the coefficients of the expansion of the support function of $K$ in spherical harmonics and the coefficients of the Legendre series of the measurable even function $f:[-1,1] \rightarrow \mathbb{R}$ such that $\tilde{f}\left(E_{1}, E_{2}\right)=$ $f\left(\left\langle u_{1}, u_{2}\right\rangle\right)$ (see Proposition 3.1).

The following result is a special case, with a different notation, of Theorem 5 in [7], whose proof is based on the Funk-Hecke Theorem ([6, p. 98]).

Theorem 4.1. Let $K$ be a compact convex set with support function $p$ given in terms of spherical harmonics by (2.1). Let $\tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}$ be an isometry-invariant measure on $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ and $f:[-1,1] \rightarrow \mathbb{R}$ an even measurable function such that $\tilde{f}\left(E_{1}, E_{2}\right)=f\left(\left\langle u_{1}, u_{2}\right\rangle\right)$ where $\pi\left(E_{i}\right)^{\perp}=\operatorname{span}\left\{u_{i}\right\}$, $i=1,2$. Then

$$
\begin{equation*}
\int_{E_{i} \cap K \neq \emptyset} \tilde{f}\left(E_{1}, E_{2}\right) d E_{1} d E_{2}=\frac{\lambda_{0}}{4 \pi} M^{2}+\sum_{n=1}^{\infty} \lambda_{2 n}\left\|\pi_{2 n}(p)\right\|^{2} \tag{4.1}
\end{equation*}
$$

where $\lambda_{2 n}=2 \pi \int_{-1}^{1} f(t) P_{2 n}(t) d t$ with $P_{2 n}$ the Legendre polynomial of degree $2 n$.

Example 1. If $f(t)=\sqrt{1-t^{2}}$ then $f\left(\left\langle u_{1}, u_{2}\right\rangle\right)=\sin \left(\theta_{12}\right)$ where $0 \leq \theta_{12} \leq \pi$ is the angle between the planes $E_{1}$ and $E_{2}$ (that is, $\cos \theta_{12}= \pm\left\langle u_{1}, u_{2}\right\rangle$ where $\left.\pi\left(E_{i}\right)^{\perp}=\operatorname{span}\left\{u_{i}\right\}, i=1,2\right)$. Applying Theorem 4.1 with the corresponding coefficients

$$
\lambda_{2 n}=2 \pi \int_{-1}^{1} f(t) P_{2 n}(t)=-\frac{\Gamma\left(n+\frac{1}{2}\right) \Gamma\left(n-\frac{1}{2}\right)}{n!(n+1)!} \frac{\pi}{2}, \lambda_{0}=\pi^{2}, \lambda_{2 n+1}=0
$$

(cf. [5], 7.132), one gets

$$
\begin{equation*}
\int_{E_{i} \cap K \neq \emptyset} \sin \left(\theta_{12}\right) d E_{1} d E_{2}=\frac{\pi}{4} M^{2}-\frac{\pi}{2}\left(\sum_{n=1}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right) \Gamma\left(n-\frac{1}{2}\right)}{n!(n+1)!}\left\|\pi_{2 n}(p)\right\|^{2}\right) . \tag{4.2}
\end{equation*}
$$

In the particular case that $f$ is a Legendre polynomial one obtains from Theorem 4.1 the following

Corollary 4.2. Let $K$ be a compact convex set with support function $p$ given in terms of spherical harmonics by (2.1). Then if $P_{2 n}$ is the Legendre polynomial of even degree $2 n$, one has

$$
\int_{E_{i} \cap K \neq \emptyset} P_{2 n}\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\frac{4 \pi}{4 n+1}\left\|\pi_{2 n}(p)\right\|^{2}
$$

Proof. In this case $\lambda_{m}=0$ for $m \neq 2 n$ and $\lambda_{2 n}=\frac{4 \pi}{4 n+1}$.
To end this section we analyze equality (4.1) when $K$ is a convex set of constant width. As said this means that $\pi_{n}(p)=0$ for $n \neq 0$ even.

Proposition 4.3. Let $K$ be a compact convex set of constant width $\mathcal{W}$ and let $f:[-1,1] \longrightarrow \mathbb{R}$ an even bounded measurable function. Then

$$
\begin{equation*}
\int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\lambda_{0} \pi \mathcal{W}^{2} \tag{4.3}
\end{equation*}
$$

where $u_{i}$ are normal unit vectors to the planes $E_{i}$ and $\lambda_{0}=2 \pi \int_{-1}^{1} f(t) d t$. Moreover if the above equality holds when $f(t)=P_{2 n}(t)$ where $P_{2 n}$ is any Legendre polynomial of degree $2 n, n \neq 0$ then $K$ is of constant width.

Proof. Since $K$ is of constant width by (4.1) one gets

$$
\int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\frac{\lambda_{0}}{4 \pi} M^{2}
$$

and remembering that $M=2 \pi \mathcal{W}$ the equality follows. If equality (4.3) holds for $f(t)=P_{2 n}(t)$ with $n \neq 0$, and since the corresponding $\lambda_{0}$ vanishes one has

$$
\int_{E_{i} \cap K \neq \emptyset} P_{2 n}\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=0 .
$$

Therefore by Corollary 4.2 it follows that $\left\|\pi_{2 n}(p)\right\|=0$ for every non zero $n$ and $K$ is of constant width.

## 5. Integrals of invariant measures in terms of the visual angle

The aim of this section is to write the integral of a isometry-invariant measure over the pairs of planes meeting a convex set $K$, given in Theorem 4.1, as an integral of an appropriate function of the visual angle.

Let us precise what we mean by the angle of a plane about a straight line $G$ and the visual angle of a convex set $K$ from a line $G$ not meeting $K$.

## Definition 5.1.

1. Given a straight line $G$ let $\left(q ; e_{1}, e_{2}\right)$ be a fixed affine orthonormal frame in $G^{\perp}$ with $q \in G$. For each plane $E$ through $G$ let $u$ be the unit normal vector to $E$ pointing from the origin to it. Then the angle $\alpha$ associated to $E$ is the one given by $u=\cos (\alpha) e_{1}+\sin (\alpha) e_{2}$.
2. The visual angle of a convex set $K$ from a line $G$ not meeting $K$ is the angle $\omega=\omega(G), 0 \leq \omega \leq \pi$, between the half-planes $E_{1}, E_{2}$ through $G$ tangents to $K$.

If $\alpha_{i}$ are the angles associated to $E_{i}, i=1,2$ then

$$
\cos (\pi-\omega)=\cos \left(\alpha_{2}-\alpha_{1}\right)=\left\langle u_{1}, u_{2}\right\rangle
$$

where $u_{1}, u_{2}$ are the normal unit vectors to $E_{1}, E_{2}$ pointing from the origin, assuming the origin inside $K$.

The measure $d E_{1} d E_{2}$ in the space $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^{3}$ can be written according to Santaló (cf. [11], section II.12.6) as

$$
\begin{equation*}
d E_{1} d E_{2}=\sin ^{2}\left(\alpha_{2}-\alpha_{1}\right) d \alpha_{1} d \alpha_{2} d G \tag{5.1}
\end{equation*}
$$

Then we can prove the following
Theorem 5.2. Let $K$ be a compact convex set and let $f:[-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let $H$ be the $\mathcal{C}^{2}$ function on $[-\pi, \pi]$ satisfying

$$
H^{\prime \prime}(x)=f(\cos (x)) \sin ^{2}(x), \quad-\pi \leq x \leq \pi, \quad H(0)=H^{\prime}(0)=0
$$

Then

$$
\begin{equation*}
\int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\pi H(\pi) F+2 \int_{G \cap K=\emptyset} H(\omega) d G \tag{5.2}
\end{equation*}
$$

where $u_{i}$ are normal unit vectors to the planes $E_{i}, \omega=\omega(G)$ is the visual angle from the line $G$ and $F$ is the area of the boundary of $K$.

Proof. Let $G=q+\operatorname{span}\{u\}$ with $u$ a unit director vector such that $K \cap G=\emptyset$. Let $E_{i}, i=1,2$ be the supporting planes of $K$ through $G$. Take now an affine orthonormal frame $\left\{q ; e_{1}, e_{2}, u\right\}$ in $\mathbb{E}^{3}$ such that $E_{1}=q+\operatorname{span}\left\{e_{1}, u\right\}$. Every plane $E$ through $G$ can be written as $E=q+\operatorname{span}\left\{v_{\alpha}, u\right\}$ where $v_{\alpha}=\cos \alpha e_{1}+\sin \alpha e_{2}$ with $\alpha \in[0, \pi)$ and the planes $E$ intersecting $K$ correspond to angles $\alpha \in[0, \omega(G)]$. Then using (5.1) one has

$$
\begin{aligned}
\int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2} & = \\
=\int_{G \cap K=\emptyset} \int_{0}^{\omega} & \int_{0}^{\omega} H^{\prime \prime}\left(\alpha_{2}-\alpha_{1}\right) d \alpha_{1} d \alpha_{2} d G+ \\
& +\int_{G \cap K \neq \emptyset} \int_{0}^{\pi} \int_{0}^{\pi} H^{\prime \prime}\left(\alpha_{2}-\alpha_{1}\right) d \alpha_{1} d \alpha_{2} d G
\end{aligned}
$$

Evaluating the inner integrals and taking into account that $\int_{G \cap K \neq \emptyset} d G=\frac{\pi}{2} F$ it follows

$$
\begin{aligned}
& \int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}= \\
&=\frac{1}{2} \pi(H(\pi)+H(-\pi)) F+\int_{G \cap K=\emptyset}(H(\omega)+H(-\omega)) d G
\end{aligned}
$$

Since $H(0)=H^{\prime}(0)=0$ and $H^{\prime \prime}(x)=H^{\prime \prime}(-x)$ it is easy to see that $H(x)=$ $H(-x)$ and the result follows.

Example 2. Let $f(t)=\sqrt{1-t^{2}}$ be the function considered in Example 1. In this case the corresponding function $H$ such that

$$
H^{\prime \prime}(x)=f(\cos (x)) \sin ^{2}(x)=\left|\sin ^{3}(x)\right|
$$

is given by

$$
H(x)=\frac{2}{3}(|x|-|\sin x|)-\frac{1}{9}\left|\sin ^{3} x\right| .
$$

Now, since $\omega \in[0, \pi]$, Theorem 5.2 and equality (4.2) leads to

$$
\begin{aligned}
\int_{G \cap K=\emptyset}(\omega- & \left.\sin \omega-\frac{1}{3!} \sin ^{3} \omega\right) d G= \\
& =\pi\left(\frac{3 M^{2}}{16}-\frac{1}{2} \pi F-\frac{3}{8} \sum_{n=1}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right) \Gamma\left(n-\frac{1}{2}\right)}{n!(n+1)!}\left\|\pi_{2 n}(p)\right\|^{2}\right)
\end{aligned}
$$

## Crofton's formula in the space

In Blaschke's work [1, p. 75] the following Crofton-Herglotz formula is given

$$
\begin{equation*}
\int_{G \cap K=\emptyset}\left(\omega^{2}-\sin ^{2} \omega\right) d G=2 M^{2}-\frac{\pi^{3} F}{2} \tag{5.3}
\end{equation*}
$$

We can easily recover (5.3) from Theorem 5.2. In fact considering $f(t)=$ 1 one gets $H(x)=\left(x^{2}-\sin ^{2} x\right) / 4$ and equality (5.2) gives

$$
M^{2}=\int_{E_{i} \cap K \neq \emptyset} d E_{1} d E_{2}=\frac{1}{4} \pi^{3} F+\frac{1}{2} \int_{G \cap K=\emptyset}\left(\omega^{2}-\sin ^{2} \omega\right) d G
$$

Blaschke's formula reveals the significance of the function of the visual angle $\omega^{2}-\sin ^{2} \omega$. One can ask what role the function $\omega-\sin \omega$ plays; this function, interpreting $\omega$ as the visual angle in the plane, is significant thanks to Crofton's formula $\int_{P \notin K}(\omega-\sin \omega) d P=L^{2} / 2-\pi F$, where $K$ is a compact convex set in the plane with area $F$ and length of its boundary $L$ (see [11]).

In [1, p. 85] Blaschke shows that

$$
\begin{equation*}
\int_{G \cap K=\emptyset}(\omega-\sin \omega) d G=\frac{1}{4} \int_{u \in S^{2}} L_{u}^{2} d u-\frac{\pi^{2}}{2} F, \tag{5.4}
\end{equation*}
$$

where $L_{u}$ is the length of the boundary of the projection of $K$ on $\operatorname{span}\{u\}^{\perp}$.

It can be easily seen that $\int_{u \in S^{2}} L_{u} d u=2 \pi M$ and from this equality, applying Schwarz's inequality, one gets

$$
\begin{equation*}
\int_{u \in S^{2}} L_{u}^{2} d u \geq \pi M^{2} \tag{5.5}
\end{equation*}
$$

Introducing (5.5) into (5.4) one obtains

$$
\begin{equation*}
\int_{G \cap K=\emptyset}(\omega-\sin \omega) d G \geq \frac{\pi}{4}\left(M^{2}-2 \pi F\right) \tag{5.6}
\end{equation*}
$$

As a consequence of Theorem 5.2 we can now evaluate the deficit in both inequalities (5.5) and (5.6).

Theorem 5.3. Let $K$ be a compact convex set with support function p, area of its boundary $F$ and mean curvature $M$. Let $L_{u}$ be the length of the boundary of the projection of $K$ on $\operatorname{span}\{u\}^{\perp}$ and let $\omega=\omega(G)$ be the visual angle of $K$ from the line $G$. Then

$$
\begin{aligned}
& \text { i) } \int_{u \in S^{2}} L_{u}^{2} d u=\pi M^{2}+4 \pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1 / 2)^{2}}{\Gamma(n+1)^{2}}\left\|\pi_{2 n}(p)\right\|^{2} \\
& \text { ii) } \int_{G \cap K=\emptyset}(\omega-\sin \omega) d G=\frac{\pi}{4}\left(M^{2}-2 \pi F\right)+\pi \sum_{n=1}^{\infty} \frac{\Gamma(n+1 / 2)^{2}}{\Gamma(n+1)^{2}}\left\|\pi_{2 n}(p)\right\|^{2}
\end{aligned}
$$

whith $\pi_{2 n}(p)$ the projection of the support function $p$ of $K$ on the vector space of spherical harmonics of degree $2 n$.

Moreover equality holds both in (5.5) and (5.6) if and only if $K$ is of constant width.

Proof. We consider $f(t)=1 / \sqrt{1-t^{2}}$. For this function the corresponding $H$ in Theorem 5.2 is $H(x)=|x|-|\sin x|$. Applying equality (5.2) and Theorem 4.1 with the corresponding $\lambda_{2 n}$ 's given by

$$
\lambda_{2 n}=2 \pi \int_{-1}^{1} f(t) P_{2 n}(t) d t=2 \pi \frac{\Gamma(n+1 / 2)^{2}}{\Gamma(n+1)^{2}}
$$

(cf. [5], 7.226), item ii) follows. Equality i) is a consequence of ii) and (5.4).
The statement about equality in (5.5) and (5.6) is a consequence of the fact that $K$ is of constant width if and only if $\pi_{2 n}(p)=0$ for $n \neq 0$.

## 6. A formulation with Fourier series

In this section we give an alternative formulation of Theorem 5.2 in terms of Fourier coefficients of the function $H^{\prime \prime}(x)$. Since $f$ is even one has that $H^{\prime \prime}(x)=f(\cos (x)) \sin ^{2}(x)$ is an even $\pi$-periodic function. Let

$$
\begin{equation*}
H^{\prime \prime}(x)=\frac{1}{2} a_{0}+\sum_{n \geq 1} a_{2 n} \cos (2 n x) \tag{6.1}
\end{equation*}
$$

be the Fourier expansion of $H^{\prime \prime}(x)$. Integrating twice and taking into account that $H(0)=H^{\prime}(0)=0$ one obtains

$$
\begin{equation*}
H(x)=\frac{a_{0}}{4} x^{2}+\sum_{n \geq 1} \frac{a_{2 n}}{4 n^{2}}(1-\cos (2 n x)) \tag{6.2}
\end{equation*}
$$

Using this expression of the function $H$, Theorem 5.2 can be written as
Proposition 6.1. Let $K$ be a compact convex set and let $f:[-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let $H$ be the $\mathcal{C}^{2}$ function on $[-\pi, \pi]$ satisfying

$$
\begin{equation*}
H^{\prime \prime}(x)=f(\cos (x)) \sin ^{2}(x), \quad-\pi \leq x \leq \pi, \quad H(0)=H^{\prime}(0)=0 \tag{6.3}
\end{equation*}
$$

If $H(x)$ is given by (6.2), then

$$
\begin{align*}
\int_{E_{i} \cap K \neq \emptyset} & f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}= \\
= & \frac{a_{0}}{4} \pi^{3} F+\frac{1}{2} \int_{G \cap K=\emptyset}\left(a_{0} \omega^{2}+\sum_{n \geq 1} \frac{a_{2 n}}{n^{2}}(1-\cos (2 n \omega))\right) d G \tag{6.4}
\end{align*}
$$

where $u_{i}$ are normal vectors to the planes $E_{i}$, the visual angle from the line $G$ is $\omega$ and $F$ denotes the area of the boundary of $K$.

The right hand side of (6.4) can be written as a linear combination of integrals of even powers of $\sin \omega$. For this purpose we will use Blaschke formula (5.3) and the known equality

$$
\begin{equation*}
\cos 2 n x=\sum_{m=0}^{n} \alpha_{n, m} \sin ^{2 m} x \quad \text { with } \quad \alpha_{n, m}=\frac{(-1)^{m} n 2^{2 m}(n+m-1)!}{(2 m)!(n-m)!}, \tag{6.5}
\end{equation*}
$$

which follows easily from the equality $\cos (2 n x)=(-1)^{n} T_{2 n}(\sin (x))$ where $T_{2 n}$ is Chebyshev's polynomial of degree $2 n$. We can state

Proposition 6.2. Let $K$ be a compact convex set and let $f:[-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let $H$ be the $\mathcal{C}^{2}$ function on $[-\pi, \pi]$ satisfying

$$
H^{\prime \prime}(x)=f(\cos (x)) \sin ^{2}(x), \quad-\pi \leq x \leq \pi, \quad H(0)=H^{\prime}(0)=0
$$

If $H(x)$ is given by (6.2) then

$$
\begin{align*}
& \int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}= \\
&=a_{0} M^{2}-\frac{1}{2} \sum_{m=2}^{\infty}\left(\sum_{n=m}^{\infty} \frac{a_{2 n}}{n^{2}} \alpha_{n, m} \int_{G \cap K=\emptyset} \sin ^{2 m} \omega d G\right) \tag{6.6}
\end{align*}
$$

where $u_{i}$ are normal vectors to the planes $E_{i}$, the visual angle from the line $G$ is $\omega, F$ denotes the area of the boundary of $K$, the coefficients $\alpha_{n, m}$ are given by (6.5) and the coefficients $a_{2 n}$ by (6.1).

Proof. Using (6.5) the right hand side of (6.4) is written as

$$
\begin{aligned}
& \quad \frac{a_{0} \pi^{3}}{4} F+\frac{1}{2} \int_{G \cap K=\emptyset}\left(a_{0} \omega^{2}-\sum_{n=1}^{\infty} \frac{a_{2 n}}{n^{2}} \alpha_{n, 1} \sin ^{2} \omega-\sum_{n=2}^{\infty} \frac{a_{2 n}}{n^{2}} \sum_{m=2}^{n} \alpha_{n, m} \sin ^{2 m} \omega\right) d G= \\
& =\frac{a_{0} \pi^{3}}{4} F+\frac{1}{2} \int_{G \cap K=\emptyset}\left(a_{0}\left(\omega^{2}-\sin ^{2} \omega\right)-\sum_{n=2}^{\infty} \frac{a_{2 n}}{n^{2}} \sum_{m=2}^{n} \alpha_{n, m} \sin ^{2 m} \omega\right) d G
\end{aligned}
$$

where we have used that $\alpha_{n, 0}=1, \alpha_{n, 1}=-2 n^{2}$ and $a_{0}=-2 \sum_{n=1}^{\infty} a_{2 n}$ which is a consequence of the fact that $H^{\prime \prime}(0)=0$. Using Blaschke formula (5.3) and reordering the double sum the result follows.

## A basis for the integrals of invariant measures

As a consequence of Proposition 6.2 we can exhibit a simple family of polynomial functions that are in some sense a basis for the integrals in Theorem 4.1. Consider the polymomials

$$
\begin{equation*}
h_{m}(t)=m\left(2 m t^{2}-1\right)\left(1-t^{2}\right)^{m-2}, m>1 . \tag{6.7}
\end{equation*}
$$

Then for $H^{\prime \prime}(x)=h_{m}(\cos (x)) \sin ^{2}(x)$ one easily checks that $H(\omega)=\frac{1}{2} \sin ^{2 m} \omega$ and Theorem 5.2 applied to $h_{m}(t)$ gives

$$
\int_{E_{i} \cap K \neq \emptyset} h_{m}\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}=\int_{G \cap K=\emptyset} \sin ^{2 m} \omega d G,
$$

that together with equation (6.6) leads to the following
Proposition 6.3. Under the same hypotheses and notation as in Proposition 6.2 one has

$$
\begin{aligned}
& \int_{E_{i} \cap K \neq \emptyset} f\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}= \\
& =a_{0} M^{2}-\frac{1}{2} \sum_{m=2}^{\infty}\left(\sum_{n=m}^{\infty} \frac{a_{2 n}}{n^{2}} \alpha_{n, m} \int_{E_{i} \cap K \neq \emptyset} h_{m}\left(\left\langle u_{1}, u_{2}\right\rangle\right) d E_{1} d E_{2}\right)
\end{aligned}
$$

where the polynomials $h_{m}$ are given in (6.7).
So every invariant integral can be written as an infinite linear combination of the integrals of the invariant measures given by the polynomials $h_{m}$.

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Julià Cufí<br>e-mail: jcufi@mat.uab.cat<br>Eduardo Gallego<br>e-mail: egallego@mat.uab.cat<br>Agustí Reventós<br>e-mail: agusti@mat.uab.cat<br>Departament de Matemàtiques<br>Universitat Autònoma de Barcelona<br>08193 Bellaterra, Barcelona<br>Catalonia


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