SOBOLEV ANISOTROPIC INEQUALITIES WITH MONOMIAL WEIGHTS

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Abstract. We derive some anisotropic Sobolev inequalities in $\mathbb{R}^n$ with a monomial weight in the general setting of rearrangement invariant spaces. Our starting point is to obtain an integral oscillation inequality in multiplicative form.

1. Introduction

The study of functional and geometric inequalities with monomial weights, *i.e.* weights defined by

$$d\mu(x) := x^A dx = |x_1|^{A_1} \cdots |x_n|^{A_n} dx. \tag{1.1}$$

where $A = (A_1, A_2, \ldots, A_n)$ is a vector in $\mathbb{R}^n$ with $A_i \geq 0$ for $i = 1, \ldots, n$, have been considered extensively recently (see for example [11], [12], [3], [9], [31] and the references quoted therein). The interest for this kind of problems appears when Cabré and Ros-Oton (motivated by an open question raised by Haim Brezis [6],[7]) studied in [13] the problem of the regularity of stable solutions to reaction-diffusion problems of double revolution in $\mathbb{R}^2$. A function u has symmetry of double revolution if $u(x, y) = u(|x|, |y|)$, with $(x, y) \in \mathbb{R}^D = \mathbb{R}^{A_1+1} \times \mathbb{R}^{A_2+1}$ (A_i are positive integers), *i.e.* the function u can be seen as a suitable function in $\mathbb{R}^2$, and it is here where the Jacobian $|x_1|^{A_1}|x_2|^{A_2}$ appears (see [13] for the details). In [12], the authors established a sharp isoperimetric inequality in $(\mathbb{R}^n, \mu)$ (see also [9]) which allows them to obtain the following weighted Sobolev inequality.

Theorem 1.1. ([12, Theorem 1.3]) *Let μ be defined in (1.1), let*

$$D = n + A_1 + \cdots + A_n \tag{1.2}$$

and $1 \leq p < D$. Then for any $f \in C_c^1(\mathbb{R}^n)$ we have[1]

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} d\mu \right)^{1/p^*} \leq C_p \left(\int_{\mathbb{R}^n} |\nabla f|^p d\mu \right)^{1/p} \tag{1.3}$$

for some positive constant C_p, where

$$p^* = \frac{Dp}{D - p}. \tag{1.4}$$

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[1] $C_c^1(\mathbb{R}^n)$ denotes the space of C^1 functions with compact support in $\mathbb{R}^n$.

As in the unweighted case a scaling argument shows that the exponent p^* is optimal, in the sense that (1.4) can not hold with any other exponent. Moreover the exponent p^* is exactly the same as in the classical Sobolev inequality, but in this case the "dimension" is given by D (instead of n). If $A_1 = ... = A_n = 0$, then exponent p^* and inequality (1.3) are exactly the classical ones.

We observe that when $p > 1$ and $A_i < p - 1$ for all $i = 1, \cdots, n$ the weight in (1.1) belongs to the Muckenhoupt class A_p, but, in general the monomial weight does not satisfy the Muckenhoupt condition.

The main purpose of this paper is to obtain some anisotropic Sobolev inequalities on $\mathbb{R}^n$ with monomial weight x^A in the general setting of rearrangement invariant spaces (*e.g.* L^p, Lorentz, Orlicz, Lorentz-Zygmund, etc...). To this end, we will use the "symmetrization by truncation principle", developed by Milman-Martín in [26] (see also [27] and [28]). This method will provide us a family of rearrangement pointwise inequalities between the special difference[2] $O_\mu(f,t) := f_\mu^{**}(t) - f_\mu^*(t)$ (called the oscillation of f) and the product of the rearrangements of the partial derivatives of f (see Theorem 3.1 below) that will be the key to obtain anisotropic inequalities. More precisely we will prove that inequality (1.3) with $p = 1$ is equivalent to the following oscillation inequality:

$$\int_0^t \left(O_\mu(f,\cdot)\,(\cdot)^{-\frac{1}{D}}\right)^*(s)ds \preceq \int_0^t \prod_{i=1}^n \left[\left(\frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}} |f_{x_i}|\,d\mu\right)^*(\tau)\right]^{\frac{A_i+1}{D}} d\tau \tag{1.5}$$

for every $t > 0$. The rearrangements without subscript μ are rearrangement with respect to Lebesgue measure on $(0, \infty)$, $f_{x_i} = \frac{\partial f}{\partial x_i}$ and symbol $f \preceq g$ means that there exists an universal constant c (independent of all parameters involved) such that $f \leq cg$.

Inequality (1.5) contains the basic information to obtain anisotropic Sobolev inequalities on rearrangement invariant spaces, since given a rearrangement invariant space X on $(\mathbb{R}^n, \mu)$, Hardy's inequality (see (2.3) below) implies[3]

$$\left\|O_\mu(f,t)t^{-\frac{1}{D}}\right\|_{\bar{X}} \preceq \left\|\prod_{i=1}^n \left[\left(\frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}} |f_{x_i}|\,d\mu\right)^*(\tau)\right]^{\frac{A_i+1}{D}}(t)\right\|_{\bar{X}}. \tag{1.6}$$

For example, given $p_1, \cdots, p_n \geq 1$, let $\overline{p}$ be the weighted harmonic mean between $p_1, \cdots, p_n$, *i.e.*

$$\frac{1}{\overline{p}} = \frac{1}{D}\sum_{i=1}^n \frac{A_i+1}{p_i}, \tag{1.7}$$

then (1.6) implies (see Theorem 4.3 below)

$$\|O_\mu(f,t)t^{-\frac{1}{D}}\|_{\bar{X}^{(\overline{p})}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}}, \tag{1.8}$$

where $X^{(p)} = \{f : |f|^p \in X\}$ endowed with the norm $\|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p}$.

[2] f_μ^* is the decreasing rearrangement of f with respect the measure μ, and $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds$ (see Subsection 2.1).

[3] The spaces $\bar{X}$ are defined in Section 2.2 below.

In the particular case that $X = L^1$ and $\overline{p} < D$, then (1.8) becomes (see Proposition 4.5 below)

$$\|f\|_{L_\mu^{\overline{p}^*}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} \quad \forall f \in C_c^1(\mathbb{R}^n).$$

where $\overline{p}^* = \frac{D\overline{p}}{D-\overline{p}}$.

In particular if $p = p_1 = \cdots = p_n$, then $\overline{p} = p$, $\overline{p}^* = p^*$, and we get

$$\|f\|_{L_\mu^{\overline{p}^*}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p}}^{\frac{A_i+1}{D}} \quad \forall f \in C_c^1(\mathbb{R}^n), \tag{1.9}$$

which implies (1.3).

In the unweighted case, *i.e.* $A_1 = \cdots, A_n = 0$, inequality (1.9) is well-known (see *e.g.* [36], [35] and [23]). Anisotropic inequalities involving Orlicz norm defined using an n-dimensional Young function were also studied in [14]. However, as far we know, our anisotropic inequalities involving rearrangement invariant spaces are new in this context.

The paper is organized as follows. In Section 2 we provide a brief review on the rearrangements of functions and the theory of rearrangement invariant spaces. In Section 3 we will prove our main result (Theorem 3.1 below) which establishes the equivalence between (1.3) with $p = 1$ and (1.5). In section 4 we use the oscillation inequality (1.5) to derive anisotropic Sobolev inequalities in $\mathbb{R}^n$ with monomial weight x^A in the general setting of rearrangement invariant spaces, with special attention in the case of Lebesgue spaces, Lorentz spaces, Lorentz-Zygmund spaces, Gamma spaces and the recent class of $G\Gamma$ spaces. Finally in section 5 we discuss the optimality of the norms appearing in the inequalities presented.

2. Notations and preliminary results

We briefly recall the basic definitions of rearrangements and of rearrangement-invariant (r.i.) spaces referring the reader to [5] and [22].

2.1. **Rearrangement of functions.** Let μ an absolutely continuous measure with respect to Lebesgue measure on $\mathbb{R}^n$. For a μ-measurable function $u : \mathbb{R}^n \to \mathbb{R}$, the distribution function of u is given by

$$\mu_u(s) = \mu\{x \in \mathbb{R}^n : |u(x)| > s\} \qquad s \geq 0.$$

The **decreasing rearrangement** u_μ^* of u is the right-continuous non-increasing function from $[0, \infty)$ into $[0, \infty]$ which is equimeasurable with u. Namely,

$$u_\mu^*(t) = \inf\{s \geq 0 : \mu_u(s) \leq t\} \qquad t \geq 0.$$

We also define $u_\mu^{**} : (0, \infty) \to (0, \infty)$ as

$$u_\mu^{**}(t) = \frac{1}{t}\int_0^t u_\mu^*(s)ds,$$

Note that u_μ^{**} is also decreasing and $u_\mu^* \leq u_\mu^{**}$, moreover

$$(u+v)_\mu^{**}(t) \leq u_\mu^{**}(t) + v_\mu^{**}(t)$$

for $t > 0$.

The **oscillation** of u is defined by

$$O_\mu(u, t) := u_\mu^{**}(t) - u_\mu^*(t).$$

Note that

$$tO_\mu(u,t)=\int_{u_\mu^*(t)}^{\infty}\mu_u(s)ds \tag{2.1}$$

is increasing.

When rearrangements are taken with respect to the Lebesgue measure on $(0,\infty)$, we may omit the measure and simply write u^* and u^{**}, etc...

2.2. **Rearrangement invariant spaces.** We say that a Banach function space $X=X(\mathbb{R}^n)$ on $(\mathbb{R}^n,\mu)$ is **rearrangement-invariant (r.i.) space**, if $g\in X$ implies that $f\in X$ for all $\mu-$measurable functions f such that $f_\mu^*=g_\mu^*$, and $\|f\|_X=\|g\|_X$.

A basic property of rearrangements is the Hardy-Littlewood inequality which tells us that, if u and w are two μ-measurable functions on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n}|u(x)w(x)|\,d\mu\le\int_0^\infty u_\mu^*(t)w_\mu^*(t)\,dt. \tag{2.2}$$

An important consequence of (2.2) is the Hardy-Littlewood-Pólya principle stating that

$$\int_0^r f_\mu^*(s)ds\le\int_0^r g_\mu^*(s)ds\quad\forall r>0\Rightarrow\|f\|_X\le\|g\|_X\quad\text{for any r.i. space } X. \tag{2.3}$$

A r.i. space $X(\mathbb{R}^n)$ can be represented by a r.i. space on $(0,+\infty)$, with Lebesgue measure, $\bar{X}=\bar{X}(0,\infty)$, such that

$$\|f\|_X=\|f_\mu^*\|_{\bar{X}},$$

for every $f\in X$. A characterization of the norm $\|\cdot\|_{\bar{X}}$ is available (see [5, Theorem 4.10 and subsequent remarks]). The space $\bar{X}$ is called the **representation space** of X.

If X is a r.i space, we have

$$L^1(0,\infty)\cap L^\infty(0,\infty)\subset\bar{X}\subset L^1(0,\infty)+L^\infty(0,\infty),$$

with continuous embeddings.

The **associate space** X' of X is the r.i. space of all measurable functions h for which the r.i. norm given by

$$\|h\|_{X'}=\sup_{g\neq0}\frac{\int_{\mathbb{R}^n}|g(x)h(x)|\,d\mu(x)}{\|g\|_X}=\sup_{g\neq0}\frac{\int_0^\infty h_\mu^*(s)g_\mu^*(s)ds}{\|g\|_X}$$

is finite. In particular the following generalized Hölder inequality

$$\int_{\mathbb{R}^n}|g(x)h(x)|\,d\mu(x)\le\|g\|_X\,\|h\|_{X'}$$

holds.

On the framework of r.i. spaces the following Hardy operators are useful:

$$Pf(t)=\frac{1}{t}\int_0^t f(s)ds;\qquad Q_af(t)=\frac{1}{t^a}\int_t^\infty s^af(s)\frac{ds}{s},\quad 0\le a<1,$$

(if $a=0$, we shall write Q instead of Q_0).

The boundedness of these operators on r.i. spaces can be described in terms of the so called **Boyd indices**[4] defined by

$$\bar{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm of the compression/dilation operator E_s on $\bar{X}$, defined for $s>0$, by $E_s f(t) = f^*(\frac{t}{s})$. For example if $X = L^p_\mu$ with $p>1$, then $\bar{\alpha}_X = \underline{\alpha}_X = \frac{1}{p}$. It is well known that

$$\begin{aligned} P \text{ is bounded on } \bar{X} &\Leftrightarrow \overline{\alpha}_X < 1, \\ Q_a \text{ is bounded on } \bar{X} &\Leftrightarrow \underline{\alpha}_X > a. \end{aligned}$$

The next two Lemmas will be used in Section 4.

Lemma 2.1. (*see* [22, Page 43]) *Let X be a r.i. space and let $0 \le \theta_i \le 1$ such that $\sum_{i=1}^n \theta_i = 1$, then*

$$\| \prod_{i=1}^n |f_i|^{\theta_i} \|_X \le \prod_{i=1}^n \|f_i\|_X^{\theta_i}. \tag{2.4}$$

Lemma 2.2. *Let g, h be two positive measurable functions on $(0,\infty)$ such that*

$$g(s) \le h^{**}(s), \text{ for all } s \in (0,\infty),$$

and

$$\int_0^t g(s) \le \int_0^t h^*(s)ds, \text{ for all } t \in (0,\infty).$$

Then

$$\int_0^t g^*(s)ds \le 4 \int_0^t h^*(s)ds \text{ for all } t \in (0,\infty).$$

Therefore for any r.i. space X

$$\|g\|_X \le 4 \|h\|_X .$$

The proof of the previous lemma is implicitly contained in the proof of Theorem 1.2 of [15], a detailed proof can be found in [27].

2.2.1. *Examples.*

Convexifications of r.i. spaces. A way to construct r.i. spaces is through the so-called p-convexification, which is the generalization of the procedure to construct L^p spaces, $1<p<\infty$, starting from L^1. If X is a r.i. space, the $p-$**convexification** $X^{(p)}$ of X, (cf. [25]) is the r.i. space defined $X^{(p)} = \{f : |f|^p \in X\}$ endowed with the following norm

$$\|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p} . \tag{2.5}$$

The same is true for the functional $\|\cdot\|_{X^{\langle p \rangle}}$ defined as

$$\|f\|_{X^{\langle p \rangle}} = \left\| \left((|f|^p)^{**} \right)^{1/p} \right\|_X . \tag{2.6}$$

Spaces $X^{\langle p \rangle}$ have been introduced in [17] in connection with the study of Sobolev embeddings into rearrangement-invariant spaces defined by a Frostman measure.

[4] Introduced by D.W. Boyd in [8].

The generalized Lorentz spaces $\Lambda^{p,q}(w)$. Given $1\leq p,q<\infty$, and w a weight (a positive locally integrable function) on $(0,\infty)$. The generalized Lorentz spaces $\Lambda^{p,q}(w)$ are defined by measurable functions on $(0,\infty)$ such that

$$\|f\|_{\Lambda^{p,q}(w)} := \left(\int_0^\infty \left(t^{1/p}f^*(t)\right)^q w(t)\frac{dt}{t}\right)^{1/q} < \infty. \tag{2.7}$$

If $p=q$, we write $\Lambda^p(w)$ instead $\Lambda^{p,p}(w)$. We denote by $W(t)=\int_0^t w(s)ds$. It is know (see [10]) that $\Lambda^{p,q}(w)=\Lambda^q(W^{q/p-1}w)$. A weight w is called a B_p-weight if three is $C>0$ such that

$$\int_t^\infty \frac{w(s)}{s^p} \leq \frac{C}{r^p}\int_0^t w(s)ds, \quad t>0.$$

The B_p class satisfies that $B_r\subset B_p$ if $r\leq p$. If $w\in B_p$ then (see [33])

$$\|f\|_{\Lambda^p(w)} \simeq \left(\int_0^\infty f^{**}(t)^p w(t)dt\right)^{1/p},$$

(the symbol $f\simeq g$ will indicate the existence of a universal constant $c>0$, independent of all parameters involved, so that $(1/c)f\leq g\leq c\,f$), therefore Lorentz spaces defined by B_p-weights are r.i. spaces. Moreover, since (see [30, Theorem 6.5])

$$w\in B_p \Rightarrow W^{q/p-1}w\in B_q$$

we get that $\Lambda^{p,q}(w)$ is a r.i. space if $w\in B_p$.

Given a $\mu-$measurable function f on $\mathbb{R}^n$ we define

$$\|f\|_{\Lambda^{p,q}_\mu(w)} = \left\{f:\ \|f^*_\mu\|_{\Lambda^{p,q}(w)}<\infty\right\}.$$

Typical examples of generalized Lorentz spaces are the L^p-spaces and the **Lorentz spaces** $L^{p,q}$, defined either for $p=q=1$ or $p=q=\infty$ or $1<p<\infty$, $1\leq q\leq\infty$ by

$$\|f\|_{L^{p,q}_\mu} := \left\|t^{1/p-1/q}f^*_\mu(t)\right\|_{L^q([0,\infty))} < +\infty,$$

and, more generally, the **Lorentz-Zygmund spaces**, defined for $1<p<\infty$, $1\leq q\leq\infty$ and $\alpha\in\mathbb{R}$ by

$$\|f\|_{L^{p,q}_\mu(\log L)^\alpha} := \left\|t^{1/p-1/q}(1+|\log t|)^{\alpha/q}f^*_\mu(t)\right\|_{L^q([0,\infty))} < +\infty.$$

Let us notice that, in spite of the notation, the quantities $\|f\|_{L^{p,q}}$ and $\|f\|_{L^{p,q}(\log L)^\alpha}$ need not be norms; however, they can be turned into equivalent norms, when $1<p<\infty$, replacing f^* by f^{**}.

The Gamma spaces $\Gamma^p(w)$. Let $1\leq p<\infty$. Let w be an admissible weight, *i.e.*

$$\int_0^t w(s)ds<\infty \text{ and } \int_t^\infty \frac{w(s)}{s^p}ds<\infty. \tag{2.8}$$

The Gamma space $\Gamma^p(w)$ is the r.i. space defined as the set of measurable functions such that

$$\|f\|_{\Gamma^p(w)} := \left(\int_0^\infty f^{**}(s)^p w(s)ds\right)^{1/p} < \infty. \tag{2.9}$$

Given f a $\mu-$measurable function on $\mathbb{R}^n$ we define

$$\|f\|_{\Gamma^p_\mu(w)} = \left\{f:\|f^*_\mu\|_{\Gamma^p(w)}<\infty\right\}.$$

The $G\Gamma^p(p,m,w)$ spaces. Let $1 \leq p, m < \infty$ and let w be a weight satisfying that

$$\int_0^\infty \min(s,t)^{m/p} w(t)dt < \infty, \quad s > 0. \tag{2.10}$$

The $G\Gamma(p,m,w)-$spaces are defined by
(2.11)

$$G\Gamma(p,m,w) = \left\{ f : \|f\|_{G\Gamma(p,m,w)} = \left(\int_0^\infty \left(\int_0^t f^*(s)^p ds \right)^{m/p} w(t)dt \right)^{1/m} < \infty \right\}.$$

These spaces has been introduced in [19] in connection with compact Sobolev type embedding results, since then its turn out to be important and several papers devoted to the study of this spaces have been published (see e.g. [18], [20], [21] and the references quoted therein).

2.3. **Some remarks about function spaces defined by oscillations.** In this subsection we analyze functional properties of function spaces whose definition involves the oscillation of f. The principal difficulty dealing with the functional $O_\mu(f,t)$ is its nonlinearity. Therefore function spaces whose definition involves this quantity are not linear spaces.

Consider the Hardy type operator defined on positive measurable function on $(0,\infty)$ by

$$\bar{Q}_a f(t) = \int_t^\infty s^{1/a} f(s) \frac{ds}{s}, \tag{2.12}$$

with $a > 0$.

Theorem 2.3. *Let X be a r.i. space on $(0,\infty)$ and let us assume that X does not contain constant functions.*
i) If $\underline{\alpha}_X > \frac{1}{a}$, then

$$\|t^{-1/a} f^{**}(t)\|_X \simeq \|t^{-1/a}[f^{**}(t) - f^*(t)]\|_X.$$

ii) If $\bar{\alpha}_X < \frac{1}{a}$, then

$$\|f\|_{L^\infty} \preceq \|t^{-1/a}[f^{**}(t) - f^*(t)]\|_X + \|f\|_{L^1_\mu + L^\infty}.$$

Proof. $i)$ Let $f \in X$. Since X does not contain constant functions, $f^{**}(\infty) = 0$. An elementary computation shows that $(-f^{**})'(t) = \frac{f^{**}(t) - f^*(t)}{t}$, thus by the fundamental theorem of Calculus we get

$$f^{**}(t) = \int_t^\infty (f^{**}(t) - f^*(t)) \frac{ds}{s}.$$

Therefore,

$$\begin{aligned} \|t^{-1/a} f^{**}(t)\|_X &= \left\| t^{-1/a} \int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s} \right\|_X \\ &= \left\| t^{-1/a} \int_t^\infty s^{1/a} s^{-1/a} (f^{**}(s) - f^*(s)) \frac{ds}{s} \right\|_X \\ &\preceq \left\| s^{-1/a} (f^{**}(s) - f^*(s)) \right\|_X \quad (\text{since } \underline{\alpha}_X > \frac{1}{a}). \end{aligned}$$

The converse inequality is obvious.

ii) Let us assume now that $\bar{\alpha}_X < \frac{1}{a}$. We get

$$\begin{aligned}\|f\|_{L^\infty} + \|f\|_{L^1_\mu + L^\infty} &= f^{**}(0) - f^{**}(1) = \int_0^1 \left(f^{**}(s) - f^*(s)\right) \frac{ds}{s} \\ &= \int_0^1 s^{1/a} \left(s^{-1/a}\left(f^{**}(s) - f^*(s)\right)\right) \frac{ds}{s} \\ &\le \|s^{-1/a}\left(f^{**}(s) - f^*(s)\right)\|_X \|s^{1/a-1}\chi_{[0,1]}(s)\|_{X'}.\end{aligned}$$

It is enough to check that $\|s^{1/a-1}\chi_{[0,1]}(s)\|_{X'} < \infty$. Since $\bar{\alpha}_X = 1 - \underline{\alpha}_{X'} < \frac{1}{a}$, we can select

$$1 - 1/a < \beta < \underline{\alpha}_{X'}. \tag{2.13}$$

By the definition of indices, there is $c > 0$ such that

$$h_{X'}(2^{-k}) \le c 2^{-k\beta} \quad \forall k \ge 0. \tag{2.14}$$

For any $k \ge 0$, write $I_k = [2^{-k-1}, 2^{-k})$. Since $\chi_{[0,1]}(s) = \sum_{k=0}^\infty \chi_{I_k}(s)$, we get

$$\begin{aligned}\|s^{1/a-1}\chi_{[0,1]}(s)\|_{X'} &\le \sum_{k=0}^\infty \|s^{1/a-1}\chi_{I_k}(s)\|_{X'} \le \sum_{k=0}^\infty 2^{k(1-1/a)}\|\chi_{I_k}(s)\|_{X'} \\ &= \sum_{k=0}^\infty 2^{k(1-1/a)}\|D_{2^{-k}}\chi_{[1/2,1)}(s)\|_{X'} \\ &\le \sum_{k=0}^\infty 2^{k(1-1/a)} h_{X'}(2^{-k})\|\chi_{[1/2,1)}(s)\|_{X'} \\ &= \sum_{k=0}^\infty 2^{k(1-1/a-\beta)} 2^{k\beta} h_{X'}(2^{-k})\|\chi_{[1/2,1)}(s)\|_{X'} \\ &\le c\sum_{k=0}^\infty 2^{k(1-1/a-\beta)}\|\chi_{[1/2,1)}(s)\|_{X'} < \infty \quad \text{(by (2.14) and (2.13))}.\end{aligned}$$

□

Lemma 2.4. *Let X, Y be two r.i. spaces on $(0,\infty)$. Assume that X does not contain constant functions. The next statements are equivalent:*
i) The Hardy type operator $\bar{Q}_a$ defined in (2.12) is bounded from X to Y.
ii) For all $g \in X$

$$\|g\|_Y \preceq \left\| t^{-1/a}\left(g^{**}(t) - g^*(t)\right)\right\|_X.$$

Proof. *i*) ⇒ *ii*) Given $g \in X$, since X does not contain constant functions, $g^{**}(\infty) = 0$, then

$$g^{**}(t) = \int_t^\infty \left(g^{**}(s) - g^*(s)\right)\frac{ds}{s} = \int_t^\infty s^{1/a}\left[s^{-1/a}\left(g^{**}(s) - g^*(s)\right)\right]\frac{ds}{s}.$$

Thus

$$\|g\|_Y \le \|g^{**}\|_Y = \left\|\int_t^\infty s^{1/a}\left[s^{-1/a}\left(g^{**}(s) - g^*(s)\right)\right]\frac{ds}{s}\right\|_Y \preceq \left\|s^{-1/a}\left(g^{**}(s) - g^*(s)\right)\right\|_X.$$

ii) $\Rightarrow$ *i)* By hypothesis

$$\begin{aligned}\left\|\bar{Q}_a f(t)\right\|_Y &\preceq \left\|t^{-1/a}\left[\frac{1}{t}\int_0^t \left(\bar{Q}_a f(\cdot)\right)^*(s)ds - \left(\bar{Q}_a f(\cdot)\right)^*(t)\right]\right\|_X \\ &= \left\|t^{-1/a}\left[\frac{1}{t}\int_0^t \bar{Q}_a f(s)ds - \bar{Q}f(t)\right]\right\|_X \quad \text{(since } \bar{Q}_a f(t) \text{ is decreasing)} \\ &= \left\|t^{-1/a}\left[\frac{1}{t}\int_0^t s^{1/a}f(s)ds + \bar{Q}_a f(t) - \bar{Q}_a f(t)\right]\right\|_X \\ &= \left\|t^{-1/a}\frac{1}{t}\int_0^t s^{1/a}f(s)ds\right\|_X \leq a\left\|f\right\|_X,\end{aligned}$$

where in the last inequality we used the fact that $\left\|t^{-\alpha}\frac{1}{t}\int_0^t f(s)ds\right\|_X \leq \frac{1}{\alpha}\left\|t^{-\alpha}f(t)\right\|_X$ for any $\alpha > 0$ (see [29, Lemma 2.7]). □

3. Self-improvement of Sobolev inequality

In this section we state our main result. Recall that given $A = (A_1, A_2, \cdots, A_n)$ a non negative vector in $\mathbb{R}^n$, *i.e.* $A_i \geq 0$, we consider the measure

$$d\mu(x) := x^A dx = |x_1|^{A_1}\cdots|x_n|^{A_n}\,dx. \tag{3.1}$$

In what follows $W_0^{1,1}(\mathbb{R}^n, \mu)$ denotes the closure of the space $C_c^1(\mathbb{R}^n)$ under the norm

$$\|u\|_{W_0^{1,1}(\mathbb{R}^n,\mu)} = \int_{\mathbb{R}^n}(|\nabla u| + |u|)d\mu.$$

Theorem 3.1. *Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$ and μ defined as in (3.1). If $f \in W_0^{1,1}(\mathbb{R}^n, \mu)$, then the following statements hold and they are equivalent:*

i) *(Poincaré inequality)*

$$\|f\|_{L_\mu^{\frac{D}{D-1}}} \preceq \sum_{i=1}^n \|f_{x_i}\|_{L_\mu^1}. \tag{3.2}$$

ii) *(Poincaré inequality in multiplicative form)*

$$\|f\|_{L_\mu^{\frac{D}{D-1}}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^1}^{\frac{A_i+1}{D}}. \tag{3.3}$$

iii) *(Mazya-Talenti's inequality in multiplicative form) The function f_μ^* is locally absolutely continuous and for all $s > 0$ we have that*

$$s^{1-1/D}\left(-f_\mu^*\right)'(s) \preceq \prod_{i=1}^n\left(\frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}}|f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}}. \tag{3.4}$$

iv) *(Oscillation inequality in multiplicative form) For all $1 \leq p < \infty$ and for all $t > 0$ we get*

(3.5)

$$\int_0^t\left(O_\mu(f,\cdot)^p(\cdot)^{-\frac{p}{D}}\right)^*(s)ds \preceq \int_0^t\prod_{i=1}^n\left[\left(\frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}}|f_{x_i}|\,d\mu\right)^*(\tau)\right]^{p\left(\frac{A_i+1}{D}\right)}d\tau,$$

where rearrangements without subscript μ *are taken with respect to Lebesgue measure on* $(0,\infty)$.

v)

$$\|f\|_{L_\mu^{\frac{D}{D-1},1}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L^1_\mu}^{\frac{A_i+1}{D}}.$$

Proof. First of all we observe that inequality (3.2) holds, because it is a consequence of (1.3) with $p=1$. In what follows we prove the equivalence of (i)-(v).

i) $\Rightarrow$ *ii)*

We follow a scaling argument as in [35, Lemma 7]. Indeed we apply (3.2) to function $w(x)=f(\lambda_1 x_1,\cdots,\lambda_n x_n)$ and we obtain (3.3) by choosing

$$\lambda_i = \prod_{j\neq i} \|f_{x_i}\|_{L^1_\mu}.$$

ii) $\Rightarrow$ *iii)*

Since $L_\mu^{\frac{D}{D-1}}$ is continuously embedded in $L_\mu^{\frac{D}{D-1},\infty}$ with constant 1, inequality (3.3) implies the weaker one

$$\sup_{t>0} t\,|\{x\in\mathbb{R}^n : |f(x)|>t\}|^{\frac{D-1}{D}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L^1_\mu}^{\frac{A_i+1}{D}}. \tag{3.6}$$

Let $0<t_1<t_2<\infty$, the truncations of f are defined by

$$f_{t_1}^{t_2}(x)=\begin{cases} t_2-t_1 & \text{if } |f(x)|>t_2,\\ |f(x)|-t_1 & \text{if } t_1<|f(x)|\le t_2,\\ 0 & \text{if } |f(x)|\le t_1.\end{cases}$$

Observe that if $f\in W_0^{1,1}(\mathbb{R}^n,\mu)$ then $f_{t_1}^{t_2}\in W_0^{1,1}(\mathbb{R}^n,\mu)$, therefore replacing f by $f_{t_1}^{t_2}$ in (3.6) we obtain

$$\sup_{t>0} t\left|\left\{x\in\mathbb{R}^n : \left|f_{t_1}^{t_2}(x)\right|>t\right\}\right|^{\frac{D-1}{D}} \preceq \prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}\left|\frac{\partial f_{t_1}^{t_2}}{\partial x_i}\right| d\mu\right)^{\frac{A_i+1}{D}}.$$

Since

$$\sup_{t>0} t\left|\left\{x\in\mathbb{R}^n : \left|f_{t_1}^{t_2}(x)\right|>t\right\}\right|^{\frac{D-1}{D}} \ge (t_2-t_1)\,|\{x\in\mathbb{R}^n : |f(x)|\ge t_2\}|^{\frac{D-1}{D}},$$

and

$$\left|\frac{\partial f_{t_1}^{t_2}}{\partial x_i}\right| = \left|\frac{\partial f}{\partial x_i}\right|\chi_{\{t_1<|f|\le t_2\}}$$

we get

$$(t_2-t_1)\,|\{x\in\mathbb{R}^n : |f(x)|\ge t_2\}|^{1-1/D} \preceq \prod_{i=1}^{n}\left(\int_{\{t_1<|f|\le t_2\}} |f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}}. \tag{3.7}$$

Using that $|f_{x_i}|\le|\nabla f|$, (3.7) implies

$$(t_2-t_1)\,|\{x\in\mathbb{R}^n : |f(x)|\ge t_2\}|^{1-1/D} \preceq \int_{\{t_1<|f|\le t_2\}} |\nabla f|\,d\mu$$

and from this inequality the locally absolutely continuity of f_μ^* follows easily using the same argument as in [26, page 137].

Let $s>0$ and $h>0$, pick $t_1=f_\mu^*(s+h)$, $t_2=f_\mu^*(s)$, then by (3.7) we get

$$\left(f_\mu^*(s)-f_\mu^*(s+h)\right)s^{1-1/D}\preceq\prod_{i=1}^n\left(\int_{\left\{f_\mu^*(s+h)<|f|\leq f_\mu^*(s)\right\}}|f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}}.$$

Thus,

$$\frac{\left(f_\mu^*(s)-f_\mu^*(s+h)\right)}{h}s^{1-1/D}\preceq\prod_{i=1}^n\left(\frac{1}{h}\int_{\left\{f_\mu^*(s+h)<|f|\leq f_\mu^*(s)\right\}}|f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}}.$$

Letting $h\to 0$ we obtain (3.4).

$iii)\Rightarrow iv)$

Let $1\leq p<\infty$. For $0<s<t$, we get

$$\begin{aligned}
O_\mu(f,t)&=\frac{1}{t}\int_0^t\left(f_\mu^*(s)-f_\mu^*(t)\right)ds=\frac{1}{t}\int_0^t\left(\int_s^t\left(-f_\mu^*\right)'(\tau)\,d\tau\right)ds=\frac{1}{t}\int_0^t s\left(-f_\mu^*\right)'(s)ds\\
&\leq\frac{t^{1/D}}{t}\int_0^t s^{1-1/D}\left(-f_\mu^*\right)'(s)ds\\
&\preceq\frac{t^{1/D}}{t}\int_0^t\prod_{i=1}^n\left(\frac{d}{d\tau}\int_{\left\{|f|>f_\mu^*(\tau)\right\}}|f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}}ds\quad\text{(by (3.4))}\\
&\preceq\frac{t^{1/D}}{t}\int_0^t\left(\prod_{i=1}^n\left[\frac{d}{d\tau}\int_{\left\{|f|>f_\mu^*(\tau)\right\}}|f_{x_i}|\,d\mu\right]^{\frac{A_i+1}{D}}\right)^*(s)ds\quad\text{(by (2.2))}\\
&\preceq t^{1/D}\left(\frac{1}{t}\int_0^t\left(\prod_{i=1}^n\left[\frac{d}{d\tau}\int_{\left\{|f|>f_\mu^*(\tau)\right\}}|f_{x_i}|\,d\mu\right]^{p\frac{A_i+1}{D}}\right)^*(s)ds\right)^{1/p}\quad\text{(by Hölder)}.
\end{aligned}$$

Thus

$$\left(t^{-1/D}O_\mu(f,t)\right)^p\preceq\frac{1}{t}\int_0^t\left(\prod_{i=1}^n\left[\frac{d}{d\tau}\int_{\left\{|f|>f_\mu^*(\tau)\right\}}|f_{x_i}|\,d\mu\right]^{p\frac{A_i+1}{D}}\right)^*(s)ds. \tag{3.8}$$

Notice that the previous computation shows that

$$O_\mu(f,t)=\frac{1}{t}\int_0^t s\left(-f_\mu^*\right)'(s)\,ds. \tag{3.9}$$

If $p=1$, then using (3.9) and (3.8) we get

$$\begin{aligned}
\int_0^t O_\mu(f,s)s^{-1/D}ds&=\int_0^t\frac{s^{-1/D}}{s}\left(\int_0^s z\left(-f_\mu^*\right)'(z)dz\right)ds\\
&=\int_0^t z\left(-f_\mu^*\right)'(z)\left(\int_z^t\frac{s^{-1/D}}{s}ds\right)dz\\
&\leq\int_0^t z\left(-f_\mu^*\right)'(z)\left(\int_z^\infty\frac{s^{-1/D}}{s}ds\right)dz=D\int_0^t z^{1-1/D}\left(-f_\mu^*\right)'(z)dz\\
&\preceq\int_0^t\left(\prod_{i=1}^n\left[\frac{d}{d\tau}\int_{\left\{|f|>f_\mu^*(\cdot)\right\}}|f_{x_i}(x)|\,d\mu(x)\right]^{\frac{A_i+1}{D}}\right)^*(s)\,ds.
\end{aligned}$$

If $1 < p < \infty$, then using (3.8), Hardy's Inequalities (see [5, page 124]) and (3.9) we get

$$\begin{aligned}\int_0^t \left(O_\mu(f,s)s^{-1/D}\right)^p ds &= \int_0^t \left(\frac{s^{-1/D}}{s}\int_0^s z\left(-f_\mu^*\right)'(z)dz\right)^p ds \\ &\le \int_0^t \left(\frac{1}{s}\int_0^s z^{1-1/D}\left(-f_\mu^*\right)'(z)dz\right)^p ds \\ &\preceq \int_0^t \left(z^{1-1/D}\left(-f_\mu^*\right)'(z)\right)^p dz \\ &\preceq \int_0^t \left(\prod_{i=1}^n \left[\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\cdot)\}} |f_{x_i}(x)|\, d\mu(x)\right]^{p\frac{A_i+1}{D}}\right)^*(s)\, ds.\end{aligned}$$

By Lemma 2.2 and [5, Exercise 10, page 88], we get

$$\begin{aligned}\int_0^t \left(O_\mu(f,\cdot)^p\,(\cdot)^{-\frac{p}{D}}\right)^*(s)\, ds \quad &\preceq \quad \int_0^t \left(\prod_{i=1}^n \left[\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\cdot)\}} |f_{x_i}(x)|\, d\mu(x)\right]^{p\frac{A_i+1}{D}}\right)^*(s)\, ds \\ &\le \quad \int_0^t \prod_{i=1}^n \left(\left[\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\cdot)\}} |f_{x_i}(x)|\, d\mu(x)\right]^{p\frac{A_i+1}{D}}\right)(s)\, ds\end{aligned}$$

iv) $\Rightarrow$ *v)*

By (2.3) and (2.4), we obtain

$$\begin{aligned}\left\|O_\mu(f,s)s^{-\frac{1}{D}}\right\|_{L^1} &\preceq \left\|\prod_{i=1}^n \left(\left[\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\tau)\}} |f_{x_i}|\, d\mu\right]^{\frac{A_i+1}{D}}\right)^*(s)\right\|_{L^1} \\ &\preceq \prod_{i=1}^n \left\|\left(\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\tau)\}} |f_{x_i}|\, d\mu\right)^*(s)\right\|_{L^1}^{\frac{A_i+1}{D}}.\end{aligned} \tag{3.10}$$

Moreover we get

$$\begin{aligned}\left\|\left(\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\tau)\}} |f_{x_i}|\, d\mu\right)^*(s)\right\|_{L^1} \quad &= \quad \int_0^\infty \left(\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\tau)\}} |f_{x_i}|\, d\mu\right)^*(s)\, ds \\ &= \quad \int_0^\infty \frac{d}{d\tau}\left(\int_{\{|f|>f_\mu^*(\tau)\}} |f_{x_i}|\, d\mu\right) d\tau \\ &= \quad \int_{\mathbb{R}^n} |f_{x_i}|\, d\mu.\end{aligned} \tag{3.11}$$

Taking into account that $\frac{\partial}{\partial t} f_\mu^{**}(t) = -(f_\mu^{**}(t) - f_\mu^*(t))/t$ and $f_\mu^{**}(\infty) = 0$ (since $f \in W_0^{1,1}(\mathbb{R}^n, \mu)$) by the Fundamental Theorem Calculus we have

$$f_\mu^{**}(t) = \int_t^\infty \left(f_\mu^{**}(\tau) - f_\mu^*(\tau)\right) \frac{d\tau}{\tau}.$$

Therefore

$$\|f\|_{L^{\frac{D}{D-1},1}} := \int_0^\infty f_\mu^{**}(s)s^{-\frac{1}{D}}\,ds = \int_0^\infty \left[\int_s^\infty \left(f_\mu^{**}(\tau)-f_\mu^{*}(\tau)\right)\frac{d\tau}{\tau}\right] s^{-\frac{1}{D}}\,ds$$
$$= \int_0^\infty \left(f_\mu^{**}(\tau)-f_\mu^{*}(\tau)\right)\left[\frac{1}{\tau}\int_0^\tau s^{-\frac{1}{D}}\,ds\right]d\tau$$
$$= \frac{D}{D-1}\int_0^\infty \left(f_\mu^{**}(\tau)-f_\mu^{*}(\tau)\right)\tau^{-\frac{1}{D}}\,d\tau.$$

Conclusion follows by (3.10) and (3.11).

v) $\Rightarrow$ *i)*

It is consequence of the continuous embedding of Lorentz space $L_\mu^{\frac{D}{D-1},1}$ into Lebesgue space $L_\mu^{\frac{D}{D-1}}$ and the relation between weighted arithmetic and geometric mean. □

Remark 3.2. *In the case $A_i = 0$ for all i, inequality* (3.4) *appears in the proof of Theorem 2 of the Appendix of* [14].

Remark 3.3. *As in the classical case we have that* (3.2) *implies a better inequality involving the Lorentz norm $\|\cdot\|_{L^{\frac{D}{D-1},1}}$(see e.g.* [16], [1], [28] *and the bibliography therein).*

Remark 3.4. *By (3.8) we get*

$$t^{-\frac{1}{D}}O_\mu(f,t) \preceq \frac{1}{t}\int_0^t \prod_{i=1}^n \left(\frac{d}{d\tau}\int_{\{|f|>f_\mu^*(\tau)\}}|f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}} ds := I(t), \tag{3.12}$$

that is a pointwise oscillation inequality in multiplicative form. Moreover by (3.12) *we recover the classical oscillation inequality (see* [28] *and* [26]*)*

$$t^{-\frac{1}{D}}O_\mu(f,t) \preceq |\nabla f|_\mu^{**}(t).$$

As matter of the fact by Hölder inequality

$$\begin{aligned} I(t) &\leq \prod_{i=1}^n \left(\frac{1}{t}\int_0^t \left(\frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}}|f_{x_i}|\,d\mu\right)ds\right)^{\frac{A_i+1}{D}} \\ &= \prod_{i=1}^n \left(\frac{1}{t}\int_{\{|f|>f_\mu^*(t)\}}|f_{x_i}|\,d\mu\right)^{\frac{A_i+1}{D}} \\ &\leq \prod_{i=1}^n \left(\frac{1}{t}\int_0^t |f_{x_i}|_\mu^*(s)ds\right)^{\frac{A_i+1}{D}} \quad \text{(by (2.2))} \\ &\preceq |\nabla f|_\mu^{**}(t). \end{aligned} \tag{3.13}$$

4. Anisotropic inequalities in rearrangement invariant spaces

In this section starting from the oscillation inequality (3.5) we derive some anisotropic inequalities in $\mathbb{R}^n$ in the general setting of rearrangement invariant spaces.

We will use throughout this Section the following notation

$$(\tilde{f}_{x_i})^*_\mu(t) := \left(\frac{d}{ds}\int_{\{|f|>f^*_\mu(s)\}} |f_{x_i}|\, d\mu\right)^*(t).$$

In order to prove these kind of results we need the following results.

Lemma 4.1. *Let $1<p<\infty$ and let v be a weight (a positive locally integrable function on $(0,\infty)$). Let*

$$u(t) = \frac{\partial}{\partial t}\left(1+\int_t^1 \frac{v^{\frac{-1}{p-1}}(s)}{s^{\frac{p}{p-1}}}ds\right)^{1-p}.$$

Then, there exists $C>0$ such that

$$\left(\int_0^1 f^{**}(s)^p u(s)ds\right)^{1/p} \leq C\left(\int_0^1 \left(O_\mu(f,s)\right)^p v(s)ds\right)^{1/p} + \left(\int_0^1 u(s)ds\right)^{1/p}\int_0^1 f^*(t)ds.$$

Proof. Since

$$\begin{aligned}\left(\int_0^t u(s)ds\right)^{1/p}\left(\int_t^1 \frac{v(s)^{\frac{-1}{p-1}}}{s^{\frac{p}{p-1}}}ds\right)^{(p-1)/p} &\leq \left(1+\int_t^1 \frac{v^{\frac{-1}{p-1}}}{s^{\frac{p}{p-1}}}ds\right)^{(1-p)/p}\left(\int_t^1 \frac{v(s)^{\frac{-1}{p-1}}}{s^{\frac{p}{p-1}}}ds\right)^{(p-1)/p} \\ &\leq 1,\end{aligned}$$

the result follows from [4, Lemma 5.4]. □

Lemma 4.2. *(see [32] and [2]) Let $f\in W_0^{1,1}(\mathbb{R}^n,\mu)$, then*

$$\int_0^t (\tilde{f}_{x_i})^*_\mu(\tau)d\tau \leq \int_0^t |f_{x_i}|^*_\mu(\tau)d\tau, \quad (t\geq 0)$$

therefore by (2.3) for any r.i space X on $(\mathbb{R}^n,\mu)$ we have that

$$\left\|(\tilde{f}_{x_i})^*_\mu\right\|_{\bar{X}} \leq \left\||f_{x_i}|^*_\mu\right\|_{\bar{X}} = \|f_{x_i}\|_X .$$

4.1. **Convexification of r.i. spaces.**

4.1.1. *The $X^{(q)}$ convexification.* Let X be a r.i. space on $\mathbb{R}^n$ and $X^{(q)}$ its $q-$convexification defined in (2.5). In the next theorem we state some anisotropic inequalities for functions f such that $f_{x_i} \in X^{(p_i)}$ with $p_i \geq 1$ for $i = 1, \cdots, n$.

Theorem 4.3. *Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$ and μ defined as in (3.1). Let X be a r.i. space on $(\mathbb{R}^n, \mu)$ and $f \in C_c^1(\mathbb{R}^n)$. If $p_1, \cdots, p_n \geq 1$, then*

$$\|t^{-1/D}[f_\mu^{**}(t) - f_\mu^*(t)]\|_{\bar{X}^{(\overline{p})}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{D}\sum_{i=1}^{n} \frac{A_i + 1}{p_i},$$

and the involved norms are defined as in (2.5). Moreover
i) if $\underline{\alpha}_X > \frac{\overline{p}}{D}$, then

$$\|t^{-1/D} f_\mu^{**}(t)\|_{\bar{X}^{(\overline{p})}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}}; \tag{4.1}$$

ii) if $\bar{\alpha}_X < \frac{\overline{p}}{D}$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1 + L^\infty}. \tag{4.2}$$

Proof. Since $\bar{X}^{(\overline{p})}$ is a r.i space, by Theorem 3.1 part *iv*) we get

$$\begin{aligned}
\left\| \left(f_\mu^{**}(s) - f_\mu^*(s)\right) s^{-\frac{1}{D}} \right\|_{\bar{X}^{(\overline{p})}} &\preceq \left\| \prod_{i=1}^{n} \left[(\tilde{f}_{x_i})_\mu^* \right]^{\frac{A_i+1}{D}} \right\|_{\bar{X}^{(\overline{p})}} \\
&= \left\| \prod_{i=1}^{n} \left[(\tilde{f}_{x_i})_\mu^* \right]^{\frac{p_i}{\overline{p}}\left(\frac{\overline{p}}{p_i}\frac{A_i+1}{D}\right)} \right\|_{\bar{X}^{(\overline{p})}} \\
&\leq \prod_{i=1}^{n} \left\| (\tilde{f}_{x_i})_\mu^*(t) \right\|_{\bar{X}^{(p_i)}}^{\frac{A_i+1}{D}} \quad \text{(by (2.4))} \\
&\leq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}} \quad \text{(by Lemma 4.2)}.
\end{aligned}$$

On the other hand, since (see [5, Proposition 5.13, Chapter 3])

$$\underline{\alpha}_{\bar{X}^{(\overline{p})}} = \frac{\underline{\alpha}_X}{\overline{p}} \text{ and } \bar{\alpha}_{\bar{X}^{(\overline{p})}} = \frac{\bar{\alpha}_X}{\overline{p}}$$

(4.1) and (4.2) follows from Theorem 2.3. □

Remark 4.4. *We stress in the previous proof if we start from* (3.13) *instead of* (3.5) *we can not consider the case when $p_1 = \cdots = p_n = 1$.*

In the particular case $X = L_\mu^1$, Theorem 4.3 can be detailed as in the following proposition.

Proposition 4.5. *Let $p_1, \cdots, p_n \geq 1$ and $f \in C_c^1(\mathbb{R}^n)$.*
i) If $\overline{p} < D$, then

$$\|f\|_{L_\mu^{\overline{p}^*,\overline{p}}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}}, \tag{4.3}$$

where $\overline{p}$ is defined as in (1.7) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\overline{p} = D$, then

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(t)}{1+\ln\frac{1}{t}}\right)^D \frac{dt}{t}\right)^{1/D} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}. \tag{4.4}$$

iii) If $\overline{p} > D$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}. \tag{4.5}$$

Proof. Since $\left(L_\mu^1\right)^{(\bar{p})} = L_\mu^{\bar{p}}$ and $\underline{\alpha}_{L^{\bar{p}}} = \bar{\alpha}_{L^{\bar{p}}} = \frac{1}{\bar{p}}$, (4.3) and (4.5) follows from Theorem 4.3. We have only to prove (4.4). If $v(t) = \frac{1}{t}$ in Lemma 4.1, then $u(t) = (D-1)\left(\frac{1}{1+\ln\left(\frac{1}{t}\right)}\right)^{-D}\frac{1}{s}$. Under this choice Lemma 4.1 allows us to get

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(t)}{1+\ln\left(\frac{1}{t}\right)}\right)^D \frac{dt}{t}\right)^{1/D} \preceq \left(\int_0^1 \left(f_\mu^{**}(t) - f_\mu^*(t)\right)^D \frac{dt}{t}\right)^{1/D} + f_\mu^{**}(1).$$

□

Remark 4.6. *Our result implies Theorem 1.1. Indeed it gives an embedding in Lorentz spaces. If $A_1 = \cdots = A_n = 0$ we obtain the same results of* [35] *and* [23] *for what concerns* (4.3).

Remark 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in C_c^1(\Omega)$. Arguing as in the proof of Proposition 4.5, we get*

$$\left(\int_0^{\mu(\Omega)} \left(\frac{f_\mu^{**}(t)}{1+\ln\frac{\mu(\Omega)}{t}}\right)^D \frac{dt}{t}\right)^{1/D} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} + \frac{1}{\mu(\Omega)}\int_\Omega |f|\, d\mu.$$

Since

$$\sup_{0<t<\mu(\Omega)} \frac{f_\mu^{**}(t)}{\left(1+\ln\frac{\mu(\Omega)}{t}\right)^{\frac{D-1}{D}}} \preceq \left(\int_0^{\mu(\Omega)} \left(\frac{f_\mu^{**}(t)}{1+\ln\frac{\mu(\Omega)}{t}}\right)^D \frac{dt}{t}\right)^{1/D},$$

we obtain the following anisotropic Trudinger inequality

$$\sup_{0<t<\mu(\Omega)} \frac{f_\mu^{**}(t)}{\left(1+\ln\frac{\mu(\Omega)}{t}\right)^{\frac{D-1}{D}}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} + \frac{1}{\mu(\Omega)}\int_\Omega |f|\, d\mu.$$

4.1.2. *The $X^{\langle q\rangle}$ convexification .*

Theorem 4.8. *Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$ and μ defined as in (3.1). Let X be a r.i. space on $(\mathbb{R}^n, \mu)$ and $f \in C_c^1(\mathbb{R}^n)$. If $p_1, \cdots, p_n \geq 1$, then*

$$\|t^{-1/D}[f_\mu^{**}(t) - f_\mu^*(t)]\|_{\bar{X}^{\langle\overline{p}\rangle}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{X^{\langle p_i\rangle}}^{\frac{A_i+1}{D}},$$

where $\overline{p}$ is defined in (1.7) *and the involved norms are defined as in* (2.6)*. Moreover*
i) if $\underline{\alpha}_{\bar{X}^{\langle\overline{p}\rangle}} > \frac{1}{D}$, then

$$\|t^{-1/D} f_\mu^{**}(t)\|_{\bar{X}^{\langle\overline{p}\rangle}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{X^{\langle p_i\rangle}}^{\frac{A_i+1}{D}}; \tag{4.6}$$

ii) if $\bar{\alpha}_{\bar{X}^{\langle\overline{p}\rangle}} < \frac{1}{D}$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^n \|f_{x_i}\|_{X^{\langle p_i\rangle}}^{\frac{A_i+1}{D}} + \|f\|_{L^1_\mu + L^\infty}. \tag{4.7}$$

Proof. By Theorem 3.1 part *iv*) with $p = \bar{p}$, we get

$$\begin{aligned}
\frac{1}{t}\int_0^t \left(O_\mu(f,\cdot)^{\bar{p}}\,(\cdot)^{-\frac{\bar{p}}{D}}\right)^*(s)ds \quad &\preceq \quad \frac{1}{t}\int_0^t \prod_{i=1}^n \left[(\tilde{f}_{x_i})_\mu^*(\tau)\right]^{\bar{p}\left(\frac{A_i+1}{D}\right)} d\tau \\
&= \quad \frac{1}{t}\int_0^t \prod_{i=1}^n \left[\left((\tilde{f}_{x_i})_\mu^*(\tau)\right)^{p_i}\right]^{\frac{\bar{p}}{p_i}\left(\frac{A_i+1}{D}\right)} d\tau \\
&\leq \quad \prod_{i=1}^n \left(\frac{1}{t}\int_0^t \left[\left((\tilde{f}_{x_i})_\mu^*(\tau)\right)^{p_i}\right] d\tau\right)^{\frac{\bar{p}}{p_i}\left(\frac{A_i+1}{D}\right)}
\end{aligned}$$

Let X a r.i. space, then

$$\begin{aligned}
\|t^{-1/D}[f_\mu^{**}(t) - f_\mu^*(t)]\|_{\bar{X}^{\langle\overline{p}\rangle}} &= \left\|\left(\frac{1}{t}\int_0^t \left(O_\mu(f,\cdot)^{\bar{p}}\,(\cdot)^{-\frac{\bar{p}}{D}}\right)^*(s)ds\right)^{1/\bar{p}}\right\|_{\bar{X}} \\
&\preceq \left\|\prod_{i=1}^n \left(\frac{1}{t}\int_0^t \left[\left((\tilde{f}_{x_i})_\mu^*(\tau)\right)^{p_i}\right] d\tau\right)^{\frac{1}{p_i}\left(\frac{A_i+1}{D}\right)}\right\|_{\bar{X}} \\
&\leq \prod_{i=1}^n \left\|\left(\frac{1}{t}\int_0^t \left[\left((\tilde{f}_{x_i})_\mu^*(\tau)\right)^{p_i}\right] d\tau\right)^{\frac{1}{p_i}}\right\|_{\bar{X}}^{\left(\frac{A_i+1}{D}\right)} \\
&= \prod_{i=1}^n \left\|(\tilde{f}_{x_i})_\mu^*(\tau)\right\|_{\bar{X}^{\langle p_i\rangle}}^{\left(\frac{A_i+1}{D}\right)} \\
&\leq \prod_{i=1}^n \|f_{x_i}\|_{\bar{X}^{\langle p_i\rangle}}^{\frac{A_i+1}{D}} \quad \text{(by Lemma 4.2).}
\end{aligned}$$

The statements (4.6) and (4.7) follows from Theorem 2.3. □

4.2. **Generalized Lorentz spaces.** In this section we state some anisotropic Sobolev-type inequalities for functions f such that the partial derivatives are in some generalized Lorentz spaces. More precisely as a consequence of Theorem 4.3 we obtain the following theorem.

Theorem 4.9. *Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$ and μ defined as in (3.1). Let $p_1, \cdots, p_n \geq 1$, w a weight in $B_{\min(p_1,\cdots,p_n)}$ and $q_1, \cdots, q_n \geq 1$. If $f \in C_c^1(\mathbb{R}^n)$, then the following inequalities hold.*
i) If $\underline{\alpha}_{\Lambda^{\bar{p},\bar{q}}(w)} > \frac{1}{D}$, then

$$\|f\|_{\Lambda_\mu^{\bar{p}^*,\bar{q}}(w)} \preceq \prod_{i=1}^n \|f_{x_i}\|_{\Lambda_\mu^{p_i,q_i}(w)}^{\frac{A_i+1}{D}},$$

where $\bar{p}$ and $\bar{q}$ are defined as in (1.7) and $\bar{p}^* = \frac{\bar{p}D}{D-\bar{p}}$.
ii) If $\bar{\alpha}_{\Lambda^{\bar{p},\bar{q}}(w)} > \frac{1}{D}$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^n \|f_{x_i}\|_{\Lambda_\mu^{p_i,q_i}(w)}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

iii) In the remaining cases we get

$$\left(\int_0^1 f^{**}(s)^{\bar{q}} u(s)ds\right)^{1/\bar{q}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{\Lambda_\mu^{p_i,q_i}(w)}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty},$$

where $u(t) = \frac{\partial}{\partial t}\left(1 + \int_t^1 \frac{\left(s^{\frac{\bar{q}}{\bar{p}}-\frac{\bar{q}}{D}-1} w(s)\right)^{\frac{-1}{\bar{q}-1}}}{s^{\frac{\bar{q}}{\bar{q}-1}}} ds\right)^{1-\bar{q}}.$

Norms $\|\cdot\|_{\Lambda_\mu^{p,q}(w)}$ are defined in (2.7).

Proof. Since $w \in B_{\min(p_1,\cdots,p_n)}$ all the spaces $\Lambda^{p_i,q_i}(w)$ are r.i spaces and from $\min(p_1, \cdots, p_n) \leq \bar{p}$ it follows that $\Lambda^{\bar{p},\bar{q}}(w)$ is a r.i. space. We note that

$$1 = \frac{\bar{q}}{D}\sum_{i=1}^n \frac{A_i+1}{q_i}. \tag{4.8}$$

By Theorem 3.1 we obtain

$$\begin{aligned}
\|t^{-1/D}[f_\mu^{**}(t) - f_\mu^*(t)]\|_{\Lambda^{\bar{p},\bar{q}}(w)} &\preceq \left\|\prod_{i=1}^n \tilde{f}_{x_i}(\tau)^{\frac{A_i+1}{D}}\right\|_{\Lambda^{\bar{p},\bar{q}}(w)} \\
&= \left(\int_0^\infty \left(t^{\frac{1}{\bar{p}}-\frac{1}{\bar{q}}}\prod_{i=1}^n \tilde{f}_{x_i}(t)^{\frac{A_i+1}{D}}\right)^{\bar{q}} w(t)\,dt\right)^{\frac{1}{\bar{q}}} \\
&\leq \prod_{i=1}^n \left(\int_0^\infty \left(t^{\frac{1}{p_i}-\frac{1}{q_i}}\tilde{f}_{x_i}(t)^{\frac{A_i+1}{D}}\right)^{q_i} w(t)\,dt\right)^{\frac{A_i+1}{q_i D}} \quad \text{(by (4.8) and (2.4))} \\
&= \prod_{i=1}^n \left\|\tilde{f}_{x_i}\right\|_{\Lambda^{p_i,q_i}(w)}^{\frac{A_i+1}{D}} \\
&\leq \prod_{i=1}^n \|f_{x_i}\|_{\Lambda_\mu^{p_i,q_i}(w)}^{\frac{A_i+1}{D}} \quad \text{(by Lemma 4.2)}.
\end{aligned}$$

Part $i)$ follows form $i)$ of Theorem 2.3, because

$$\begin{aligned}\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\Lambda^{\bar p,\bar q}(w)} &\simeq \|t^{-1/D}f_\mu^{**}(t)\|_{\Lambda^{\bar p,\bar q}(w)}\\ &\simeq \|f_\mu^{**}(t)\|_{\Lambda^{\bar p^*,\bar q}(w)}\\ &\simeq \|f\|_{\Lambda_\mu^{\bar p^*,\bar q}(w)}.\end{aligned}$$

Part $ii)$ is consequence of $i)$ of Theorem 2.3.

Let us prove now $iii)$. If we denote $I := \|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\Lambda^{\bar p,\bar q}(w)}$ we get

$$\begin{aligned}I &= \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}}\left(t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\right)^*(s)\right)^{\bar q} w(s)\,ds\right)^{1/\bar q}\\
&\simeq \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}}\frac1s\int_0^s \left(t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\right)^*(z)\,dz\right)^{\bar q} w(s)\,ds\right)^{1/\bar q} \qquad (\text{since } w\in B_{\bar p})\\
&\ge \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}}\frac1{2s}\int_0^{2s} \left(t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\right)^*(z)\,dz\right)^{\bar q} w(s)\,ds\right)^{1/\bar q}\\
&\ge \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}}\frac1{2s}\int_0^{2s} \left(z^{-1/D}[f_\mu^{**}(z)-f_\mu^*(z)]\right)dz\right)^{\bar q} w(s)\,ds\right)^{1/\bar q}\\
&\ge \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}}\frac1{2s}\int_s^{2s} \left(z^{-1/D}[f_\mu^{**}(z)-f_\mu^*(z)]\right)dz\right)^{\bar q} w(s)\,ds\right)^{1/\bar q}\\
&\ge \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}}\frac{s[f_\mu^{**}(s)-f_\mu^*(s)]}{2s}\int_s^{2s} \left(z^{-1/D-1}\right)dz\right)^{\bar q} w(s)\,ds\right)^{1/\bar q} \qquad (\text{by } (2.1))\\
&\simeq \left(\int_0^\infty \left(s^{\frac1{\bar p}-\frac1{\bar q}-\frac1D}[f_\mu^{**}(s)-f_\mu^*(s)]\right)^{\bar q} w(s)\,ds\right)^{1/\bar q}\\
&= \left(\int_0^\infty [f_\mu^{**}(s)-f_\mu^*(s)]^{\bar q} s^{\frac{\bar q}{\bar p}-\frac{\bar q}{D}-1} w(s)\,ds\right)^{1/\bar q}.\end{aligned}$$

Considering as weights

$$v(s) := s^{\frac{\bar q}{\bar p}-\frac{\bar q}{D}-1}w(s) \ \text{ and } u(t) = \frac{\partial}{\partial t}\left(1+\int_t^1 \frac{\left(s^{\frac{\bar q}{\bar p}-\frac{\bar q}{D}-1}w(s)\right)^{\frac{-1}{\bar q-1}}}{s^{\frac{\bar q}{\bar q-1}}}ds\right)^{1-\bar q},$$

the result follows from Lemma 4.1. □

The following corollaries follow from the previous theorem considering $w=1$ and $w(t)=(1+|\ln t|)^\alpha$ with $\alpha\in\mathbb{R}$, respectively, and recalling that

$$\bar\alpha_{L^{p,q}(\log L)^\alpha} = \underline{\alpha}_{L^{p,q}(\log L)^\alpha} = \bar\alpha_{L^{p,q}} = \underline{\alpha}_{L^{p,q}} = \frac1p.$$

Corollary 4.10. *Let $f\in C_c^1(\mathbb{R}^n)$, $p_1,\cdots,p_n\ge 1$ and $q_1,\cdots,q_n\ge 1$.*
i) If $\bar p < D$, then

$$\|f\|_{L_\mu^{\bar p^*,\bar q}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i,q_i}}^{\frac{A_i+1}{D}},$$

where $\bar p$ and $\bar q$ are defined as in (1.7) and $\bar p^* = \frac{\bar pD}{D-\bar p}$.

ii) If $\bar{p} > D$, then

$$\|f\|_{L^\infty(\mathbb{R}^n)} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i,q_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

iii) If $\bar{p} = D$, then

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(s)}{1+\ln\frac{1}{s}}\right)^{\bar{q}} \frac{ds}{s}\right)^{1/\bar{q}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i,q_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

Corollary 4.11. *Let $f \in C_c^1(\mathbb{R}^n)$, $p_1, \cdots, p_n \geq 1$, $q_1, \cdots, q_n \geq 1$ and $\alpha \in \mathbb{R}$.*
i) If $\bar{p} < D$, then

$$\|f\|_{L_\mu^{\overline{p}^*,\bar{q}}(\log L)^\alpha} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i,q_i}(\log L)^\alpha}^{\frac{A_i+1}{D}}.$$

where $\bar{p}$ and $\bar{q}$ are defined as in (1.7) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\bar{p} > D$, then

$$\|f\|_{L^\infty(\mathbb{R}^n)} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i,q_i}(\log L)^\alpha}^{\frac{A_i+1}{D}} + \|f\|_{L^1\mu+L^\infty}.$$

iii) If $\bar{p} = D$, then

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(s)}{1+\ln\frac{1}{s}}\right)^{\bar{q}} \left(1+\ln\frac{1}{s}\right)^\alpha \frac{ds}{s}\right)^{1/\bar{q}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i,q_i}(\log L)^\alpha}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

When $A_1 = \cdots = A_n = 0$, $i)$ of Corollary 4.10 is contained in [35] and [23]. For our knowledge the other ones are new.

4.3. **The Gamma spaces.**

Theorem 4.12. *Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$ and μ defined as in (3.1). Let w be a weight satisfying condition (2.8), $f \in C_c^1(\mathbb{R}^n)$, $p_1, \cdots, p_n \geq 1$ and the involved norms are defined as in* (2.9).
i) If $\underline{\alpha}_{\Gamma^{\overline{p}^*}(w)} > \frac{1}{D}$, then

$$\|t^{-1/D} f_\mu^*(t)\|_{\Gamma^{\overline{p}^*}(w)} \preceq \prod_{i=1}^n \|f_{x_i}\|_{\Gamma_\mu^{p_i}(w)}^{\frac{A_i+1}{D}},$$

where $\bar{p}$ is defined as in (1.7) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\bar{\alpha}_{\Gamma^{\overline{p}^*}(w)} < \frac{1}{D}$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^n \|f_{x_i}\|_{\Gamma_\mu^{p_i}(w)}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

iii) In the remaining cases we get

$$\left(\int_0^1 f^{**}(s)^{\overline{p}^*} u(s)ds\right)^{1/\overline{p}^*} \preceq \prod_{i=1}^n \|f_{x_i}\|_{\Gamma_\mu^{p_i}(w)}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty},$$

where $u(t) = \frac{\partial}{\partial t}\left(1 + \int_t^1 \frac{\left(s^{-\frac{\overline{p}^*}{D}} w(s)\right)^{\frac{-1}{\overline{p}^*-1}}}{s^{\frac{\overline{p}^*}{\overline{p}^*-1}}} ds\right)^{1-\overline{p}^*}.$

Proof.

$$\begin{aligned}
\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\Gamma^{\overline{p}^*}(w)} &\preceq \left\| \prod_{i=1}^n (\tilde{f}_{x_i})_\mu^*(t)^{\frac{A_i+1}{D}} \right\|_{\Gamma^{\overline{p}^*}(w)} \\
&= \left(\int_0^\infty \left(\frac{1}{t}\int_0^t \prod_{i=1}^n (\tilde{f}_{x_i})_\mu^*(s)^{\frac{A_i+1}{D}} ds \right)^{\overline{p}^*} w(t)\,dt \right)^{\frac{1}{\overline{p}^*}} \\
&\leq \prod_{i=1}^n \left(\int_0^\infty \left(\frac{1}{t}\int_0^t (\tilde{f}_{x_i})_\mu^*(s) ds \right)^{p_i} w(t)\,dt \right)^{\frac{A_i+1}{p_i D}} \quad \text{(by (4.8) and (2.4))} \\
&= \prod_{i=1}^n \left\| (\tilde{f}_{x_i})_\mu^* \right\|_{\Gamma_\mu^{p_i}(w)}^{\frac{A_i+1}{D}} \\
&\leq \prod_{i=1}^n \|f_{x_i}\|_{\Gamma_\mu^{p_i}(w)}^{\frac{A_i+1}{D}} \quad \text{(by Lemma 4.2).}
\end{aligned}$$

Part i) follows form i) of Theorem 2.3, since

$$\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\Gamma^{\overline{p}^*}(w)} \simeq \|t^{-1/D} f_\mu^{**}(t)\|_{\Gamma^{\overline{p}^*}(w)} \simeq \|t^{-1/D} f_\mu^{*}(t)\|_{\Gamma^{\overline{p}^*}(w)}.$$

Part ii) is consequence of i) of Theorem 2.3.

Let us prove now iii). If we denote $I := \|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\Gamma^{\overline{p}^*}(w)}$ using the same argument that in part iii) of Theorem 4.9 we easily obtain

$$I \geq \left(\int_0^\infty [f_\mu^{**}(s)-f_\mu^*(s)]^{\bar{q}} s^{-\frac{\overline{p}^*}{D}} w(s)\,ds \right)^{1/\overline{p}^*}.$$

Considering as weights

$$v(s) := s^{-\frac{\overline{p}^*}{D}} w(s) \quad \text{and } u(t) = \frac{\partial}{\partial t}\left(1 + \int_t^1 \frac{\left(s^{-\frac{\overline{p}^*}{D}} w(s)\right)^{\frac{-1}{\overline{p}^*-1}}}{s^{\frac{\overline{p}^*}{\overline{p}^*-1}}} ds \right)^{1-\overline{p}^*},$$

the result follows from Lemma 4.1. □

4.4. The $G\Gamma(m,p,w)$-spaces.

Theorem 4.13. *Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$ and μ defined as in (3.1). Let w be a satisfying (2.10), $f \in C_c^1(\mathbb{R}^n)$, $p_1, \cdots, p_n \geq 1$ and the involved norms are defined as in (2.11).*
i) If $\underline{\alpha}_{G\Gamma(\overline{p}^*,m,w)} > \frac{1}{D}$, then

$$\|t^{-1/D} f_\mu^*(t)\|_{G\Gamma(\overline{p}^*,m,w)} \preceq \prod_{i=1}^n \|f_{x_i}\|_{G\Gamma(p_i,m,w)}^{\frac{A_i+1}{D}},$$

where $\bar{p}$ is defined as in (1.7) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\bar{\alpha}_{G\Gamma(\overline{p}^*,m,w)} < \frac{1}{D}$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^n \|f_{x_i}\|_{G\Gamma(p_i,m,w)}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1 + L^\infty}.$$

iii) In the remaining cases we get

$$\left(\int_0^1 f^{**}(s)^m u(s)ds\right)^{1/m} \preceq \prod_{i=1}^n \|f_{x_i}\|_{G\Gamma(p_i,m,w)}^{\frac{A_i+1}{D}} + \|f\|_{L^1_\mu+L^\infty},$$

where $u(t) = \frac{\partial}{\partial t}\left(1+\int_t^1 \frac{\left(s^{-\frac{m}{D}}w(s)\right)^{\frac{-1}{m-1}}}{s^{\frac{m}{m-1}}}ds\right)^{1-m}$.

Proof. By Theorem 3.1 we get

$$\begin{aligned}
\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{G\Gamma(\overline{p}^*,m,w)} &\preceq \left\|\prod_{i=1}^n (\tilde{f}_{x_i})_\mu^*(t)^{\frac{A_i+1}{D}}\right\|_{G\Gamma(\overline{p}^*,m,w)} \\
&= \left(\int_0^\infty \left(\int_0^t \prod_{i=1}^n (\tilde{f}_{x_i})_\mu^*(s)^{\overline{p}^*\frac{A_i+1}{D}}ds\right)^{m/\overline{p}^*} w(t)\,dt\right)^{\frac{1}{m}} \\
&\leq \left(\int_0^\infty \prod_{i=1}^n \left(\int_0^t (\tilde{f}_{x_i})_\mu^*(s)^{p_i}ds\right)^{\frac{m}{p_i}\frac{A_i+1}{D}} w(t)\,dt\right)^{\frac{1}{m}} \\
&\leq \prod_{i=1}^n \left(\int_0^\infty \left(\left(\int_0^t (\tilde{f}_{x_i})_\mu^*(s)^{p_i}ds\right)^{\frac{m}{p_i}} w(t)\right)dt\right)^{\frac{1}{m}\frac{A_i+1}{D}} \\
&= \prod_{i=1}^n \left\|(\tilde{f}_{x_i})_\mu^*\right\|_{G\Gamma(p_i,m,w)}^{\frac{A_i+1}{D}} \\
&\leq \prod_{i=1}^n \|f_{x_i}\|_{G\Gamma(p_i,m,w)}^{\frac{A_i+1}{D}} \quad \text{(by Lemma 4.2)}.
\end{aligned}$$

Part $i)$ follows form $i)$ of Theorem 2.3, because

$$\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{G\Gamma(\overline{p}^*,m,w)} \simeq \|t^{-1/D}f_\mu^{**}(t)\|_{G\Gamma(\overline{p}^*,m,w)}$$

Part $ii)$ is consequence of $i)$ of Theorem 2.3.

Let us prove now $iii)$. If we denote $I := \|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{G\Gamma(\overline{p}^*,m,w)}$ using the method of part iii) of Theorem 4.9 we get

$$I \geq \left(\int_0^\infty [f_\mu^{**}(s)-f_\mu^*(s)]^m s^{-\frac{m}{D}} w(s)\,ds\right)^{1/m}.$$

Considering as weights

$$v(s) := s^{-\frac{m}{D}}w(s) \quad \text{and} \quad u(t) = \frac{\partial}{\partial t}\left(1+\int_t^1 \frac{\left(s^{-\frac{m}{D}}w(s)\right)^{\frac{-1}{m-1}}}{s^{\frac{m}{m-1}}}ds\right)^{1-m},$$

the result follows from Lemma 4.1. □

5. Optimality

In this section we discuss the optimality of the norms appearing in the inequalities presented.

Let $A=(A_1,A_2,\dots,A_n)$ be a non negative vector in $\mathbb{R}^n$, $D=A_1+A_2+\cdots+A_n+n$ and μ defined as in (3.1). Given $p\geq 1$, let us consider

$$A(p)=\left\{\hat q=(q_1,\cdots,q_n)\in\mathbb{R}^n: q_i\geq 1 \text{ for } i=1,\cdots,n \text{ and } \frac{1}{D}\sum_{i=1}^n\frac{A_i+1}{q_i}=\frac{1}{p}\right\}.$$

Given $\hat q\in A(p)$, with the same proof of Theorem 4.3, we obtain

$$\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\bar X^{(p)}}\leq c\prod_{i=1}^n\|f_{x_i}\|_{X^{(q_i)}}^{\frac{A_i+1}{D}}, \tag{5.1}$$

where $c=c(\overline{p},n,C_1)$ is a constant that just depends on p,n and C_1, the constant that appears in (1.3) with $p=1$.

Our aim will be to show that the oscillation space that appears on the left side of (5.1) is optimal, in the sense that if Y is an r.i. space on $(\mathbb{R}^n,\mu)$ such that that for all $\hat q\in A(p)$ we have that

$$\|f\|_Y\leq c\prod_{i=1}^n\|f_{x_i}\|_{X^{(q_i)}}^{\frac{A_i+1}{D}}, \tag{5.2}$$

where c is a constant that does not depend on $\hat q$, then the following embedding holds:

$$\|f\|_Y\preceq\|t^{-1/D}[f_\mu^{**}(t)-f_\mu^*(t)]\|_{\bar X^{(p)}}.$$

Moreover inequality (5.2) is equivalent to the boundedness of the Hardy type operator $\bar Q_Df(t)=\int_t^\infty s^{1/D}f(s)\frac{ds}{s}$ from $\bar X^{(p)}$ to $\bar Y$.

In what follows we denote

$$\mathbb{R}^n_*=\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n:\ x_i>0 \text{ if } A_i>0\}.$$

First of all we collect some useful properties that we need in the next lemma.

Lemma 5.1. *Let u be a Lipschitz continuous function in $\mathbb{R}^n_*$ with compact support in $\overline{\mathbb{R}^n_*}$. Then, there exists a radial rearrangement $u^\star$ of u such that*
i) $\mu(\{|u|>t\})=\mu\left(\{u^\star>t\}\right)$ for all $t>0$;
ii) $u^\star$ is radially decreasing;
iii) for every Young function Φ (i.e., convex and increasing function that vanishes at 0),

$$\int_{\mathbb{R}^n}\Phi\left(|\nabla u^\star|\right)d\mu(x)\leq\int_{\mathbb{R}^n}\Phi\left(|\nabla u|\right)d\mu(x); \tag{5.3}$$

iv) there is a positive constant[5] *C such that for every Young function Φ*

$$\int_{\mathbb{R}^n}\Phi\left(|\nabla u^\star|\right)d\mu(x)=\int_0^\infty\Phi\left(Cs^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right)ds. \tag{5.4}$$

Proof. Parts $i)$, $ii)$ and $iii)$ are proved in [12, Proposition 4.2]. Part $iv)$ is contained in the proof of the weighted version of a rearrangement inequality due to Talenti (see [34, Formula (3.8)]). □

[5] $C=D\mu^{\frac{1}{D}}(B_1^*)$, where $B_1^*=B_1(0)\cap\mathbb{R}^n_*$ (see [12]).

Theorem 5.2. *Let $A = (A_1, A_2, \dots, A_n)$ be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2 + \cdots + A_n + n$ and μ defined as in (3.1). Let X, Y be two r.i. spaces on $(\mathbb{R}^n, \mu)$ and $p \geq 1$. The next statements are equivalent:*
i) The Hardy type operator $\bar{Q}_D$ is bounded from $\bar{X}^{(p)}$ to $\bar{Y}$.
ii) There is a positive constant c, such that for all $\hat{q} \in A(p)$ and $u \in C_c^1\left(\overline{\mathbb{R}^n_*}\right)$ we get

$$\|u\|_Y \leq c \prod_{i=1}^n \|u_{x_i}\|_{X^{(q_i)}}^{\frac{A_i+1}{D}}. \tag{5.5}$$

iii) For all $u \in C_c^1\left(\overline{\mathbb{R}^n_*}\right)$ we get

$$\|u\|_Y \preceq \|t^{-1/D}[u_\mu^{**}(t) - u_\mu^*(t)]\|_{\bar{X}^{(p)}}.$$

Proof. *i) $\Rightarrow$ ii)*

Since u has compact support, we get that $u_\mu^{**}(\infty) = 0$, then

$$u_\mu^{**}(t) = \int_t^\infty \left(u_\mu^{**}(s) - u_\mu^*(s)\right) \frac{ds}{s} = \int_t^\infty s^{1/D} \left[s^{-1/D}\left(u_\mu^{**}(s) - u_\mu^*(s)\right)\right] \frac{ds}{s}.$$

Therefore

$$\begin{aligned} \|u\|_Y \leq \left\|u_\mu^{**}\right\|_{\bar{Y}} &= \left\|\int_t^\infty s^{1/D}\left[s^{-1/D}\left(u_\mu^{**}(s) - u_\mu^*(s)\right)\right]\frac{ds}{s}\right\|_{\bar{Y}} \\ &\preceq \|t^{-1/D}[u_\mu^{**}(t) - u_\mu^*(t)]\|_{\bar{X}^{(p)}} \end{aligned}$$

and (5.5) follows from (5.1).

ii) $\Rightarrow$ i)

Let us consider $u \in C_c^1\left(\overline{\mathbb{R}^n_*}\right)$ and let $u^\bigstar$ be the radial rearrangement of u given in Lemma 5.1. Since

$$\frac{1}{p} = \frac{1}{D}\sum_{i=1}^n \frac{A_i+1}{p}$$

it follows that $(p, \cdots, p) \in A(p)$, thus by hypothesis

$$\left\|u^\bigstar\right\|_Y \leq c\prod_{i=1}^n \left\|\left(u^\bigstar\right)_{x_i}\right\|_{X^{(p)}}^{\frac{A_i+1}{D}} \leq c\prod_{i=1}^n \left\|\left|\nabla u^\bigstar\right|\right\|_{X^{(p)}}^{\frac{A_i+1}{D}} = c\left\|\left|\nabla u^\bigstar\right|\right\|_{X^{(p)}}. \tag{5.6}$$

Since Φ is increasing (see [5, exercise 3 pag. 87]), we have that

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi\left(\left|\nabla u^\bigstar\right|\right) d\mu(x) &= \int_0^\infty \Phi\left(\left|\nabla u^\bigstar\right|_\mu^*(s)\right) ds \\ &= \int_0^\infty \Phi\left(Cs^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right) ds \quad \text{by (5.4),} \end{aligned}$$

which (see [5, exercise 5 pag. 88]) implies that

$$\int_0^t \left|\nabla u^\bigstar\right|_\mu^*(s) ds = \int_0^t \left(C\,(\cdot)^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(\cdot)\right)\right)^* ds. \tag{5.7}$$

Taking into account that u has compact support, we can write

$$u_\mu^*(t) = \int_t^\infty -\frac{\partial u_\mu^*}{\partial s}(s) ds = \int_t^\infty \frac{s^{1/D-1}}{C}\left[Cs^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right] ds.$$

Consequently, using (5.6), (5.7) and (2.3) we get

$$\left\|\int_t^\infty \frac{s^{1/D-1}}{C}\left[Cs^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right]ds\right\|_{\bar{Y}} = \|u_\mu^*\|_{\bar{Y}} = \|u\|_Y \preceq \left\||\nabla u^\bigstar|\right\|_{X^{(p)}}$$
$$= \left\||\nabla u^\bigstar|_\mu^*\right\|_{\bar{X}^{(p)}} = \left\|Cs^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right\|_{\bar{X}^{(p)}}.$$

In summary, we have proved that,

$$\left\|\int_t^\infty s^{1/D-1}\left[s^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right]\right\|_{\bar{Y}} \preceq \left\|s^{1-1/D}\left(-\frac{\partial u_\mu^*}{\partial s}(s)\right)\right\|_{\bar{X}^{(p)}}. \tag{5.8}$$

Now let h be a positive measurable function with compact support on $(0,\infty)$ and let us consider

$$g(t) = \int_t^\infty s^{1/D-1}h(s)ds.$$

The function g is decreasing, positive and continuous on $(0,\infty)$, hence there is a function u defined in $\mathbb{R}_*^n$ (see [5, Corollary 7.8 page 86]) such that $u_\mu^* = g$, hence

$$-\frac{\partial u_\mu^*(s)}{\partial s} = s^{1/D-1}h(s). \tag{5.9}$$

Replacing (5.9) in (5.8) we obtain

$$\left\|\int_t^\infty s^{1/D-1}h(s)\right\|_{\bar{Y}} \preceq \|h\|_{\bar{X}^{(p)}},$$

i.e. the operator $\bar{Q}_D$ is bounded from $\bar{X}^{(p)}$ to $\bar{Y}$.

ii)⇔ *iii*)

The equivalence between *ii*) and *iii*) is given by Lemma 2.4. □

Remark 5.3. *The optimality result presented in Theorem 5.2, can be easily adapted to inequalities that involve the spaces presented in sections 4.1.2, 4.2, 4.3 and 4.4.*

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