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ALGEBRAIC AND TOPOLOGICAL CLASSIFICATION OF HOMOGENEOUS QUARTIC VECTOR FIELDS IN THE PLANE

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Abstract. We provide canonical forms for the homogeneous polynomials of degree five. Then we characterize all the phase portraits in the Poincaré disk for all quartic homogeneous polynomial differential systems. More precisely, there are exactly 24 different topological phase portraits for the quartic homogeneous polynomial differential systems.

1. Introduction

We consider a family of polynomial vector fields in the plane of the form

$$\dot{x} = P(x,y), \ \ \dot{y} = Q(x,y), \tag{1}$$

where P and Q are homogeneous polynomials of degree four (shortly, they will be called *quartic systems*). This work is divided in two parts. First we are going to give all the possible canonical forms for the homogeneous polynomials of degree five, and secondly, we will characterize all the phase portraits in the Poincaré disk of all homogeneous quartic polynomial differential systems (1). For a definition of the Poincaré disk and the local charts we are going to work, see for instance Chapter 5 of [7].

In general polynomials vector fields are a current topic of research (see for instance [2, 5, 7, 8]). The study of homogenenous polynomial vector fields was initiated by Markus [10] in 1960. He gave a classification for quadratic homogeneous vector fields $\mathcal{X} = (P, Q)$ with P and Q with no common factor. Argemí [1] completed the classification of Markus in 1968, and provided the classification of cubic homogeneous vector fields that have no common factor. Furthermore, for planar homogeneous polynomial vector fields of degree m that have no common factor Argemí gave upper and lower bounds of the numbers of phase portraits.

An algebraic classification of the differential systems (1) when P and Q are homogeneous polynomials of degree 2 was given by Date and Iri in [6]. Their classification of linear binary and cubic forms was obtained using the theory of algebraic invariants (according to Gurevich [9] a binary form can be seen as an homogeneous polynomial in two variables). Gurevich [9] did the classification for third and fourth-order binary forms on the field of complex numbers, and Cima and Llibre in [3] obtained a classification of the fourth-order binary in the real domain using Caley's method. In that paper we find an algebraic classification of homogeneous systems of degree three, and Collins [4] extended this to homogeneous polynomial vector field of degree m.

Our classification of all the possible canonical forms for homogeneous polynomials of degree 5 is based in the results of [3], where the classification of quartic binary forms in the real domain was given. In fact, since a quintic polynomial has a real root we look a homogeneous polynomial of degree five as the product of a linear factor and a homogeneous polynomial of degree 4. The classification of the canonical forms for the homogeneous vector fields (1) of degree 4 will be based in the canonical forms of the homogeneous polynomial of degree 5. Our first result in this direction is the following.

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Proposition 1. *For each real quintic binary form*

$$f(x,y) = a_0x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4 + a_5y^5, \tag{2}$$

there exists some $\sigma \in GL(2,\mathbb{R})$ which transforms f in one and only one of the following canonical forms:

(i) $\alpha(bx+cy)(x^4+ax^2y^2+y^4)$ *with* $a > -2$,
(ii) $\alpha(bx+cy)(x^2+y^2)^2$,
(iii) $\alpha y^3(x^2+y^2)$,
(iv) $\alpha(bx+cy)y^2(x^2+y^2)$,
(v) $(bx+cy)(x^4+ax^2y^2-y^4)$,
(vi) αx^5,
(vii) $\alpha(bx+cy)x^4$,
(viii) αx^3y^2,
(ix) $\alpha y^3(x^2-y^2)$,
(x) $\alpha(bx+cy)x^2y^2$,
(xi) $\alpha(bx+cy)y^2(x^2-y^2)$,
(xii) $(bx+cy)(x^4+ax^2y^2+y^4)$, *with* $a < -2$,
(xiii) $\left(x-\beta_-^-\, y\right)^i \left(x+\beta_-^-\, y\right)^j \left(x-\beta_+^-\, y\right)^k \left(x+\beta_+^-\, y\right)^l$ ***for*** $a < -2$, *with* $i+j+k+l = 5$ *and* $i,j,k,l = \{1,2\}$,
(xiv) $(x-\beta^+y)^i\,(x+\beta^+y)^j\left(x^2-(\beta^+)^2\, y^2\right)$, *with* $i+j = 3$ *and* $i,j = \{1,2\}$

where $\alpha = \pm 1$, $\beta_\pm^- = \sqrt{(-a \pm \sqrt{a^2-4})/2}$, $\beta^+ = \sqrt{(-a-\sqrt{a^2+4})/2}$ *and* $a,b,c \in \mathbb{R}$.

The proof of this proposition will be given in Section 3.

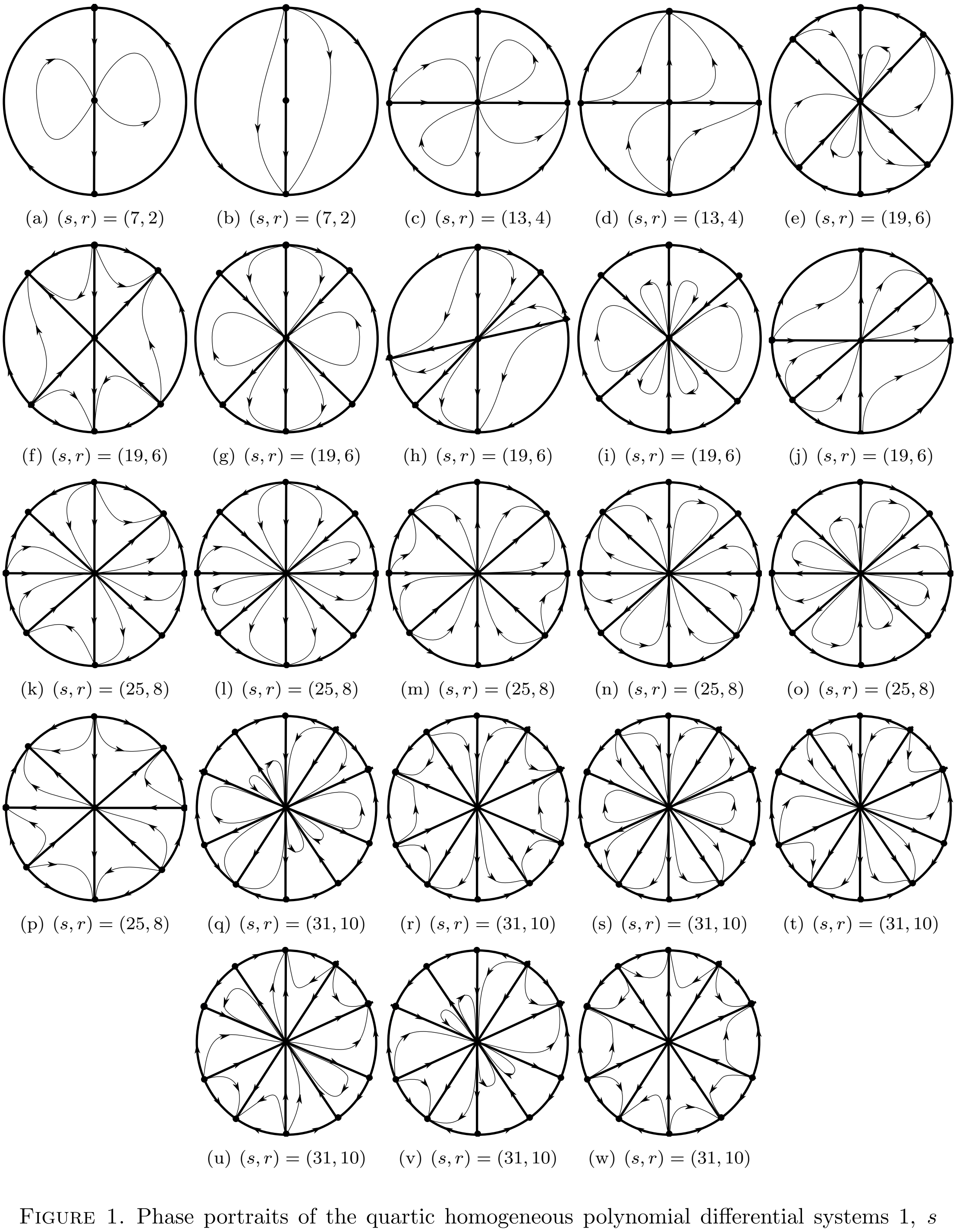
Using the classification of the binary forms of degree five, as in the previous proposition, we study the global phase portraits of all quartic homogeneous polynomial differential systems (1), and our main result is the following.

Theorem 2. *Let $X = (P,Q)$ be a quartic homogeneous polynomial vector field in the plane. Assume that P and Q have no common factors. Then the phase portrait of X is topologically equivalent to one of the* 23 *phase portraits of Figure 1.*

In Section 5 the proof of this theorem is given. The principal idea is to study the associated function $F = xQ - yP$ and the invariant straight lines, in order to analize the infinite equilibrium points and their stability, the infinite equilibrium points determine the phase portrait of the systems $X = (P,Q)$ as we will see in subsection 2.2. We separate the proof in five cases according to the number of straight lines of F, this number determines the number of separatrices s and of the canonical regions r of the phase portraits.

We must mention that all the topologically different phase portraits for the quadratic homogeneous polynomial differential systems were given by [6] and [15]. While all the topologically different phase portraits for the cubic homogeneous polynomial differential systems were classified in [3].

This work is organized as follow. In Section 2 we present basic definitions and results necessary to prove Proposition 2. In Section 3 the proof of Proposition 1 is given. For this purpose we consider a quintic polynomial as the product of two factors, one of order one and the other of order four. Section 4 is dedicated to point out the algebraic classification of the homogeneous quartic vector fields, here we present all the homogeneous systems of degree four taking into account the forms of the polynomials of degree five given in Porposition 1. Finally in Section 5, the proof of Theorem 2 is given, studying the phase portraits associated to the algebraic classification of the homogeneous quartic vector fields given in Section 4.



(a) $(s, r) = (7, 2)$ (b) $(s, r) = (7, 2)$ (c) $(s, r) = (13, 4)$ (d) $(s, r) = (13, 4)$ (e) $(s, r) = (19, 6)$

(f) $(s, r) = (19, 6)$ (g) $(s, r) = (19, 6)$ (h) $(s, r) = (19, 6)$ (i) $(s, r) = (19, 6)$ (j) $(s, r) = (19, 6)$

(k) $(s, r) = (25, 8)$ (l) $(s, r) = (25, 8)$ (m) $(s, r) = (25, 8)$ (n) $(s, r) = (25, 8)$ (o) $(s, r) = (25, 8)$

(p) $(s, r) = (25, 8)$ (q) $(s, r) = (31, 10)$ (r) $(s, r) = (31, 10)$ (s) $(s, r) = (31, 10)$ (t) $(s, r) = (31, 10)$

(u) $(s, r) = (31, 10)$ (v) $(s, r) = (31, 10)$ (w) $(s, r) = (31, 10)$

FIGURE 1. Phase portraits of the quartic homogeneous polynomial differential systems 1, s denotes the number of separatrices and r the number of canonical regions.

2. Preliminaries

In order to give a detailed proof of Theorem 2 we give some definitions and results that will be useful.

2.1. Phase portraits in the Poincaré disk.

Let $p(X)$ denotes the compactified vector field of the polynomial vector field X in the Poincaré disk $\mathbb{D}^2$, see for more details Chapter 5 of [7]. In this subsection we shall see how to characterize the phase portrait of the compactified vector field $p(X)$ in the Poincaré disk.

A *separatrix* of $p(X)$ being X a polynomial vector field defined in the whole $\mathbb{R}^2$ is an orbit which is either an equilibrium point, or a trajectory which lies in the boundary of a hyperbolic sector of a finite, or an infinite equilibrium point, or any orbit contained at the infinity of the Poincaré disk, or a limit cycle. Neumann [12] proved that the set formed by all separatrices of $p(X)$, denoted by $S(p(X))$ is closed.

The open connected components of $\mathbb{D}^2 \setminus S(p(X))$ are called canonical regions of X or of $p(X)$. A *separatrix configuration* is the union of $S(p(X))$ plus one solution chosen in each canonical region. Two separatrix configurations $S(p(X))$ and $S(p(\mathcal{Y}))$ are *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(X))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [11], Neumann [12] and Peixoto [13], who found it independently.

Theorem 3. *The phase portraits in the Poincaré disk $\mathbb{D}^2$ of two compactified polynomial vector fields $p(X)$ and $p(\mathcal{Y})$ are topologically equivalent, if and only if, their separatrix configurations $S(p(X))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

2.2. General results for homogeneous polynomial differential systems.

If the polynomial differential system (1) is homogeneous of degree 4, then the results of section 4 in [3] can be applied. Next, there are presented some results of [3] that we shall use.

For this purpose, consider system (1) and let $F(x, y) = xQ(x, y) - yP(x, y)$. To study the infinite critical points of X we consider the induced vector field $p(X)$ on the Poincaré two-sphere.

The Poincaré compactification permits to study the dynamics in a neighborhood of infinity, and to describe the compactified vector field in local coordinates, we consider the maps $\phi_i : U_i \longrightarrow \mathbb{R}^2$ and $\varphi_i : V_i \longrightarrow \mathbb{R}^2$ where $U_i = \{y \in S^2/y_i > 0\}$, $V_i = \{y \in S^2/y_i < 0\}$ $(i = 1, 2, 3)$ and $\phi_i(y) = \varphi_i(y) = \left(\frac{y_j}{y_i}, \frac{y_k}{y_i}\right)$, with $i, j, k = 1, 2, 3$; $j < k$. We will denote by $z = (z_1, z_2)$ the value of $\phi_i(y)$ or $\varphi_i(y)$ for any i, so that z represents different things according to the local chart under consideration. Making straightforward computations we arrive to the final expression for the vector field on U_1

$$\frac{z_2^\kappa}{\Delta(z)^{\kappa-1}} \left(Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) \right) \tag{3}$$

analogously, on U_2 we have

$$\frac{z_2^\kappa}{\Delta(z)^{\kappa-1}} \left(P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right)$$

and finally, on U_3

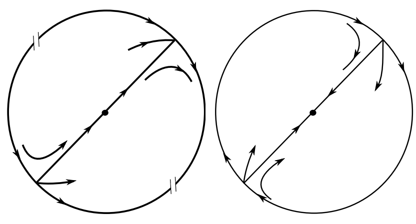
$$(z_2^\kappa)/(\Delta(z)^{\kappa-1})\ (P(z_1, z_2), Q(z_1, z_2)),$$

where κ is the maximum of the degrees of P and Q, of course in this work $\kappa = 4$. For the local charts V_i for $i = 1, 2, 3$ we obtain the same expressions (3), (2.2), (2.2) but multiplied by (-1), respectively.

Proposition 4. *Let $X = (P, Q)$ be a homogeneous polynomial vector field in the plane with degree $(P) =$ degree$(Q) = n$ and assume that P and Q have no common factors. Assume that $F(x, y) = xQ(x, y) - yP(x, y)$ has some real linear factor. Then the following holds.*

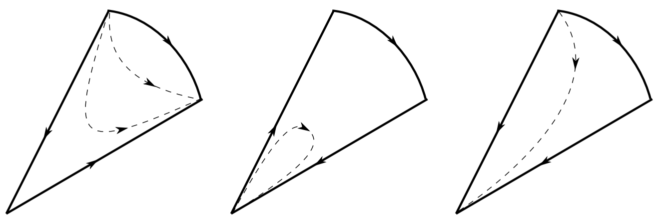
(a) *The linear factor $ax + by$ of $F(x, y)$ provides the invariant straight line $ax + by = 0$ for the flow of X.*

(b) *X has no limit cycles.*
(c) *The singular points at infinity are all elemental and they are nodes, saddles, or saddles-nodes. An infinite singular point on the local chart U_1, $(z_1, z_2) = (\lambda_i, 0)$ shall be a saddle-node if and only if λ_i is a root of $f(\lambda) = F(1, \lambda) = Q(1, \lambda) - \lambda P(1, \lambda)$ of even multiplicity. Furthermore, the orbits in the Poincaré disc near a saddle-node are drawn in Fig. 2.*
(d) *The behavior of the flow of $p(X)$ in a neighborhood of infinity determines the phase portrait of X (Fig. 3 shows the possible behavior at infinity between two consecutive invariant rays of X).*



(a) (b)

FIGURE 2. Behavior of the orbits of $p(X)$ near a saddle-node at infinity (we can reverse the orientation of the orbits). (a): n even, (b): n odd.



(a) (b) (c)

FIGURE 3. The behaviour in a neighbourhood of the infinity determines the phase portrait of X.

Furthermore, let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the real roots of $f(\lambda) = 0$. By the Poincaré compactification (see [7, 8, 14]) and considering (3), $(z_1, z_2) = (\lambda_i, 0)$ are the singular points of $p(X)$ in the local chart U_1. For the local chart U_2 we consider $g(\lambda) = F(\lambda, 1)$ and the unique singular point of interest is the origin $(0, 0)$ of the local chart U_2. The linear parts at these singular points are

$$\begin{pmatrix} f'(\lambda_i) & * \\ 0 & -P(1, \lambda_i) \end{pmatrix} \text{ or } \begin{pmatrix} g'(0) & * \\ 0 & -Q(0, 1) \end{pmatrix}, \tag{4}$$

respectively. Then the sign of the product of their eigenvalues determines if a hyperbolic infinite singular point is a saddle (negative) or a node (positive).

3. PROOF OF PROPOSITION 1

In order to provide the canonical forms of the binary forms of degree five we explicit the result given in Theorem 2.6 of [3] for the binary forms of degree four in the real domain. Let

$$f_4(x, y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4.$$

be a homogeneous polynomial of degree four with $a_j \in \mathbb{R}$.

Theorem 5. *For each real fourth-order binary form f_4 there exists some $\sigma \in GL(2,\mathbb{R})$ which transform f_4 in one and only one of the following canonical forms:*

(i) $x^4 + 6\mu x^2y^2 + y^4$ *with* $\mu < -1/3$,
(ii) $\alpha(x^4 + 6\mu x^2y^2 + y^4)$ *with* $\mu > -1/3$,
(iii) $x^4 + 6\mu x^2y^2 - y^4$,
(iv) $\alpha y^2(6x^2 + y^2)$,
(v) $\alpha y^2(6x^2 - y^2)$,
(vi) $\alpha(x^2 + y^2)^2$,
(vii) $6\alpha x^2y^2$,
(viii) $4x^3y$,
(ix) αx^4,
(x) 0,

where $\alpha = \pm 1$.

We note that in order to simplify our study the numerical coefficients 6 and 4 in any of the previous canonical binary forms can be eliminated easily. For example for the canonical form (viii) we do the change of variables $(x,y) \to (X, Y/4)$ then we have X^3Y. The canonical binary forms of degree four that we consider are given in the next corollary.

Corollary 6. *For each non null real fourth-order binary form f_4, there exists some $\sigma \in GL(2,\mathbb{R})$ which transforms f in one and only one of the following simplified canonical forms:*

(i) $x^4 + ax^2y^2 + y^4$ *with* $a < -2$,
(ii) $\alpha(x^4 + ax^2y^2 + y^4)$ *with* $a > -2$,
(iii) $x^4 + ax^2y^2 - y^4$,
(iv) $\alpha y^2(x^2 + y^2)$,
(v) $\alpha y^2(x^2 - y^2)$,
(vi) $\alpha(x^2 + y^2)^2$,
(vii) αx^2y^2,
(viii) x^3y,
(ix) αx^4,
(x) 0,

where $\alpha = \pm 1$.

Since we are interested in the study of polynomials differential systems of degree 4, it is clear that we do not need to consider the binary form identically null.

Proof of Proposition 1. Consider the homogeneous polynomial of degree 5 given in (2). Since the polynomial is of odd degree, we have that f always has a real linear factor, then we can write f as a product of a quartic homogeneous polynomial with a homogeneous polynomial of degree one, i.e. there exists a linear transformation σ such that $f(\sigma(x,y)) = (\tilde{a}x + \tilde{b}y)f_4(x,y)$, where f_4 is a homogeneous polynomial of degree four. Now, for the quartic factor we consider the canonical forms of the homogeneous polynomials degree 4 given in Corollary 6. For each canonical form of degree four we add the linear factor to get the canonical forms of degree five, we need to put attention in the nature of this linear factor in the sense that if this factor coincides or not with one of the factors that the quartic canonical form can have.

If the quartic factor presents the canonical polynomial form $x^4+ax^2y^2+y^4$ with $a < -2$, then the associated quintic canonical form is given by $(bx+cy)(x^4+ax^2y^2+y^4)$ with $a < -2$ and $(b,c) \neq \left\{m\left(1, \pm\beta_{\pm}^{-}\right) : m \in \mathbb{Z}\right\}$ if the linear factor does not coincide with the factor of the quartic form, or in the opposite case the quintic form will be $\left(x - \beta_{-}^{-}\ y\right)^i \left(x + \beta_{-}^{-}\ y\right)^j \left(x - \beta_{+}^{-}\ y\right)^k (x + \beta_{+}^{-}\ y)^l$ with $i+j+k+l = 5$ and $i,j = \{1,2\}$.

For the quartic factor $\alpha(x^4 + ax^2y^2 + y^4)$ with $a > -2$, the associated quintic canonical form is $\alpha(bx + cy)(x^4 + ax^2y^2 + y^4)$ with $a > -2$, (note that the quartic form has not real roots, so the linear factor added can not coincide with a factor of this quartic form).

In the case when the quartic factor can be put in the canonical form $x^4+ax^2y^2-y^4$, the quintic canonical form associated is $(bx+cy)(x^4+ax^2y^2-y^4)$ with $(b,c) \neq \{m(1,\pm\beta^+) : m \in \mathbb{Z}\}$ if the linear factor do not coincide with the factor of the quartic form, or for the opposite case the quintic will be $(x-\beta^+\ y)^i\ (x+\beta^+\ y)^j(x^2+(\beta^+)^2\, y^2)$ with $i+j = 3$ and $i,j = \{1,2\}$.

From (iv) when the quartic factor has the form $\alpha y^2(x^2+y^2)$, we have two different quintic canonical forms associated which depends on the additional simple root of the polynomial of degree five. In fact, if it coincides or not with the real root of the quartic form. Thus the canonical forms are $\alpha(bx+cy)y^2(x^2+y^2)$ with $b\neq 0$ or $\alpha y^3(x^2+y^2)$. Note that the number and multiplicity of the roots of $\alpha(bx+cy)y^2(x^2+y^2)$ with $b\neq 0$ coincide with the two polynomials given in the previous case whose roots are two complex roots, one simple root and one root with multiplicity two. However it is easily verified that it does not exist a linear transformation which allows to transform one of these binary forms into the other.

For the quartic binary form $\alpha y^2(x^2-y^2)$, we have associated four homogeneous polynomials of degree 5 given by $\alpha(bx+cy)y^2(x^2-y^2)$ with $(b,c)\neq m(b^*,c*)$ with $(b^*,c^*)\in\{(0,1),(1,1),(1,-1)\}$ for $m\in\mathbb{R}$ when the linear factor $(bx+cy)$ does not coincide with the factors of the quartic form, in the case when the linear factor $(bx+cy)$ coincides with one of the factor of the quartic canonical form we have the forms $\alpha y^3(x^2-y^2)$, $\alpha y^2(x+y)^2(x-y)$ and $\alpha y^2(x+y)(x-y)^2$. Note that the form $\alpha(bx+cy)y^2(x^2-y^2)$ has three real roots with multiplicity one and one real root of the polynomial, $\left(x-\beta_-^-\ y\right)^i\left(x+\beta_-^-\ y\right)^j\left(x-\beta_+^-\ y\right)^k\left(x+\beta_+^-\ y\right)^l$ with $i+j+k+l=5$ and $i,j,k,l=\{1,2\}$. However it is not possible to do a linear transformation between these quintic forms with the same type of roots.

When the quartic factor has two complex roots with multiplicity two, i.e. it is of the form $\alpha(x^2+y^2)^2$, then the quintic polynomial canonical form is $\alpha(bx+cy)(x^2+y^2)^2$.

If the quartic factor has the form αx^2y^2, then in the associated quintic form the real root of the factor of degree one can coincides or not with one of the roots of the quartic part. If it coincides with one of the roots of αx^2y^2, we get the quintic canonical form αx^3y^2, and if the linear factor has a different root, we have the quintic form $\alpha(bx+cy)x^2y^2$. This last form has the same properties of the roots that the two forms $\alpha y^2(x+y)^2(x-y)$ and $\alpha y^2(x+y)(x-y)^2$. We can apply to the form $\alpha y^2(x+y)(x-y)^2$ the change of variables $(x,y)\to(a_1X, a_3X+a_4Y)$ with $a_1=(b^3/(2^3c^2))^{1/5}$, $a_3=-a_1$ and $a_4=-(2^2c^3/b^2)^{1/5}$ (respectively $a_1=(b^3/(2^3c^2))^{1/5}, a_3=a_1$ and $a_4=(2^2c^3/b^2)^{1/5} respectively$) to obtain the quintic form $\alpha(bX+cY)X^2Y^2$ ($\alpha y^2(x+y)^2(x-y)$ respectively). We will consider the form $\alpha(bx+cy)x^2y^2$ as representative of this class.

If we are in the case where the quartic factor is of the form x^3y, then a new quintic canonical form appears when the real root of the polynomial of degree one coincides with the triple root of the quartic factor and it provides the quintic canonical form x^4y. Other quintic homogeneous polynomials with this quartic factor are $(bx+cy)x^3y$ and x^3y^2, but these last forms have the same properties that the forms previously presented: $\alpha y^3(x^2-y^2)$ and αx^3y^2 respectively. We choose to consider the forms $\alpha y^3(x^2-y^2)$ and αx^3y^2 as representatives of each class. Note that if for $\alpha y^3(x^2-y^2)$ we apply the change of variables $(x,y)\to(a_1X, a_3X+a_4Y)$ with $a_1=-(b^2/(2^2c))^{1/5}, a_3=-a_1$ and $a_4=-(2^3c^4/b^3)^{1/5}$ to obtain $(bX+cY)X^3Y$. And, for αx^3y^2 we can, without lost of generality, assume that $\alpha=1$ (doing $x\to x/\sqrt[3]{\alpha}$).

Finally for the last non null polynomial of degree four in Corollary 6, we note that it has a real root of multiplicity four, i.e. of the form αx^4, we have a unique new quintic canonical form given when the real root of the polynomial of degree one coincides with the previous one and we get αx^5, and the generic case $\alpha(bx+cy)x^4$ is related with the previous x^4y. We consider the most general form $\alpha(bx+cy)x^4$ with $c\neq 0$ as representative of this class.

This completes the proof of Proposition 1. □

4. Algebraic classification of homogeneous quartic vector fields

In this section we obtain the algebraic classification of homogeneous polynomials systems $\dot{x}=P(x,y), \dot{y}=Q(x,y)$ of degree 4. For an arbitrary system $X=(P,Q)$ with P and Q homogeneous polynomials of degree 4, we can know through the algebraic characteristics the equivalence-class at which it belongs.

Let $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ be the system of differential equations associated to the vector field $X = (P,Q)$, i.e.

$$\begin{array}{llll} (5) & \dot{x} & = P(x,y) & = P_{10}x^4 + 4P_{11}x^3y + 6P_{12}x^2y^2 + 4P_{13}xy^3 + P_{14}y^4 \\ & \dot{y} & = Q(x,y) & = Q_{10}x^4 + 4Q_{11}x^3y + 6Q_{12}x^2y^2 + 4Q_{13}xy^3 + Q_{14}y^4. \end{array}$$

According to Proposition 4, in this case F has the form:

$$\begin{array}{lrl} (6) & F(x,y) = & xQ(x,y) - yP(x,y) \\ & = & Q_{10}x^5 + (Q_{11} - Q_{10})x^4y + (Q_{12} - P_{11})x^3y^2 + (Q_{13} - P_{12})x^2y^3 + \\ & & (Q_{14} - P_{13})xy^4 - P_{14}y^5, \end{array}$$

From [8] and [14] it is known that the only possible directions at which the orbits of X go to or go back from infinity are determined by the real linear factor of the homogeneous polynomial $F(x,y)$.

Lemma 7. *Let $X = (P,Q)$ be a homogeneous quartic vector field on the plane as in* (5)*. If*

$$F(x,y) = a_0x^5 + 5a_1x^4y + 10a_2x^3y^2 + 10a_3x^2y^3 + 5a_4xy^4 + a_5y^5, \tag{7}$$

then there exist constants $p_1, p_2, p_3, p_4 \in \mathbb{R}$ such that system (5) takes the form

$$\begin{array}{lll} (8) & \dot{x} & = (p_1 - a_1)x^4 + (p_2 - 4a_2)x^3y + (p_3 - 6a_3)x^2y^2 + (p_4 - 4a_4)xy^3 - a_5y^4, \\ & \dot{y} & = a_0x^4 + (4a_1 + p_1)x^3y + (6a_2 + p_2)x^2y^2 + (4a_3 + p_3)xy^3 + (a_4 + p_4)y^4. \end{array}$$

Proof. From (6) and comparing with (7), we arrive to

$$\begin{array}{lll} a_0 = Q_{10}, & a_1 = (4Q_{11} - P_{10})/5, & a_2 = (6Q_{12} - 4P_{11})/10, \\ a_3 = (4Q_{13} - 6P_{12})/10, & a_4 = (Q_{14} - 4P_{13})/5, & a_5 = -P_{14}. \end{array}$$

Substituting a_j in (8) and comparing with (5) we obtain

$$p_1 = \frac{4}{5}(P_{10} + Q_{11}), \quad p_2 = \frac{12}{5}(P_{11} + Q_{12}), \quad p_3 = \frac{12}{5}(Q_{13} + P_{12}), \quad p_4 = \frac{4}{5}(P_{13} + Q_{14}).$$

Therefore, system (5) can be expressed as in (8). □

Since our objective is to classify all the phase portrait in the Poincaré disk of the quartic homogeneous polynomial vector fields in (5), the next result provides the canonical forms of this class of vector fields.

Theorem 8. *For each homogeneous vector field $X = (P,Q)$ of degree 4, there exists some $\sigma \in GL(2,\mathbb{R})$ and a change of time scale which transforms X in one and only one of the following canonical forms:*

$$
\begin{aligned}
\textit{(I)}\quad \dot{x} &= (p_1-\alpha c/5)x^4+(p_2-2ab\alpha/5)x^3y+(p_3-3\alpha ac/5)x^2y^2+(p_4-4b\alpha/5)xy^3-\alpha cy^4,\\
\dot{y} &= \alpha bx^4+(p_1+4\alpha c/5)x^3y+(3ab\alpha/5+p_2)x^2y^2+(p_3+2\alpha ac/5)xy^3+(\alpha b/5+p_4)y^4,\quad a<-2;\\
\textit{(II)}\quad \dot{x} &= (p_1-\alpha c/5)x^4+(p_2-4\alpha b/5)x^3y+(p_3-6\alpha c/5)x^2y^2+(p_4-4\alpha b/5)xy^3-\alpha cy^4,\\
\dot{y} &= \alpha bx^4+(p_1+4\alpha c/5)x^3y+(6\alpha b/5+p_2)x^2y^2+(p_3+4\alpha c/5)xy^3+(\alpha b/5+p_4)y^4;\\
\textit{(III)}\quad \dot{x} &= p_1x^4+p_2x^3y+(p_3-3\alpha/5)x^2y^2+p_4xy^3-\alpha y^4,\\
\dot{y} &= p_1x^3y+p_2x^2y^2+(2\alpha/5+p_3)xy^3+p_4y^4;\\
\textit{(IV)}\quad \dot{x} &= p_1x^4+(p_2-2\alpha b/5)x^3y+(p_3-3\alpha c/5)x^2y^2+(p_4-4\alpha b/5)xy^3-\alpha cy^4,\\
\dot{y} &= p_1x^3y+(3\alpha b/5+p_2)x^2y^2+(p_3+2\alpha c/5)xy^3+(\alpha b/5+p_4)y^4;\\
\textit{(V)}\quad \dot{x} &= (p_1-c/5)x^4+(p_2-2ab/5)x^3y+(p_3-3ac/5)x^2y^2+(p_4+4b/5)xy^3+cy^4,\\
\dot{y} &= bx^4+(p_1+4c/5)x^3y+(3ab/5+p_2)x^2y^2+(p_3+2ac/5)xy^3+(p_4-b/5)y^4;\\
\textit{(VI)}\quad \dot{x} &= p_1x^4+p_2x^3y+p_3x^2y^2+p_4xy^3,\\
\dot{y} &= \alpha x^4+p_1x^3y+p_2x^2y^2+p_3xy^3+p_4y^4;\\
\textit{(VII)}\quad \dot{x} &= (p_1-\alpha c/5)x^4+p_2x^3y+p_3x^2y^2+p_4xy^3,\\
\dot{y} &= \alpha bx^4+(p_1+4\alpha c/5)x^3y+p_2x^2y^2+p_3xy^3+p_4y^4;\\
\textit{(VIII)}\quad \dot{x} &= p_1x^4+(p_2-2\alpha/5)x^3y+p_3x^2y^2+p_4xy^3,\\
\dot{y} &= p_1x^3y+(3\alpha/5+p_2)x^2y^2+p_3xy^3+p_4y^4;\\
\textit{(IX)}\quad \dot{x} &= p_1x^4+p_2x^3y+(p_3-3\alpha/5)x^2y^2+p_4xy^3+\alpha y^4,\\
\dot{y} &= p_1x^3y+p_2x^2y^2+(2\alpha/5+p_3)xy^3+p_4y^4;\\
\textit{(X)}\quad \dot{x} &= p_1x^4+(p_2-2\alpha b/5)x^3y+(p_3-3\alpha c/5)x^2y^2+p_4xy^3,\\
\dot{y} &= p_1x^3y+(3\alpha b/5+p_2)x^2y^2+(p_3+2\alpha c/5)xy^3+p_4y^4;\\
\textit{(XI)}\quad \dot{x} &= p_1x^4+(p_2-2\alpha b/5)x^3y+(p_3-3\alpha c/5)x^2y^2+(p_4+4\alpha b/5)xy^3+\alpha cy^4,\\
\dot{y} &= p_1x^3y+(3\alpha b/5+p_2)x^2y^2+(p_3+2\alpha c/5)xy^3+(p_4-\alpha b/5)y^4;\\
\textit{(XII)}\quad \dot{x} &= (p_1-c/5)x^4+(p_2-2ab/5)x^3y+(p_3-3ac/5)x^2y^2+(p_4-4b/5)xy^3-cy^4,\\
\dot{y} &= bx^4+(p_1+4c/5)x^3y+(3ab/5+p_2)x^2y^2+(p_3+2ac/5)xy^3+(b/5+p_4)y^4,\ \ a>-2;\\
\textit{(XIII)}\quad \dot{x} &= xy^3\left(p_4-4b(\beta_+^-)^2(\beta_-^-)^2/5\right)+x^4\left(\beta_-^-/5+p_1\right)+x^3y\left(p_2-2/5\left(-(\beta_-^-)^2-(\beta_+^-)^2\right)\right)+\\
&\quad x^2y^2\left(p_3-3/5\left((\beta_-^-)^3+(\beta_-^-)(\beta_+^-)^2\right)\right)+(\beta_-^-)^3(\beta_+^-)^2y^4,\\
\dot{y} &= y^4\left(b(\beta_+^-)^2(\beta_-^-)^2/5+p_4\right)+x^3y\left(p_1-4\beta_-^-/5\right)+x^2y^2\left(3/5\left(-(\beta_-^-)^2-(\beta_+^-)^2\right)+p_2\right)+\\
&\quad xy^3\left(2/5\left((\beta_-^-)^3+(\beta_-^-)(\beta_+^-)^2\right)+p_3\right)+x^4;\\
\textit{(XIV)}\quad \dot{x} &= x^2y^2\left(6(\beta^+)^3c/5+p_3\right)+x^4\left(p_1-\beta^+/5\right)+x^3y\left((4(\beta^+)^2)/5+p_2\right)+xy^3\left(p_4-4(\beta^+)^4/5\right)+\\
&\quad (\beta^+)^5\left(-y^4\right),\\
\dot{y} &= xy^3\left(-4(\beta^+)^3c/5\right)+x^3y\left(4\beta^+/5+p_1\right)+x^2y^2\left(p_2-6(\beta^+)^2/5\right)+y^4\left((\beta^+)^4/5+p_4\right)+x^4
\end{aligned}
$$

where $\alpha=\pm1$.

Proof. Applying the canonical forms of the quintic homogeneous polynomials (i)-(xiv) given in Proposition 1 to system (8) defined in Lemma 7 we arrive to systems (I)-(XIV). Systems $(XIII)$ and (XIV) consider the exponent $i=2$ for the canonical forms $(xiii)$ and (xiv) in Proposition 1.

Note that system (XII) coincides with system (I) for $\alpha=1$ except for the intervals of the parameter a. Systems (II) can be obtained from system (I) with $a=2$. □

5. Proof of Theorem 2

In this section we characterize the global phase portraits on the Poincaré disk of all homogeneous polynomial differential systems of degree four of system (1), given in Theorem 8. Our analysis of the phase portraits depends essentially on the number and type of real roots of the quintic binary forms associated to each system $(I)--(XIV)$. Note that we do not need to study systems $(XIII)$ and (XIV) because their analysis is the same than for the systems (XI) and (IV), respectively. On the other hand, system (II) is equals to systems (I) for $a = 2$, thus is not necessary study systems (II), is enough consider the analysis for system I considering $a = 2$.

For each system given in Theorem 8 the origin is an equilibrium point. Note that each invariant straight line of system (8) divides the Poincaré disk in two regions, then if $F(x, y)$ has k invariant real linear factors, then the Poincaré disk has $2k$ canonical regions. We separate the study according to the number of invariant real linear factors of the homogeneous polynomial (6). In each case, we present all the possibilities following the clockwise order in which they appear at infinity. For example if we have eight infinite equilibria, we need only to characterize four consecutive equilibria, because the others four are the antipodal points and have the same local phase portrait. Here, we introduce the notation $s - s - sn - n$ for the four consecutive equilibria, meaning that the first equilibrium point is a saddle, the second is a saddle, the third is a saddle-node and the four is a node.

5.1. $F(x, y)$ **has a unique invariant real linear factor.** First we assume that $F(x, y)$ has only one invariant real linear factor. Note that systems (I) for $a > -2$, II, (III) and (VI) of Theorem 8 have this property.

5.1.1. *Phase portraits of system* (I). System (I) ((II) and (XII)) has associated the quintic form $F(x, y) = \alpha(bx+cy)(x^4+ax^2y^2+y^4)$, where the straight line $y = -bx/c$ is a factor of F. Note that if $c = 0$ then this curve is $x = 0$, then the associated infinite equilibrium point is in U_2 and will be studied later. Note that $\lambda = -b/c$ is a simple root of the function $f(\lambda) = \alpha\left(a\lambda^2 + \lambda^4 + 1\right)(b+c\lambda)$, defined in Proposition 4(c). Then, the associated infinite equilibrium point $(\lambda, 0)$ in U_1 is a saddle or a node and their local phase portrait depends, according to (4), of the sign of $e_1 = -\alpha b(ab^2c^2 + b^4 + c^4)(a\alpha b^2c^2 + \alpha b^4 + 5b^3p_4 - 5b^2cp_3 + 5bc^2p_2 + \alpha c^4 - 5c^3p_1)/(5c^7)$ given by the determinant of the matrix at the left in (4), so if $e_1 > 0$ it is a node and if $e_1 < 0$ it is a saddle. Therefore, when the infinite equilibrium point in U_1 is a saddle (then the corresponding infinite equilibria in V_1 is a saddle too), we have that the canonical regions are elliptic (see Figure 3 (b)), and when they are nodes the canonical regions are hyperbolic close to the origin.

In the particular case when $c = 0$ it is necessary to study the local chart U_2, considering the polynomial $g(x) = F(x, 1) = \alpha bx\left(ax^2 + x^4 + 1\right)$ where $x = 0$ is a simple root, and for $\alpha = 1$ from (4) we have that the origin of the local chart U_2 is a node if $b > 0$ and $p_4 < -b/5$, or $b < 0$ and $-b/5 < p_4$, and a saddle if $b < 0$ and $p_4 < -b/5$, or $b > 0$ and $-b/5 < p_4$. In the case $\alpha = -1$ the origin of U_2 is a node if $b > 0$ and $p_4 > b/5$, or $b < 0$ and $p_4 < b/5$, and a saddle if $b < 0$ and $p_4 > b/5$, or $b > 0$ and $p_4 < b/5$. We consider $a > -2$, then there are no more invariant straight lines. Thus the infinite equilibria on the Poincaré disk are two and can be both saddles or both nodes, because the system has even degree and the other infinite equilibria are in U_1 and V_1, and of course in the origin of V_2. Then the global phase portrait of system (I) with $a > -2$ is topologically equivalent to Figure 1(a) if $e_1 < 0$ or for $c = 0$ if $\alpha = 1$, $b < 0$ and $p_4 < -b/5$, or $b > 0$ and $-b/5 < p_4$ or $\alpha = -1$, $b < 0$ and $p_4 > b/5$, or $b > 0$ and $p_4 < b/5$; and to Figure 1(b) if $e_1 > 0$ or for $c = 0$ and either $\alpha = 1$, $b > 0$ and $p_4 < -b/5$, or $b < 0$ and $-b/5 < p_4$ or $\alpha = -1$, $b > 0$ and $p_4 > b/5$, or $b < 0$ and $p_4 < b/5$. Note that if $e_1 \neq 0$ (or $c = 0$ and $p_4 \neq 1/5$), then system (I) has no a common real linear factor. Without loss of generality for the global phase portrait we consider $c = 0$.

5.1.2. *Phase portraits of system* (III). For system (III) the canonical binary form is $F(x, y) = \alpha y^3(x^2 + y^2)$, which presents only one invariant straight line given by $y = 0$. It is clear that there no are infinite equilibria in U_2 (because the polynomial $g(x) = F(x, 1) = \alpha(x^2 + 1)$ does not have real roots). In U_1 we have that $y = 0$ is a root of the polynomial $f_1(y) = F(1, y)$ with multiplicity three, then by Proposition 4(c) the origin of the local chart U_1 can be a saddle or a node. More precisely, since it is not a simple root we need to study in the local chart U_1 their local phase portrait. The system in the local chart U_1 of system (III) is

$$\dot{z_1} = \alpha z_1^3(z_1^2 + 1), \quad \dot{z_2} = -p_1 z_2 - p_2 z_1 z_2 - p_3 z_1^2 z_2 - p_4 z_1^3 z_2 + 3/5 z_1^2 z_2 \alpha + z_1^4 z_2 \alpha.$$

Thus the origin of U_1 is the unique equilibrium and it is semi-hyperbolic with the non-zero eigenvalue equal to $-p_1$ (because the two equations of systems (III) have not a common factor). Assuming $p_1 < 0$ we can apply Theorem 2.19 of [7] and we get that if $\alpha = 1$ it is a saddle, and for $\alpha = -1$ it is an unstable node (if $p_1 > 0$ we do a re-parametrization on the time $dt/d\tau = -1$ to change the sign of the eigenvalue).

With these information we can complete the global phase portrait of system (III) (see Proposition 4(d)). Then the global phase portrait of system (III) is topologically equivalent to Figure 1(a) or 1(b).

5.1.3. *Phase portraits of system* (VI). System (VI) has associated the quintic canonical form $F_6(x, y) = \alpha x^5$. This system only has the invariant straight line $x = 0$. Then the unique infinite equilibrium points are the origin of the local charts U_2 and V_2. The system in the local chart U_2 becomes

$$\dot{z_1} = -\alpha z_1^5, \quad \dot{z_2} = -z_2(p_4 + p_3 z_1 + p_2 z_1^2 + p_1 z_1^3 + \alpha z_1^4).$$

So the origin is a semi-hyperbolic equilibrium with eigenvalues 0 and $-p_4$ (where p_4 is not null because the system (VI) has no common factors). Then, by Theorem 2.19 of [7] we can study its local phase portraits. We can suppose that $p_4 < 0$ (if not, we do a rescaling in the time $dt/d\tau = -1$). Using the notation and results of the theorem previously mentioned we have that $g(x) = -\alpha x^5$, and then the local phase portrait at the origin of U_2 is a saddle if $\alpha = 1$, and it is an unstable node if $\alpha = -1$. Therefore the local phase portrait is as in Figures 1(a) or 1(b).

5.2. $F(x, y)$ **has two invariant real linear factors.** Three systems present exactly two invariant straight lines, these are systems (IV), (VII) and $(VIII)$.

5.2.1. *Phase portraits of system* (IV). The canonical binary form $xQ(x, y) - yP(x, y)$ of system (IV) presents one simple real linear factor, one double real factor, and two complex linear factors, i.e. it has the form $F(x, y) = \alpha(bx + cy)y^2(x^2 + y^2)$. So system (IV) has two invariant invariant straight lines $y = 0$ and either $y = -bx/c$ if $c \neq 0$, or $x = 0$ if $c = 0$.

In the local chart U_1 we have that $y = 0$ is a double root of the polynomial $f_1(y) = F(1, y)$, then the origin of the local chart U_1 is a saddle-node. Furthermore, the straight line $y = -bx/c$ is a simple root of $f_1(y)$, analyzing the lineal part of the associated function $f(\lambda) = \alpha\lambda^2\left(\lambda^2 + 1\right)(b + c\lambda)$ we obtain that if $e_4 = -((b^3(b^2 + c^2)\alpha(-5c^3 p_1 + 5bc^2 p_2 - 5b^2 c p_3 + 5b^3 p_4 + b^4\alpha + b^2 c^2\alpha))/(5c^7)) > 0$, then the corresponding infinite equilibrium is a node, and if $e_4 < 0$ it is a saddle.

If $c = 0$ then the origin of the local chart U_2 is an equilibrium. We suppose without loss of generality that $b = 1$. Then the linear part of the simple root $x = 0$ of the polynomial $g(x) = F(x, 1)$ is given in (4). The sign of the determinant $-g'(0)P(0, 1) = (-p_4 - \alpha/5)\alpha$ determines the local phase portrait at the origin of U_2, precisely it is a saddle if $\alpha = -1$ and $p_4 < 1/5$, or $\alpha = 1$ and $p_4 > -1/5$, or an unstable node if $\alpha = -1$ and $p_4 < 1/5$, or a stable node if $\alpha = 1$ and $p_4 > -1/5$.

We can obtain the global phase portraits of system (IV) using the information of the infinite equilibria. Then system (IV) is topologically equivalent to Figure 1(c) if $e_4 > 0$, or $c = 0$ and $\alpha = -1$, $p_4 < 1/5$, or $c = 0$ and $\alpha = 1$, $p_4 > -1/5$, and topologically equivalent to Figure 1(d) if $e_4 < 0$, or $c = 0$ and $\alpha = -1$, $p_4 < 1/5$, or $c = 0$ adn $\alpha = 1$, $p_4 > -1/5$.

5.2.2. *Phase portrait of system* (VII). The polynomial $xQ(x,y) - yP(x,y)$ for system (VII) is $F(x,y) = \alpha(bx+cy)x^4$ with $c \neq 0$, i.e. it has one simple real linear factor and one real linear factor of multiplicity four, which provide the invariant straight lines $x = 0$ and $y = -bx/c$.

Since $x = 0$ is a root of even multiplicity of the polynomial $g(y) = F(x,1) = \alpha(bx+c)x^4$, by Proposition 4(c) we have a saddle-node in the local chart U_2. On the other hand, $y = -bx/c$ is a simple root of $f_1(x) = F(1,y) = \alpha(b+cy)$, analyzing $e_7 = -f_1'(0)P(0,1) = -b\alpha(-5c^3p_1 + 5bc^2p_2 - 5b^2cp_3 + 5b^3p_4 + c^4\alpha)/(5c^3)$ we have that the local phase portrait at the infinity of this invariant straight line is a node if $e_7 > 0$, or a saddle if $e_7 < 0$.

Note that if $b = 0$ then this straight line becomes the straight line $y = 0$, taking by simplicity $c = 1$ (the general case is analogous) the study in the local chart U_1 give us that the origin of U_1 is a hyperbolic equilibrium with eigenvalues α and $\alpha/5 - p_1$, so it is a node if $\alpha = 1$ and $p_4 > -1/5$, or if $\alpha = -1$ and $p_4 < 1/5$; and it is a saddle if $\alpha = 1$ and $p_4 < -1/5$, or if $\alpha = -1$ and $p_4 > 1/5$. Note that for $\alpha = 1$ and $p_4 = -1/5$, or for $\alpha = -1$ and $p_4 = 1/5$ the polynomials $P(x,y)$ and $Q(x,y)$ have the common factor $x = 0$, it happens similarly with $e_7 = 0$.

The global phase portrait of system (VII) is topologically equivalent to either Figure 1(c), or Figure 1(d).

5.2.3. *Phase portraits of system* $(VIII)$. For system $(VIII)$ we have the polynomial $F(x,y) = \alpha x^3y^2$, i.e. it has one double real linear factor and one triple real linear factor. So system $(VIII)$ has the two invariant straight lines $x = 0$ and $y = 0$. Since $y = 0$ is a double root of the polynomial $f_1(y) = F(1,y) = \alpha y^2$, by Proposition 4(c) this straight line provides a saddle-node in the local chart U_1. System $(VIII)$ in the local chart U_2 writes

$$\dot{z_1} = -\alpha z_1^3, \quad \dot{z_2} = -z_2(5p_4 + 5p_3z_1 + 5p_2z_1^2 + 5p_1z_1^3 + 3\alpha z_1^2)/5. \tag{9}$$

The origin of this system is a semi-hyperbolic equilibria with the non null eigenvalue equal to $-p_4$, (note that $p_4 \neq 0$, otherwise the system $(VIII)$ would have the a common factor x), using Theorem 2.19 of [7] (supposing $p_4 < 0$) we get that the origin of system (9) is a saddle if $\alpha = 1$ and it is a node if $\alpha = -1$.

The global phase portrait of systems (VII) coincides with the global phase portrait of systems IV and VII and they are topologically equivalent to Figures 1(c) or 1(d).

5.3. $F(x,y)$ **has three invariant real linear factors.** Three systems present exactly three invariant straight lines, these are systems (V), (IX) and (X).

5.3.1. *Phase portrait of system* (V). For system (V) the canonical form is $F(x,y) = \alpha(bx+cy)(x^4+ax^2y^2-y^4)$ with $(b,c) \neq (1, \pm\sqrt{(-a+\sqrt{a^2+4})/2})$, i.e. has three simple real linear factors and two complex linear factors, providing the invariant straight lines $y = bx/c$ (or $x = 0$ if $c = 0$) and $y = \pm\sqrt{(\sqrt{a^2+4}+a)/2}\ x$.

In the local chart U_1 we have the equilibria $\left(\pm\sqrt{a/2+\sqrt{4+a^2}/2}, 0\right)$ and $(-b/c, 0)$. Since the first coordinate of these equilibria are simple roots of the polynomial $f_1(y) = F(1,y)$, these equilibria are saddles or nodes. The study of the linear part of the system at these equilibria states that $\left(-\sqrt{a/2+\sqrt{4+a^2}/2}, 0\right)$ is a saddle if

$$\begin{aligned} e_5^1 &= (-20p_1 + 2\sqrt{2}(-2ab+5p_2)\sqrt{\alpha_1} - 10p_3\alpha_1 + \sqrt{2}(4b+5p_4)\alpha_1^{3/2} + c(4+6a\alpha_1 - 5\alpha_1^2))(4\sqrt{2}b\sqrt{\alpha_1}(-a+ \\ &\quad \alpha_1) + c(4+6a\alpha_1 - 5\alpha_1^2))/80 < 0, \end{aligned}$$

where $\alpha_1 = a + \sqrt{4+a^2}$, and it is a node if $e_5^1 > 0$. The equilibrium $\left(\sqrt{a/2+\sqrt{4+a^2}/2}, 0\right)$ is a saddle if

$$\begin{aligned} e_5^2 &= (20p_1 + 2\sqrt{2}(-2ab+5p_2)\sqrt{\alpha_1} + 10p_3\alpha_1 + \sqrt{2}(4b+5p_4)\alpha_1^{3/2} + c(-4-6a\alpha_1 + 5\alpha_1^2))(4\sqrt{2}b\sqrt{\alpha_1}(-a+ \\ &\quad \alpha_1) + c(-4-6a\alpha_1 + 5\alpha_1^2)) < 0, \end{aligned}$$

or it is a node for $e_5^2 > 0$, and the equilibrium $(-b/c, 0)$ is a saddle if

$$e_5^3 = (1/(5c^6))(-b^4 + ab^2c^2 + c^4)(-b^4 + c^3(c - 5p_1) + 5bc^2p_2 + b^2c(ac - 5p_3) + 5b^3p_4) < 0,$$

or a node if $e_5^3 > 0$.

In the case $c = 0$, the real linear factor of $F(x, y)$ that depends on b and c is transformed into $bx = 0$, and $x = 0$ is a simple root of the function $g(x) = F(x, 1)$. Therefore from the linear part of system (V) at the origin of U_2 we get that this origin is a node if $\tilde{e}_5^3 = -(-p_4 + \alpha b/5)\alpha b > 0$, and it is a saddle if $\tilde{e}_5^3 < 0$.

In short we have that the global phase portrait of system (II) is topologically equivalent to Figure 1(g) for n-n-n, Figure 1(h) for n-s-s, Figure 1(i) for s-n-n, and Figure 1(j) for s-s-s.

5.3.2. *Phase portraits of system* (IX). This system has associated the canonical form $F(x, y) = \alpha y^3(x^2 - y^2)$ which has three real linear factors, two simple and one of multiplicity three. We study only the local chart U_1 for obtaining the phase portrait in a neighborhood of the infinity, because the origin of U_2 is not an equilibrium point.

We note that the roots of the polynomial $f_1(y) = F(1, y) = \alpha y^3(1 - y^2)$ are the triple root $y = 0$, and the simple roots $y = \pm 1$ which are saddles or nodes. Actually, the sign of $-f_1'(y_i)P(1, y_i)$ for $i = 1, 2$ with $y_1 = 1$ and $y_2 = -1$ depend on $c_1 = -2(-p1-p2-p3-p4+2\alpha/5)\alpha$ for y_1, and on $c_2 = -2(-p1+p2-p3+p4-2\alpha/5)\alpha$ for y_2. For determining the local phase portrait at the origin of U_1 it is necessary to study the system in the local chart U_1 which is

$$\dot{z_1} = \alpha z_1^3(1 - z_1^2), \quad \dot{z_2} = -p_1z_2 - p_2z_1z_2 - p_3z_1^2z_2 - p_4z_1^3z_2 + 3/5z_1^2z_2\alpha - z_1^4z_2\alpha. \tag{10}$$

The origin of system (10) is semi-hyperbolic and applying the Theorem 2.19 of [7] we obtain that for $p_1 < 0$ it is a saddle if $\alpha = 1$, or a node if $\alpha = -1$.

The combinations of the three correlative infinite equilibria can be n-n-n, n-s-s, s-n-n and s-s-s. According to these combinations we have that the global phase portrait of system (IX) ia topologically equivalent to Figures 1(g)-(j).

5.3.3. *Phase portraits of system* (X). System (X) has three invariant straight lines $x = 0$, $y = 0$ and $y = bx/c$, which correspond to the real linear factors of the canonical form $F(x, y) = \alpha x^2y^2(bx + cy)$ with $bc \neq 0$.

The invariant straight line $x = 0$ says that the origin of U_2 is an infinite equilibrium point. Since $x = 0$ is double root of the polynomial $g(x) = F(x, 1)$, the local phase portrait of the origin of U_2 is a saddle-node. In a similar way we obtain that the origin of U_1 is also a saddle-node. The infinite equilibrium corresponding to the simple root $y = -b/c$ of $f_1(y) = F(1, y) = \alpha y^2(b + cy)$ from (4) has the linear part

$$\begin{pmatrix} (2ab + 3c)\alpha/a^2 & * \\ 0 & -(5a^3p_1 + 5a^2p_2 + 5ap_3 + 5p_4 + 3a^2b\alpha + 2ac\alpha)/(5a^4) \end{pmatrix}.$$

Therefore it is a node if $e_{10} = -(b^3\alpha(-5c^3p_1 - 5b^2cp_3 + 5b^3p_4 + bc^2(5p_2 + b\alpha))/(5c^5) > 0$, and it is a saddle if $e_{10} < 0$.

According to the possible combinations of the equilibria at the infinity we have that the global phase portrait of system (X) is topologically equivalent to Figure 1(k) for sn-n-sn, or 1(e) for sn-s-sn.

5.4. $F(x, y)$ **has four real linear factors.** System (XI) is the only one that has four invariant straight lines, these are $y = 0$, $y = x$, $y = -x$ and $y = -b/c$ (if $c = 0$ the straight line is $x = 0$). The polynomial $F(x, y)$ is $F(x, y) = \alpha(bx + cy)y^2(x^2 - y^2)$.

In the local chart U_1 we have four different infinite equilibria and their coordinates are $(0, 0)$, $(1, 0)$, $(-1, 0)$ and $(-b/c, 0)$. Since $y = 0$ is a double root of the polynomial $f_1(y) = F(1, y) = \alpha(b + cy)y^2(1 - y^2)$, the origin of U_1 is a saddle-node. Due to the fact that $y = \pm 1$ and $y = -b/c$ are simple roots of the polynomial $f_1(y)$

the corresponding equilibria $(\pm 1, 0)$ and $(-b/c, 0)$ can be saddles or nodes. More precisely, $(-1, 0)$ is a node if $e_{11}^1 = 2(b-c)\alpha(-p_1+p_2-p_3+p_4+(2b\alpha)/5-(2c\alpha)/5) > 0 > 0$, and it is a saddle if $e_{11}^2 < 0$. The equilibrium $(1, 0)$ is a node if $c_3 = -2(b+c)\alpha(-p_1-p_2-p_3-p_4-(2b\alpha)/5-(2c\alpha)/5) > 0$, or a saddle if $c_3 < 0$. The equilibrium $(-b/c, 0)$ is a saddle if $e_{11}^3 = b^2(b-c)(b+c)\alpha(5c^3p_1+5b^2cp_3-bc^2(5p_2+b\alpha)+b^3(-5p_4+b\alpha))/(5c^6) < 0$, or a node if $e_{11}^3 > 0$.

If $c = 0$ then we have the invariant straight line $x = 0$ instead of $y = -bx/c$, which implies that the origin of U_2 is an infinite equilibrium. Since $x = 0$ is a simple root of the polynomial $p_2(x) = F(x, 1) = \alpha x(x^2-1)$, from the linear part at the origin of the system in the local chart U_2 we obtain that if $\tilde{e}_{11}^3 = -2(-p_1+p_2-p_3+p_4+2\alpha b/5)\alpha b > 0$, the origin is a node, and if $\tilde{e}_{11}^3 < 0$ it is a saddle.

According to the combinations of the equilibria at infinity we have that the global phase portrait of system (XI) is topologically equivalent to Figure 1(l) for s-n-n-sn, Figure 1(m) for s-n-s-sn, Figure 1(n) for n-s-n-sn, Figure 1(o) for n-s-s-sn, Figure 1(p) for s-s-s-sn, or Figure 1(q) for n-n-n-sn.

Note that the order n-n-s-sn coincides with the order s-n-n-sn doing the reflection $(x, y) \to (x, -y)$, and the same happens for s-s-n-sn with respect to n-s-s-sn.

5.5. $F(x, y)$ **has five real linear factors.** Finally we study the phase portraits of system (XII) (or (I) for $a > -2$ and $\alpha = 1$) which have five invariant straight lines. The analysis of the invariant straight line $y = -bx/c$ (or $x = 0$ if $c = 0$) already has been done in the first case of the study of system (I) in 5.1.1. In the case $a < -2$ there exists four additional invariant straight lines (to $y = -bx/c$) given by $y = \pm\sqrt{(\pm\sqrt{a^2-4}x^2 - ax^2)/2}\, x$. Then here exists four additional equilibrium points at infinity in the local chart U_1.

We study the roots of the polynomial $f_1(y) = F(1, y) = \alpha(1+ay^2+y^4)(b+cy)$ for $a < -2$, which are $y_1 = -\sqrt{(-\sqrt{a^2-4}x^2 - ax^2)/2}$, $y_2 = \sqrt{(-\sqrt{a^2-4}x^2 - ax^2)/2}$, $y_3 = -\sqrt{(\sqrt{a^2-4}x^2 - ax^2)/2}$ and $y_4 = \sqrt{(\sqrt{a^2-4}x^2 - ax^2)/2}$. All these roots are simple, so they can be saddles o nodes. From the linear part of the system given in (4) at these roots we obtain: for $y = y_i$ with $i = \overline{1,4}$, we have that if $-f'(y_i)P(1, y_i) > 0$ (respectively $-f'(y_i)P(1, y_i) < 0$) the associated equilibrium is a node (respectively a saddle). Doing all the combinations of the local phase portrait at the infinite equilibria in U_1 and the infinite equilibria $(0, 1)$ on the Poincaré disk we have seven combinations: n-n-n-n-n, n-n-n-n-s, n-n-n-s-s, n-n-s-n-s, n-n-s-s-s, n-s-s-s-n and s-s-s-s-n.

The case where all the infinite equilibria are saddles cannot occurs. Indeed, we write $F(x, y) = a(y-r_1x)(y-r_2x)(y-r_3x)(y-r_4x)(y-r_5x)$ with $r_1 < r_2 < r_3 < r_4 < r_5$. Then the conditions over the first four roots r_i of $F(1, y)$ are $-f'(r_i)P(1, r_i) = -\phi_i^2$ $i = \overline{1,4}$, because they must be saddles. We get

$$\begin{aligned}
p_1 \;&= (-a^2\Delta(-r_2r_3r_4r_5 - r_1(r_3r_4r_5 + r_2(-4r_3r_4 + r_3r_5 + r_4r_5))) - r_1(r_1-r_3)r_3(r_1-r_4)(r_3-r_4)r_4\phi_2^2 + \\
&\quad r_2^3(r_1(r_1-r_4)r_4\phi_3^2 - r_3^2(r_4\phi_1^2 + r_1\phi_4^2) + r_3(r_4^2\phi_1^2 + r_1^2\phi_4^2)) + r_2^2(r_1r_4(-r_1^2 + r_4^2)\phi_3^2 + r_3^3(r_4\phi_1^2 + r_1\phi_4^2) - \\
&\quad r_3(r_4^3\phi_1^2 + r_1^3\phi_4^2)) + r_2(r_1^2(r_1-r_4)r_4^2\phi_3^2 - r_3^3(r_4^2\phi_1^2 + r_1^2\phi_4^2) + r_3^2(r_4^3\phi_1^2 + r_1^3\phi_4^2)))/(5a\Delta), \\
p_2 \;&= (a^2\Delta(-2r_3r_4r_5 + r_2(3r_3r_4 - 2r_3r_5 - 2r_4r_5) + r_1(3r_3r_4 + r_2(3r_3 + 3r_4 - 2r_5) - 2r_3r_5 - 2r_4r_5)) + \\
&\quad r_3^3r_4^2\phi_1^2 - r_3^2r_4^3\phi_1^2 + r_1^3r_3^2\phi_2^2 - r_1^2r_3^3\phi_2^2 - r_1^3r_4^2\phi_2^2 + r_3^3r_4^2\phi_2^2 + r_1^2r_4^3\phi_2^2 - r_3^2r_4^3\phi_2^2 - r_1^3r_4^2\phi_3^2 + r_1^2r_4^3\phi_3^2 - \\
&\quad r_1^3r_3^2\phi_4^2 + r_1^2r_3^3\phi_4^2 + r_2^3(r_4^2(-\phi_1^2 + \phi_3^2) + r_3^2(\phi_1^2 + \phi_4^2) - r_1^2(\phi_3^2 + \phi_4^2)) + r_2^2(r_4^3(\phi_1^2 - \phi_3^2) - r_3^3(\phi_1^2 + \phi_4^2) + \\
&\quad r_1^3(\phi_3^2 + \phi_4^2)))/(5a\Delta), \\
p_3 \;&= (-a^2\Delta(2r_3r_4 + r_2(2r_3 + 2r_4 - 3r_5) + r_1(2r_2 + 2r_3 + 2r_4 - 3r_5) - 3r_3r_5 - 3r_4r_5) - r_3^3r_4\phi_1^2 + \\
&\quad r_3r_4^3\phi_1^2 - r_1^3r_3\phi_2^2 + r_1r_3^3\phi_2^2 + r_1^3r_4\phi_2^2 - r_3^3r_4\phi_2^2 - r_1r_4^3\phi_2^2 + r_3r_4^3\phi_2^2 + r_1^3r_4\phi_3^2 - r_1r_4^3\phi_3^2 + \\
&\quad r_1^3r_3\phi_4^2 - r_1r_3^3\phi_4^2 + r_2^3(r_4(\phi_1^2 - \phi_3^2) - r_3(\phi_1^2 + \phi_4^2) + r_1(\phi_3^2 + \phi_4^2)) + r_2(r_4^3(-\phi_1^2 + \phi_3^2) + r_3^3(\phi_1^2 + \phi_4^2) - \\
&\quad r_1^3(\phi_3^2 + \phi_4^2)))/(5a\Delta),
\end{aligned}$$

$$\begin{aligned} p_4 &= (a^2\Delta(r_1+r_2+r_3+r_4-4r_5)+r_3^2r_4\phi_1^2-r_3r_4^2\phi_1^2+r_1^2r_3\phi_2^2-r_1r_3^2\phi_2^2-r_1^2r_4\phi_2^2+\\ &r_3^2r_4\phi_2^2+r_1r_4^2\phi_2^2-r_3r_4^2\phi_2^2-r_1^2r_4\phi_3^2+r_1r_4^2\phi_3^2-r_1^2r_3\phi_4^2+r_1r_3^2\phi_4^2+r_2^2(r_4(-\phi_1^2+\phi_3^2)+\\ &r_3(\phi_1^2+\phi_4^2)-r_1(\phi_3^2+\phi_4^2))+r_2(r_4^2(\phi_1^2-\phi_3^2)-r_3^2(\phi_1^2+\phi_4^2)+r_1^2(\phi_3^2+\phi_4^2)))/(5a\Delta). \end{aligned}$$

where $\Delta=(r_1-r_2)(r_1-r_3)(r_2-r_3)(r_1-r_4)(r_2-r_4)(r_3-r_4)$. These conditions implies that $-f'(r_5)P(1,r_5)>0$, because $-f'(r_5)P(1,r_5)+\phi_5^2=0$ is equivalent to $\kappa_1a^2+\kappa_2=0$, and this equation has a unique solution if $\kappa_1\kappa_2<0$, but

$$\kappa_2=5(r_1-r_5)(r_2-r_5)(r_3-r_5)(r_4-r_5)>0,$$

and

$$\kappa_1=\frac{c_1\phi_1^2+c_2\phi_2^2+c_3\phi_3^2+c_4\phi_4^2+c_5\phi_5^2}{(r_1-r_2)(r_1-r_3)(r_2-r_3)(r_1-r_4)(r_2-r_4)(r_3-r_4)},$$

with

$$\begin{aligned} c_1 &= (r_2-r_3)(r_2-r_4)(r_3-r_4)(r_2-r_5)(r_3-r_5)(r_4-r_5)>0,\\ c_2 &= (r_1-r_3)(r_1-r_4)(r_3-r_4)(r_1-r_5)(r_3-r_5)(r_4-r_5)>0,\\ c_3 &= (r_1-r_2)(r_1-r_4)(r_2-r_4)(r_1-r_5)(r_2-r_5)(r_4-r_5)>0,\\ c_4 &= (r_1-r_2)(r_1-r_3)(r_2-r_3)(r_1-r_5)(r_2-r_5)(r_3-r_5)>0,\\ c_5 &= (r_1-r_2)(r_1-r_3)(r_2-r_3)(r_1-r_4)(r_2-r_4)(r_3-r_4)>0. \end{aligned}$$

Thus $\kappa_1\kappa_2>0$ and the infinite equilibria associated to r_5 cannot be a saddle. (for $c=0$ we can prove following the same argument that the infinite equilibria cannot be all saddles).

According to the possible combinations of the equilibria at infinity we have that the global phase portrait of system (X) is topologically equivalent to Figure 1(r) if s-n-s-s-s, Figure 1(s) if s-n-n-n-n, Figure 1(t) if s-n-s-s-n, Figure 1(u) if s-n-n-s-n, Figure 1(v) if n-n-n-s-s, Figure 1(w) if s-n-n-s-s and Figure 1(x) if n-n-n-n-n.

This completes the proof of Theorem 2.

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References

[1] J. Argemí *Sur les points singuliers multiples de systemes dynamiques dans* $\mathbb{R}^2$, J. Ann. Mat. Pura Appl. **4** (1968), 35–70.
[2] A. Cima,A. Gasull, Armengol and F. Mañosas, it On polynomial Hamiltonian planar vector fields, J. Differential Equations, **106**(2), (1993), 367–383.
[3] A. Cima and J. Llibre, *Algebraic and topological classification of the homogeneous cubic vector fields*, J. of Math. Anal. and Appl. **147** (1990), 420–448.
[4] C. B. Collins *Algebraic classification of polynomial vector fields in the plane*, Japan J. Indust. Appl. Math. (1996) 13–63-
[5] A. Ferragut and J. Llibre, *Cofactors and equilibria for polynomial vector fields*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, **144**(4) (2014), 753-760.
[6] T. Date and M. Iri, *Canonical forms of real homogeneous quadratic transformations*, J. Math. Appl. **56** (1976), 650–682.
[7] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Spring-Verlag, 2006.
[8] E. A. Gonzalez, *Generic properties of polynomial vector fields at infinity*, Trans. Amer. Math. Soc. **143** (1969), 201–222
[9] G. Gurevich, *Foundations of the theory of Algebraic Invariants*, Noordhoff, Groningen, 1964.
[10] L. Markus *Quadratic differential equations and non-associative algebras*, Annals of Math. Studies, **45** (1960), 185–213.
[11] L. Markus, *Global structure of ordinary differential equations in the plane:* Trans. Amer. Math Soc. **76** (1954), 127–148.
[12] D. A. Neumann, *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.
[13] M.M. Peixoto, *Dynamical Systems. Proccedings of a Symposium held at the University of Bahia*, 389–420, Acad. Press, New York, 1973.
[14] J. Sotomayor, *Curvas definidas por equacoes diferenciais no plano*, Instituto de Matematica Pura y Aplicada, Rio de Janeiro, 1981.

[15] Y.Q. Ye, S. Cai, L. Chen, K. Huang, D. Luo, Z. Ma, E. Wang, M. Wang and X. Yang, *Theory of limit cycles.* Translated from the Chinese by Chi Y. Lo. Second edition. Translations of Mathematical Monographs, **66**, American Mathematical Society, Providence, RI, (1986).

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