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HAMILTONIAN NILPOTENT SADDLES OF LINEAR PLUS CUBIC HOMOGENEOUS POLYNOMIAL VECTOR FIELDS

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Abstract. We completely characterize the global phase portraits in the Poincaré disk for all planar Hamiltonian vector fields with linear plus cubic homogeneous terms having a nilpotent saddle at the origin.

1. Introduction and statement of the results

Let (P, Q) be an analytic map from $\mathbb{R}^2$ into itself. The qualitative theory of ordinary differential equations in the plane provide a qualitative description of the behavior of each orbit instead of giving explicitly (or quantitatively) the solutions. In this paper we describe the local phase portraits of singular points for a wide general class of systems being of great interest due to their connection with physical systems.

Quadratic systems having a center at the origin have been widely studied in the last 100 years, and more than 1.000 papers have been published about them (see [10, pages 3 and 4 and 13] for a brief history of the problem of the center in general, and where it includes a list of 300 papers covering this topic.) There are also some partial results for the centers of planar polynomial differential systems of degree larger than two. Recently Colak, Llibre and Valls [3, 4, 5, 6] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams.

Dulak [8] was the first to detect that centers can pass to saddles through a complex change of variables, see for more details [9], and so it is natural to ask whether such kind of studies can also be done for saddles. It is interesting to observe that despite the fact that the classification of phase portraits of Hamiltonian planar polynomial vector fields having a ceter at the origin have been widely studied very few results exist in the case of saddles. For the case of quadratic systems having an integrable saddle, its phase portraits were provided in [2]. As far as the authors know, for the case in which there exists a nilpotent saddle at the origin and the degree of the system is greater than two no result exists on the classification of the phase portraits. This is the objective pursuit by this paper. This is a huge class of systems with too many parameters and so, in this paper we restrict to classifying the global

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phase portraits of all Hamiltonian planar polynomial systems of degree three of the form linear plus cubic homogeneous with a nilpotent saddle at the origin. We recall that a *Hamiltonian planar polynomial system of degree d* is a system of the form

$$x' = H_y \quad y' = -H_x$$

(here the prime denotes derivative with respect to the time t) where the maximum of the degrees of H_y and H_x is d.

To do this we will use the Poincaré compactification of polynomial vector fields. The Poincaré compactification that we shall use for describing the global phase portraits of our Hamiltonian systems is standard. For all the definitions and results on the Poincaré compactification see Chapter 5 of [7]. We say that two vector fields on the Poincaré disk are topologically equivalent if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow. Our main result is the following one.

Theorem 1. *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent saddle at the origin if and only if, after a linear change of variables and a rescaling of its independent variable it can be written as one of the following six classes:*

(I) $x' = ax + by$, $y' = -\frac{a^2}{b}x - ay + x^3$ *with* $b > 0$;
(II) $x' = ax + by - x^3$, $y' = -\frac{a^2}{b}x - ay + 3x^2y$ *with* $a < 0$;
(III) $x' = ax + by - 3x^2y + y^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + 3xy^2$ *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $ab \neq 0$ *and* $a^2/b - 6b < 0$;
(IV) $x' = ax + by - 3x^2y - y^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + 3xy^2$ *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $ab \neq 0$ *and* $b > 0$;
(V) $x' = ax+by-3\mu x^2y+y^3$, $y' = \left(c-\frac{a^2}{b+c}\right)x-ay+x^3+3\mu xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $b \neq 0$ *and* $(a^4 - b^4 - 6a^2b^2\mu)/b < 0$;
(VI) $x' = ax+by-3\mu x^2y-y^3$, $y' = \left(c-\frac{a^2}{b+c}\right)x-ay+x^3+3\mu xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $b \neq 0$ *and* $(a^4 + b^4 + 6a^2b^2\mu)/b > 0$;

where $a, b, c, \mu \in \mathbb{R}$. *Moreover systems (I), (III), (IV), (V) and (VI) are invariant under the transformation* $a \to -a$, $(x, y) \to (-x, y)$ *and reversing the time (i.e.* $t \to -t$*), therefore it is not restrictive to consider* $a \geq 0$.

The proof of Theorem 1 is given in Section 3.

Theorem 2. *The global phase portraits of the six families of systems provided by Theorem 1 are topologically equivalent to the following of Figure 1:*

- 1.1 *for systems (I); or systems (IV) with* $a = b = 0$, $c < 0$; *or systems (IV) with* $c = 0$, $ab \neq 0$, $b > 0$ *and* $b < 2|a|/\sqrt{3}$;
- 1.2 *for systems (II);*
- 1.3 *for systems (III) with* $a = b = 0$ *and* $c > 0$; *or* $c = 0$, $ab \neq 0$, $a^2/b - 6b < 0$ *and* $b < 0$;
- 1.4 *for systems (III), with* $c = 0$, $ab \neq 0$, $a^2/b - 6b < 0$ *and* $b > 0$;

- 1.5 *for systems (IV) with* $c = 0$*,* $ab \neq 0$ *and* $b > 2|a|/\sqrt{3}$*;*
- 1.6 *for systems (IV) with* $c = 0$*,* $ab \neq 0$ *and* $b = 2|a|/\sqrt{3}$*;*
- 1.7 *for systems (V) with* $a = b = 0$*,* $c > 0$*; or* $c = a = 0$*,* $b > 0$*; or* $c = 0$*,* $a \neq 0$*,* $(\bar{b}, \mu) \in R_1 \cup R_2 \cup R_3$*; or* $c = 0$*,* $a \neq 0$*,* $(\bar{b}, \mu) \in (\{\bar{b}_2^-(\mu), \mu\} \cup \{\bar{b}_1(\mu), \mu\}) \cap S$*, where* $R_1, R_2, R_3, S, \bar{b}_2^-(\mu), \bar{b}_1(\mu)$ *are given in Subsection 4.5;*
- 1.8 *for systems (V) with* $c = 0$*,* $a \neq 0$ *and* $(\bar{b}, \mu) \in R_4$*, where* R_4 *is given in Subsection 4.5;*
- 1.9 *for systems (V) with* $c = 0$*,* $a \neq 0$ *and* $(\bar{b}, \mu) \in \{\bar{b}_2^+(\mu), \mu\} \cap S$*, where* S *and* $\bar{b}_2^+(\mu)$ *are given in Subsection 4.5;*
- 1.10 *for systems (VI) with* $a = b = 0$*,* $c < 0$ *and* $\mu < -1/3$*, or with* $c = a = 0$*,* $b > 0$ *and* $\mu < -1/3$*, or with* $c = 0$*,* $(\bar{b}, \mu) \in R_1 \cup R_2 \cup R_3 \cup R_5 \cup (\{-\bar{b}_1(\mu), \mu\} \cap S)$*, where* $R_1, R_2, R_3, R_5, \bar{b}_1(\mu)$*, and* S *are given in Subsection 4.6;*
- 1.11 *for systems (VI) with* $a = b = 0$*,* $c < 0$ *and* $\mu = -1/3$*, or with* $c = 0$*,* $b > 0$*,* $a \neq \pm b$ *and* $\mu = -1/3$*;*
- 1.12 *for systems (VI) with* $a = b = 0$*,* $c < 0$ *and* $\mu \in (-1/3, 0)$*, or with* $c = a = 0$*,* $b > 0$ *and* $\mu \in (-1/3, 0)$*, or with* $c = 0$*,* $(\bar{b}, \mu) \in R_4 \cup R_6 \cup R_{10}$*, where* R_4, R_6, R_{10} *are given in Subsection 4.6;*
- 1.13 *for systems (VI) with* $a = b = 0$*,* $c < 0$ *and* $\mu \geq 0$*, or with* $a = c = 0$*,* $b > 0$ *and* $\mu \geq 0$*, or* $c = 0$*,* $b > 0$ *and* $\mu = 1/3$*, or with* $c = 0$ *and* $(\bar{b}, \mu) \in R_7 \cup (\{\bar{b}_1(\mu), \mu\} \cap S) \cup R_8 \cup R_9$*, or* $c = 0$*,* $b = a$ *and* $\mu = 1$*, where* $R_7, R_8, R_9, \bar{b}_1(\mu)$ *and* S *are given in Subsection 4.6;*
- 1.14 *for systems (VI) with* $c = 0$ *and* $(\bar{b}, \mu) \in (\{\bar{b}_2^+(\mu), \mu\} \cup \{\bar{b}_2^-(\mu), \mu\}) \cap S$*, where* $\bar{b}_2^{\pm}(\mu)$ *and* S *are given in Subsection 4.6.*

Here $\bar{b} = b/|a|$*.*

The proof of Theorem 2 is given in section 4. Note that Theorem 2 also provides the bifurcation diagrams.

2. Preliminary results

A vector field is said to have the *finite sectorial decomposition property* at a singular point q if either q is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors. We note that all the isolated singular points of a polynomial differential system satisfy the finite vectorial decomposition property.

Theorem 3 (Poincaré Formula)**.** *Let* q *be an isolated singular point having the finite sectorial decomposition property. Let* e*,* h *an* p *denote the number of elliptic, hyperbolic and parabolic sectors of* q*, respectively. Then the index of* q *is* $(e - h)/2 + 1$*.*

The indices of a saddle, a center and a cusp are $-1, 1$ and 0, respectively.

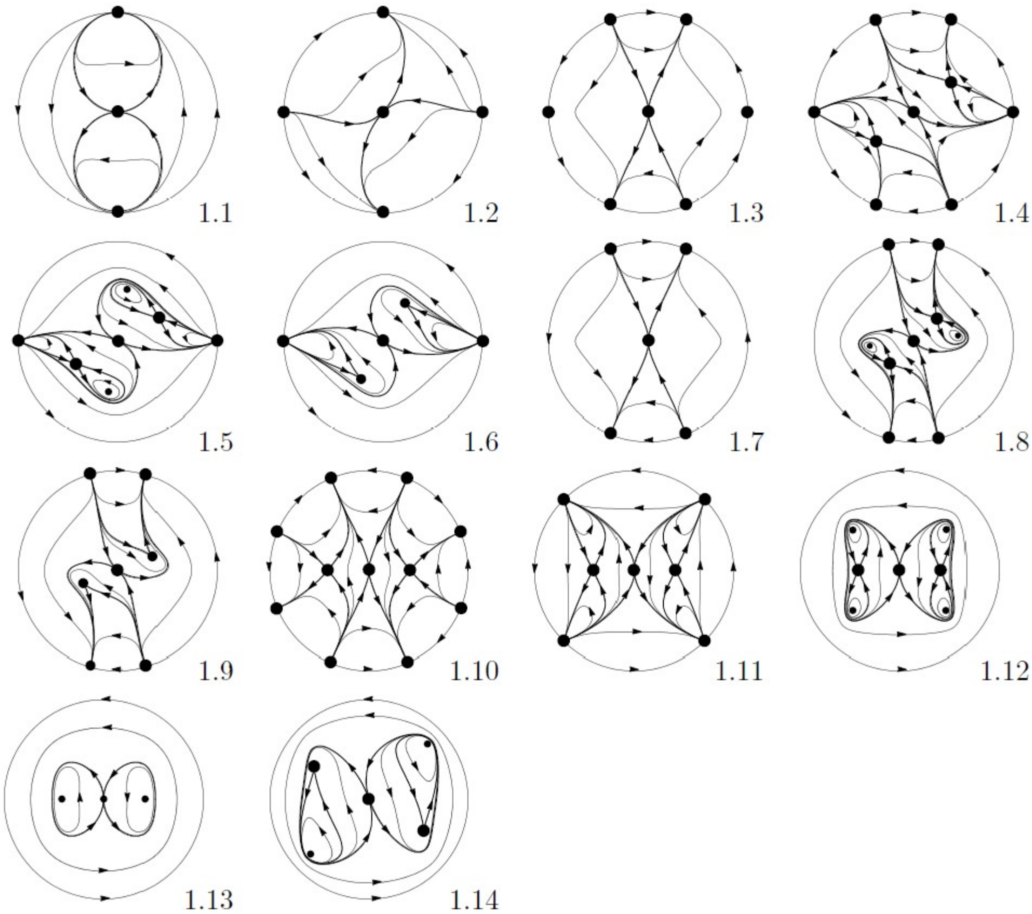

FIGURE 1. Global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms with a nilpotent saddle at the origin. The separatrices are in bold.

Theorem 4 (Poincaré–Hopf Theorem)**.** *For every vector field on the sphere* $\mathbb{S}^2$ *with a finite number of singular points, the sum of the indices of these singular points is* 2*.*

Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see Chapter Theorem 3.5 of [7] taking into account that Hamiltonian systems cannot have foci).

We define *center–loop* as a hyperbolic saddle with a loop and a center inside the loop as in Figure 2.

Proceeding as in the proof of Lemma 12 in [4] we can show that if p is an isolated singular point which is non-elementary then it must be nilpotent. Hence, the unique possible isolated finite singular points that we can have are either centers, saddles or cusps.

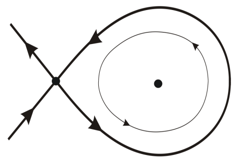

Figure 2. A center–loop.

3. Proof of Theorem 1

It was proved in [4] that a Hamiltonian vector field with linear plus cubic homogeneous terms which has a nilpotent term at the origin, after a linear change of variables and a rescaling of its independent variable it can be written as one of the following six classes:

(I') $x' = ax + by$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + x^3$;

(II') $x' = ax + by - x^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + 3x^2y$;

(III') $x' = ax + by - 3x^2y + y^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + 3xy^2$;

(IV') $x' = ax + by - 3x^2y - y^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + 3xy^2$;

(V') $x' = ax + by - 3\mu x^2y + y^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$;

(VI') $x' = ax + by - 3\mu x^2y - y^3$, $y' = \left(c - \frac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$;

where $a, b, c, \mu \in \mathbb{R}$ with either $a = b = 0$ and $c \neq 0$, or $c = 0$ and $b \neq 0$.

Now we use Theorem 3.5 in [7] to find necessary and sufficient conditions so that the origins of systems (I')–(VI') are saddles. For this we need to make a change of variables so that systems (I')–(VI') can be written in such a way that the linear part is in Jordan form. Note that applying the change of variables

$$X = x, \quad Y = ax + by \tag{1}$$

when $c = 0$ and $b \neq 0$, or

$$X = y, \quad Y = cx \tag{2}$$

when $a = b = 0$ and $c \neq 0$, systems in classes (I')–(VI') can be written in the form

$$\dot{X} = Y + P(X, Y), \quad \dot{Y} = Q(X, Y).$$

More precisely, system (I') with $a = b = 0$ and $c \neq 0$ by the change of variables (2) can be written in the form

$$\dot{X} = Y + \frac{1}{c^3}Y^3, \quad \dot{Y} = 0,$$

and applying Theorem 3.5 in [7] we obtain that the origin is never a nilpotent saddle, so this case is not possible. On the other hand, system (I') with $c = 0$ and $b \neq 0$, by the change of variables (1) can be written in the form

$$\dot{X} = Y, \quad \dot{Y} = bX^3,$$

and the condition so that the origin is a nilpotent saddle is $b > 0$ (see Theorem 3.5 in [7]). This provides system (I).

System (II') with $a = b = 0$ and $c \neq 0$ by the change of variables (2) can be written in the form

$$\dot{X} = Y + \frac{3}{c^2} X Y^2, \quad \dot{Y} = -\frac{1}{c^2} Y^3,$$

and applying Theorem 3.5 in [7] we obtain that the origin is a center, so this case is not possible. On the other hand, system (II') with $c = 0$ and $b \neq 0$, by the change of variables (1) can be written in the form

$$\dot{X} = Y - X^3, \quad \dot{Y} = -4aX^3 + 3X^2 Y,$$

and the condition so that the origin is a nilpotent saddle is $a < 0$ (see Theorem 3.5 in [7]). This provides system (II).

System (III') with $a = b = 0$ and $c \neq 0$ by the change of variables (2) can be written in the form

$$\dot{X} = Y + \frac{3}{c} X^2 Y, \quad \dot{Y} = cX^3 - \frac{3}{c} X Y^2.$$

Applying Theorem 3.5 in [7] the condition so that the origin is a nilpotent saddle is $c > 0$. Moreover, system (III') with $c = 0$ and $b \neq 0$, by the change of variables (1) can be written in the form

$$\begin{aligned}
\dot{X} &= Y - \frac{a(a^2 - 3b^2)}{b^3} X^3 + \frac{3(a^2 - b^2)}{b^3} X^2 Y - \frac{3a}{b^3} X Y^2 + \frac{1}{b^3} Y^3, \\
\dot{Y} &= -\frac{a^2(a^2 - 6b^2)}{b^3} X^3 + \frac{3a(a^2 - 3b^2)}{b^3} X^2 Y - \frac{3(a^2 - b^2)}{b^3} X Y^2 + \frac{a}{b^3} Y^3,
\end{aligned}$$

and by Theorem 3.5 in [7], it has a nilpotent saddle at the origin if and only if

$$a^2/b - 6b < 0 \quad \text{and} \quad a \neq 0. \tag{3}$$

We have thus obtained system (III).

System (IV') with $a = b = 0$ and $c \neq 0$ by the change of variables (2) can be written in the form

$$\dot{X} = Y + \frac{3}{c} X^2 Y, \quad \dot{Y} = -cX^3 - \frac{3}{c} X Y^2.$$

By Theorem 3.5 in [7] it has a saddle at the origin if and only if $c < 0$. Furthermore, system (IV') with $c = 0$ and $b \neq 0$, by the change of variables (1) can be written in the form

$$\begin{aligned}
\dot{X} &= Y + \frac{a(a^2 + 3b^2)}{b^3} X^3 - \frac{3(a^2 + b^2)}{b^3} X^2 Y + \frac{3a}{b^3} X Y^2 - \frac{1}{b^3} Y^3, \\
\dot{Y} &= \frac{a^2(a^2 + 6b^2)}{b^3} X^3 - \frac{3a(a^2 + 3b^2)}{b^3} X^2 Y + \frac{3(a^2 + b^2)}{b^3} X Y^2 - \frac{a}{b^3} Y^3
\end{aligned}$$

and by Theorem 3.5 in [7], the origin is a saddle if and only if $b > 0$ and $a \neq 0$, obtaining the normal form (IV).

System (V') with $a = b = 0$ and $c \neq 0$ by the change of variables (2) can be written in the form

$$\dot{X} = Y + \frac{3\mu}{c} X^2 Y + \frac{Y^3}{c^3}, \quad \dot{Y} = cX^3 - \frac{3\mu}{c} X Y^2,$$

and by Theorem 3.5 in [7], the origin is a saddle if and only if $c > 0$. Furthermore, system (V') with $c = 0$ and $b \neq 0$, by the change of variables (1) can be written in the form

$$\begin{aligned}\dot{X} &= Y - \frac{a(a^2 - 3b^2\mu)}{b^3}X^3 + \frac{3(a^2 - b^2\mu)}{b^3}X^2Y - \frac{3a}{b^3}XY^2 + \frac{1}{b^3}Y^3,\\ \dot{Y} &= -\frac{(a^4 - b^4 - 6a^2b^2\mu)}{b^3}X^3 + \frac{3a(a^2 - 3b^2\mu)}{b^3}X^2Y - \frac{3(a^2 - b^2\mu)}{b^3}XY^2\\ &\quad + \frac{a}{b^3}Y^3,\end{aligned}$$

and by Theorem 3.5 in [7], the origin is a saddle if and only if

$$(a^4 - b^4 - 6a^2b^2\mu)/b < 0, \tag{4}$$

providing the normal form (V).

Finally, System (VI') with $a = b = 0$ and $c \neq 0$ by the change of variables (2) can be written in the form

$$\dot{X} = Y + \frac{3\mu}{c}X^2Y + \frac{1}{c^3}Y^3, \quad \dot{Y} = -cX^3 - \frac{3\mu}{c}XY^2.$$

By Theorem 3.5 in [7] the origin is a saddle if and only if $c < 0$. Furthermore, system (VI') with $c = 0$ and $b \neq 0$, by the change of variables (1) can be written in the form

$$\begin{aligned}\dot{X} &= Y + \frac{a(a^2 + 3b^2\mu)}{b^3}X^3 - \frac{3(a^2 + b^2\mu)}{b^3}X^2Y + \frac{3a}{b^3}XY^2 - \frac{1}{b^3}Y^3,\\ \dot{Y} &= \frac{(a^4 + b^4 + 6a^2b^2\mu)}{b^3}X^3 - \frac{3a(a^2 + 3b^2\mu)}{b^3}X^2Y + \frac{3(a^2 + b^2\mu)}{b^3}XY^2\\ &\quad - \frac{a}{b^3}Y^3,\end{aligned}$$

and by Theorem 3.5 in [7], the origin is a saddle if and only if

$$\frac{a^4 + b^4 + 6a^2b^2\mu}{b} > 0, \tag{5}$$

providing the normal form (VI).

If $a < 0$, setting $a = -|a|$, doing the linear transformation $(x, y) \to (-x, y)$ and reversing the time (by setting $t \to -t$) system (I) becomes

$$\begin{aligned}\dot{x} &= |a|x + by,\\ \dot{y} &= -(a^2/b)x - |a|y + x^3.\end{aligned} \tag{6}$$

So, system (6) is system (I) with $a > 0$ (and reversing the time). The same occurs with systems (III), (IV), (V) and (VI). This completes the proof of Theorem 1.

4. Proof of Theorem 2

We will study each of the global phase portraits of systems (I)–(VI) provided by the normal form in Theorem 1. We note that the right-hand side of these six families are odd functions, and so their phase portraits are symmetric with respect to the origin (that is, invariant by the change $(x, y) \mapsto (-x, -y)$).

4.1. **Global phase portraits of systems (I).** To find the global phase portraits of system (I) we first investigate the local charts U_1 and U_2 and after that, we study the finite singular points.

In the local chart U_1, systems (I) become

$$\dot{u} = 1 - v^2(bu+a)^2/b, \quad \dot{v} = -v^3(bu+a),$$

and when $v = 0$ there are no singular points on U_1.

Now we study the origin of U_2. In the local chart U_2 systems (I) can be written as

$$\dot{u} = -u^4 + v^2(b+au)^2/b, \quad \dot{v} = -u^3v + v^3a(b+au)/b,$$

and the origin is a singular point whose linear part is zero. We need to apply blow-up techniques (see for instance [1]) to obtain the local behavior of this point. Doing so, we get that the origin of U_2 is formed by two elliptic and four parabolic sectors.

We now look at the finite singular points of system (I) and we see the origin is the unique finite singular point, which is a saddle. Therefore the global phase portraits of system (I) are topologically equivalent to 1.1 of Figure 1.

4.2. **Global phase portraits of system (II).** Note that for system (II) we can assume that $b > 0$ because the change of variables $y \mapsto -y$ gives exactly the same systems with the opposite sign of the parameter b.

In the local chart U_1 system (II) becomes

$$\dot{u} = 4u - v^2(bu+a)^2/b, \quad \dot{v} = -v^3(bu+a) + v.$$

When $v = 0$ the origin of U_1 is the unique singular point. The eigenvalues at this point are 4 and 1 and so it is a repelling node.

Now we check if the origin of U_2 is a singular point. Systems (II) on the local chart U_2 can be written as

$$\dot{u} = -4u^3 + v^2(au+b)^2/b, \quad \dot{v} = -3u^2v + v^3a(au+b)/b.$$

The origin is a singular point whose linear part is zero. We need to do blow-up to analyze the local behavior of this point. Doing so, we get that the origin of U_2 is formed by four stable parabolic sectors.

Now we study the finite singular points. Taking into account that $a < 0$ we get that the origin is the unique finite singular point. According to this local information we obtain that the global phase portrait is topologically equivalent to 1.2 of Figure 1.

4.3. **Global phase portraits of system (III).**

4.3.1. *Case $a = b = 0$ and $c > 0$.* First we study system (III) when $a = b = 0$ and $c > 0$. In the local chart U_1 we get

$$\dot{u} = cv^2 - u^2(u^2-6), \quad \dot{v} = -uv(u^2-3). \tag{7}$$

When $v = 0$ there are three singular points: $(0,0)$ and $(\pm\sqrt{6},0)$. The eigenvalues of the linear part of systems (7) on the points $(\pm\sqrt{6},0)$ are $\mp 12\sqrt{6}$ and $\mp 3\sqrt{6}$, respectively. So, $(\sqrt{6},0)$ is an attracting node and $(-\sqrt{6},0)$ is a repelling node. At the origin the linear part of the jacobian matrix is zero, and so it is linearly zero. To describe its local behavior we use blow up techniques. Doing so, we get that the origin is formed by two hyperbolic sectors.

Now we look for the origin in the local chart U_2. Note that system (III) on the chart U_2 is

$$\dot{u} = -cu^2v^2 - 6u^2 + 1 \quad \dot{v} = -uv(cv^2+3),$$

and so the origin of U_2 is not a singular point.

The unique finite singular point of system (III) with $a = b = 0$ and $c > 0$ is the origin. Consequently the global phase portrait is topologically equivalent to 1.3 in Figure 1.

4.3.2. *Case $c = 0$, $ab \neq 0$, $a^2/b - 6b < 0$ and $a > 0$.* Now we study system (III) when $c = 0$, $ab \neq 0$, $a > 0$ and condition (3) holds. Under these assumptions we first study the infinite singular points. In U_1 system (III) become

$$\begin{aligned} \dot{u} &= -(a+bu)^2v^2/b - u^2(u^2-6), \\ \dot{v} &= -(bu+a)v^3 - uv(u^2-3). \end{aligned}$$

When $v = 0$ the singular points are $(0,0)$ and $(\pm\sqrt{6},0)$. Just like in the case $a = b = 0$, the eigenvalues of the linear part of system (III) in the local chart U_1 on the points $(\pm\sqrt{6},0)$ are $\mp 12\sqrt{6}$ and $\mp 3\sqrt{6}$, respectively. So, $(\sqrt{6},0)$ is an attracting node and $(-\sqrt{6},0)$ is a repelling node. At the origin, the linear part is zero. Again we do a blow-up and we get that when $b < 0$ the origin of U_1 consists of two hyperbolic sectors and when $b > 0$ it is formed by two elliptic and four parabolic sectors.

We now look at the origin of U_2, in which system (III) write as

$$\begin{aligned} \dot{u} &= (b+au)^2v^2/b - 6u^2 + 1, \\ \dot{v} &= a(b+au)v^3/b - 3uv. \end{aligned}$$

Hence the origin is not a singular point.

The finite singular points of system (III) are the origin and the points

$$\begin{aligned} p_{1,2} &= \pm\left(\frac{(3b-A)\sqrt{B-A}}{6\sqrt{6}\,a}, \frac{\sqrt{B-A}}{\sqrt{6}}\right), \\ p_{3,4} &= \pm\left(\frac{(3b+A)\sqrt{B+A}}{6\sqrt{6}\,a}, \frac{\sqrt{B+A}}{\sqrt{6}}\right), \end{aligned}$$

where $A = \sqrt{12a^2+9b^2}$ and $B = 2a^2/b - 3b$.

Consider the expression

$$B^2 - A^2 = (2a^2/b - 3b)^2 - (12a^2 + 9b^2) = \frac{4a^2}{b}(a^2/b - 6b). \tag{8}$$

Note that since $\sqrt{12a^2+9b^2} > \sqrt{9b^2} = 3|b|$, we have

$$B - A = 2\frac{a^2}{b} - 3b - \sqrt{12a^2+9b^2} < 2\frac{a^2}{b} - 3b - 3|b| \\ = \begin{cases} 2a^2/b & \text{if } b<0, \\ 2a^2/b - 6b & \text{if } b>0. \end{cases} \tag{9}$$

If $b < 0$, from (8) and (9) we get that $B^2 - A^2$ is positive due to (3) and $B - A < 0$. So $B - A < 0$ and $B + A < 0$. Hence, the only finite singular point is the origin. Consequently we obtain phase portrait 1.3 of Figure 1.

If $b > 0$, from (8), (9) and (3), we get that $B^2 - A^2 < 0$ and $B - A < 0$. So $B + A > 0$. Therefore, system (III) has, among the origin, only the two finite singular points p_3 and p_4. The eigenvalues of the linear part of this system at these points are

$$\pm\frac{\sqrt{4a^4+15a^2b^2+9b^4+b(5a^2-3b^2)\sqrt{12a^2+9b^2}}}{\sqrt{3}b}.$$

Note that

$$(4a^4+15a^2b^2+9b^4)^2 - (b(5a^2-3b^2)\sqrt{12a^2+9b^2})^2 \\ = 4a^2(4a^2+3b^2)(a^2-6b^2)^2 > 0.$$

Since $4a^4 + 15a^2b^2 + 9b^4 > 0$, this condition implies that

$$4a^4+15a^2b^2+9b^4 > |b(5a^2-3b^2)\sqrt{12a^2+9b^2}|,$$

and so p_3 and p_4 are both saddles.

Now we investigate the possible saddle connections. The Hamiltonian of system (III) with $c = 0$ is

$$H(x,y) = \frac{a^2x^2}{2b} + axy + \frac{1}{2}y^2\left(b - 3x^2\right) + \frac{y^4}{4}.$$

Clearly $H(0,0) = 0$ and $H(p_3) = H(p_4) = \widetilde{H}$ with

$$\widetilde{H} = \frac{\left(b\left(\sqrt{12a^2+9b^2} - 3b\right) + 2a^2\right)\left(3b\left(\sqrt{12a^2+9b^2} + 5b\right) + 2a^2\right)}{144b^2}.$$

Solving $\widetilde{H} = 0$ we get the solutions $b = \pm a/\sqrt{6}$ which do not satisfy condition $a^2/b - 6b < 0$. So the saddle p_3 (respectively, p_4) and the saddle at the origin belong to different energy levels and therefore cannot be connected. The separatrices of the saddle at the origin decompose the Poincaré disc into several disjoint connected components. Since the system posses a symmetry with respect to the origin (i.e. $(x,y) \to (-x,-y)$), the saddles p_3 and p_4 belong to two different connected components. Therefore they cannot be connected because if they were connected their separatrices would cross at some point a separatrix of the saddle at the origin which is not possible. Since there are no saddle connections, all phase portraits of system (I) with $c = 0$, $a, b > 0$ and $a^2/b - 6b < 0$ are topologically equivalent to 1.4 of Figure 1. This phase portrait is realized for instance when $a = 1$, $b = 1$ and $c = 0$.

4.4. **Global phase portraits of system (IV).**

4.4.1. *Case $a = b = 0$ and $c < 0$.* We first we study systems (IV) when $a = b = 0$ and $c < 0$. In the local chart U_1 we get

$$\dot{u} = cv^2 + u^2(u^2 + 6), \quad \dot{v} = uv(u^2 + 3).$$

When $v = 0$ the only singular point is the origin whose linear part is zero. Applying blow-up techniques we get that the origin of U_1 is formed by two elliptic and four parabolic sectors.

In the local chart U_2 system (IV) has the form

$$\dot{u} = -cu^2v^2 - 6u^2 - 1, \quad \dot{v} = -uv(cv^2 + 3).$$

Hence, the origin of U_2 is not a singular point.

The finite singular points are $(0, 0)$ and $(\pm\sqrt{c/3}, \pm\sqrt{-c/3})$ which are not real because $c < 0$. As a result we conclude that the global phase portrait of system (IV) with $a = b = 0$ are topologically equivalent to 1.1 of Figure 1.

4.4.2. *Case $c = 0$, $b > 0$ and $a > 0$.* Now we study systems (IV) when $c = 0$, $b > 0$ and $a > 0$. On the local chart U_1 we rewrite system (IV) in the form

$$\begin{aligned} \dot{u} &= -v^2(a + bu)^2/b + u^2(u^2 + 6), \\ \dot{v} &= -v^3(a + bu) + uv(u^2 + 3). \end{aligned}$$

When $v = 0$ only the origin is a singular point whose linear part is zero. Doing blow-ups we obtain that the origin of U_1 consists in two elliptic and four parabolic sectors.

On the local chart U_2 we rewrite system (IV) in the form

$$\begin{aligned} \dot{u} &= v^2(b + au)^2/b - 6u^2 - 1, \\ \dot{v} &= v^3a(b + au)/b - 3uv. \end{aligned}$$

Clearly the origin of U_2 is not a singular point.

Now we consider the finite singular points, besides the origin, which are

$$\begin{aligned} p_{1,2} &= \pm\left(\frac{(3b + A)\sqrt{B - A}}{6\sqrt{6}\,a}, \frac{\sqrt{B - A}}{\sqrt{6}}\right), \\ p_{3,4} &= \pm\left(\frac{(3b - A)\sqrt{B + A}}{6\sqrt{6}\,a}, \frac{\sqrt{B + A}}{\sqrt{6}}\right), \end{aligned}$$

where $A = \sqrt{9b^2 - 12a^2}$ and $B = 2a^2/b + 3b$ with $b > 0$. Hence, they can exist only when $b \geq 2a/\sqrt{3}$. If $0 < b < 2a/\sqrt{3}$ there are no finite singular points besides the origin and the global phase portraits are topologically equivalent to 1.1 of Figure 1.

If $b \geq 2a/\sqrt{3}$ then all four points exist due to the fact that since $b > 0$ we have $B > 0$, so $B + A > 0$ and

$$B^2 - A^2 = (B - A)(B + A) = (2a^2/b + 3b)^2 - (-12a^2 + 9b^2) = 4a^4/b^2 + 24a^2 > 0.$$

Moreover for $b = 2|a|/\sqrt{3}$ the points p_1 and p_3 (respectively, p_2 and p_4) coincide. The eigenvalues of the linear part of system (IV) at the points $p_{1,2}$

are

$$\pm\frac{\sqrt{-4a^4+15a^2b^2-9b^4+\sqrt{3}(5a^2+3b^2)b\sqrt{-4a^2+3b^2}}}{\sqrt{3}b}.$$

We observe that

$$\begin{aligned}&(-4a^4+15a^2b^2-9b^4)^2-(\sqrt{3}(5a^2+3b^2)b\sqrt{-4a^2+3b^2})^2\\&=4a^2(4a^2-3b^2)(a^2+6b^2)^2\le 0,\end{aligned}\tag{10}$$

because $3b^2-4a^2\ge 0$. Since $b>0$, inequality (10) implies that

$$\sqrt{3}(5a^2+3b^2)b\sqrt{-4a^2+3b^2})\ge|-4a^4+15a^2b^2-9b^4|.$$

Therefore, when $b>2a/\sqrt{3}$, p_1 and p_2 are both saddles.

On the other hand, the eigenvalues of the linear part of systems (IV) at the points $p_{3,4}$ are

$$\pm\frac{\sqrt{-4a^4+15a^2b^2-9b^4-\sqrt{3}(5a^2+3b^2)b\sqrt{-4a^2+3b^2}}}{\sqrt{3}b}.$$

In view of (10) we get that, when $b>2a/\sqrt{3}$, p_3 and p_4 are both centers.

Next we investigate the possible saddle connections. The Hamiltonian of system (IV) with $c=0$ is

$$H(x,y)=\frac{a^2x^2}{2b}+axy+\frac{1}{2}y^2\left(b-3x^2\right)-\frac{y^4}{4}.$$

Clearly $H(0,0)=0$ and $H(p_1)=H(p_2)=\widetilde{H}$ with

$$\widetilde{H}=\frac{\left(b\left(3b-\sqrt{9b^2-12a^2}\right)+2a^2\right)\left(3b\left(\sqrt{9b^2-12a^2}+5b\right)-2a^2\right)}{144b^2}.$$

Since $\widetilde{H}\neq 0$ when $a\neq 0$, the saddle p_1 (respectively, p_2) and the saddle at the origin belong to different energy levels and therefore they cannot be connected. As in system (III), the saddles p_1 and p_2 cannot be connected because, by the symmetry $(x,y)\to(-x,-y)$, they belong to two different connected components of the Poincaré disc minus the separatrices of the saddle at the origin. Since there are no saddle connections, all possible global phase portraits of systems (IV) with $c=0$, $a>0$ and $b>2a/\sqrt{3}$ are topologically equivalent to 1.5 of Figure 1, which is realized for instance when $a=1$, $b=2$, $c=0$.

If $b=2a/\sqrt{3}$, both singular points p_1, p_2 are nilpotent. Note that the sum of the indices of p_1 and p_2 must be zero (see Theorems 3 and 4). Using the symmetry of the system with respect to the origin, we conclude that both singular points must have the same index and so both of them must be cusps. Therefore the global phase portraits is topologically equivalent to 1.6 of Figure 1.

4.5. **Global phase portraits of system (V).**

4.5.1. *Case $a = b = 0$ and $c > 0$.* We first study system (V) when $a = b = 0$ and $c > 0$ which become

$$\dot{x} = -3\mu x^2 y + y^3, \quad \dot{y} = cx + x^3 + 3\mu x y^2 \tag{11}$$

On the local chart U_1 we have

$$\dot{u} = cv^2 - u^4 + 6\mu u^2 + 1, \quad \dot{v} = -vu(u^2 - 3\mu).$$

When $v = 0$ the real singular points are $(\pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0)$ and the eigenvalues of the linear part are

$$\mp 4\sqrt{9\mu^2 + 1}\sqrt{3\mu + \sqrt{9\mu^2 + 1}} \text{ and } \mp \sqrt{9\mu^2 + 1}\sqrt{3\mu + \sqrt{9\mu^2 + 1}}.$$

Hence $(\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0)$ and $(-\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0)$ are an attracting node and a repelling node respectively.

In U_2 system (V) becomes

$$\dot{u} = -cu^2v^2 - u^4 - 6\mu u^2 + 1, \quad \dot{v} = -uv(cv^2 + u^2 + 3\mu)$$

and so the origin is not a singular point.

The finite singular points are the origin, $\pm(\sqrt{-c}, 0)$ (which are not real because $c > 0$) and

$$p_i = \pm(\pm\sqrt{-c/(1 + 9\mu^2)}, \sqrt{-3c\mu/(1 + 9\mu^2)}),$$

which again are not real because $c > 0$ (independently on μ being positive, negative or zero). Therefore, the unique real singular point is the origin. Hence, the global phase portrait is topologically equivalent to 1.7 of Figure 1.

4.5.2. *Case $c = 0$, $b \neq 0$, $a \geq 0$, and $(a^4 - b^4 - 6a^2b^2\mu)/b < 0$.* Having established the case $a = b = 0$ and $c > 0$, we now investigate the case $c = 0$, $b \neq 0$, $a \geq 0$, and $(a^4 - b^4 - 6a^2b^2\mu)/b < 0$ in which system (V) can be written as

$$\begin{aligned} \dot{x} &= ax + by - 3\mu x^2 y + y^3, \\ \dot{y} &= -(a^2/b)x - ay + x^3 + 3\mu x y^2. \end{aligned} \tag{12}$$

We note that without loss of generality we can assume that $b > 0$. Indeed, if we do the linear transformation $(x, y) \mapsto (-y, -x)$ system (V) becomes

$$\begin{aligned} \dot{x} &= -ax - (a^2/b)y + 3\mu x^2 y + y^3, \\ \dot{y} &= bx + ay + x^3 - 3\mu x y^2. \end{aligned} \tag{13}$$

After defining $\bar{a} = -a$, $\bar{\mu} = -\mu$ and $\bar{b} = a^2/b$ we see that system (13) is system (V) with $b \mapsto -b$ whenever $a \neq 0$. But when $a = 0$, system (12) is just system (11) with the axes interchanged, $c = b$ and $\mu \mapsto -\mu$. We know that it has a nilpotent saddle at the origin if and only if $b > 0$. So this system has already been studied and the global phase portraits are topologically equivalent to 1.7 of Figure 1. Moreover by Theorem 1 it is not restrictive to consider $a \geq 0$ and so we need to investigate the case $a > 0$ with $b > 0$. In this case condition (4) can be written as

$$a^4 - b^4 - 6a^2b^2\mu < 0 \quad \text{and} \quad b > 0. \tag{14}$$

On the local chart U_1 we have

$$\dot{u} = -v^2(a + bu)^2/b + 6u^2\mu - u^4 + 1, \quad \dot{v} = -v^3(a + bu) - uv(u^2 - 3\mu).$$

So the infinite singular points of system (12) on U_1 are the same as those of system (11): the point $(\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0)$ which is an attracting node and the point $(-\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0)$ which is a repelling node .

On U_2 we have

$$\dot{u} = v^2(b + au)^2/b - 6u^2\mu - u^4 + 1, \quad \dot{v} = v^3a(b + au)/b - uv(u^2 + 3\mu).$$

Thus the origin of U_2 is not a singular point.

The explicit expressions for the finite singular points are lengthy. We have the following lemma.

Lemma 5. *Assume that (a, b, μ) with $a > 0$ satisfy condition (14). Then, there exists at most six finite singular points for system (12). The values of (a, b, μ) for which the number of different real solutions of system (12) can change are when either $C_1(a, b, \mu) = 0$, or $C_2(a, b, \mu) = 0$ being*

$$\begin{aligned} C_1(a, b, \mu) &= -(1 - 9\mu^2)a^2 - 54b^2\mu^3, \\ C_2(a, b, \mu) &= -(1 + 6\mu^2 - 3\mu^4)a^2b^2 + 4a^4\mu^3 - 4b^4\mu^3. \end{aligned} \tag{15}$$

Proof. We compute the Groebner basis for the polynomials in (12) and we obtain nine polynomials. Recall that $y \neq 0$, because the unique singular point with $y = 0$ is the origin. One polynomial of the Groebner basis, after dividing by $-y^3/b^2$, is

$$\begin{aligned} &a^4b - b^5 - 6a^2b^3\mu - 3(b^4 + 3a^4\mu^2 + 6b^4\mu^2 - 18a^2b^2\mu^3)y^2 \\ &\quad - 3b(1 + 9\mu^2)(b^2 - 2a^2\mu + 3b^2\mu^2)y^4 - b^2(1 + 9\mu^2)^2y^6. \end{aligned}$$

Introducing the variable $z = y^2$ we get the cubic equation

$$\begin{aligned} C = b(a^4 - b^4 - 6a^2b^2\mu) - 3(b^4 + 3a^4\mu^2 + 6b^4\mu^2 - 18a^2b^2\mu^3)z \\ - 3b(1 + 9\mu^2)(b^2 - 2a^2\mu + 3b^2\mu^2)z^2 - b^2(1 + 9\mu^2)^2z^3. \end{aligned} \tag{16}$$

Note that C is always a cubic equation because the coefficient of z^3 is never zero. The discriminant of the cubic C is

$$27a^2b^4(1 + 9\mu^2)^2C_1(a, b, \mu)^2C_2(a, b, \mu),$$

where $C_1(a, b, \mu)$ and $C_2(a, b, \mu)$ are given in (15). The sign of the discriminant is the sign of $C_2(a, b, \mu)$, so if $C_2(a, b, \mu) < 0$ there is a unique real solution for $C = 0$ and if $C_2(a, b, \mu) > 0$ there are three different real solutions for $C = 0$. Moreover, if either $C_2(a, b, \mu) = 0$, or $C_1(a, b, \mu) = 0$ there are two real solutions (at least one which is double). Each positive root of $C = 0$ will give two solutions in y for systems (V). Now we study how many solutions in x we can have for each solution in y.

The Groebner basis has a polynomial in the variables x and y which is linear in x and has a coefficient in x equal to $aC_1(a,b,\mu)$ where $C_1(a,b,\mu)$ is the one defined above. If $C_1(a,b,\mu) \neq 0$, then for each y coming from a solution of the cubic equation $C = 0$ in (16), there exists a unique solution in x. When $C_1(a,b,\mu) = 0$, we take the polynomial in the Groebner basis which is also linear in x and whose coefficient in x is $a\widetilde{C}_1(a,b,\mu)$ with $\widetilde{C}_1(a,b,\mu) = 2y^2 - 3a^2\mu/b + 2b(9\mu^2+1)$. If $C_1(a,b,\mu) = 0$ and $\widetilde{C}_1(a,b,\mu) \neq 0$, there exists also a unique solution in x for each solution in y coming from the cubic equation $C = 0$. Finally, we analyze the case $C_1(a,b,\mu) = 0$ and $\widetilde{C}_1(a,b,\mu) = 0$. Solving system $C_1(a,b,\mu) = 0$ and $\widetilde{C}_1(a,b,\mu) = 0$ we get the solutions

$$b = b_1 = \frac{a}{3\sqrt{6}}\sqrt{\frac{-1+9\mu^2}{\mu^3}}, \qquad y = y_1 = \pm\frac{a}{\sqrt{3\sqrt{6}}}\sqrt{\frac{1}{\mu^3 a}\sqrt{\frac{\mu^3}{-1+9\mu^2}}},$$

which are defined for $\mu > 1/3$ (note that if $\mu = 0$, then $C_1(a,b,\mu) \neq 0$). We substitute each solution in y into the remaining polynomials of the Groebner basis and we obtain two different solutions for x

$$x = \frac{a\mu(-1+9\mu^2)y_1 \pm \sqrt{a^2\mu^2(1+9\mu^2)b_1}}{6\mu^2 b_1}.$$

In short, there exists at most six finite singular points for system (V). We also point out that the unique values of (a,b,μ) for which the number of different real solutions of system (12) can change is when either $C_2(a,b,\mu) = 0$, or $C_1(a,b,\mu) = \widetilde{C}_1(a,b,\mu) = 0$ (for this last case one must have $\mu > 1/3$). This proves the lemma. □

In view of section 2, the finite singular points are either elementary or nilpotent. Hence they are either centers, saddles or cusps. We have the following lemma.

Lemma 6. *Assume that* (a,b,μ) *with* $a > 0$ *satisfy condition* (14). *Then, there exist at most two non–elementary finite singular points of systems* (12). *They are nilpotent, different from the origin and occur when the parameters* (a,b,μ) *satisfy* $C_2(a,b,\mu) = 0$ *with* $C_2(a,b,\mu)$ *as in* (15).

Proof. We compute the Groebner basis for the polynomials in (12) together with the determinant of the linear parts of (12) and we obtain sixteen polynomials. Four of them suffices to prove our claim. The first one is

$$y^2(a^4 - b^4 - 6a^2b^2\mu)^2 C_2(a,b,\mu).$$

Hence, since by assumptions $a^4 - b^4 - 6a^2b^2\mu < 0$ in order that the solution is non–elementary we must have $(a,b,\mu) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ such that $C_2(a,b,\mu) = 0$.

The second one, once divided by y^2, is a quadratic polynomial only in the variable y, where the coefficient of y^2 is

$$5(1+\mu^2)(-1+3\mu^2)^2(1+9\mu^2)^2$$

and the third is a polynomial in the variables x and y, linear in x with the coefficient $a(1+9\mu^2)$ which is always different from zero because $a \neq 0$. Hence, when $3\mu^2 \neq 1$ system (V) can has at most two nilpotent finite singular points and they must satisfy $C_2(a,b,\mu)=0$. When $\mu=\pm 1/\sqrt{3}$, condition $C_2(a,b,\mu)=0$ implies $b=a/\sqrt{2\pm\sqrt{3}}$, and substituting these values of b and μ into the polynomials of the Groebner basis we find the polynomial

$$\pm\frac{6400y^4\left(2\sqrt{2\pm\sqrt{3}}\,(209\sqrt{3}\pm 362)\,y^2\mp(627\pm 362\sqrt{3})\,|a|\right)}{\left(2\pm\sqrt{3}\right)^{11/2}},$$

which also has at most two solutions different from zero. In short, system (V) can has at most two nilpotent finite singular points different from the origin and they must satisfy $C_2(a,b,\mu)=0$. This ends the proof. □

We introduce the variable $\bar b=b/a$. Note that the curves $C_1(a,b,\mu)=0$, $C_2(a,b,\mu)=0$ and the condition in (14) can be written as

$$C_1(\bar b,\mu)=-1+9\mu^2-54\bar b^2\mu^3=0,$$
$$C_2(\bar b,\mu)=-(1+6\mu^2-3\mu^4)\bar b^2+4\mu^3-4\bar b^4\mu^3=0,$$

and

$$C_3(\bar b,\mu)=1-\bar b^4-6\bar b^2\mu<0,\quad \bar b>0,$$

respectively.

Let

$$\bar b=\bar b_1(\mu)=\frac{1}{3\sqrt{6}}\sqrt{\frac{9\mu^2-1}{\mu^3}},$$
$$\bar b=\bar b_2^{\pm}(\mu)=\frac{1}{2\sqrt{2}}\sqrt{\frac{-1-6\mu^2+3\mu^4\pm\sqrt{(1+\mu^2)^3(1+9\mu^2)}}{\mu^3}},$$
$$\bar b=\bar b_3(\mu)=\sqrt{-3\mu+\sqrt{1+9\mu^2}},$$

be the solutions of $C_1(\bar b,\mu)=0$, $C_2(\bar b,\mu)=0$ and $C_3(\bar b,\mu)=0$ respectively. Studying the behavior of these curves $C_1(\bar b,\mu)=0$, $C_2(\bar b,\mu)=0$ and $C_3(\bar b,\mu)=0$ on the plane $(\bar b,\mu)$ we see that these curves divide the region S into four different regions (see Figure 3) which are

$$R_1=R_{11}\cup R_{12},\qquad R_2=R_{21}\cup R_{22},\qquad R_3=R_{31}\cup R_{32}\cup R_{33},$$
$$R_4=\{(\bar b,\mu):\bar b_3(\mu)<\bar b<\bar b_2^+(\mu),\mu\in(1/\sqrt{3},+\infty)\},$$

where

$$R_{11}=\{(\bar b,\mu):\bar b>\bar b_3(\mu),\mu\in(-\infty,-1/\sqrt{3})\},$$
$$R_{12}=\{(\bar b,\mu):\bar b>\bar b_2^-(\mu),\mu\in[-1/\sqrt{3},0)\},$$
$$R_{21}=\{(\bar b,\mu):\bar b_3(\mu)<\bar b<\bar b_2^-(\mu),\mu\in(-1/\sqrt{3},-1/(3\sqrt{3}))\},$$
$$R_{22}=\{(\bar b,\mu):\bar b_1(\mu)<\bar b<\bar b_2^-(\mu),\mu\in[-1/(3\sqrt{3}),0)\},$$
$$R_{31}=\{(\bar b,\mu):\bar b_3(\mu)<\bar b<\bar b_1(\mu),\mu\in(-1/(3\sqrt{3}),0)\},$$
$$R_{32}=\{(\bar b,\mu):\bar b>\bar b_3(\mu),\mu\in[0,1/\sqrt{3})\},$$
$$R_{33}=\{(\bar b,\mu):\bar b>\bar b_2^+(\mu),\mu\in[1/\sqrt{3},+\infty)\}.$$

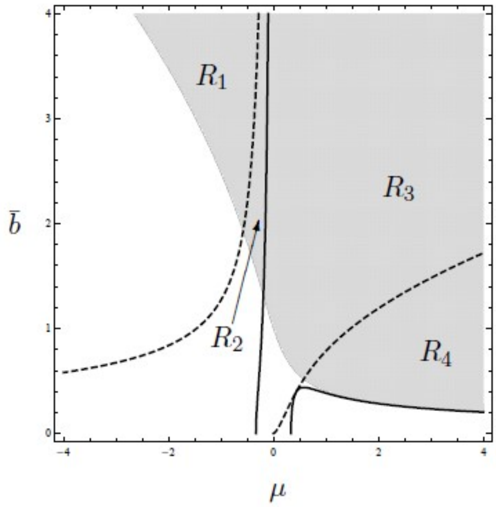

FIGURE 3. The region S in gray. The curves $C_1(\bar{b}, \mu) = 0$ (thick continuous line), $C_2(\bar{b}, \mu) = 0$ (dashed lines) and $C_3(\bar{b}, \mu) = 0$ (thin continuous line).

We compute the number of finite singular points of system (12) other than the origin in each region R_i and on the curves that delimit these regions. We get: no finite singular points in the regions R_1, R_2 and R_3; two centers and two saddles in the region R_4; no finite singular points on $\{(\bar{b}_2^-(\mu), \mu)\} \cap S$ and $\{(\bar{b}_1(\mu), \mu)\} \cap S$; and two cusps on $\{(\bar{b}_2^+(\mu), \mu)\} \cap S$ (the last assertion comes from Lemma 6, the symmetry and the fact that the sum of the indices of the two finite singular points must be zero due to Theorems 3 and 4). So the unique phase portrait of system (V) on the regions R_1, R_2, R_3 and on $\{(\bar{b}_2^-(\mu), \mu)\} \cap S$ and $\{(\bar{b}_1(\mu), \mu)\} \cap S$ is topologically equivalent to 1.7 of Figure 1. On the region R_4 the saddle at the origin cannot be connected with the other saddles because they belong to different energy levels. Indeed, the Hamiltonian of system (V) with $c = 0$ is

$$H(x, y) = \frac{a^2x^2}{2b} + axy + \frac{1}{2}y^2\left(b - 3\mu x^2\right) - \frac{x^4}{4} + \frac{y^4}{4}.$$

Computing the Groëbner Basis of the polynomials x', y' and $H(x, y)$, we get 9 polynomials, one of them is

$$y^3\left(a^4 - b^4 - 6a^2b^2\mu\right)^2.$$

Since $\left(a^4 - b^4 - 6a^2b^2\mu\right)/b < 0$, $H(0, 0) = 0$, and the unique singular point with $y = 0$ is the origin, we conclude that the saddles that are not at the origin belong to an energy level different from zero. As in the previous systems, the saddles that are not at the origin cannot be connected to each other because they belong to two different connected components of the Poincaré disc minus the separatrices of the saddles at the origin. In short, there is no saddle connections. Then the only realizable global phase portrait on R_4 is topologically equivalent to 1.8 of Figure 1 and it is realized for instance when $a = 1$, $b = 0.8$, $c = 0$ and $\mu = 2.5$.

Note that the two cusps on $\{(\bar{b}_2^+(\mu), \mu)\} \cap S$ are formed when the two centers and the two saddles in the region R_4 coalesce and so the unique possible global phase portrait is topologically equivalent to 1.9 of Figure 1.

4.6. **Global phase portraits of system (VI).**

4.6.1. *Case $a = b = 0$ and $c < 0$.* First we consider system (VI) when $a = b = 0$ and $c < 0$. In the local chart U_1 system (VI) can be written as

$$\dot{u} = cv^2 + u^4 + 6u^2\mu + 1, \quad \dot{v} = uv(u^2 + 3\mu).$$

When $v = 0$, the possible singular points of system (VI) are the four points $(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2 - 1}}, 0)$. When $\mu < -1/3$ the four points exist. Computing the eigenvalues of the jacobian matrix at these points we get that $(\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0)$ and $(-\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0)$ are repelling nodes while $(-\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0)$ and $(\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0)$ are attracting nodes.

If $\mu = -1/3$ there are only two singular points which are $(\pm 1, 0)$ and they are linearly zero. Doing blow ups we get that both singular points are formed by two elliptic and four parabolic sectors.

Finally, if $\mu > -1/3$ there are no infinite singular points in the local chart U_1.

In U_2 system (VI) becomes

$$\dot{u} = -cu^2v^2 - u^4 - 6u^2\mu - 1, \quad \dot{v} = -uv(u^2 + cv^2 + 3\mu),$$

and we see that the origin is not a singular point.

The finite singular points of system (VI) are the origin, $q_i = \pm(\sqrt{-c}, 0)$ for $i = 1, 2$ and

$$p_i = \pm\big(\pm \sqrt{c/(9\mu^2 - 1)}, \sqrt{-3c\mu/(9\mu^2 - 1)}\big),$$

for $i = 1, 2, 3, 4$ and $\mu \neq \pm 1/3$. The points q_1, q_2 are always real because $c < 0$. Moreover, computing the eigenvalues at q_1, q_2 we get that they are $\pm i\sqrt{6}c\sqrt{\mu}$. So, they are centers if $\mu > 0$ and saddles if $\mu < 0$. If $\mu = 0$, q_1 and q_2 are nilpotent centers. The singular points p_i are real whenever $\mu \in (-1/3, 0)$, otherwise they are not real. In this case, they are all centers since the eigenvalues are $\pm 2c\sqrt{3\mu/(1 - 9\mu^2)}$.

As in the previous cases, we can prove that there is no saddle connections and so the unique possible global phase portraits that are realized are topologically equivalent to the following ones of Figure 1: 1.10 when $\mu < -1/3$ (it is realized for instance when $a = b = 0$, $c = -1$ and $\mu = -1$); 1.11 when $\mu = -1/3$ (it is realized for instance when $a = b = 0$, $c = -1$); 1.12 when $\mu \in (-1/3, 0)$ (it is realized for instance when $a = b = 0$, $c = -1$ and $\mu = -1/4$); and 1.13 when $\mu \geq 0$ (it is realized for instance when $a = b = 0$, $c = -1$ and $\mu = 0$).

4.6.2. *Case $c = 0$, $b \neq 0$, $a \geq 0$, and $(a^4 + b^4 + 6a^2b^2\mu)/b > 0$.* Now we assume that $c = 0$, $b \neq 0$, $a \geq 0$ and condition (5) holds. In this case system (VI) is

$$\begin{aligned} \dot{x} &= ax + by - 3\mu x^2 y - y^3, \\ \dot{y} &= -(a^2/b)x - ay + x^3 + 3\mu x y^2. \end{aligned} \tag{17}$$

The infinite singular points in this case are the same as those for the case $a = b = 0$ and $c < 0$.

We will study the finite singular points in the same way as we did for system (V) in case $b \neq 0$.

First we study the cases $a = 0$, $\mu = -1/3$ and $\mu = 1/3$. When $a = 0$ the finite singular points of system (VI) are the origin, $q_i = \pm(0, \sqrt{b})$ for $i = 1, 2$ and

$$p_i = \pm\big(\sqrt{3b\mu/(9\mu^2 - 1)}, \pm\sqrt{-b/(9\mu^2 - 1)}\big)$$

for $i = 1, 2, 3, 4$ and $\mu \neq \pm 1/3$. Notice that condition (5) for $a = 0$ becomes $b > 0$. So we are only interested in singular points with $b > 0$. The points q_1, q_2 are real for $b > 0$. Computing the eigenvalues at q_1, q_2 we get that they are $\pm b\sqrt{-6\mu}$. So q_1, q_2 are centers when $\mu > 0$ and saddles when $\mu < 0$. If $\mu = 0$, q_1, q_2 are nilpotent centers. On the other hand, the points p_i are real for $b > 0$ and $\mu \in (-1/3, 0]$. The eigenvalues at p_i are $\pm 2\sqrt{3}b\sqrt{\mu/(1 - 9\mu^2)}$, so they are centers if $\mu \neq 0$. When $\mu = 0$ the points p_i coincide with the points q_i. Proceeding in a similar way than in system (VI) with $a = b = 0$ and $c < 0$, we conclude that the unique global phase portraits are topologically equivalent to the following ones of Figure 1: 1.10 when $\mu < -1/3$; to 1.12 when $\mu \in (-1/3, 0)$ and to 1.13 when $\mu \geq 0$.

When $\mu = -1/3$ the finite singular points of system (VI) are the origin and $q_i = \pm(-a/\sqrt{b}, \sqrt{b})$ for $i = 1, 2$, which are real for $b > 0$. Condition (5) for $\mu = -1/3$ becomes $(a^2 - b^2)^2/b > 0$, so $b > 0$ and $a \neq \pm b$. Computing the eigenvalues at q_1, q_2 we get that they are $\pm\sqrt{2(a^2 - b^2)^2/b^2}$ and so they are saddles. Proceeding as in the previous cases, we analyze the realizable phase portraits taking into account the possible saddle connections and we conclude that the unique global phase portraits are topologically equivalent to 1.11 of Figure 1 which is realized for instance when $a = 1$, $b = 2$ $c = 0$ and $\mu = -1/3$.

When $\mu = 1/3$ the finite singular points of system (VI) are the origin and $q_i = \pm(a/\sqrt{b}, \sqrt{b})$ for $i = 1, 2$, which are real for $b > 0$. Condition (5) for $\mu = 1/3$ becomes $(a^2 + b^2)^2/b > 0$, so $b > 0$. Computing the eigenvalues at q_1, q_2 we get that they are $\pm\sqrt{-2(a^2 + b^2)^2/b^2}$ and so they are centers. All global phase portraits are topologically equivalent to 1.13 of Figure 1.

Assume now that $a > 0$ and $\mu \notin \{-1/3; 1/3\}$. Proceeding as we did for the finite singular points of systems (V) we have the following lemma.

Lemma 7. *Assume that (a, b, μ) with $a > 0$ and $\mu \notin \{-1/3; 1/3\}$ satisfy* (5). *Then, there exists at most six finite singular points, different from the origin, for system (VI). The values of (a, b, μ) for which the number of different real solutions of system* (17) *can change are when either $C_1(a, b, \mu) = 0$, or*

$C_2(a,b,\mu)=0$ *with*

$$
\begin{aligned}
C_1(a,b,\mu) &= a^2\left(9\mu^2+1\right)+54b^2\mu^3,\\
C_2(a,b,\mu) &= -\left(1-6\mu^2-3\mu^4\right)a^2b^2-4a^4\mu^3-4b^4\mu^3.
\end{aligned}
\tag{18}
$$

Proof. We compute the Groebner basis for the polynomials in (17) and we obtain nine polynomials. One of the polynomials, after dividing by $-y^3/b^2$ ($y\neq 0$, because the unique singular point with $y=0$ is the origin) is

$$
\begin{aligned}
&b\left(a^4+6a^2b^2\mu+b^4\right)-3\left(3a^4\mu^2+18a^2b^2\mu^3+b^4\left(1-6\mu^2\right)\right)y^2\\
&\quad+3b\left(9\mu^2-1\right)\left(2a^2\mu+b^2\left(3\mu^2-1\right)\right)y^4-b^2\left(1-9\mu^2\right)^2y^6.
\end{aligned}
$$

Introducing the variable $z=y^2$ we get the cubic equation

$$
\begin{aligned}
C &= b\left(a^4+6a^2b^2\mu+b^4\right)-3\left(3a^4\mu^2+18a^2b^2\mu^3+b^4\left(1-6\mu^2\right)\right)z\\
&\quad+3b\left(9\mu^2-1\right)\left(2a^2\mu+b^2\left(3\mu^2-1\right)\right)z^2-b^2\left(1-9\mu^2\right)^2z^3.
\end{aligned}
\tag{19}
$$

Since we are in the case $\mu\neq\pm 1/3$, C is always a cubic equation. Recall that the number of finite singular points can change when $\mu=\pm 1/3$. The discriminant of the cubic C is

$$27a^2b^4(1-9\mu^2)^2(C_1(a,b,\mu))^2C_2(a,b,\mu),$$

where $C_1(a,b,\mu)$ and $C_2(a,b,\mu)$ are given in (18). The sign of the discriminant is the sign of $C_2(a,b,\mu)$. Hence, if $C_2(a,b,\mu)<0$ there is a unique real solution for $C=0$ and if $C_2(a,b,\mu)>0$ there are three different real solutions for $C=0$. Moreover, if either $C_2(a,b,\mu)=0$ or $C_1(a,b,\mu)=0$ there are two real solutions (at least one which is double). Each positive root of $C=0$ will give two solutions in y for system (VI). Now we find how many solutions we can have in the variable x for each solution in the variable y.

The Groebner basis has a polynomial in the variables x and y, linear in x with the coefficient in x equal to $aC_1(a,b,\mu)$ where $C_1(a,b,\mu)$ is given in (18). If $C_1(a,b,\mu)\neq 0$, for each y coming from a solution of the cubic equation $C=0$ in (19), there exists a unique solution in x. When $C_1(a,b,\mu)=0$, we take the polynomial in the Groebner basis which is also linear in x and whose coefficient in x is $a\widetilde{C}_1(a,b,\mu)$ with $\widetilde{C}_1(a,b,\mu)=2y^2+3a^2\mu/b+2b(9\mu^2-1)$. If $C_1(a,b,\mu)=0$ and $\widetilde{C}_1(a,b,\mu)\neq 0$, there exists also a unique solution in x for each solution y coming from the cubic equation $C=0$. Finally, we analyze the case $C_1(a,b,\mu)=0$ and $\widetilde{C}_1(a,b,\mu)=0$. Solving system $C_1(a,b,\mu)=\widetilde{C}_1(a,b,\mu)=0$ we get the solution

$$b=b_1=\frac{a}{3\sqrt{6}}\sqrt{-\frac{1+9\mu^2}{\mu^3}},\qquad y=y_1=\pm\sqrt{-\frac{a}{3\sqrt{6}\mu^3}\sqrt{-\frac{\mu^3}{1+9\mu^2}}},$$

defined for $\mu<0$ (note that if $\mu=0$, then $C_1(a,b,\mu)\neq 0$) and the solution

$$b=-\frac{a}{3\sqrt{6}}\sqrt{-\frac{1+9\mu^2}{\mu^3}},\qquad y=\pm\sqrt{\frac{a}{3\sqrt{6}\mu^3}\sqrt{-\frac{\mu^3}{1+9\mu^2}}},$$

which is always complex. We substitute each solution y into the remaining polynomials of the Groebner basis and we obtain the two different solutions for x

$$x = \frac{a(\mu(1+9\mu^2)y_1 \pm \sqrt{\mu^2(-1+9\mu^2)b_1})}{6\mu^2 b_1}.$$

In short, there exists at most six finite singular points for system (VI). This proves the claim. We observe that the number of different real solutions of system (17) can change when $\mu = \pm 1/3$, or when (a,b,μ) satisfy $C_2(a,b,\mu) = 0$, or $C_1(a,b,\mu) = \widetilde{C}_1(a,b,\mu) = 0$ (this last case corresponds to $\mu < 0$). This ends the proof. □

In view of Section 2, the finite singular points are either elementary or nilpotent. Hence they are either centers, saddles or cusps.

Lemma 8. *Assume that* (a,b,μ) *with* $a > 0$ *and* $\mu \notin \{-1/3; 1/3\}$ *satisfy* (5). *Then, there exist at most two non–elementary finite singular points of systems* (17) *which are nilpotent, different from the origin and occur when the parameters* (a,b,μ) *satisfy* $C_2(a,b,\mu) = 0$*, with* $C_2(a,b,\mu)$ *given in* (18).

Proof. We compute the Groebner basis for the polynomials in (17) together with the determinant of the linear parts of (17) and we obtain sixteen polynomials. One of the polynomials of the Groebner basis is

$$-y^2(bC_3(a,b,\mu))^2 C_2(a,b,\mu),$$

where $C_3(a,b,\mu) = (a^4 + b^4 + 6a^2b^2\mu)/b$. Hence, since by condition (5) $C_3(a,b,\mu) \neq 0$ in order that the solution is non–elementary we must have $(a,b,\mu) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$ such that $C_2(a,b,\mu) = 0$.

The Groebner basis has a polynomial in the variable y that, once divided by y^2, becomes $Ay^2 + B$ where

$$A = 5(-1+\mu^2)(-1+9\mu^2)^2(1+3\mu^2)^2,$$

and

$$\begin{aligned} b^9 B = {} & 16a^{10}\left(2\mu^7 - 8\mu^5 + \mu^3\right) + 4a^8b^2\left(90\mu^8 - 392\mu^6 + 95\mu^4 - 14\mu^2 + 1\right) \\ & + 16a^6b^4\mu\left(54\mu^8 - 308\mu^6 + 161\mu^4 - 40\mu^2 + 3\right) \\ & + a^4b^6\left(-864\mu^{10} - 735\mu^8 + 5132\mu^6 - 1490\mu^4 + 28\mu^2 + 9\right) \\ & + 2a^2b^8\mu\left(99\mu^8 - 284\mu^6 + 938\mu^4 - 340\mu^2 + 27\right) + \\ & b^{10}\left(57\mu^8 - 68\mu^6 + 146\mu^4 - 60\mu^2 + 5\right). \end{aligned}$$

It has also a polynomial in the variables x and y, linear in x with the coefficient $a(-1+9\mu^2)$. Recall that we have assumed that $\mu \neq \pm 1/3$. Hence if $\mu \neq \pm 1$ systems (VI) can have at most two nilpotent finite singular points and they must satisfy $C_2(a,b,\mu) = 0$. If $\mu = 1$, then $A = 0$ and $B = -80(bC_3(a,b,\mu))^2(a^2 - b^2)$. Assume $b = \pm a$ (this condition comes from $C_2(a,b,1) = 0$). Then, we find the following polynomial in the Groebner basis

$$6400y^2\left(a \mp 2y^2\right)^2.$$

Hence if $\mu = 1$ systems (VI) can have at most two nilpotent finite singular points and they must satisfy $C_2(a,b,\mu) = 0$. If $\mu = -1$ then $B =$

$80(bC_3(a,b,\mu))^2(a^2+b^2)$. Since by condition (5) we have $C_3 \neq 0$ and $B \neq 0$ we conclude that there are no solutions in this case. This ends the proof of the lemma. □

We introduce the variable $\bar{b} = b/a$ (recall $a > 0$). Note that the curves $C_1(a,b,\mu) = 0$, $C_2(a,b,\mu) = 0$ and the condition in (5) ($C_3(a,b,\mu) > 0$) can be written as

$$C_1(\bar{b},\mu) = 1+9\mu^2 54\bar{b}^2\mu^3 = 0,\ C_2(\bar{b},\mu) = -(1-6\mu^2-3\mu^4)\bar{b}^2-4\mu^3-4\bar{b}^4\mu^3 = 0,$$

and

$$C_3(\bar{b},\mu) = \frac{1+\bar{b}^4+6\bar{b}^2\mu}{\bar{b}} > 0.$$

Let

$$\bar{b} = \pm\bar{b}_1(\mu) = \pm\frac{1}{3\sqrt{6}}\sqrt{-\frac{9\mu^2+1}{\mu^3}},$$

$$\bar{b} = \pm\bar{b}_2^{\pm}(\mu) = \pm\frac{1}{2\sqrt{2}}\sqrt{\frac{-1+6\mu^2+3\mu^4 \pm \sqrt{(-1+\mu^2)^3(-1+9\mu^2)}}{\mu^3}},$$

$$\bar{b} = \pm\bar{b}_3^{\pm}(\mu) = \pm\sqrt{-3\mu \pm \sqrt{-1+9\mu^2}},$$

be the solutions of $C_1(\bar{b},\mu) = 0$, $C_2(\bar{b},\mu) = 0$ and $C_3(\bar{b},\mu) = 0$ respectively. After studying the behavior of the curves $C_1(\bar{b},\mu) = 0$, $C_2(\bar{b},\mu) = 0$ and $C_3(\bar{b},\mu) = 0$ on the plane $(\bar{b},\mu)$ we conclude that these curves divide the region S into the point $p_0 = \{(\bar{b},\mu) = (1,1)\}$ and ten different regions (see Figure 4) which are

$$\begin{aligned}
R_1 &= \{(\bar{b},\mu) : -\bar{b}_3^{+}(\mu) < \bar{b} < \bar{b}_1(\mu), \mu < -1/3\},\\
R_2 &= \{(\bar{b},\mu) : -\bar{b}_1(\mu) < \bar{b} < -\bar{b}_3^{-}(\mu), \mu < -1/3\},\\
R_3 &= \{(\bar{b},\mu) : 0 < \bar{b} < \bar{b}_3^{-}(\mu), \mu < -1/3\},\\
R_4 &= \{(\bar{b},\mu) : 0 < \bar{b} < \bar{b}_2^{+}(\mu), -1/3 < \mu < 0\},\\
R_5 &= \{(\bar{b},\mu) : \bar{b} > \bar{b}_3^{+}(\mu), \mu < -1/3\},\\
R_6 &= \{(\bar{b},\mu) : \bar{b} > \bar{b}_2^{-}(\mu), -1/3 < \mu < 0\},\\
R_7 &= \{(\bar{b},\mu) : \bar{b}_1(\mu) < \bar{b} < \bar{b}_2^{-}(\mu), -1/3 < \mu < 0\},\\
R_8 &= R_{81} \cup R_{82}, \qquad R_9 = R_{91} \cup R_{92} \cup R_{93},\\
R_{10} &= \{(\bar{b},\mu) : \bar{b}_2^{-}(\mu) < \bar{b} < \bar{b}_2^{+}(\mu), 1 < \mu\},
\end{aligned}$$

where

$$\begin{aligned}
R_{81} &= \{(\bar{b},\mu) : \bar{b}_2^{+}(\mu) < \bar{b} < \bar{b}_1(\mu), -1/3 < \mu < 0\},\\
R_{82} &= \{(\bar{b},\mu) : \bar{b} > 0, 0 \leq \mu < 1/3\},\\
R_{91} &= \{(\bar{b},\mu) : \bar{b} > 0, 1/3 < \mu < 1\},\\
R_{92} &= \{(\bar{b},\mu) : 0 < \bar{b} < \bar{b}_2^{-}(\mu), 1 \leq \mu < +\infty\},\\
R_{93} &= \{(\bar{b},\mu) : \bar{b} > \bar{b}_2^{+}(\mu), 1 \leq \mu < +\infty\},
\end{aligned}$$

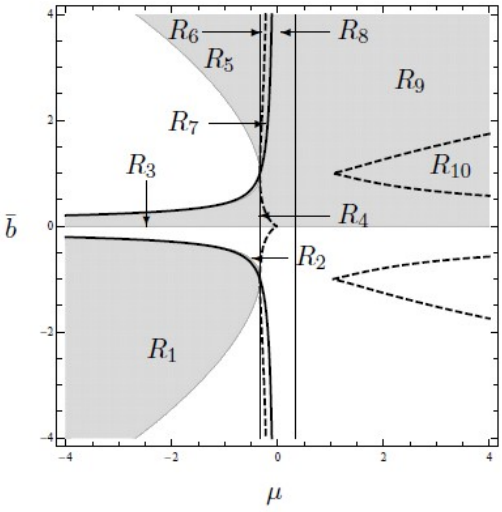

FIGURE 4. The region S in gray. The curves $C_1(\bar{b}, \mu) = 0$ (thick continuous line), $C_2(\bar{b}, \mu) = 0$ (dashed lines) and $C_3(\bar{b}, \mu) = 0$ (thin continuous line). The regions R_i for $i = 1, \dots, 10$.

We compute the number of finite singular points of systems (VI) other than the origin in each region R_i and on the curves that delimit these regions. We get: two saddles on R_1, R_2, R_3, R_5, and $\{(-\bar{b}_1(\mu), \mu)\} \cap S$ (proceeding as in the previous cases with possible saddle connections we conclude that the global phase portraits are topologically equivalent to 1.10 of Figure 1); two centers on R_7, R_8, R_9 and $\{(\bar{b}_1(\mu), \mu)\} \cap S$ (the global phase portraits in these regions are topologically equivalent to 1.13 in Figure 1); two saddles and four centers on R_4, R_6 and R_{10} (proceeding again as in the previous cases with possible saddle connections we conclude that the global phase portraits are topologically equivalent to 1.12 of Figure 1); and two centers and two cusps on $\{(\bar{b}_2^{\pm}(\mu), \mu)\} \cap S$ (this last case comes from the fact that the sum of the indices of the four finite singular points (none of them being the origin) must be four (see Theorems 3 and 4), that at most we have two cusps (see Lemma 8) and the symmetry with respect to the origin). Hence, on $\{(\bar{b}_2^{\pm}(\mu), \mu)\} \cap S$, the unique global phase portrait is topologically equivalent to 1.14 in Figure 1). Finally, when $(\bar{b}, \mu) = p_0$ that is, $b = a$ and $\mu = 1$, by direct computations we get the two finite singular points $(\pm\sqrt{a}/\sqrt{2}, \pm\sqrt{a}/\sqrt{2})$ besides the origin. They are nilpotent singular points and using Theorem 3.5 in [7] we conclude that they are nilpotent centers. Then, the unique global phase portrait is topologically equivalent to 1.13 in Figure 1.

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