

# ON THE LIMIT CYCLE OF A BELOUSOV–ZABOTINSKY DIFFERENTIAL SYSTEMS

JAUME LLIBRE AND REGILENE OLIVEIRA

**ABSTRACT.** The authors in [7] shown numerically the existence of a limit cycle surrounding the unstable node that system (1) has in the positive quadrant for specific values of the parameters. System (1) is one of the Belousov–Zabotinsky dynamical models. The objective of this paper is to prove that system (1), when in the positive quadrant  $Q$  has an unstable node or focus, has at least one limit cycle and, when  $f = 2/3$ ,  $q = \epsilon^2/2$  and  $\epsilon > 0$  sufficiently small this limit cycle is unique.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the most studied chemical oscillation systems is the Belousov-Zhabotinsky (BZ) reaction, which was elucidated by 20 chemical equations to explain the reaction mechanism and was simplified to three-variable differential equations. Many works about physical and chemical mechanism, numerical simulation and experimental research on BZ reaction appeared, see for instance [1, 3, 4, 9]. After the 1990s, the slow-fast oscillation was found in many chemical reactions the reason was that the catalyst could make the reaction process involve in different time scales with large gap. But most of the researches on the slow-fast oscillation in chemical reaction were limited to numerical simulation and experimental investigation.

One of the BZ dynamical models given as a slow-fast system is the following

$$(1) \quad \epsilon \dot{x} = x(1-x) + \frac{f(g-x)}{q+x}y, \quad \dot{y} = x-y,$$

where the parameters  $f$  and  $q$  are positive and  $\epsilon > 0$  is sufficiently small. As usual the dot denotes derivative with respect to the time  $t$ .

In [7] the authors shown numerically the existence of a limit cycle surrounding the unstable node that this system has in the positive quadrant  $Q = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$ , for the values of the parameters  $f = 2/3$ ,  $\epsilon = 1/25$  and  $q = \epsilon^2/3$ .

The objective of this paper is to prove that system (1), when in the positive quadrant  $Q$  has an unstable node or focus, then first it has at least one limit cycle, and second, that when  $f = 2/3$ ,  $q = \epsilon^2/2$  and  $\epsilon > 0$  sufficiently small this limit cycle is unique. More precisely, our main results are the following two theorems:

**Theorem 1.** *The BZ differential system (1) when for  $f, q$  and  $\epsilon > 0$  has an unstable focus or node in the first quadrant  $Q$ , then it has at least one limit cycle in  $Q$ .*

**Theorem 2.** *The BZ differential system (1) for  $f = 2/3$ ,  $q = \epsilon^2/3$  and  $\epsilon > 0$  sufficiently small has a unique stable limit cycle in the quadrant  $Q$ , which is the unique limit cycle of the system.*

---

*Key words and phrases.* limit cycles, planar differential systems, Poincaré compactification, slow-fast systems.

Note that Theorem 2 provides an analytic proof of the existence and uniqueness of the limit cycle detected numerically by Leonov and Kutnetsov in their example 1.3 of [7].

Theorems 1 and 2 are proved in sections 2 and 3, respectively.

## 2. PROOF OF THEOREM 1

With the change of time  $t = \epsilon\tau$  system (1) becomes

$$(2) \quad x' = x(1-x) + \frac{f(g-x)}{q+x}y, \quad y' = \epsilon(x-y),$$

where the prime denotes the derivative with respect to the new time  $\tau$ . Now doing other change of time  $d\tau = (q+x)ds$  system (2) can be written as

$$(3) \quad \dot{x} = x(1-x)(q+x) + f(g-x)y, \quad \dot{y} = \epsilon(x-y)(q+x),$$

where the dot denotes derivative with respect the times  $s$ .

Since  $q+x > 0$  when  $x \geq 0$ , the orbits of system (2) and (3) in the quadrant  $Q$  are the same, they only are run in different times. So to prove Theorem 1 it is sufficient to show that system (3) has a limit cycle in the quadrant  $Q$ .

Due to the fact that system (3) is polynomial we can compactify it in the Poincaré disc  $\mathbb{D}$ . This disk is the closed disc of radius one and center in the origin of coordinates. The plane  $\mathbb{R}^2$  where is defined system (3) is diffeomorphic to the interior of  $\mathbb{D}$ , and its boundary  $\mathbb{S}^1$  corresponds to the infinity of  $\mathbb{R}^2$ . System (3) is defined in the interior of  $\mathbb{D}$ . System (3) through a diffeomorphism is defined in the interior of  $\mathbb{D}$  and can be extended to the closed disc  $\mathbb{D}$  in a unique analytic way in such manner that  $\mathbb{S}^1$  boundary of  $\mathbb{D}$  is invariant by the extended flow i.e. if an orbit of the extend flow has a point in  $\mathbb{S}^1$  the whole orbit is contained in  $\mathbb{S}^1$ . This extension is called the Poincaré compactification, for more details see Chapter 5 of [2].

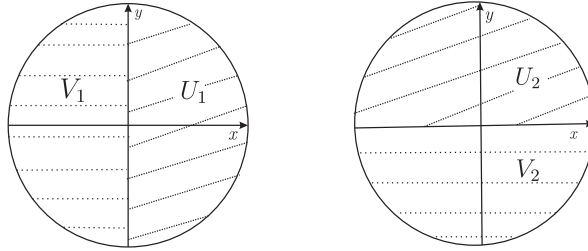


FIGURE 1. The local charts  $U_i$  and  $V_i$ , for  $i = 1, 2$  on the Poincaré disc  $\mathbb{D}$ .

In order to work with the Poincaré disc we need four local charts. Let

$$\begin{aligned} U_1 &= \{(x, y) \in \mathbb{R}^2 : x > 0\}, & U_2 &= \{(x, y) \in \mathbb{R}^2 : x < 0\}, \\ V_1 &= \{(x, y) \in \mathbb{R}^2 : y > 0\}, & V_2 &= \{(x, y) \in \mathbb{R}^2 : y < 0\}. \end{aligned}$$

We define  $\phi_k : U_k \rightarrow \mathbb{D}$  and  $\psi_k : V_k \rightarrow \mathbb{D}$  for  $k = 1, 2$  as follows

$$\phi_1(x, y) = \left( \frac{y}{x}, \frac{1}{x} \right) = (u, v), \quad \phi_2(x, y) = \left( \frac{1}{y}, \frac{x}{y} \right) = (u, v),$$

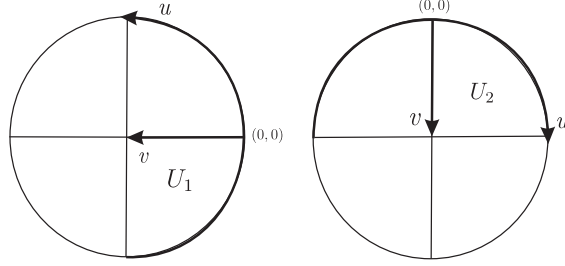


FIGURE 2. The coordinates  $(u, v)$  in the local charts  $U_1$  and  $U_2$ .

and  $\psi_k = -\phi_k$ ,  $k = 1, 2$ .

If  $\dot{x} = P(x, y)$ ,  $\dot{y} = Q(x, y)$  is a polynomial differential system and  $d$  is the maximum of the degrees of  $P$  and  $Q$ , then the expression of the extended flow on the local chart  $(U_1, \phi_1)$  is

$$(4) \quad \dot{u} = v^d \left[ -uP\left(\frac{y}{x}, \frac{1}{x}\right) + Q\left(\frac{y}{x}, \frac{1}{x}\right) \right], \quad \dot{v} = -v^{d+1}P\left(\frac{y}{x}, \frac{1}{x}\right).$$

The expression on the local chart  $(U_2, \phi_2)$  is

$$(5) \quad \dot{u} = v^d \left[ P\left(\frac{1}{y}, \frac{x}{y}\right) - uQ\left(\frac{1}{y}, \frac{x}{y}\right) \right], \quad \dot{v} = -v^{d+1}Q\left(\frac{1}{y}, \frac{x}{y}\right).$$

The expressions in the local charts  $(V_k, \psi_k)$ , for  $k = 1, 2$  are the same than in the chart  $(U_k, \phi_k)$  multiplied by  $(-1)^{d-1}$ . We note that  $v = 0$  corresponds to the infinity  $\mathbb{S}^1$  in all the local charts.

We shall study the dynamics of the extended system (3) in the closed first quadrant of the Poincaré disc  $\mathbb{D}$ , i.e. in  $Q_+ = \{(x, y) \in \mathbb{D} : x \geq 0 \text{ and } y \geq 0\}$ .

We start studying the infinite singular points on  $U_1 \cup \mathbb{Q}_+ \cup \mathbb{S}^1$ . From (4) the extended system (3) in the local chart  $U_1$  is

$$(6) \quad \begin{aligned} \dot{u} &= u + \epsilon v + (q - \epsilon - 1)uv + \epsilon qv^2 + fu^2v - q(1 + \epsilon)uv^2 - fqu^2v^2, \\ \dot{v} &= v(1 + (q - 1)v + fuv - qv^2 - fquv^2). \end{aligned}$$

The unique infinite singular point on  $v = 0$  is  $(0, 0)$  is the origin. Since the eigenvalues of the Jacobian matrix evaluated in this singular point are  $1, 1$  we conclude that this is an unstable node (see for instance Theorem 2.15 of [2]).

The extended system (3) in the local chart  $U_2$  is

$$(7) \quad \begin{aligned} \dot{u} &= -fuv + fqv^2 - u^3 + (1 + \epsilon - q)u^2v + q(1 + \epsilon)uv^2 - \epsilon u^3v - \epsilon qu^2v^2, \\ \dot{v} &= \epsilon(uv^2 + qv^3 - u^2v^2 - quv^3). \end{aligned}$$

The origin of the local chart  $U_2$  is an infinity singular point. The linear part of  $(0, 0)$  is the matrix zero. In order to obtain it local phase portrait we apply blow up. Making the change of coordinates

$u \rightarrow u, v \rightarrow W$ , where  $W = u/v$  and rescaling the common factor  $u$  we get the differential system

$$(8) \quad \begin{aligned} \dot{u} &= -u^2(c+w)(-1+uw), \\ \dot{w} &= bc - (b-acu)w + au(1-c)w^2 - auw^3. \end{aligned}$$

So,  $(u, w) = (0, c)$  is the unique critical point of system (8) on the straight line  $u = 0$ . Moreover, the eigenvalues of the Jacobian matrix evaluated in the critical point  $(0, c)$  are 0 and  $-b$ . In order to conclude the topological type of such point we can move the singular point at origin and apply a linear change of coordinates and a rescaling such that system (8) in this new coordinates is written in the normal form of Theorem 2.19 of [2]. Since  $c \neq 0$  we get that  $(0, c)$  is a saddle-node point. From the blowing down we get the local behavior of all orbits around  $(0, 0)$  of system (7). See Figure 3 for details about this process.

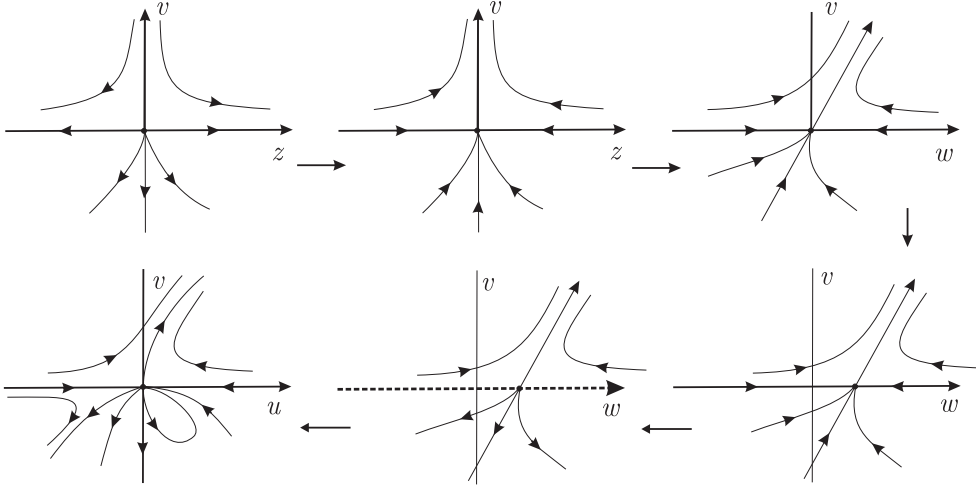


FIGURE 3. The blowing down process to get the local phase portrait at origin of the local chart  $U_2$ .

Now, as the flow of system (3) on the  $y$ -axis of  $Q$  satisfies  $\dot{x}|_{x=0} = fgy$  and on the  $x$ -axes of  $Q$  we have  $\dot{y}|_{y=0} = \epsilon x(q+x)$  we conclude that the flow on  $Q$  is qualitatively described in Figure 4.

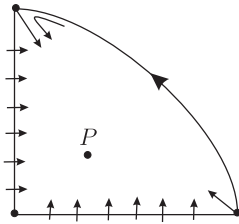


FIGURE 4. The flow in the quadrant  $Q$ .

Now it follows from the Poincaré-Bendixon Theorem (see for instance Theorem 1.25 of [2]) that if in the quadrant  $Q$  there is a unique singular point  $p$ , which is an unstable focus or node than it exists a limit cycle surrounding the point  $p$ . So Theorem 1 is proved.

### 3. PROOF OF THEOREM 2

Now we have that  $q = \epsilon^2/2$  and  $\epsilon > 0$  is sufficiently small. Therefore following the theory of singularly perturbed differential systems, system (1) is a slow system and system (2) is a fast system. For details on the theory of singularly perturbed differential system see [8, 5, 6, 10, 11]. Then the time  $t$  of system (1) in the slow system time and the time  $\tau$  of system (2) is the fast time.

From system (2) we get that near the straight line  $y = c$ , where  $c$  is a real constant, the motion of the system is fast. Now we shall compute the slow invariant manifold of system (1) (see for instance [8]). Assume that the slow invariant manifold is

$$(9) \quad F(x, y) = y - F_0(x) + \epsilon F_1(x) + O(\epsilon^2) = y - f(x).$$

Then it must satisfy on the orbits  $\gamma(t) = (x, y) = (x(t), y(t))$  of system (2) that

$$\frac{dF}{dt}(x, y) \Big|_{y=f(x)} = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial y} \dot{y} \Big|_{y=f(x)} = G_0(x) + \epsilon G_1(x) + O(\epsilon^2) = 0,$$

where  $G_0(x) = \frac{1}{3}(2F_0(x) + 3x(x-1))F'_0(x)$ . Solving  $G_0(x) = 0$  we get that either  $F'_0(x) = 0$  or  $2F_0(x) + 3x(x-1) = 0$ . Since  $F'_0(x) = 0$  implies that the motion takes near  $y = c$ , and its corresponds to the fast motion, the option of the slow motion is  $F_0(x) = 3/2(1-x)x$ . So the slow invariant manifold for  $\epsilon = 0$  pass through the origin of coordinates which is a singular point of system (1).

Now we solve  $G_1(x) = 0$ , and we obtain  $F_1(x) = \frac{x(6x-2)}{4(2x-1)}$ . Therefore the slow invariant manifold (9) is defined, in  $0 < x < 1/2$ , and has the expression

$$y = \frac{3}{2}x(x-1) + \epsilon \frac{x(6x-2)}{4(2x-1)} + O(\epsilon^2).$$

System (1) has three finite singular points, the  $(0, 0)$  and

$$p_{\pm} = \left( \frac{2 - 3\epsilon^2 \pm \sqrt{4 + 108\epsilon^2 + 9\epsilon^4}}{12}, \frac{2 - 3\epsilon^2 \pm \sqrt{4 + 108\epsilon^2 + 9\epsilon^4}}{12} \right).$$

Only the point  $p_+$  is in the interior of the quadrant  $Q$ , and its eigenvalues are  $\epsilon + O(\epsilon^2)$  and  $\frac{1}{3} - 2\epsilon + O(\epsilon^2)$ . So  $p_+$  is an unstable node.

It easy to check that  $(0, 0)$  and  $p_-$  are saddles. So there are no periodic cycles sunrrounding the some of these singularities, because in the region limited by a periodic solution the sum of the indices of the singularities contained in that region must to be 1, and a saddle has index  $-1$ . For more details on the (topological) index of a singularity see Chapter 6 of [2].

Due to the slow-fast dynamics if system (1) has a periodic solution  $((x(t), y(t)))$  of period  $T$  sunrrounding the unstable node  $p_+$  it must have a piece of it close to the slow invariant manifold. Then due to the sense that a periodic solution is run, the integral

$$(10) \quad \int_0^T \text{div}((x(t), y(t))) dt$$

(where  $\text{div}(x, y) = 1 - 2x - \epsilon + O(\epsilon^2)$  is the divergent of system (1)) is negative near the slow invariant manifold where the periodic orbit passes the major part of the time of its period  $T$ .

When the integral (10) is negative the periodic orbit is stable (see for instance Theorem 1.23 of [2]). Due to the fact that two stable periodic orbits cannot be consecutive sunrounding the unstable node at most there is one periodic solution. So the uniqueness of the periodic solution in the quadrant  $Q$  is proved, and consequently, Theorem 2 follows.

#### ACKNOWLEDGEMENTS

The first author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by Projeto Temático FAPESP number 2014/00304-2 and FAPESP grant number 2017/20854-5.

#### REFERENCES

- [1] B. P. BELOUSOV A periodic reaction and its mechanism, *Collection of Short Papers on Radiation Medicine for 1958* (1959). (Moscow: Med. Publ.) [in Russian].
- [2] F. DUMORTIER, J. LLIBRE AND J. C. ARTÉS, *Qualitative theory of planar differential systems*, Universitext, Springer, New York, 2006.
- [3] L. GYORGYI AND R. J. FIELD A three-variable model of deterministic chaos in the Belousov-Zhabotinsky reaction. *Nature* **355** (1992), 808–810.
- [4] D. S. HUH, Y. M. CHOE, D. Y. PARK, S. H. PARK, Y. S. ZHAO, Y. J. KIM AND T. YAMAGUCHI Controlling the Ru-catalyzed Belousov-Zhabotinsky reaction by addition of hydroquinone. *Chemical Physics Letters* **417** (2006), 555–560.
- [5] C. K. R. T. JONES Geometric singular perturbation theory Dynamical Systems, Montecatini Terme *Lecture Notes in Mathematics* **1609** ed. L Arnold. Berlin: Springer (1994), 44–118.
- [6] T. KAPER, An introduction to geometric methods and dynamical systems theory for singular perturbation problems. Analyzing Multiscale Phenomena Using Singular Perturbation Methods, *Baltimore, MD*, 1998, *Providence, RI: American Mathematical Society* (1999), 85–131.
- [7] G. A. LEONOV, N. V. KUZNETSOV AND S. M. SELEDZHI, Nonlinear analysis and design of phase-locked loops, *Automation Control Theory and Practice (In-Tech)* (2009), 89–114.
- [8] J. LLIBRE AND JEAN-MARC GINOUX, The flow curvature method applied to canard explosion, *J. Phys. A: Math. Theor.* **44** (2011), 465203 (13pp).
- [9] T. RITA, S. CHRISTOPHER, Y. A. ADAMATZK, B. D. L. COSTELLO AND L. BULL Dynamic control and information processing in the Belousov-Zhabotinsky reaction using a coevolutionary algorithm. *Journal of Chemical Physics*, **129** (2008), 184708-1–9.
- [10] F. TAKENS Constrained Equations: A Study of Implicit Differential Equations and Their Discontinuous Solutions *Lecture Notes in Mathematics* **525**, Berlin: Springer, 1976.
- [11] A. N. TIKHONOV On the dependence of the solutions of differential equations on a small parameter *Mat. Sbornik N.S.* **22** (1948), 193–204.

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

*E-mail address:* jllibre@mat.uab.cat

INSTITUTO DE CIÊNCIAS MATEMÁTICAS E COMPUTAÇÃO, UNIVERSIDADE DE SÃO PAULO, AVENIDA TRABALHADOR SÃO CARLENSE, 400, 13566-590, SÃO CARLOS, SP, BRAZIL.

*E-mail address:* regilene@icmc.usp.br