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INVARIANT ALGEBRAIC CURVES OF GENERALIZED LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this note we focus on the invariant algebraic curves of generalized Liénard polynomial differential systems $x' = y$, $y' = -f_m(x)y - g_n(x)$ where the degrees of the polynomials f and g are m and n respectively, and we correct some results previously stated.

1. Introduction and statement of the main results

In this work we study the generalized Liénard polynomial differential systems of the form

$$x' = y, \qquad y' = -f_m(x)y - g_n(x), \tag{1}$$

where the degrees of the polynomials f and g are given by the subscripts m and n respectively.

Consider $F(x,y) = 0$ an *invariant algebraic curve* of the differential system (1) where $F(x,y)$ is a polynomial, then there exists a polynomial $K(x,y)$ such that

$$\frac{\partial F}{\partial x} y + \frac{\partial F}{\partial y}(-f_m(x)y - g_n(x)) = KF \tag{2}$$

The knowledge of the algebraic curves of system (1) allows to study the modern Darboux and Liouvillian theories of integrability, see [6] and references therein. In fact the existence of invariant algebraic curves is a measure of integrability in such theories. Another problem is finding a bound on the degree of irreducible invariant algebraic curves of system (1). This problem goes back to Poincaré for any differential system and is known as the *Poincaré problem.*

In 1996 Hayashi [8] stated the following result.

Theorem 1. *The generalized Liénard polynomial differential system* (1) *with* $f_m \not\equiv 0$ *and* $m+1 \geq n$ *has an invariant algebraic curve if and only if there is an invariant curve* $y - P(x) = 0$ *satisfying* $g_n(x) = -(f_m(x) + P'(x))P(x)$*, where* $P(x)$ *or* $P(x) + F(x)$ *is a polynomial of degree at most one, such that* $F(x) = \int_0^x f(s)ds$*.*

Given P and Q polynomials, an algebraic curve of the form $(y+P(x))^2 - Q(x) = 0$ is called *hyperelliptic curve*, see for instance [9, 13, 14, 17]. In such works and others the hyperelliptic curves are used to determine algebraic limit cycles of the generalized Liénard systems (1).

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Theorem 1 is also announced in [16] where the author seems not aware that the theorem is false. Theorem 1 is not correct as the following proposition shows. More precisely it shows the existence of hyperelliptic invariant algebraic curves for the generalized Liénard systems (1).

Proposition 2. *Under the assumptions of Theorem 1 the generalized Liénard polynomial differential system* (1) *has the following hyperelliptic invariant algebraic curves:*

(a) $F(x,y) = -(b+ax)\lambda + (y-b-ax)^2 = 0$ *for* $f_0(x) = -3a/2$ *and* $g_1(x) = a(b+ax-\lambda)/2$ *with* $a \neq 0$.
(b) $F(x,y) = -Ax^2 + (y-ax)^2$ *for* $f_0(x) = -2a$ *and* $g_1(x) = (a^2-A)x$ *with* $aA \neq 0$.
(c) $F(x,y) = -bc/(2a) - cx - acx^2/(2b) + (b+ax-y)^2 = 0$ *for* $f_0(x) = -2a$ *and* $g_1(x) = (2ab-c)(b+ax)/(2b)$ *with* $ab \neq 0$.

Proposition 2 is proved in section 2.

In fact the correct statement of Theorem 1 is the following.

Theorem 3. *The generalized Liénard polynomial differential system* (1) *with* $f_m \not\equiv 0$ *and* $m+1 \geq n$ *has the invariant algebraic curve* $y - P(x) = 0$ *if* $g_n(x) = -(f_m(x)+P'(x))P(x)$, *being* $P(x)$ *or* $P(x)+F(x)$ *a polynomial of degree at most one, where* $F(x) = \int_0^x f(s)ds$.

Theorem 3 is proved in section 2.

Note that the mistake in the statement of Theorem 1 is the claim that the unique invariant algebraic curves are of the form $y - P(x) = 0$.

Demina in [3] also detected that Theorem 1 was not correct. She found counterexamples to Theorem 1 with invariant algebraic curves of degree 2 and 3 in the variable y.

Singer in [15] found the characterization of the systems that are Liouvillian integrable. Christopher [2] rewrite this result stating that if a polynomial differential system in $\mathbb{R}^2$ has an inverse integrating factor of the form

$$V = \exp\left(\frac{D}{E}\right)\prod_{i=1}^{p} F_i^{\alpha_i}, \tag{3}$$

where D, E and F_i are polynomials in $\mathbb{C}[x,y]$ and $\alpha_i \in \mathbb{C}$, then this differential system is *Liouvillian integrable*. For a definition of (inverse) integrating factor see for instance section 8.3 of [6].

We say that $\exp(g/h)$, with g and $h \in \mathbb{C}[x,y]$, is an *exponential factor* of the polynomial differential system (1) if there exists a polynomial $L(x,y)$ of degree at most d where $d = \max\{m, n-1\}$ such that

$$\frac{\partial \exp(g/h)}{\partial x} y + \frac{\partial \exp(g/h)}{\partial y}(-f_m(x)y - g_n(x)) = K \exp(g/h).$$

More information on exponential factors can be found in section 8.5 of [6].

The existence of an inverse integrating factor (3) for a polyomial differential system in $\mathbb{R}^2$ is equivalent to the existence of λ_i and $\mu_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = \mathrm{div}(P,Q)$, where K_i and L_i are the cofactors of some invariant algebraic curves and exponential factors of the given polynomial differential system, respectively. See for more details statement (iv) of Theorem 8.7 of [6].

We remark that the two kind of invariant algebraic curves mentioned in Theorem 3 can appear simultaneously in some generalized Liénard polynomial differential systems (1) as the following example shows which already appeared in [7].

The generalized polynomial Liénard differential system

$$x' = y, \qquad y' = -ex^3 - e^2/3x - (3x^2 + 4e/3)y, \tag{4}$$

has the invariant algebraic curves $f_1 = y + ex/3 = 0$ and $f_2 = y + x^3 + ex/3 = 0$. Moreover system (4) is Liouvillian integrable because it has the inverse integrating factor $V = f_1 f_2^{1/3}$.

Let U be an open subset of $\mathbb{R}^2$. A C^1 function $H : U \to \mathbb{R}$ is a *first integral* of system (1) if it is constant on the orbits of the system contained in U, or equivalently if

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} y + \frac{\partial H}{\partial y}(-f_m(x)y - g_n(x)) = 0 \ \text{ on } U. \tag{5}$$

Consider W as an open subset of $\mathbb{R}^2 \times R$. A C^1 function $I : W \to \mathbb{R}$ is a *Darboux invariant* of system (1) if it is constant on the orbits of the system contained in W, or equivalently if

$$\frac{dI}{dt} + \frac{\partial I}{\partial x} y + \frac{\partial I}{\partial y}(-f_m(x)y - g_n(x)) = 0 \ \text{ on } W. \tag{6}$$

Moreover, given λ_i and $\mu_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s$ for some $s \in \mathbb{C} \setminus \{0\}$, then the (multivalued) funtion

$$I = \prod_{i=1}^{p} F_i^{\alpha_i} \prod_{j=1}^{q} \left(\exp\left(\frac{g_j}{h_j} \right) \right)^{\mu_j} \exp(st) \tag{7}$$

is a *Darboux invariant* of the differential system, see for more details statement (vi) of Theorem 8.7 of [6].

Under the assumptions of Theorem 3 there are generalized Liénard polynomial differential systems (1) which are Liouvillian integrable as it is shown in the next result.

Proposition 4. *Under the assumptions of Theorem* 3 *if the generalized Liénard polynomial differential system* (1) *has the invariant algebraic curve* $y - P(x) = 0$. *Then the following statements hold.*

(a) *If* $P(x) = -F(x) + ax + b$, *then system* (1) *has the Darboux invariant* $(y - P(x))e^{at}$.
(b) *If* $P(x) = b$, *then system* (1) *is Liouvillian integrable with the first integral* $H = e^{y+F(x)}(y - b)^b$ *if* $b \neq 0$, *and the first integral* $H = y + F(x)$ *if* $b = 0$.

Proposition 4 is proved in section 2.

We note that Proposition 4 shows that Theorem 2 of [7] and Theorem 4 of [11] are not correct because their proofs are based in the wrong Theorem 1.

Proposition 5. *Consider the generalized Liénard polynomial differential system* (1)*. Let $P(x)$ be a polynomial, then $y - P(x) = 0$ is an invariant algebraic curve of system* (1) *if and only if $g_n(x) = -(f_m(x) + P'(x))P(x)$.*

Proposition 5 is proved in section 2. In fact the statement of Proposition 5 already appears in [17] without proof.

Note that in Proposition 5 there are no restrictions on the degrees of the polynomials f_m, g_n and $P(x)$.

The Liouvillian integrability of the generalized Liénard polynomial differential system has been studied by several authors. The main result of [10] is that under the restriction $2 \le n \le m$, then system (1) has a Liouvillian first integral if and only if $g_n(x) = af_m(x)$, where $a \in \mathbb{C}$, see also [1] for a shorter proof. Later on it was studied the Liouvillian integrability of the differential systems (1) having hyperelliptic curves of the form $(y + Q(x)P(x))^2 - Q(x)^2 = 0$, see [12].

In summary, the Liouvillian integrability in the case $n > m$ is still open. In fact the characterization of the invariant algebraic curves of system (1) for this case is not complete. Recently it has been solved the case $m = 1$ and $n = 2$, see [5].

The case $n = m + 1$ is the still objective of several recent works. Thus, for instance in [3, 4] some particular cases for $m = 2$ and $n = 3$ have been solved.

2. Proofs

Proof of Proposition 2. Assume that system (1) has the hyperelliptic invariant curve $F = (y + P(x))^2 - Q(x) = 0$. Then from (2) denoting by $K = K(x, y)$ the cofactor of $F = 0$ we get

$$\begin{aligned} &2g_n(x)P(x) + K(-P(x)^2 + Q(x)) \\ &+ y(-2g_n(x) + 2P(x)(K + f_m(x) + P'(x)) - Q'(x)) \\ &- y^2(K + 2f_m(x) + 2P'(x)) = 0. \end{aligned}$$

From this equality we see that $K = K(x) = -2(f_m(x) + P'(x))$,

$$f_m(x) = -P'(x) - \frac{P(x)Q'(x)}{2Q(x)}, \text{ and } g_n(x) = -\frac{1}{2}Q'(x) + \frac{P^2(x)Q'(x)}{2Q(x)},$$

where $fm(x)$ and $g_n(x)$ must be polynomials.

If we assume that $\deg P = p$ and $\deg Q = q$, we get that $\deg f_m = p - 1$ and $\deg g_n = \max\{q - 1, p^2 - 1\}$. Since $m + 1 \ge n$ we obtain $p \ge \max\{q - 1, p^2 - 1\}$ which implies $p = 1$. Consequently $1 \ge q - 1$, which implies $q = 1, 2$.

If $q = 1$ then $P(x) = ax + b$ with $a \neq 0$ and $Q(x)$ must be proportional to $P(x)$, that is, $Q(x) = \lambda P(x)$. So $f_m = -3a/2$ and $g_n = a(ax + b - \lambda)/2$, and $F = (b + ax - y)^2 - (b + ax)\lambda$. So statement (a) follows.

If $q = 2$ then we have $P(x) = ax+b$ and $Q(x) = Ax^2+Bx+C$ with $aA \neq 0$, and since f_m must be a polynomial we obtain that $f_m = -2a$, and in order that g_m be a polynomial we obtain either $b = B = C = 0$, or $A = aB/(2b)$ and $C = (bB)/(2a)$ with $ab \neq 0$.

If $b = B = C = 0$ then $g_n = (a^2 - A)x$ and $F = -Ax^2 + (y - ax)^2$. Therefore statement (b) is proved.

If $A = aB/(2b)$ and $C = (bB)/(2a)$ then $g(x) = (2ab - B)(b + ax)/(2b)$ and $F(x, y) = -bB/(2a) - Bx - aBx^2/(2b) + (b + ax - y)^2$. Renaming B by c we get statement (c). □

Proof of Proposition 5. First we suppose that $g_n(x) = -(f_m(x)+P'(x))P(x)$, and we shall prove that $y - P(x) = 0$ is an invariant algebraic curve. From equation (7) we have

$$-P'(x)y - yf_m(x) - g_n(x) = K(y - P(x)).$$

Substituting $g_n(x)$ we obtain

$$-P'(x)(y - P(x)) - f_m(x)(y - P(x)) = K(y - P(x)).$$

Dividing by $y - P(x)$ the previous equality we get $K = -P'(x) - f_m(x)$, which is a cofactor of degree $p - 1 + m$ of system (1). Note that the degree of the polynomial Liénard differential system is the degree of, i.e. the maximum of $m + p$ and $2p - 1$.

Now we assume that $y - P(x)$ is an invariant algebraic curve of system (1) with cofactor K, then from (7) we have

$$-P'(x)y - yf_m(x) - g_n(x) = K(y - P(x)).$$

From this equality we obtain that $K = K(x)$, then we have

$$(K(x) + f_m(x) + P'(x))y = K(x)P(x) - g_n(x).$$

Therefore $K(x) = -(f_m(x) + P'(x))$ and $g_n(x) = K(x)P(x)$. Hence $g_n(x) = -(f_m(x) + P'(x))P(x)$, and the proposition is proved. □

Proof of Theorem 3. By Proposition 5 we only need to prove that $P(x)$ or $P(x) + F(x)$ are polynomial of degree at most one. Since $m+1 \geq n$ and n is the maximum of $m + p$ and $2p - 1$ where p is the degree of the polynomial $P(x)$, we have that $m + 1 \geq m + p$, consequently $p \leq 1$, and the theorem is proved. □

Proof of Proposition 4. We have system (1) with $g_n(x) = -(f_m(x) + P'(x))P(x)$ and with the invariant algebraic curve $y - P(x) = 0$ being $P(x) = -F(x) + ax + b$. Then using equation (2) we obtain that the cofactor of the invariant algebraic curve $y - P(x) = 0$ is $K = -a$. Consequently system (1) has the Darboux invariant (7), which in our case becomes $I = (y - P(x))e^{at}$. Hence statement (a) is proved. Assume now that $P(x) = b$. Therefore $g(x) = -bf(x)$ and the differential system becomes $\dot{x} = y$ and $\dot{x} = -(y + b)f(x)$, which has the Darboux first integral $H = e^{y+F(x)}(y - b)^b$ if $b \neq 0$, and the Darboux first integral $H = y + F(x)$ if $b = 0$, as it is easy to check using (6). □

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References

[1] G. Chèze, T. Cluzeau, *On the nonexistence of Liouvillian first integrals for generalized Liénard polynomial differential systems*, J. Nonlinear Math. Phys. **20** (2013), no. 4, 475–479.

[2] C.J. Christopher, *Liouvillian first integrals of second order polynomial differential equations*, Electron. J. Differential Equations **1999**, No. 49, 7 pp.

[3] M.V. Demina, *Invariant algebraic curves for Liénard dynamical systems revisited*, Appl. Math. Lett. **84** (2018), 42–48.

[4] M.V. Demina, *Novel algebraic aspects of Liouvillian integrability for two-dimensional polynomial dynamical systems*, Phys. Lett. A **382** (2018), no. 20, 1353–1360.

[5] M.V. Demina, C. Valls, *On the Poincaré problem and Liouvillian integrability of quadratic Liénard differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 6, 3231–3251.

[6] F. Dumortier, J. Llibre, J.C. Artés, *Qualitative theory of planar differential systems*, UniversiText, Springer–Verlag, New York, 2006.

[7] J. Giné, *A note on: The generalized Liénard polynomial differential systems $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$, with $\deg g = \deg f + 1$, are not Liouvillian integrable*, Bull. Sci. math. **139** (2015), 214–227; Bull. Sci. Math. **161** (2020), 102857.

[8] M. Hayashi, *On polynomial Liénard systems which have invariant algebraic curves*, Funkc. Ekvacioj, **39** (1996), 403–408.

[9] C. Liu, G. Chen, J. Yang, *On the hyperelliptic limit cycles of Liéenard systems*, Nonlinearity **25** (2012), 1601–1611.

[10] J. Llibre, C. Valls, *Liouvillian first integrals for generalized Liénard polynomial differential systems*, Adv. Nonlinear Stud. **13** (2013), no. 4, 825–835.

[11] J. Llibre, C. Valls, *The generalized Liénard polynomial differential systems $x' = y$, $y' = -g(x) - f(x)y$, with $\deg g = \deg f + 1$, are not Liouvillian integrable*, Bull. Sci. Math. **139** (2015), 214–227.

[12] J. Llibre, C. Valls, *Liouvillian first integrals for a class of generalized Liénard polynomial differential systems*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), no. 6, 1195–1210.

[13] X. Qian, J. Yang, *On the number of hyperelliptic limit cycles of Liénard systems*, Qual. Theory Dyn. Syst. (2020) 19:43.

[14] X. Qian, Y. Shen, J. Yang, *Invariant algebraic curves and hyperelliptic limit cycles of Liénard systems*, Qual. Theory Dyn. Syst. 20 (2021), no. 2, Paper No. 44, 14 pp.

[15] M.F. Singer, *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

[16] X. Zhang, *Integrability of dynamical systems: algebra and analysis*, Developments in Mathematics, 47. Springer, Singapore, 2017.

[17] H. Żołądek, *Algebraic invariant curves for the Liénard equation*, Trans. Amer. Math. Soc. **350** (1998), no. 4, 1681–1701.

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