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Invariant Algebraic Curves of Generalized Liénard Polynomial Differential Systems

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Abstract: In this study, we focus on invariant algebraic curves of generalized Liénard polynomial differential systems $x' = y$, $y' = -f_m(x)y - g_n(x)$, where the degrees of the polynomials f and g are m and n, respectively, and we correct some results previously stated.

Keywords: Liénard differential systems; invariant algebraic curve; first integrals

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1. Introduction and Statement of the Main Results

In this work, we study the generalized Liénard polynomial differential systems of the following form:

$$x' = y, \qquad y' = -f_m(x)y - g_n(x), \tag{1}$$

where the degrees of the polynomials f and g are given by the subscripts m and n, respectively. These generalized Liénard systems are used to model different problems in numerous areas of knowledge and have been intensively studied in the last decades (see for instance [1,2] and references therein).

Consider $F(x, y) = 0$ an *invariant algebraic curve* of the differential system (1) where $F(x, y)$ is a polynomial, then there exists a polynomial $K(x, y)$ such that the following is the case.

$$\frac{\partial F}{\partial x}y + \frac{\partial F}{\partial y}(-f_m(x)y - g_n(x)) = KF \tag{2}$$

The knowledge of the algebraic curves of system (1) allows studying modern Darboux and Liouvillian theories of integrability (see [3] and references therein). In fact the existence of invariant algebraic curves is a measure of integrability in such theories. Another problem is finding a bound on the degree of irreducible invariant algebraic curves of system (1). This problem goes back to Poincaré for any differential system and is known as *Poincaré problem*.

In 1996, Hayashi [4] stated the following result.

Theorem 1. *The generalized Liénard polynomial differential system (1) with $f_m \not\equiv 0$ and $m + 1 \geq n$ has an invariant algebraic curve if and only if there is an invariant curve $y - P(x) = 0$ satisfying $g_n(x) = -(f_m(x) + P'(x))P(x)$, where $P(x)$ or $P(x) + F(x)$ is a polynomial with a degree of at most one, such that $F(x) = \int_0^x f(s)ds$.*

Given P and Q polynomials, an algebraic curve of the form $(y + P(x))^2 - Q(x) = 0$ is called *hyperelliptic curve* (see for instance [5–8]). In such works, hyperelliptic curves are used to determine the algebraic limit cycles of generalized Liénard systems (1).

Theorem 1 is also announced in [9], where the author seems to not be aware that the theorem is false. Theorem 1 is not correct as the following proposition shows. More precisely, it shows the existence of hyperelliptic invariant algebraic curves for generalized Liénard systems (1).

Proposition 1. *Under the assumptions of Theorem 1, the generalized Liénard polynomial differential system (1) has the following hyperelliptic invariant algebraic curves:*

(a) $F(x,y) = -(b+ax)\lambda + (y-b-ax)^2 = 0$ *for* $f_0(x) = -3a/2$ *and* $g_1(x) = a(b+ax-\lambda)/2$ *with* $a \neq 0$*;*

(b) $F(x,y) = -Ax^2 + (y-ax)^2$ *for* $f_0(x) = -2a$ *and* $g_1(x) = (a^2-A)x$ *with* $aA \neq 0$*;*

(c) $F(x,y) = -bc/(2a) - cx - acx^2/(2b) + (b+ax-y)^2 = 0$ *for* $f_0(x) = -2a$ *and* $g_1(x) = (2ab-c)(b+ax)/(2b)$ *with* $ab \neq 0$.

Proposition 1 is proved in Section 2.

In fact, the correct statement of Theorem 1 is the following.

Theorem 2. *The generalized Liénard polynomial differential system (1) with* $f_m \not\equiv 0$ *and* $m+1 \geq n$ *has the invariant algebraic curve* $y - P(x) = 0$ *if* $g_n(x) = -(f_m(x) + P'(x))P(x)$*, being* $P(x)$ *or* $P(x) + F(x)$ *a polynomial of degree at most one, where* $F(x) = \int_0^x f(s)ds$.

Theorem 2 is proved in Section 2.

Note that the mistake in the statement of Theorem 1 is the claim that unique invariant algebraic curves are of the following form $y - P(x) = 0$.

Demina in [10] also detected that Theorem 1 was not correct. She found counterexamples to Theorem 1 with invariant algebraic curves of degree 2 and 3 in the variable y.

Singer in [11] found the characterization of systems that are Liouvillian integrable. Christopher [12] rewrote this result stating that if a polynomial differential system in $\mathbb{R}^2$ has an inverse integrating factor of the following form:

$$V = \exp\left(\frac{D}{E}\right) \prod_{i=1}^{p} F_i^{\alpha_i}, \tag{3}$$

where D, E and F_i are polynomials in $\mathbb{C}[x,y]$ and $\alpha_i \in \mathbb{C}$, then this differential system is *Liouvillian integrable*. For a definition of (inverse) integrating factor, see for instance Section 8.3 of [3].

We say that $\exp(g/h)$, with g and $h \in \mathbb{C}[x,y]$, is an *exponential factor* of the polynomial differential system (1) if there exists a polynomial $L(x,y)$ of a degree with at most d where $d = \max\{m, n-1\}$ such that the following is the case.

$$\frac{\partial \exp(g/h)}{\partial x} y + \frac{\partial \exp(g/h)}{\partial y}(-f_m(x)y - g_n(x)) = K \exp(g/h).$$

More information on exponential factors can be found in Section 8.5 of [3].

The existence of an inverse integrating factor (3) for a polyomial differential system in $\mathbb{R}^2$ is equivalent to the existence of λ_i and $\mu_i \in \mathbb{C}$ is not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = \operatorname{div}(P,Q)$, where K_i and L_i are the cofactors of some invariant algebraic curves and exponential factors of the given polynomial differential system, respectively. See, for more details, statement (iv) of Theorem 8.7 of [3].

We remark that the two kinds of invariant algebraic curves mentioned in Theorem 2 can appear simultaneously in some generalized Liénard polynomial differential systems (1) as the following example shows, which already appeared in [13].

The generalized polynomial Liénard differential system of the following:

$$x' = y, \qquad y' = -ex^3 - e^2/3x - (3x^2 + 4e/3)y, \tag{4}$$

has invariant algebraic curves $f_1 = y + ex/3 = 0$ and $f_2 = y + x^3 + ex/3 = 0$. Moreover, system (4) is Liouvillian integrable because it has an inverse integrating factor $V = f_1 f_2^{1/3}$.

Let U be an open subset of $\mathbb{R}^2$. A C^1 function $H : U \to \mathbb{R}$ is a *first integral* of system (1) if it is constant on the orbits of the system contained in U, or equivalently if the following is the case.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} y + \frac{\partial H}{\partial y}(-f_m(x)y - g_n(x)) = 0 \text{ on } U. \tag{5}$$

Consider W as an open subset of $\mathbb{R}^2 \times R$. A C^1 function $I : W \to \mathbb{R}$ is a *Darboux invariant* of system (1) if it is constant on the orbits of the system contained in W, or equivalently if the following is the case.

$$\frac{dI}{dt} + \frac{\partial I}{\partial x} y + \frac{\partial I}{\partial y}(-f_m(x)y - g_n(x)) = 0 \text{ on } W. \tag{6}$$

Moreover, given λ_i and $\mu_i \in \mathbb{C}$ that is not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s$ for some $s \in \mathbb{C} \setminus \{0\}$, then the (multivalued) function of the following:

$$I = \prod_{i=1}^{p} F_i^{\alpha_i} \prod_{j=1}^{q} \left(\exp\left(\frac{g_j}{h_j} \right) \right)^{\mu_j} \exp(st) \tag{7}$$

is a *Darboux invariant* of the differential system (see for more details statement (vi) of Theorem 8.7 of [3]).

Under the assumptions of Theorem 2, there are generalized Liénard polynomial differential systems (1) that are Liouvillian integrable, as it is shown in the next result.

Proposition 2. *Under the assumptions of Theorem 2, if the generalized Liénard polynomial differential system (1) has an invariant algebraic curve $y - P(x) = 0$, then the following statements hold:*

(a) If $P(x) = -F(x) + ax + b$, then system (1) has the Darboux invariant $(y - P(x))e^{at}$;

(b) If $P(x) = b$, then system (1) is Liouvillian integrable with the first integral $H = e^{y+F(x)}(y - b)^b$ if $b \neq 0$, and the first integral $H = y + F(x)$ if $b = 0$.

Proposition 2 is proved in Section 2.

We note that Proposition 2 shows that Theorem 2 of [13] and Theorem 4 of [14] are not correct because their proofs are based on the incorrect Theorem 1.

Proposition 3. *Consider the generalized Liénard polynomial differential system (1). Let $P(x)$ be a polynomial, then $y - P(x) = 0$ is an invariant algebraic curve of system (1) if and only if $g_n(x) = -(f_m(x) + P'(x))P(x)$.*

Proposition 3 is proved in Section 2. In fact, the statement of Proposition 3 already appears in [8] without proof.

Note that, in Proposition 3, there are no restrictions on the degrees of the polynomials f_m, g_n and $P(x)$.

The Liouvillian integrability of the generalized Liénard polynomial differential system has been studied by several authors. The main result of [15] is that under restriction $2 \leq n \leq m$, system (1) has a Liouvillian first integral if and only if $g_n(x) = af_m(x)$, where $a \in \mathbb{C}$ (see also [16] for a shorter proof). Later on, the Liouvillian integrability of differential systems (1) having hyperelliptic curves of the form $(y + Q(x)P(x))^2 - Q(x)^2 = 0$ was studied (see [17]).

In summary, the Liouvillian integrability in the case where $n > m$ is still open. In fact, the characterization of the invariant algebraic curves of system (1) for this case is not complete. Recently, cases $m = 1$ and $n = 2$ have been solved (see [18]).

Case $n = m + 1$ is the still the objective of several recent works. Thus, for instance in [10,19], some particular cases for $m = 2$ and $n = 3$ have been solved.

2. Proofs

Proof of Proposition 1. Assume that system (1) has an hyperelliptic invariant curve $F = (y + P(x))^2 - Q(x) = 0$. Then, from (2), denoting by $K = K(x, y)$ the cofactor of $F = 0$, we obtain the following.

$$\begin{aligned} &2g_n(x)P(x) + K(-P(x)^2 + Q(x)) \\ &+ y(-2g_n(x) + 2P(x)(K + f_m(x) + P'(x)) - Q'(x)) \\ &- y^2(K + 2f_m(x) + 2P'(x)) = 0. \end{aligned}$$

From this equality, we observe that $K = K(x) = -2(f_m(x) + P'(x))$:

$$f_m(x) = -P'(x) - \frac{P(x)Q'(x)}{2Q(x)}, \text{ and } g_n(x) = -\frac{1}{2}Q'(x) + \frac{P^2(x)Q'(x)}{2Q(x)},$$

where $fm(x)$ and $g_n(x)$ must be polynomials.

If we assume that $\deg P = p$ and $\deg Q = q$, we obtain $\deg f_m = p - 1$ and $\deg g_n = \max\{q - 1, p^2 - 1\}$. Since $m + 1 \geq n$, we obtain $p \geq \max\{q - 1, p^2 - 1\}$, which implies $p = 1$. Consequently, $1 \geq q - 1$, which implies $q = 1, 2$.

If $q = 1$, then $P(x) = ax + b$ with $a \neq 0$ and $Q(x)$ must be proportional to $P(x)$; that is, $Q(x) = \lambda P(x)$. Thus, $f_m = -3a/2$ and $g_n = a(ax + b - \lambda)/2$, and $F = (b + ax - y)^2 - (b + ax)\lambda$. Thus, statement (a) follows.

If $q = 2$, then we have $P(x) = ax + b$ and $Q(x) = Ax^2 + Bx + C$ with $aA \neq 0$, and since f_m must be a polynomial, we obtain $f_m = -2a$; moreover, in order for g_m to be a polynomial, we obtain either $b = B = C = 0$ or $A = aB/(2b)$ and $C = (bB)/(2a)$ with $ab \neq 0$.

If $b = B = C = 0$, then $g_n = (a^2 - A)x$ and $F = -Ax^2 + (y - ax)^2$. Therefore, statement (b) is proven.

If $A = aB/(2b)$ and $C = (bB)/(2a)$, then $g(x) = (2ab - B)(b + ax)/(2b)$ and $F(x, y) = -bB/(2a) - Bx - aBx^2/(2b) + (b + ax - y)^2$. By renaming B by c, we obtain statement (c). □

Proof of Proposition 2. We have system (1) with $g_n(x) = -(f_m(x) + P'(x))P(x)$ and the invariant algebraic curve $y - P(x) = 0$ is $P(x) = -F(x) + ax + b$. Then, by using Equation (2), we obtain the result where the cofactor of the invariant algebraic curve $y - P(x) = 0$ is $K = -a$. Consequently, system (1) has the Darboux invariant (7), which in our case becomes $I = (y - P(x))e^{at}$. Hence, statement (a) is proved. Assume now that $P(x) = b$. Therefore, $g(x) = -bf(x)$, and the differential system becomes $\dot{x} = y$ and $\dot{x} = -(y + b)f(x)$, which has the Darboux first integral $H = e^{y+F(x)}(y - b)^b$ if $b \neq 0$, and the Darboux first integral $H = y + F(x)$ if $b = 0$, as it is easy to verify using (6). □

Proof of Proposition 3. First, we suppose that $g_n(x) = -(f_m(x) + P'(x))P(x)$, and we shall prove that $y - P(x) = 0$ is an invariant algebraic curve. From Equation (7), we have the following.

$$-P'(x)y - yf_m(x) - g_n(x) = K(y - P(x)).$$

By substituting $g_n(x)$, we obtain the following.

$$-P'(x)(y - P(x)) - f_m(x)(y - P(x)) = K(y - P(x)).$$

Dividing the previous equality by $y - P(x)$, we obtain $K = -P'(x) - f_m(x)$, which is a cofactor of degree $p - 1 + m$ of system (1). Note that the degree of the polynomial Liénard differential system is the degree of, i.e., the maximum of $m + p$ and $2p - 1$.

Now, we assume that $y - P(x)$ is an invariant algebraic curve of system (1) with cofactor K. Then, from (7), we obtain

$$-P'(x)y - yf_m(x) - g_n(x) = K(y - P(x)).$$

From this equality, we obtain $K = K(x)$; then, we have

$$(K(x) + f_m(x) + P'(x))y = K(x)P(x) - g_n(x).$$

Therefore, $K(x) = -(f_m(x) + P'(x))$ and $g_n(x) = K(x)P(x)$. Hence, $g_n(x) = -(f_m(x) + P'(x))P(x)$, and the proposition is proved. □

Proof of Theorem 2. By Proposition 3, we only need to prove that $P(x)$ or $P(x) + F(x)$ are polynomials with a degree of at most one. Since $m + 1 \geq n$ and n are the maxima of $m + p$ and $2p - 1$ where p is the degree of the polynomial $P(x)$, we have $m + 1 \geq m + p$; consequently, $p \leq 1$, and the theorem is proved. □

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