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CROSSING LIMIT CYCLES FOR DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY LINEAR HAMILTONIAN SADDLES OR LINEAR CENTERS SEPARATED BY A CONIC[1]

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Abstract. The extension of the 16th Hilbert problem to discontinuous piecewise linear differential systems asks for an upper bound for the maximum number of crossing limit cycles that such systems can exhibit. The study of this problem is being very active, specially for discontinuous piecewise linear differential systems defined in two zones and separated by one straight line. In the case that the differential systems in these zones are formed either by linear centers or linear Hamiltonian saddles it is known that there are no crossing limit cycles. However it is also known that the number of crossing limit cycles can change if we change the shape of the discontinuity curve. In this paper we study the maximum number of crossing limit cycles of discontinuous piecewise differential systems formed by either linear Hamiltonian saddles or linear centers and separated by a conic which intersect the conic in two points. For this class of discontinuous piecewise differential systems we solve the extended 16th Hilbert problem.

1. Introduction and statement of the main results

Poincaré [22, 23] was the first in introducing the notion of *limit cycle* of a differential system, i.e. a periodic orbit isolated in the set of all periodic orbits of the differential system. After the limit cycles became of great importance because they model many real world phenomena. This caused that the study of their existence, their number and their properties became very active, see for instance [3, 5, 12, 19, 20, 21, 26, 27].

In general the problem of finding the limit cycles of a given class of differential systems is very difficult, in especial to provide an upper bound on the

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maximal number of limit cycles that a given class differential systems can exhibit. One of these classes is the class of discontinuous piecewise linear differential systems. Such systems were studied by first time by Andronov, Vitt and Khaikin in [1], and after their appearance it became clear that they have many applications in different areas, modeling real phenomena in a quite accurate way (see for instance [5, 25]). So now there is a great activity in studying these systems.

A *discontinuous piecewise differential system* on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth codimension one manifold Σ. The *line of discontinuity* Σ of the discontinuous piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) > 0\}$ and $\Sigma^- = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) < 0\}$. So the piecewise C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } h(x,y) \geq 0, \\ Y(x,y), & \text{if } h(x,y) \leq 0. \end{cases} \tag{1}$$

As usual the vector field associated to system (1) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$, when the separation line Σ is well understood. In order to establish a definition for the trajectories of Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field X, i.e.

$$Xh(p) = \langle \nabla h(p), X(p) \rangle \,,$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [7]. We can divide the line of discontinuity Σ in the following sets:

(a) *Crossing set*: $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.
(b) *Escaping set*: $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.
(c) *Sliding set*: $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

The *escaping* Σ^e or *sliding* Σ^s regions are respectively defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the *crossing region* Σ^c (see Figure 1). The complementary of the union of these regions is the set formed by the *tangency* points between X or Y with Σ.

Our goal is to study the so-called crossing limit cycles of the discontinuous piecewise differential systems formed with linear centers or linear Hamiltonian saddles which are separated by conics. A *crossing limit cycles* is a limit cycle that have isolated crossing points of intersection with the discontinuity curve.

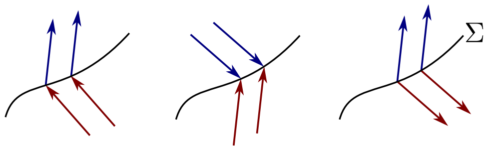


Figure 1. Crossing, sliding and escaping regions, respectively.

The case of finding an upper bound for the number of crossing limit cycles for discontinuous piecewise linear differential systems separated by a straight line has been studied by many authors (see for instance [2, 6, 8, 9, 24] and there is a conjecture claiming that discontinuous piecewise linear systems in the plane separated by one straight line have at most three limit cycles, but although there are examples with three limit cycles (the first ones were [10, 13]) the conjecture is still open.

Here we will work with two classes of Hamiltonian linear differential systems the linear centers and the linear Hamiltonian saddles. In the case in which the linear systems are either centers or Hamiltonian saddles and are separated by a straight line it was proved in [16, 17] that they do not have crossing limit cycles, however it is known that the number of crossing limit cycles can change if we change the shape of the discontinuity curve. In [11, 15, 18] it was studied the number of limit cycles of discontinuous piecewise differential systems formed by linear centers, separated by a conic.

In the present paper we will study the number of limit cycles of discontinuous piecewise differential systems formed by linear Hamiltonian saddles or linear centers and separated by a conic Σ.

Using an affine change of coordinates, i.e. $(x, y) \to (ax+by+c, \alpha x+\beta y+\gamma)$ with $a\beta - b\alpha \neq 0$, it is well known that any conic that separates the plane in connected regions can be written in one of following six canonical forms:
(DL): $x^2 = 0$ one double real straight line;
(PL): $x^2 - 1 = 0$ two real parallel straight lines;
(LV): $xy = 0$ two real straight lines intersecting at a real point;
(E): $x^2 + y^2 - 1 = 0$ ellipse;
(H): $x^2 - y^2 - 1 = 0$, hyperbola;
(P): $y - x^2 = 0$ parabola.
For more details see [4].

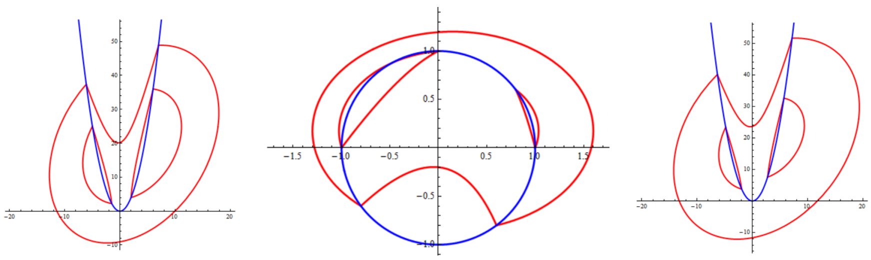
Of course any conic that does not separate the plane in connected regions can be either two complex straight lines intersecting at areal point, two complex parallel straight lines, and the complex ellipse, but these conics will not be considered.

We observe that we have two options for crossing limit cycles of discontinuous piecewise linear differential Hamiltonian saddles separated by a conic Σ. First we have the crossing limit cycles that intersect the discontinuity curve in exactly two points and second we have the crossing limit cycles that intersect the discontinuity curve Σ in four points. In this paper we study the crossing limit cycles such that intersect the discontinuity curve in exactly two points and we denote by $\mathcal{F}$ the class of piecewise differential systems separated by a conic such that in any region of the conic we can have either a linear Hamiltonian saddle or a linear center.

The maximum number of crossing limit cycles of piecewise linear differential systems in class $\mathcal{F}$ separated by a conic Σ such that intersect Σ in exactly two points is given in the following theorems.

Theorem 1. *Consider a planar discontinuous piecewise differential system in class $\mathcal{F}$ where Σ is a conic. If Σ is of the type (LV), (PL) or (DL), then there are no crossing limit cycles.*

Analyzing the case of discontinuous piecewise linear differential systems in class $\mathcal{F}$ with discontinuity curve a conic of the type (LV), (PL) or (DL) the maximum number of crossing limit cycles is equal to the maximum number of crossing limit cycles in discontinuous piecewise linear differential of class $\mathcal{F}$ in the plane separated by a single straight line which was studied in [16]. In this paper it was proved that such class of piecewise differential systems have no crossing limit cycles. This proves Theorem 1.



(a) (b) (c)

Figure 2. The three limit cycles of the discontinuous piecewise differential systems: (a) (10)-(11) the discontinuous line is the parabola $y = x^2$, (b) (14)-(15) the discontinous line is the circle $x^2 + y^2 = 1$, (c) (14)-(15) the discontinuous line is a branch of the hyperbola $y - x^2 + xy/5000 = 0$. The three limit cycles are travelled in counterclockwise sense.

Now we consider the other conics.

Theorem 2. *Consider a planar discontinuous piecewise differential system in class $\mathcal{F}$, where Σ is either a parabola (P), or an ellipse (E), or a hyperbola (H). Then the following statements hold.*

(a) *For this family of systems the maximum number of crossing limit cycles that intersect Σ in two points is three.*

(b) *There are systems having exactly three crossing limit cycles that intersect Σ in two points, see (a), (b) and (c) of Figure 2 for the cases of (P), (E) and (H), respectively.*

The proofs of Theorem 2 for the parabola, ellipse and hyperbola are given in sections 2, 3 and 4, respectively.

2. Proof of Theorem 2 for the parabola

For the proof of Theorem 2 we will use the following two results which provide a normal form for a linear differential Hamiltonian saddle (for a proof see [16, 17]) and for a linear center (for a proof see [14]).

Proposition 3. *Any linear differential system having a Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \delta y + d, \quad \dot{y} = \alpha x + by + c, \tag{2}$$

with $\alpha \in \{0,1\}$, $b, \delta, c, d \in \mathbb{R}$. Moreover, if $\alpha = 1$ then $\delta = b^2 - \omega$ with $\omega > 0$ and if $\alpha = 0$ then $b = 1$. A first integral of this system is

$$H(x,y) = -\frac{\alpha}{2}x^2 - bxy - \frac{\delta}{2}y^2 - cx + dy. \tag{3}$$

Proposition 4. *Any linear differential system having a center can be written as*

$$\dot{x} = -\overline{b}x - \overline{\delta}y + \overline{d}, \quad \dot{y} = x + \overline{b}y + \overline{c}, \tag{4}$$

where $\overline{\delta} = \overline{b}^2 + \overline{\omega}$ with $\overline{\omega} > 0$. A first integral of system (4) is

$$F(x,y) = -\frac{1}{2}x^2 - \overline{b}xy - \frac{\overline{\delta}}{2}y^2 - \overline{c}x + \overline{d}y. \tag{5}$$

Note that any of the Hamiltonians (3) and (5) can be written as

$$G(x,y) = -\frac{A}{2}x^2 - Bxy - \frac{\Delta}{2}y^2 - Cx + Dy,$$

where $A = 1$ and $\Delta = B^2 + \omega$ with $\omega > 0$ if we have a linear center and in case we have a linear Hamiltonian saddle then $A \in \{0,1\}$, so that if $A = 1$ then $\Delta = B^2 - \omega$ with $\omega > 0$ and if $A = 0$ then $B = 1$ and $\Delta \in \mathbb{R}$.

2.1. Proof of Theorem 2 for the parabola.

For the systems of the class $\mathcal{F}_0$ we have following regions in the plane:

$$R_1 = \{(x,y) \in \mathbb{R}^2 : y < x^2\},$$

which is the bounded region, and the region

$$R_2 = \{(x,y) \in \mathbb{R}^2 : y > x^2\},$$

which is the unbounded region.

Without loss of generality we can assume that in R_1 we have either a linear center or a linear Hamiltonian saddle with first integral

$$G_1(x,y) = -\frac{A_1}{2}x^2 - B_1xy - \frac{\Delta_1}{2}y^2 - C_1x + D_1y \tag{6}$$

and in the region R_2 we have either a linear center or a linear Hamiltonian saddle with first integral

$$G_2(x,y) = -\frac{A_2}{2}x^2 - B_2xy - \frac{\Delta_2}{2}y^2 - C_2x + D_2y \tag{7}$$

To have a crossing limit cycle, which intersects the parabola $y = x^2$ in two different points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, these points must satisfy the *closing equations*

$$\begin{aligned} G_1(x_1,y_1) &= G_1(x_2,y_2), \\ G_2(x_2,y_2) &= G_2(x_1,y_1), \\ y_1 - x_1^2 &= 0, \\ y_2 - x_2^2 &= 0. \end{aligned} \tag{8}$$

that can be written as

$$e_1 := G_1(x_1,x_1^2) - G_1(x_2,x_2^2) = 0, \quad e_2 := G_2(x_1,x_1^2) - G_2(x_2,x_2^2) = 0. \tag{9}$$

Proof of statement (a) of Theorem 2 *for the parabola.* To study the number of limit cycles it is necessary to compute the common zeros of e_1 and e_2 in (9). For doing so we will compute $\operatorname{Res}(e_1,e_2,x_1)$ and $\operatorname{Res}(e_1,e_2,x_2)$, that is, the resultant of e_1 and e_2 with respect to x_1 and x_2, respectively. By the symmetry of e_1 and e_2 we know that both resultants have the same expression and so we only need to compute one of them. We compute $\mathcal{R} = \operatorname{Res}(e_1,e_2,x_2)$. Doing so we get

$$\mathcal{R} = C_0 + C_1x_1 + C_2x_1^2 + C_3x_1^3 + C_4x_1^4 + C_5x_1^5 + C_6x_1^6,$$

where

$$C_6 = \frac{1}{8}(B_2\Delta_1 - B_1\Delta_2)^3,$$

$$C_5 = \frac{1}{8}(B_2\Delta_1 - B_1\Delta_2)^2(-A_1\Delta_2 + A_2\Delta_1 + 2D_1\Delta_2 - 2\Delta_1D_2),$$

$$\begin{aligned} C_4 = -\frac{1}{16}&(B_1\Delta_2 - B_2\Delta_1)\big(A_1^2\Delta_2^2 + A_2^2\Delta_1^2 - 2A_1A_2\Delta_1\Delta_2 + 2A_2B_1^2\Delta_2 - 2A_2B_1B_2\Delta_1 + 4A_2D_1\Delta_1\Delta_2 \\ &- 4A_2\Delta_1^2D_2 - 4B_1^2\Delta_2D_2 - 2A_1B_1B_2\Delta_2 + 4B_1B_2D_1\Delta_2 + 4B_1B_2\Delta_1D_2 + 2B_1C_1\Delta_2^2 - 2B_1C_2\Delta_1\Delta_2 \\ &+ 2A_1B_2^2\Delta_1 - 4B_2^2D_1\Delta_1 - 2B_2C_1\Delta_1\Delta_2 + 2B_2C_2\Delta_1^2 + 4D_1^2\Delta_2^2 - 4A_1D_1\Delta_2^2 - 8D_1\Delta_1\Delta_2D_2 + 4\Delta_1^2D_2^2 \\ &+ 4A_1\Delta_1\Delta_2D_2\big), \end{aligned}$$

$$\begin{aligned} C_3 = \frac{1}{4}&(B_2\Delta_1 - B_1\Delta_2)\big(-A_2C_1\Delta_1\Delta_2 + A_2C_2\Delta_1^2 + 2B_1^2C_2\Delta_2 - 2B_1B_2C_1\Delta_2 - 2B_1B_2C_2\Delta_1 + 2B_2^2C_1\Delta_1 \\ &+ A_1C_1\Delta_2^2 - 2C_1D_1\Delta_2^2 + 2C_1\Delta_1\Delta_2D_2 - A_1C_2\Delta_1\Delta_2 + 2C_2D_1\Delta_1\Delta_2 - 2C_2\Delta_1^2C_2\big), \end{aligned}$$

$$C_2 = \frac{1}{32}\big(-B_1\Delta_1^2A_2^3 + 2C_2\Delta_1^3A_2^2 - 2B_2D_1\Delta_1^2A_2^2 + 6B_1D_2\Delta_1^2A_2^2 + B_2A_1\Delta_1^2A_2^2 + 4B_1^2B_2\Delta_1A_2^2 - 4B_1^3\Delta_2A_2^2$$

$$
\begin{aligned}
&- 2C_1\Delta_1^2\Delta_2A_2^2 - 4B_1D_1\Delta_1\Delta_2A_2^2 + 2B_1A_1\Delta_1\Delta_2A_2^2 - 8C_2D_2\Delta_1^3A_2 - 12B_1D_2^2\Delta_1^2A_2 + 8B_2^2C_1\Delta_1^2A_2\\
&- 4B_1B_2C_2\Delta_1^2A_2 + 8B_2D_1D_2\Delta_1^2A_2 - 4B_2D_2A_1\Delta_1^2A_2 - 4B_1D_1^2\Delta_2^2A_2 - B_1A_1^2\Delta_2^2A_2 + 4B_1^2C_1\Delta_2^2A_2\\
&+ 4B_1D_1A_1\Delta_2^2A_2 - 8C_1D_1\Delta_1\Delta_2^2A_2 + 4C_1A_1\Delta_1\Delta_2^2A_2 + 16B_1B_2^2D_1\Delta_1A_2 - 16B_1^2B_2D_2\Delta_1A_2\\
&- 8B_1B_2^2A_1\Delta_1A_2 + 8C_2D_1\Delta_1^2\Delta_2A_2 + 8C_1D_2\Delta_1^2\Delta_2A_2 - 4C_2A_1\Delta_1^2\Delta_2A_2 - 16B_1^2B_2D_1\Delta_2A_2\\
&+ 16B_1^3D_2\Delta_2A_2 + 8B_1^2B_2A_1\Delta_2A_2 - 8B_2D_1^2\Delta_1\Delta_2A_2 - 2B_2A_1^2\Delta_1\Delta_2A_2 - 12B_1B_2C_1\Delta_1\Delta_2A_2\\
&+ 4B_1^2C_2\Delta_1\Delta_2A_2 + 16B_1D_1CD_2\Delta_1\Delta_2A_2 + 8B_2D_1A_1\Delta_1\Delta_2A_2 - 8B_1D_2A_1\Delta_1\Delta_2A_2 + 4B_2C_2^2\Delta_1^3\\
&+ 8C_2D_2^2\Delta_1^3 - 4B_1C_1^2\Delta_2^3 - 8C_1D_1^2\Delta_2^3 - 2C_1A_1^2\Delta_2^3 + 8C_1D_1A_1\Delta_2^3 + 8B_1D_2^3\Delta_1^2 - 8B_2D_1D_2^2\Delta_1^2\\
&+ 8B_2^2C_2D_1\Delta_1^2 - 16B_2^2C_1D_2\Delta_1^2 + 8B_1B_2C_2D_2\Delta_1^2 + 4B_2D_2^2A_1\Delta_1^2 - 4B_2^2C_2A_1\Delta_1^2 - 8B_2D_1^3\Delta_2^2\\
&+ B_2A_1^3\Delta_2^2 - 6B_2D_1A_1^2\Delta_2^2 + 2B_1D_2A_1^2\Delta_2^2 - 8B_1B_2C_1D_1\Delta_2^2 + 16B_1^2C_2D_1\Delta_2^2 + 8B_1D_1^2D_2\Delta_2^2\\
&- 8B_1^2C_1D_2\Delta_2^2 + 12B_2D_1^2A_1\Delta_2^2 + 4B_1B_2C_1A_1\Delta_2^2 - 8B_1^2C_2A_1\Delta_2^2 - 8B_1D_1D_2A_1\Delta_2^2 + 4B_2C_1^2\Delta_1\Delta_2^2\\
&+ 8C_2D_1^2\Delta_1\Delta_2^2 + 2C_2A_1^2\Delta_1\Delta_2^2 + 8B_1C_1C_2\Delta_1\Delta_2^2 + 16C_1D_1D_2\Delta_1\Delta_2^2 - 8C_2D_1A_1\Delta_1\Delta_2^2\\
&- 8C_1D_2A_1\Delta_1\Delta_2^2 + 16B_2^3D_1^2\Delta_1 + 16B_1^2B_2D_2\Delta_1 + 4B_2^3A_1^2\Delta_1 - 32B_1B_2^2D_1D_2\Delta_1 - 16B_2^3D_1A_1\Delta_1\\
&+ 16B_1B_2^2D_2A_1\Delta_1 - 16B_1B_2^2D_1^2\Delta_2 - 16B_1^3D_2^2\Delta_2 - 4B_1B_2^2A_1^2\Delta_2 - 4B_1C_2^2\Delta_1^2\Delta_2 - 8C_1D_2^2\Delta_1^2\Delta_2\\
&- 8B_2C_1C_2\Delta_1^2\Delta_2 - 16C_2D_1D_2\Delta_1^2\Delta_2 + 8C_2D_2A_1\Delta_1^2\Delta_2 + 32B_1^2B_2D_1D_2\Delta_2 + 16B_1B_2^2D_1A_1\Delta_2\\
&- 16B_1^2B_2D_2A_1\Delta_2 - 16B_1D_1D_2^2\Delta_1\Delta_2 + 4B_2D_2A_1^2\Delta_1\Delta_2 + 8B_2^2C_1D_1\Delta_1\Delta_2 - 24B_1B_2C_2D_1\Delta_1\Delta_2\\
&+ 16B_2D_1^2D_2\Delta_1\Delta_2 + 24B_1B_2C_1D_2\Delta_1\Delta_2 - 8B_1^2C_2D_2\Delta_1\Delta_2 + 8B_1D_2^2A_1\Delta_1\Delta_2 - 4B_2^2C_1A_1\Delta_1\Delta_2\\
&+ 12B_1B_2C_2A_1\Delta_1\Delta_2 - 16B_2D_1D_2A_1\Delta_1\Delta_2\big),
\end{aligned}
$$

$$
\begin{aligned}
C_1 = &\frac{1}{16}\big(- A_2^2B_1C_2\Delta_1^2 + A_2^2B_2C_1\Delta_1^2 - 4A_2B_1^3C_2\Delta_2 + 4A_2B_1^2B_2C_1\Delta_2 + 4A_2B_1^2B_2C_2\Delta_1\\
&- 4A_2B_1B_2^2C_1\Delta_1 + 2A_1A_2B_1C_2\Delta_1\Delta_2 - 4A_2B_1C_2D_1\Delta_1\Delta_2 + 4A_2B_1C_2\Delta_1^2D_2 - 2A_1A_2B_2C_1\Delta_1\Delta_2\\
&+ 4A_2B_2C_1D_1\Delta_1\Delta_2 - 4A_2B_2C_1\Delta_1^2D_2 + 2A_2C_1^2\Delta_1\Delta_2^2 - 4A_2C_1C_2\Delta_1^2\Delta_2 + 2A_2C_2^2\Delta_1^3 + 8B_1^3C_2\Delta_2D_2\\
&- 8B_1^2B_2C_1\Delta_2D_2 + 4A_1B_1^2B_2C_2\Delta_2 - 8B_1^2B_2C_2D_1\Delta_2 - 8B_1^2B_2C_2\Delta_1D_2 - 4B_1^2C_1C_2\Delta_2^2 + 4B_1^2C_2^2\Delta_1\Delta_2\\
&- 4A_1B_1B_2^2C_1\Delta_2 + 8B_1B_2^2C_1D_1\Delta_2 + 8B_1B_2^2C_1\Delta_1D_2 - 4A_1B_1B_2^2C_2\Delta_1 + 8B_1B_2^2C_2D_1\Delta_1\\
&+ 4B_1B_2C_1^2\Delta_2^2 - 4B_1B_2C_2^2\Delta_1^2 - A_1^2B_1C_2\Delta_2^2 - 4B_1C_2D_1^2\Delta_2^2 + 4A_1B_1C_2D_1\Delta_2^2 + 8B_1C_2D_1\Delta_1\Delta_2D_2\\
&- 4B_1C_2\Delta_1^2D_2^2 - 4A_1B_1C_2\Delta_1\Delta_2D_2 + 4A_1B_2^3C_1\Delta_1 - 8B_2^3C_1D_1\Delta_1 - 4B_2^2C_1^2\Delta_1\Delta_2 + 4B_2^2C_1C_2\Delta_1^2\\
&+ A_1^2B_2C_1\Delta_2^2 + 4B_2C_1D_1^2\Delta_2^2 - 4A_1B_2C_1D_1\Delta_2^2 - 8B_2C_1D_1\Delta_1\Delta_2D_2 + 4B_2C_1\Delta_1^2D_2^2 + 4A_1B_2C_1\Delta_1\Delta_2D_2\\
&- 2A_1C_1^2\Delta_2^3 + 4C_1^2D_1\Delta_2^3 - 4C_1^2\Delta_1\Delta_2^2D_2 + 4A_1C_1C_2\Delta_1\Delta_2^2 - 8C_1C_2D_1\Delta_1\Delta_2^2 + 8C_1C_2\Delta_1^2\Delta_2D_2\\
&- 2A_1C_2^2\Delta_1^2\Delta_2 + 4C_2^2D_1\Delta_1^2\Delta_2 - 4C_2^2\Delta_1^3D_2\big),
\end{aligned}
$$

$$
\begin{aligned}
C_0 = &\frac{1}{32}\big(- C_1\Delta_1^2A_2^3 - 2C_2D_1\Delta_1^2A_2^2 + 6C_1D_2\Delta_1^2A_2^2 + C_2A_1\Delta_1^2A_2^2 + 4B_1B_2C_1\Delta_1A_2^2 - 4B_1^2C_1\Delta_2A_2^2\\
&- 4C_1D_1\Delta_1\Delta_2A_2^2 + 2C_1A_1\Delta_1\Delta_2A_2^2 - 4B_1C_2^2\Delta_1^2A_2 - 12C_1D_2^2\Delta_1^2A_2 + 12B_2C_1C_2\Delta_1^2A_2 + 8C_2D_1D_2\Delta_1^2A_2\\
&- 4C_2D_2A_1\Delta_1^2A_2 + 8B_1C_1^2\Delta_2^2A_2 - 4C_1D_1^2\Delta_2^2A_2 - C_1A_1^2\Delta_2^2A_2 + 4C_1D_1A_1\Delta_2^2A_2 + 8B_2^2C_1D_1\Delta_1A_2\\
&+ 8B_1B_2C_2D_1\Delta_1A_2 - 16B_1B_2C_1D_2\Delta_1A_2 - 4B_2^2C_1A_1\Delta_1A_2 - 4B_1B_2C_2A_1\Delta_1A_2 - 8B_1B_2C_1D_1\Delta_2A_2\\
&- 8B_1^2C_2D_1\Delta_2A_2 + 16B_1^2C_1D_2\Delta_2A_2 + 4B_1B_2C_1A_1\Delta_2A_2 + 4B_1^2C_2A_1\Delta_2A_2 - 12B_2C_1^2\Delta_1\Delta_2A_2\\
&- 8C_2D_1^2\Delta_1\Delta_2A_2 - 2C_2A_1^2\Delta_1\Delta_2A_2 - 4B_1C_1C_2\Delta_1\Delta_2A_2 + 16C_1D_1D_2\Delta_1\Delta_2A_2 + 8C_2D_1A_1\Delta_1\Delta_2A_2\\
&- 8C_1D_2A_1\Delta_1\Delta_2A_2 + 4C_2^3\Delta_1^3 - 4C_1^3\Delta_2^3 + 8C_1D_2^3\Delta_1^2 - 8C_2D_1D_2^2\Delta_1^2 + 16B_2C_2^2D_1\Delta_1^2 + 8B_1C_2^2D_2\Delta_1^2
\end{aligned}
$$

$$
\begin{aligned}
&- 24B_2C_1C_2D_2\Delta_1^2 - 8B_2C_2^2A_1\Delta_1^2 + 4C_2D_2^2A_1\Delta_1^2 - 8C_2D_1^3\Delta_2^2 + C_2A_1^3\Delta_2^2 - 6C_2D_1A_1^2\Delta_2^2 + 2C_1D_2A_1^2\Delta_2^2\\
&- 8B_2C_1^2D_1\Delta_2^2 + 24B_1C_1C_2D_1\Delta_2^2 - 16B_1C_1^2D_2\Delta_2^2 + 8C_1D_1^2D_2\Delta_2^2 + 4B_2C_1^2A_1\Delta_2^2 + 12C_2D_1^2A_1\Delta_2^2\\
&- 12B_1C_1C_2A_1\Delta_2^2 - 8C_1D_1D_2A_1\Delta_2^2 + 12C_1^2C_2\Delta_1\Delta_2^2 + 16B_2^3C_1^2\Delta_1 + 16B_1^2B_2C_2^2\Delta_1 + 16B_2^2C_2D_1^2\Delta_1\\
&+ 16B_1B_2C_1D_2^2\Delta_1 + 4B_2^2C_2A_1^2\Delta_1 - 32B_1B_2^2C_1C_2\Delta_1 - 16B_2^2C_1D_1D_2\Delta_1 - 16B_1B_2C_2D_1D_2\Delta_1\\
&- 16B_2^2C_2D_1A_1\Delta_1 + 8B_2^2C_1D_2A_1\Delta_1 + 8B_1B_2C_2D_2A_1\Delta_1 - 16B_1B_2^2C_1^2\Delta_2 - 16B_1^3C_2^2\Delta_2\\
&- 16B_1B_2C_2D_1^2\Delta_2 - 16B_1^2C_1D_2^2\Delta_2 - 4B_1B_2C_2A_1^2\Delta_2 - 12C_1C_2^2\Delta_1^2\Delta_2 + 32B_1^2B_2C_1C_2\Delta_2\\
&+ 16B_1B_2C_1D_1D_2\Delta_2 + 16B_1^2C_2D_1D_2\Delta_2 + 16B_1B_2C_2D_1A_1\Delta_2 - 8B_1B_2C_1D_2A_1\Delta_2 - 8B_1^2C_2D_2A_1\Delta_2\\
&- 16C_1D_1D_2^2\Delta_1\Delta_2 + 4C_2D_2A_1^2\Delta_1\Delta_2 - 24B_1C_2^2D_1\Delta_1\Delta_2 - 8B_2C_1C_2D_1\Delta_1\Delta_2 + 24B_2C_1^2D_2\Delta_1\Delta_2\\
&+ 16C_2D_1^2D_2\Delta_1\Delta_2 + 8B_1C_1C_2D_2\Delta_1\Delta_2 + 12B_1C_2^2A_1\Delta_1\Delta_2 + 8C_1D_2^2A_1\Delta_1\Delta_2 + 4B_2C_1C_2A_1\Delta_1\Delta_2\\
&- 16C_2D_1D_2A_1\Delta_1\Delta_2\big).
\end{aligned}
$$

Note that if $x_1 \neq x_2$ is a solution of the polynomial system $e_1 = e_2 = 0$ then x_1 is a root of the resultants above, but both resultants have the same roots, because these two polynomials are the same so we can pass from one to another interchanging the variables x_1 and x_2. So the values of x_1 and x_2 are the same. Consequently we only have at most 6 points (x_1, x_1^2) and (x_2, x_2^2) which are points where the crossing limit cycle intersects the parabola $y = x^2$, but due to the symmetry explained above there can not be more than 3 limit cycles. This completes the proof of Theorem 2(a) for the parabola. $\square$

Proof of statement (b) of Theorem 2 *for the parabola.* We give an example with three crossing limit cycles. More precisely, in the region R_1 we consider the linear Hamiltonian saddle

$$\dot{x} = 2x - \frac{2464}{663}y, \quad \dot{y} = -\frac{81322}{663}x - 2y \tag{10}$$

with the first integral

$$H(x,y) = -\frac{40661}{663}x^2 - 2xy + \frac{1232}{663}y^2.$$

and in the region R_2 we consider the linear center

$$\dot{x} = \frac{21145}{522} + \frac{4}{3}x - \frac{20}{9}y, \quad \dot{y} = \frac{508}{87} + 8x - \frac{4}{3}y, \tag{11}$$

with the first integral

$$F(x,y) = \frac{1}{522}(3048x + 2088x^2 - 21145y - 696xy + 580y^2).$$

This discontinuous piecewise differential system formed by the linear differential Hamiltonian saddle (10) and the linear center (11) has three crossing limit cycles, because the unique real solutions (p,q) of system (8) are $(6, 36, 2, 4)$, $(-5, 25, -3/2, 9/4)$ and (y_1, y_1^2, y_2, y_2^2) where

$$y_1 = \frac{1}{116}(51 - \sqrt{577921}) \quad \text{and} \quad y_2 = \frac{1}{116}(51 + \sqrt{577921}).$$

Therefore the intersection points of the three crossing limit cycles with the parabola are the pairs $(6, 36)$, $(2, 4)$; $(-5, 25)$, $(-3/2, 9/4)$ and (y_1, y_1^2), (y_2, y_2^2). See these three crossing limit cycles in Figure 2(a). These crossing limit cycles are travelled in couterclockwise sense. □

3. Proof of Theorem 2 for the ellipse

For these systems

$$R_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

which is the bounded region, and the region

$$R_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\},$$

which is the unbounded region. Without loss of generality we can assume that in the region R_1 we have either a linear center or a linear Hamiltonian saddle with first integral (6) and in the region R_2 we have either a linear center or a linear Hamiltonian saddle with first integral (7).

To have a crossing limit cycle, which intersects the ellipse $x^2 + y^2 = 1$ in two different points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, these points must satisfy the *closing equations*

$$\begin{aligned} G_1(x_1, y_1) =&\ \ G_1(x_2, y_2), \\ G_2(x_2, y_2) =&\ \ G_2(x_1, y_1), \\ x_1^2 + y_1^2 =&\ \ 1, \\ x_2^2 + y_2^2 =&\ \ 1 \end{aligned} \tag{12}$$

that can be written as

$$\begin{aligned} e_1 := G_1(x_1, y_1) - G_1(x_2, y_2) = 0, \\ e_1 := G_2(x_1, y_1) - G_2(x_2, y_2) = 0, \\ e_3 := x_1^2 + y_1^2 - 1 = 0, \\ e_4 := x_2^2 + y_2^2 - 1 = 0. \end{aligned} \tag{13}$$

Proof of statement (a) of Theorem 2 *for the ellipse.* To study the number of limit cycles it is necessary to compute the common zeros of e_1 and e_2 in (13) together satisfying e_3 and e_4. In order to be able to compute it we use the rational parameterization of the circle, or in other words, we introduce the change of variables

$$x_1 = \frac{2t_1}{1 + t_1^2}, \quad y_1 = \frac{1 - t_1^2}{1 + t_1^2}, \quad x_2 = \frac{2t_2}{1 + t_2^2}, \quad y_2 = \frac{1 - t_2^2}{1 + t_2^2}.$$

In these new variables equations the numerator of e_1 and the numerator of e_2 become, respectively

$$\begin{aligned} E_1 = &-2(t_1 - t_2)(-B_1 - C_1 - A_1t_1 - D_1t_1 + B_1t_1^2 - C_1t_1^2 - D_1t_1^3 - A_1t_2 - D_1t_2 + 3B_1t_1t_2 \\ &+ C_1t_1t_2 - D_1t_1^2t_2 + B_1t_1^3t_2 + C_1t_1^3t_2 + B_1t_2^2 - C_1t_2^2 - D_1t_1t_2^2 + 3B_1t_1^2t_2^2 - C_1t_1^2t_2^2 + A_1t_1^3t_2^2 \\ &- D_1t_1^3t_2^2 - D_1t_2^3 + B_1t_1t_2^3 + C_1t_1t_2^3 + A_1t_1^2t_2^3 - D_1t_1^2t_2^3 - B_1t_1^3t_2^3 + C_1t_1^3t_2^3 + t_1\Delta_1 + t_2\Delta_1 \end{aligned}$$

$$- t_1^3 t_2^2 \Delta_1 - t_1^2 t_2^3 \Delta_1)$$

and

$$\begin{aligned} E_2 = & -2(t_1 - t_2)(-B_2 - C_2 - A_2 t_1 - D_2 t_1 + B_2 t_1^2 - C_2 t_1^2 - D_2 t_1^3 - A_2 t_2 - D_2 t_2 + 3B_2 t_1 t_2 \\ & + C_2 t_1 t_2 - D_2 t_1^2 t_2 + B_2 t_1^3 t_2 + C_2 t_1^3 t_2 + B_2 t_2^2 - C_2 t_2^2 - D_2 t_1 t_2^2 + 3B_2 t_1^2 t_2^2 - C_2 t_1^2 t_2^2 + A_2 t_1^3 t_2^2 \\ & - D_2 t_1^3 t_2^2 - D_2 t_2^3 + B_2 t_1 t_2^3 + C_2 t_1 t_2^3 + A_2 t_1^2 t_2^3 - D_2 t_1^2 t_2^3 - B_2 t_1^3 t_2^3 + C_2 t_1^3 t_2^3 + t_1 \Delta_2 + t_2 \Delta_2 \\ & - t_1^3 t_2^2 \Delta_2 - t_1^2 t_2^3 \Delta_2), \end{aligned}$$

respectively. We also consider the new variables

$$E_3 = \frac{E_1}{2(t_1 - t_2)} \quad \text{and} \quad E_4 = \frac{E_2}{2(t_1 - t_2)}.$$

As in the proof of Theorem 2 we compute the resultant between E_3 and E_4 in the variable t_2 (since the resultant in the variable t_1 is the same). Doing so, we obtain a polynomial of degree six in the variable t_1. This polynomial is very large and so we do not write it here. Using again the symmetry of the solutions as in the proof of Theorem 2 we conclude that there are at most three crossing limit cycles intersecting $x^2 + y^2 = 1$. This completes the proof of Theorem 2(a) for the ellipse. □

Proof of statement (b) of Theorem 2 *for the ellipse.* We give an example with three crossing limit cycles. More precisely, in the region R_1 we consider the linear Hamiltonian saddle

$$\dot{x} = 2 - x, \quad \dot{y} = -4x + y, \tag{14}$$

with the first integral

$$H(x, y) = 2y + 2x^2 - xy,$$

and in the region R_2 we consider the linear center

$$\dot{x} = 7 - 40y, \quad \dot{y} = -3 + 20x, \tag{15}$$

with the first integral

$$F(x, y) = -3x - 7y + 10x^2 + 20y^2.$$

This discontinuous piecewise differential system formed by the linear differential Hamiltonian saddle (14) and the linear center (15) has three crossing limit cycles, because the unique real solutions (p, q) of system (12) are $(3/5, -4/5, -4/5, -3/5)$, $(1, 0, 4/5, 3/5)$ and $(-1, 0, 0, 1)$. Therefore the intersection points of the three crossing limit cycles with the ellipse are the pairs $(3/5, -4/5)$, $(-4/5, -3/5)$; $(1, 0)$, $(4/5, 3/5)$ and $(-1, 0)$, $(0, 1)$. See these three crossing limit cycles in Figure 2(b). These crossing limit cycles are travelled in couterclockwise sense. □

4. Proof of Theorem 2 for the hyperbola

For these systems we have following regions in the plane:

$$R_1 = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 > 1\},$$

which is a region that consist of two connected components, and the region

$$R_2 = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 < 1\}.$$

Without loss of generality we can assume that in the region R_1 we have either a linear center or a linear Hamiltonian saddle with first integral (6) and in the region R_2 we have either a linear center or a linear Hamiltonian saddle with first integral (7).

To have a crossing limit cycle, which intersects the hyperbola $x^2 - y^2 = 1$ in two different points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, these points must satisfy the *closing equations*

$$\begin{aligned} H_1(x_1, y_1) = &\ H_1(x_2, y_2), \\ H_2(x_2, y_2) = &\ H_2(x_1, y_1), \\ x_1^2 - y_1^2 = &\ 1, \\ x_2^2 - y_2^2 = &\ 1, \end{aligned} \tag{16}$$

or equivalently

$$\begin{aligned} e_1 := G_1(x_1, y_1) - G_1(x_2, y_2) = 0, \\ e_1 := G_2(x_1, y_1) - G_2(x_2, y_2) = 0, \\ e_3 := x_1^2 - y_1^2 - 1 = 0, \\ e_4 := x_2^2 - y_2^2 - 1 = 0. \end{aligned} \tag{17}$$

Proof of statement (a) of Theorem 2 for the hyperbola. To study the number of limit cycles it is necessary to compute the common zeros of e_1 and e_2 in (17) together satisfying e_3 and e_4. In order to be able to compute it we use the rational parameterization of the hyperbola, or in other words, we introduce the change of variables

$$x_1 = \frac{1 + t_1^2}{1 - t_1^2}, \quad y_1 = \frac{2t_1}{1 - t_1^2}, \quad x_2 = \frac{1 + t_2^2}{1 - t_2^2}, \quad y_2 = \frac{2t_2}{1 - t_2^2}.$$

In these new variables equations the numerator of e_1 and the numerator of e_2 become, respectively,

$$\begin{aligned} E_1 = -2(t_1 - t_2)(&-B_1 + D_1 - A_1t_1 - C_1t_1 - B_1t_1^2 - D_1t_1^2 + C_1t_1^3 - A_1t_2 - C_1t_2 - 3B_1t_1t_2 \\ &+ D_1t_1t_2 + C_1t_1^2t_2 + B_1t_1^3t_2 - D_1t_1^3t_2 - B_1t_2^2 - D_1t_2^2 + C_1t_1t_2^2 + 3B_1t_1^2t_2^2 + D_1t_1^2t_2^2 + A_1t_1^3t_2^2 \\ &- C_1t_1^3t_2^2 + C_1t_2^3 + B_1t_1t_2^3 - D_1t_1t_2^3 + A_1t_1^2t_2^3 - C_1t_1^2t_2^3 + B_1t_1^3t_2^3 + D_1t_1^3t_2^3 - t_1\Delta_1 - t_2\Delta_1 \\ &+ t_1^3t_2^2\Delta_1 + t_1^2t_2^3\Delta_1) \end{aligned}$$

$$\begin{aligned} E_2 = -2(t_1 - t_2)(&-B_2 + D_2 - A_2t_1 - C_2t_1 - B_2t_1^2 - D_2t_1^2 + C_2t_1^3 - A_2t_2 - C_2t_2 - 3B_2t_1t_2 \\ &+ D_2t_1t_2 + C_2t_1^2t_2 + B_2t_1^3t_2 - D_2t_1^3t_2 - B_2t_2^2 - D_2t_2^2 + C_2t_1t_2^2 + 3B_2t_1^2t_2^2 + D_2t_1^2t_2^2 + A_2t_1^3t_2^2 \end{aligned}$$

$$- C_2t_1^3t_2^2 + C_2t_2^3 + B_2t_1t_2^3 - D_2t_1t_2^3 + A_2t_1^2t_2^3 - C_2t_1^2t_2^3 + B_2t_1^3t_2^3 + D_2t_1^3t_2^3 - t_1\Delta_2 - t_2\Delta_2 + t_1^3t_2^2\Delta_2 + t_1^2t_2^3\Delta_2),$$

respectively. We also consider the new variables

$$E_3 = \frac{E_1}{2(t_1 - t_2)} \quad \text{and} \quad E_4 = \frac{E_2}{2(t_1 - t_2)}.$$

As in the proof of Theorem 2 we compute the resultant between E_3 and E_4 in the variable t_2 (since the resultant in the variable t_1 is the same). Doing so, we obtain a polynomial of degree six in the variable t_1. This polynomial is very large and so we do not write it here. Using again the symmetry of the solutions as in the proof of Theorem 2 we conclude that there are at most three crossing limit cycles intersecting $x^2 - y^2 = 1$. This completes the proof of Theorem 2(a) for the hyperbola. □

Proof of statement (b) of Theorem 2 *for the hyperbola.* We give an example with three crossing limit cycles for the discontinuous piecewise differential system (14)-(15) but with the discontinuous line the hyperbola $y - x^2 + xy/5000 = 0$.

The unique real solutions (p, q) of system (16) are (x_1, y_1, x_2, y_2) given by

(−6.32869587046.., 40.1031515507.., 7.19045317418.., 51.6283705739..),
(2.749783799607.., 7.55715483619.., 5.71306538283.., 32.6018647515..),
(−4.8335367422.., 23.3856845520.., −1.944028944857.., 3.780718503683..).

We have the exact expressions of these three solutions but they are very big, and we only give here their approximations. See these three crossing limit cycles in Figure 2(c). These crossing limit cycles are travelled in couterclockwise sense. □

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