fractal and fractional

Article

FOPI/FOPID Tuning Rule Based on a Fractional Order Model for the Process

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Abstract: This paper deals with the design of a control system based on fractional order models and fractional order proportional-integral-derivative (FOPID) controllers and fractional-order proportional-integral (FOPI) controllers. The controller design takes into account the trade-off between robustness and performance as well as the trade-off between the load disturbance rejection and set-point tracking tasks. The fractional order process model is able to represent an extensive range of dynamics, including over-damped and oscillatory behaviors and this simplifies the process modelling. The tuning of the FOPID and FOPI controllers is achieved by using an optimization, as a first step, and in a second step, several fitting functions were used to capture the behavior of the optimal parameters of the controllers. In this way, a new set of tuning rules called *FOMCoRoT* (*Fractional Order Model and Controllers Robust Tuning*) is obtained for both FOPID and FOPI controllers. Simulation examples show the effectiveness of the proposed control strategy based on fractional calculus.

Keywords: PID control; fractional order; automatic tuning; performance analysis; robustness

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1. Introduction

Theoretical and practical studies have demonstrated the advantages of using fractional calculus in the modelling and control of dynamic systems, mainly from an industrial process control point of view [1,2]. For instance, in process modelling, several works (see, for example [3–11]) have shown that fractional order models with one or two fractional parameters can represent the process dynamics better than integer transfer functions of low order, such as the well-known first order plus dead time (FOPDT) and second order plus dead time (SOPDT) models. Therefore, a wider range of real-world processes can be modeled and an improved control system design can be achieved.

In particular, this paper considers a fractional first order plus dead time (FFOPDT) model that is able to represent a wide range of dynamics, including non-oscillatory, as the typical first order and oscillatory ones, as those exhibited by under-damped processes. Moreover, the effectiveness of this model has been proved through research (see for example [12–16]). This model is useful in practice because it addresses a frequent shortcoming of tuning rules available in the literature, that is, they are based on a specific low-order integer transfer function, such that they can only be applied to a restricted range of dynamics. Thanks to this feature of the FFOPDT model, it is not necessary to change the method used to tune the integer or non-integer controller, because the structure of the model remains the same irrespective of the over- or under-damped nature of the process.

Another interesting application of fractional calculus is the design of the control algorithm. It has been demonstrated that the fractional proportional-integral-derivative

(FOPID) controllers provide more flexibility and accuracy in the adjustment of the feedback system. This can be used to guarantee more stringent specifications related to relative stability—phase, gain margins, and maximum sensitivity—and performance—set-point tracking and load-disturbance rejection—in comparison to those achievable with the classical PID controller (see, for instance [3,5,6,17–19]).

In the literature, different approaches to tune FOPID controllers have been devised. Some of them take into account the robustness of the control system with respect to process variations and model uncertainty, and therefore specifications such as gain crossover frequency, phase margin, or robustness to variations of the gain are imposed, see [3,5,20,21]; while others consider the maximum sensitivity index M_S as a measure of robustness, as in [6,16,22–25].

Other approaches have applied artificial intelligence based on fuzzy logic control to adjust the FOPID controller parameters [7,17,26], while others focus exclusively on integral performance criteria as in [27].

Finally, some approaches take into consideration the robustness and the performance of the closed-loop system at the same time; among them, ref [28] optimizes the load-disturbance rejection, ref [29] minimises an objective function in the frequency response, and [18] considers either an optimal performance for set-point tracking task or for load-disturbance rejection.

Despite the wealth of results available in the literature, it is difficult to find a set of rules that consider the trade-off between performance and robustness, and between the set-point tracking task, also known as servo-control operation, and load-disturbance rejection, also known as regulatory-control mode. To overcome these limitations, in this paper we propose a new approach, the *FOMCoRoT*, to design fractional-order PID and PI controllers. The main novelty of the proposed method is that it explicitly considers the above mentioned trade-offs and uses an FFOPDT model, thereby taking advantage of both a fractional controller and the flexibility of a fractional model. Furthermore, given the considered fractional-order process model, the devised tuning rules are also more general than those found in the literature.

This paper is organized as follows. Section 2 is devoted to the problem formulation, the description of the process model, as well as of the control algorithm and of the performance and robustness indices. Section 3 focuses on the design of the *FOMCoRoT* method for FOPID and FOPI controllers. An analysis of the robustness and performance is presented in Section 4 and bounds are established to decide when the use of fractional controllers is recommended to guarantee a minimum improvement of the performance compared to the one obtainable with classical PID/PI controllers. Then, in Section 5, simulation examples with the corresponding results are presented. Finally, conclusions are drawn in Section 6.

2. Problem Formulation

In order to obtain the tuning rule for FOPI/FOPID controllers, we consider the closed-loop control system shown in Figure 1, where $P(s)$ is the controlled process model and $C(s)$ the controller to be tuned. In this system, $r(s)$, $u(s)$, $d(s)$, and $y(s)$, are the set-point signal, the controller output, the load-disturbance, and the feedback signal, respectively.

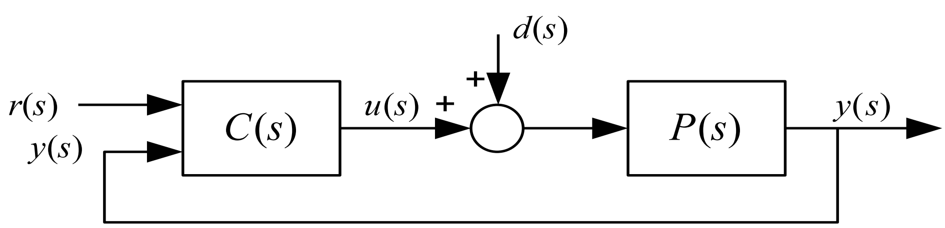

Figure 1. Closed-loop control system.

2.1. Controlled Process Model

The controlled process is a Fractional First Order Plus Dead Time model, referred here as a FFOPDT model, whose dynamic is described by the following transfer function:

$$P(s) = \frac{Ke^{-Ls}}{Ts^{\alpha} + 1}, \tag{1}$$

where K is the static gain, T is the time constant, L is the dead-time, and α is the fractional-order parameter. The FFOPDT dynamics are fully characterized by using two dimensionless parameters: the fractional order α and the normalized fractional dead-time τ_0, defined as $\tau_0 = \frac{L}{T^{\frac{1}{\alpha}}}$.

This model with values of α between 1 and 2 was selected because it can easily represent the most typical industrial processes with either over- or under-damped dynamics. In Figure 2, it can be seen how the fractional order α modifies the shape of the step response. It is important to highlight that when $\alpha = 1$, the classical FOPDT model is obtained, and when $\alpha = 2$ a pair of pure imaginary poles are found (undamped oscillatory behavior).

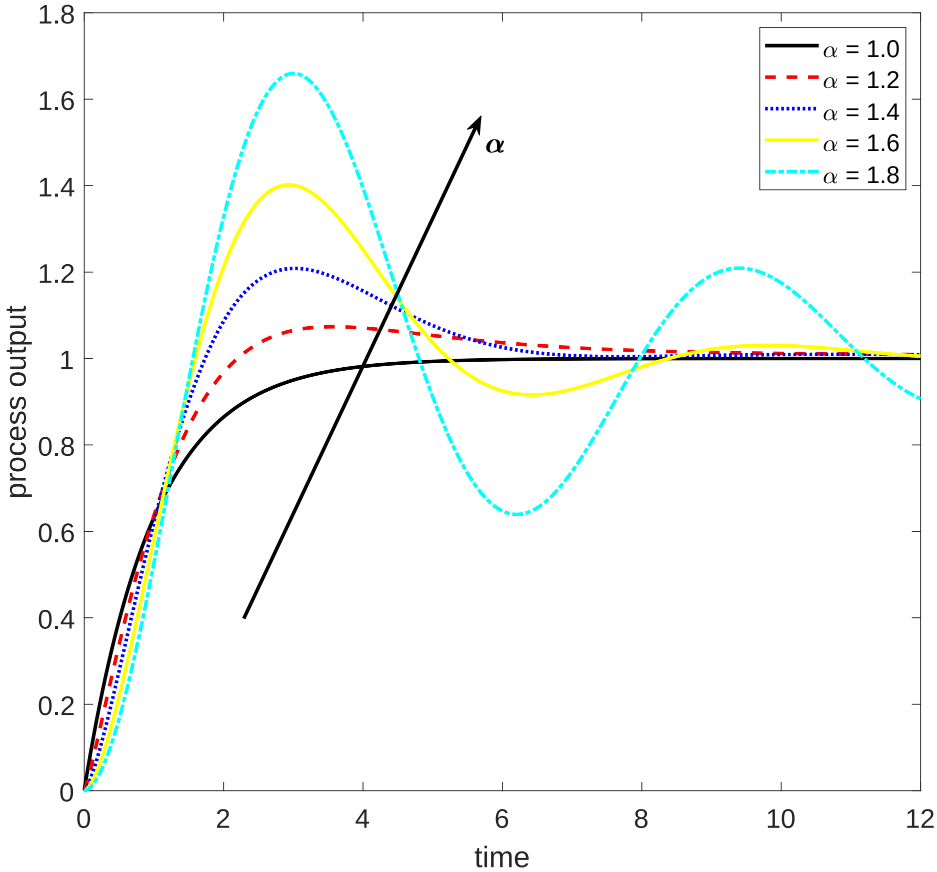

Figure 2. Step responses of the fractional system for $\alpha \in [1.0, 1.8]$.

When a fractional system is considered, the application of a rational approximation for the non-integer term is required. For this tuning rule design, the integer representation of the fractional term s^{α} is obtained by applying the so-called CRONE approach [30], defined as:

$$s^{\alpha} \to s^{\alpha}_{[\omega l, \omega h]} \approx C_o \prod_{k=1}^{N} \frac{1 + \frac{s}{w_{z,k}}}{1 + \frac{s}{w_{p,k}}}, \alpha > 0, \tag{2}$$

where $[\omega l, \omega h]$ (selected as $[0.001, 1000]$ in this work) is the frequency range where the approximation is valid and the term C_o is adjusted so that the approximation has a unity gain at the logarithmic midpoint of the approximation range. Furthermore, the parameter

N in (2) (in this case $N = 8$ to meet the minimum value $N = log(\frac{\omega h}{\omega l})$ recommended by [30]) is used to select the number of poles and zeros of the real-rational transfer function that approximates the fractional term.

2.2. FOPI(D) Controller Equation

The output signal of the considered one-degree-of-freedom (1DoF) PI(D) controller is:

$$u(s) = K_p\{e_p(s) + e_i(s) + e_d(s)\}, \tag{3}$$

with

$$e_p(s) = r(s) - y(s), \tag{4}$$

$$e_i(s) = \frac{1}{T_i s^{\lambda}}[r(s) - y(s)], \tag{5}$$

$$e_d(s) = -\frac{T_d s^{\mu}}{\frac{T_d}{\zeta}s + 1}y(s), \tag{6}$$

where K_p is the proportional gain, T_i is the integral time, T_d is the derivative time, and λ and μ are the fractional orders for the integral and derivative part, respectively. Moreover, $\frac{T_d}{\zeta}$ is the derivative filter time constant (traditionally selected by fixing $\zeta = 10$ in integer PID controllers) [31]. In order to preserve the effectiveness of this parameter on the performance of the closed-loop system, mainly in the rejection of the high frequency noise and without changing the controller dynamics significantly, it was selected as:

$$\zeta = 10T_d^{\frac{\mu-1}{\mu}}, \tag{7}$$

for FOPID controllers with the purpose of placing the pole corner frequency one decade after the zero frequency of the derivative action as it is often considered in the case of the series PID controller. Notice that an integer order filter has been included because, when using the CRONE approximation, this will be sufficient to guarantee the properness of the controller irrespective of the derivative order μ.

Finally, as shown in (6), the derivative action is only applied to the feedback signal, in order to avoid abrupt changes in the controller output signal, known as derivative kick, when a set-point step change occurs [32]. In the case of a FOPI controller, we simply set $e_d(s) = 0$.

2.3. Performance and Robustness

With the purpose of designing the tuning rule for the FOPI and FOPID controllers, the trade-off between performance and robustness has been considered along the lines of [33–36]. The function to be optimized is a multi-objective performance index given by:

$$J_t = J_{er} + J_{ed}, \tag{8}$$

where J_{er} quantifies the set-point tracking performance and J_{ed} measures the load-disturbance rejection performance. Both indices are computed as the integral of the absolute value of the error, given by:

$$J_e = \int_0^{\infty} |e(t)|\mathrm{d}t = \int_0^{\infty} |r(t) - y(t)|\mathrm{d}t. \tag{9}$$

The integral absolute error (IAE) is commonly used because, generally, it guarantees a low overshoot and a low settling time at the same time [37]. Note that the IAE index is finite only if $\lambda \geq 1$ [38]. The design procedure is usually based on a low order model identified at the closed-loop operating point, thereby disregarding non-linearities, which are found in most of the real-world industrial processes. Therefore, it is critical to consider certain stability degree or robustness requirements. For these reasons, the devised tuning rule was obtained by minimising the integral absolute of the error (9), subject to a constraint imposed on the maximum value of the sensitivity function, which is defined as:

$$M_S = \max_{\omega} |S(j\omega)| = \max_{\omega} \frac{1}{|1 + C(j\omega)P(j\omega)|}. \tag{10}$$

This index represents the inverse of the maximum distance of the Nyquist plot from the critical point $(1 + j0)$ and therefore provides a measure of the stability margin of the closed-loop system. The M_S value should remain, at least for stable processes, in the range $1.4 \leq M_S \leq 2.0$ [32]. In this paper the limits of this range ($M_S^t = 1.4$ and $M_S^t = 2.0$) are considered, where M_S^t denotes the target value of the maximum value of the magnitude of the sensitivity function.

3. Optimal Tuning

In order to find the optimal parameters for the FOPID controller, the range $1.0 \leq \alpha \leq 1.8$ for the fractional order α of the model has been considered. Note that for values of α greater than 1.8, the process becomes practically an undamped second order system which is rarely encountered in practical applications.

In the case of FOPI controllers, the fractional order α of the model in the range $1.0 \leq \alpha \leq 1.6$ has been established. This is because values of α greater than 1.6 leads to extremely low values of the normalized proportional gain κ_p, which is in line with the long-held perception that PI regulators are unsuitable to control highly under-damped processes.

In both cases, the fractional order α was varied in steps of 0.1 and the normalized fractional dead-time τ_0 was considered in the range $0.1 \leq \tau_0 \leq 2.0$, in steps of 0.1. Note that such a range includes both lag-dominant and dead-time dominant processes for which (FO)PID controllers can achieve a reasonable performance. For larger values of the normalized dead time, more complex control structures, e.g., Smith predictor, should be used.

The optimal parameters for FOPI and FOPID controllers for both $M_S^t = 1.4$ and $M_S^t = 2.0$ were obtained by optimizing the cost function (8) constrained to $M_S = M_S^t$, where the maximum sensitivity M_S is defined in (10). To solve the optimization problem, the MATLAB© solver `fminimax` and the active-set algorithm were used. Once the optimal parameters were found, different fitting functions were used to obtain simple tuning rules based on the fractional order model (1).

As an example of how the tuning rule has been obtained, in Figure 3 the optimal values of the normalized proportional gain κ_p have been plotted for different values of the normalized fractional dead-time τ_0, as well as the corresponding interpolating function in the case of the FOPID controller.

As the final result, the following general structure for the normalized FOPID controller parameters has been devised:

$$\kappa_p = K_p K = a_1 \tau_0^{a_2} + a_3, \tag{11}$$

$$\tau_i = \frac{T_i}{T^{\frac{\lambda}{\alpha}}} = b_1\tau_0^4 + b_2\tau_0^3 + b_3\tau_0^2 + b_4\tau_0 + b_5, \tag{12}$$

$$\lambda = 1, \tag{13}$$

$$\tau_d = \frac{T_d}{T^{\frac{\mu}{\alpha}}} = c_1\tau_0^3 + c_2\tau_0^2 + c_3\tau_0 + c_4, \tag{14}$$

$$\mu = d_1\tau_0^3 + d_2\tau_0^2 + d_3\tau_0 + d_4, \tag{15}$$

Values of the constants are presented in Table 1 for $M_S^t = 1.4$ and in Table 2 for $M_S^t = 2.0$.

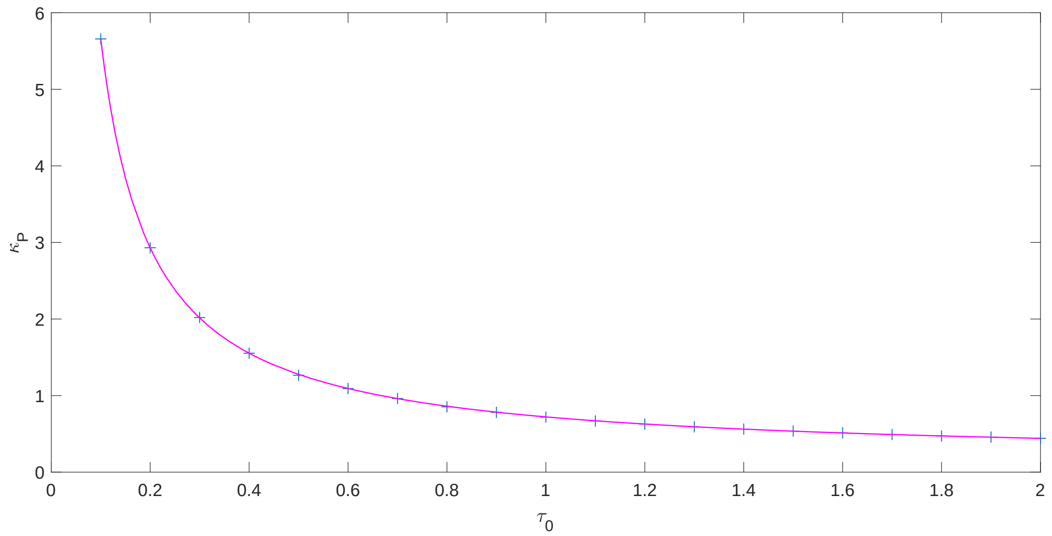

Figure 3. Determination of κ_p tuning rule for FOPID ($M_S^t = 1.4$). + are the optimal values for κ_p . Solid line is the interpolating function (11).

In the case of the FOPI controller, the following general structure was devised:

$$\kappa_p = K_p K = a_1 \tau_0^{a_2} + a_3, \quad (16)$$

$$\tau_i = \frac{T_i}{T^{\frac{\lambda}{\alpha}}} = b_1 \tau_0^{b_2} + b_3, \quad (17)$$

$$\lambda = c_1 \tau_0^{c_2} + c_3. \quad (18)$$

Table 1. FOPID tuning for servo and regulatory control operation, $M_S^t = 1.4$.

	Fractional Model Order α								
	1.0	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**
					$M_S^t = 1.4$				
a_1	0.5638	0.5653	0.5432	0.4328	0.3650	0.3351	0.2532	0.2378	0.1849
a_2	−0.9893	−1.0791	−1.1283	−1.2442	−1.2734	−1.2932	−1.4021	−1.3649	−1.5736
a_3	0.1577	0.1491	0.1554	0.2161	0.1832	0.1343	0.1207	0.0284	0.0013
b_1	−0.4000	0.1654	0.1383	−0.3206	−0.2715	−0.2702	−0.6129	−0.7134	0.0637
b_2	1.9196	−0.5511	−0.4646	1.5184	1.4090	1.5126	2.9236	3.4332	−0.6934
b_3	−3.2939	0.3352	0.2729	−2.6022	−2.6646	−2.9467	−4.5127	−5.3134	2.6015
b_4	2.7735	0.5874	0.4581	1.9910	2.0294	2.0369	2.1310	2.2545	−4.2468
b_5	0.2315	0.7783	0.9765	0.8235	0.8424	0.9472	1.1278	1.2233	2.9495
c_1	−0.0183	−0.0267	−0.0041	0.0529	0.0613	0.0813	0.0620	−0.1829	−0.1222
c_2	0.0419	0.0705	0.0059	−0.1789	−0.1725	−0.2300	−0.3381	0.4952	0.5186
c_3	0.2780	0.3095	0.4054	0.5958	0.6102	0.7256	1.1030	0.7696	1.3970
c_4	−0.0092	−0.0056	−0.0044	−0.0035	0.0658	0.1053	0.0961	0.2357	0.1155
d_1	−0.0060	0.0229	0.0805	0.0095	0.0647	0.0903	0.1043	0.0827	0.0215
d_2	0.0277	−0.0861	−0.3340	−0.1421	−0.3946	−0.4802	−0.5093	−0.4150	−0.1752
d_3	−0.0967	0.0275	0.3543	0.3105	0.6515	0.7309	0.7412	0.6236	0.3322
d_4	1.2090	1.2080	1.1290	1.0780	0.9217	0.9046	0.9145	0.9566	1.1100

Table 2. FOPID tuning for servo and regulatory control operation, $M_S^t = 2.0$.

	Fractional Model Order α								
	1.0	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**
					M_S^t = **2.0**				
a_1	0.9955	0.8489	0.5728	0.5794	0.5374	0.3551	0.2700	0.2994	0.2658
a_2	−1.0001	−1.1607	−1.3786	−1.3572	−1.4250	−1.6330	−1.7331	−1.7088	−1.7546
a_3	0.2899	0.3599	0.4664	0.2440	0.1421	0.2829	0.2587	0.1533	0.0646
b_1	−0.4951	0.1149	−0.2805	−0.1981	−0.1695	−0.0510	−0.0156	−0.3510	−0.4147
b_2	2.5286	−0.4875	1.4698	1.0982	0.9569	0.2631	0.0347	1.7148	1.9312
b_3	−4.5373	0.4419	−2.6825	−2.1250	−1.9021	−0.4175	0.1649	−2.6742	−2.7057
b_4	3.7920	0.7243	2.4579	1.8162	1.4605	0.2979	−0.4271	1.1461	0.4894
b_5	0.1189	0.6500	0.4359	0.6983	0.8765	1.1982	1.4558	1.3167	1.7059
c_1	−0.0322	−0.0168	0.0593	0.0413	0.1046	0.1654	0.2576	0.0775	−0.0033
c_2	0.0834	0.0282	−0.2785	−0.2214	−0.3584	−0.7399	−1.1615	−0.6308	−0.2688
c_3	0.2353	0.3495	0.7055	0.7685	1.0178	1.4743	2.0619	1.8940	1.8795
c_4	−0.0063	−0.0142	−0.0373	−0.0171	−0.0228	−0.0411	−0.0693	−0.0314	−0.0009
d_1	0.0000	−0.0694	−0.1143	0.0000	0.0000	−0.0637	−0.0665	−0.0027	0.0089
d_2	0.0105	0.2257	0.4496	0.0000	0.0000	0.2103	0.2108	−0.0069	−0.0543
d_3	−0.0872	−0.2320	−0.4362	0.0000	0.0000	−0.0330	0.0234	0.2409	0.2475
d_4	1.2125	1.2165	1.1699	1.0000	1.0000	0.9900	0.9861	1.0054	1.0287

Values of the constants are presented in Table 3 for $M_S^t = 1.4$ and in Table 4 for $M_S^t = 2.0$. It is important to highlight that when the fractional order α of the model was also equal to one, the optimal value for the fractional term λ of the integral mode was also equal to one. The same happens for values of the normalized fractional dead-time τ_0 greater than 0.6 and $M_S^t = 2.0$ and therefore, in those cases, there is not any advantage in using a more complex structure for the controller as the FOPI. For that reason, the fitting functions presented in Equations (16)–(18) were limited to the mentioned range.

Table 3. FOPI tuning for servo and regulatory control operation, $M_S^t = 1.4$.

	Fractional Model Order α					
	1.1	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**
			M_S^t = **1.4**			
a_1	0.1646	0.1461	0.1132	0.06155	0.06375	0.3025
a_2	−1.413	−1.32	−1.313	−1.535	−1.245	−0.4086
a_3	0.1514	0.1339	0.116	0.1153	0.07255	−0.1073
b_1	0.08874	0.03202	0.01904	0.06183	0.1728	−0.9853
b_2	1.83	2.729	−1.338	−1.173	−0.986	0.263
b_3	0.9161	1.016	1.017	0.8984	0.6747	2.295
c_1	0	0	0	1.8×10^{-10}	0.003039	−0.09594
c_2	0	0	0	−9.306	−2.057	1.694
c_3	1.01	1.03	1.043	1.059	1.068	1.445

Table 4. FOPI tuning for control and regulatory control operation, $M_S^t = 2.0$.

	Fractional Model Order α					
	1.1	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**
			M_S^t = **2.0**			
a_1	0.4784	0.3539	0.1929	0.07267	0.1209	0.1949
a_2	−1.16	−1.289	−1.524	−1.901	−1.439	−1.036
a_3	0.2257	0.2792	0.4144	0.504	0.227	0.2001
b_1	1.2	−0.1899	0.4381	−0.4154	−1.421	−1.13
b_2	0.3124	−0.8787	0.2076	1.891	0.5037	0.09457
b_3	0.1598	2.258	1.01	1.586	2.43	2.849
c_1	0.2798	−0.04119	5.739×10^{-4}	1.673×10^{-4}	5.772×10^{-5}	0.6522
c_2	0.002471	0.3394	−1.891	−3.252	−3.51	−0.1015
c_3	0.727	1.041	1.011	1.014	1.017	0.7017

4. Robustness and Performance Analysis for the Tuning Rule

4.1. FOPID Controllers

For FOPID controllers, the devised tuning rule for κ_p, τ_i, τ_d, λ, and μ are given by the expressions (11)–(15), where the values of the coefficients are presented in Table 1 for $M_S^t = 1.4$ and in Table 2 for $M_S^t = 2.0$. Note that the optimal value for λ is always equal to one, and therefore the advantages in performance and robustness of using a FOPID controller are given by the fractional order μ associated with the derivative part. This is in line with the results shown in [18], where it was proven that the fractional order associated with the integral part did not provide any advantage when a FOPID controller is designed by minimizing the integral absolute error.

To show the effectiveness of the *FOMCoRoT* tuning rule, two aspects were evaluated. The first was the maximum sensitivity M_S. The second was a comparison against the performance obtained with PID controllers designed by the same constraint on the maximum sensitivity and minimizing the same performance index (8). The optimal performance indexes for PID controllers, denoted by J_{tPID}, were taken from [39] and the J_t performance index was obtained by applying the *FOMCoRoT* tuning rule developed in this work using FOPID controllers. For this comparison, the index η is defined as:

$$\eta = \frac{J_t}{J_{tPID}}, \tag{19}$$

was used. Note that η less than one denotes and improvement in the performance provided by the tuning rule *FOMCoRoT*.

The evaluation of the maximum sensitivity value was made considering the fractional order α from 1.0 to 1.8 and the normalized fractional dead-time τ_0 from 0.1 to 2.0, in steps of 0.05 for both parameters. The results are shown in Figure 4 for both $M_S^t = 1.4$ (blue surface) and $M_S^t = 2.0$ (yellow surface). As it can be seen in the figure, the maximum sensitivity has a stable behavior very close to the target one.

The index η for $M_S^t = 1.4$ in Figure 5 shows that the improvement in the performance provided by the use of a FOPID controllers is greater than 5 % (black dashed line in Figure 5) in two specific regions when $1.0 \leq \alpha \leq 1.2$ and $0.1 \leq \tau_0 \leq 1.3$, approximately, and when $1.7 \leq \alpha \leq 1.8$. It is important to highlight that the first region encompasses most of the dynamics present in practical control applications, including first order and overdamped dynamics. In this region, the performance improvement achievable with the *FOMCoRoT* method can be as high as 10.5% larger than the one obtained with PID controllers. The second region comprises highly underdamped dynamics and, according to [40], for the fractional order range $1.7 \leq \alpha \leq 1.8$ the corresponding overshoot should roughly be between 53.2% and 67.2% and the improvement in the performance can reach 31.2%.

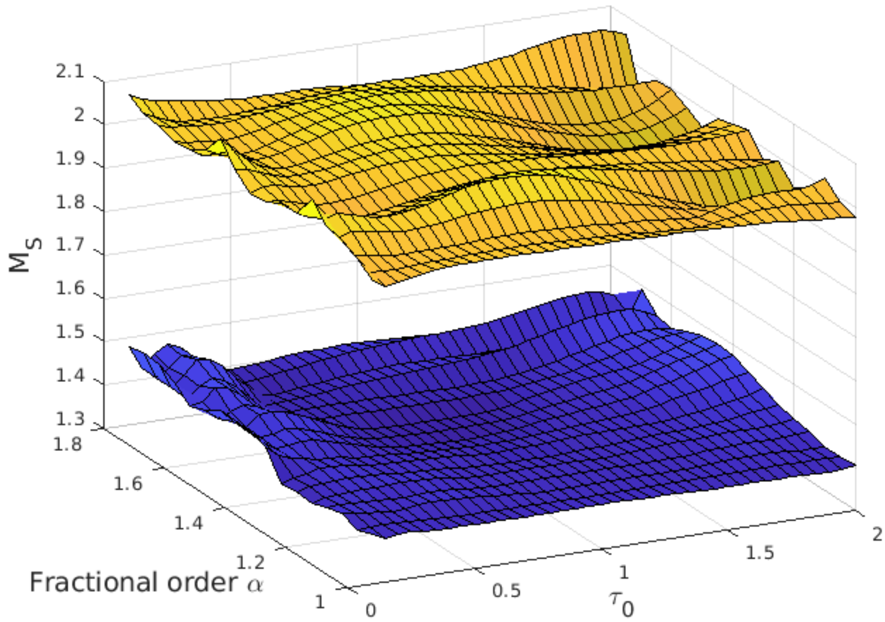

Figure 4. Evaluation of M_S using the FOPID controller.

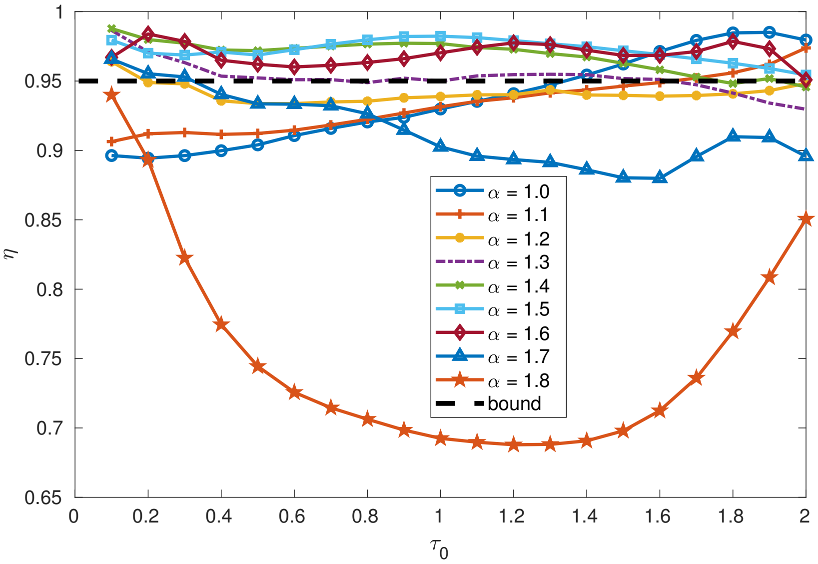

Figure 5. Index η for FOPID controllers with $M_S^t = 1.4$.

The index η for $M_S^t = 2.0$ is shown in Figure 6. As in the case $M_S^t = 1.4$, some regions can be defined where the increase in the performance provided by the use of the *FOMCoRoT* method for FOPID over the optimal performance obtained with PID controllers is larger than or equal to 5%. The first region is characterized by $\alpha = 1.8$ and $\tau_0 \geq 0.4$, and an improvement in the performance of up to 34% can be achieved. The second region is defined by $\alpha = 1.7$ and $\tau_0 \geq 0.6$, and in this case, the improvement can reach values of 21%. The third region is for $\alpha = 1.6$ and $\tau_0 \geq 1.2$, with an amelioration of up to 11% in the J_t index.

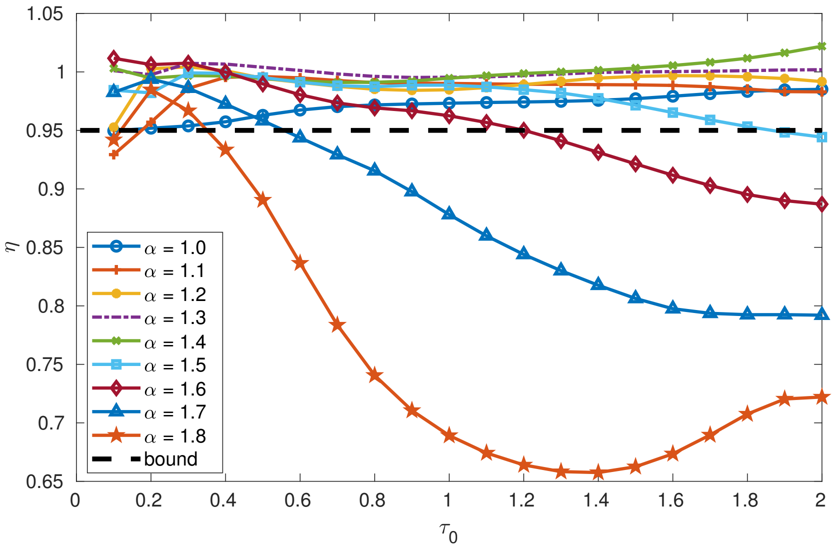

Figure 6. Index η for FOPID controllers with $M_S^t = 2.0$.

4.2. FOPI Controllers

For FOPI controllers, the *FOMCoRoT* tuning rule for κ_p, τ_i, and λ are given by the expressions (16)–(18), where the values of the parameters are presented in Table 3 for $M_S^t = 1.4$ and in Table 4 for $M_S^t = 2.0$. Note that the advantages in performance and robustness of using a FOPI controller are given by the fractional order λ associated to the integral part.

The evaluation of the maximum sensitivity was made considering the fractional order α from 1.1 to 1.6 and the normalized fractional dead-time τ_0 from 0.1 to 2.0 for $M_S^t = 1.4$ and from 0.1 to 0.6 for $M_S^t = 2.0$, both parameters were varied in steps of 0.05. Note that these cover a wider range than the one used to devise the tuning rules. The results are shown in Figure 7. The maximum sensitivity index is very close to the target one.

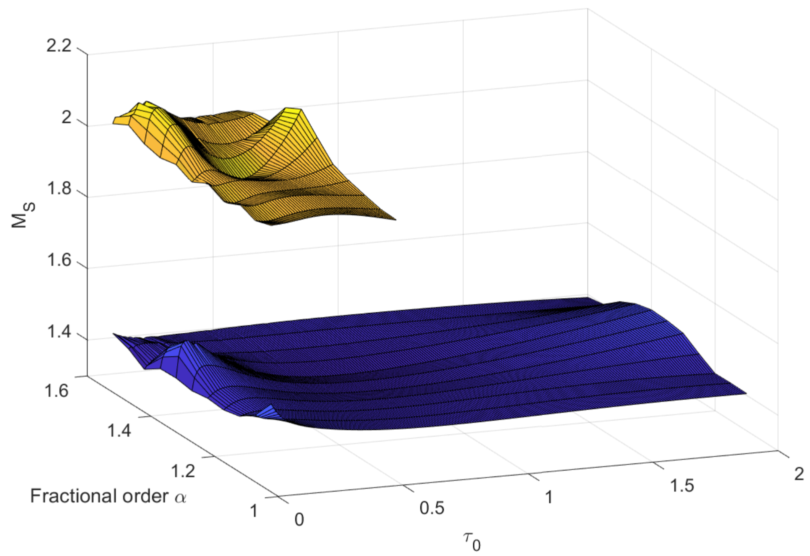

Figure 7. Evaluation of M_S for FOPI.

To evaluate the performance obtained by applying the *FOMCoRoT* tuning along the line comparison carried out for FOPID controllers, it is defined the performance index:

$$\eta = \frac{J_t}{J_{tPI}}, \tag{20}$$

where J_{tPI} is obtained from [39].

As expected, this index is always less or equal than one because an improvement in the performance of the closed-loop system is always obtained using a fractional controller.

The index η is shown in Figure 8 for $M_S^t = 1.4$, where black dashed line delimits the regions where a 5% performance improvement can be achieved. The performance improvement for τ_0 in the interval [0.1 1.3] and $\alpha = 1.6$ can be up to 24%; for τ_0 in the range [0.1 2.0] and $\alpha = 1.5$, up to 10%; for τ_0 in [0.1 0.4] and [1.3 2.0] and $\alpha = 1.4$, up to 14%; for τ_0 in [0.2 0.3] and [1.4 2.0] and $\alpha = 1.3$, up to 12%, and for τ_0 in [0.1 0.2] and $\alpha = 1.1$, up to 13%.

The performance analysis based on index η for FOPI controllers with $M_S^t = 2.0$ is shown in Figure 9. The interval where the performance improves by more than 5% is given by $\tau_0 \in [0.1\ 0.3]$ and $\alpha = 1.6$, and the improvement in performance can be up to 14%.

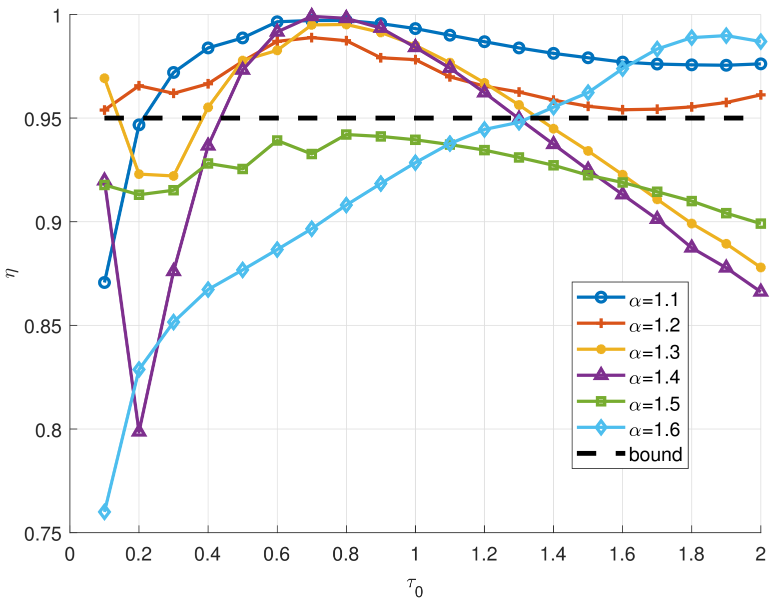

Figure 8. Index η for FOPI controllers with $M_S^t = 1.4$.

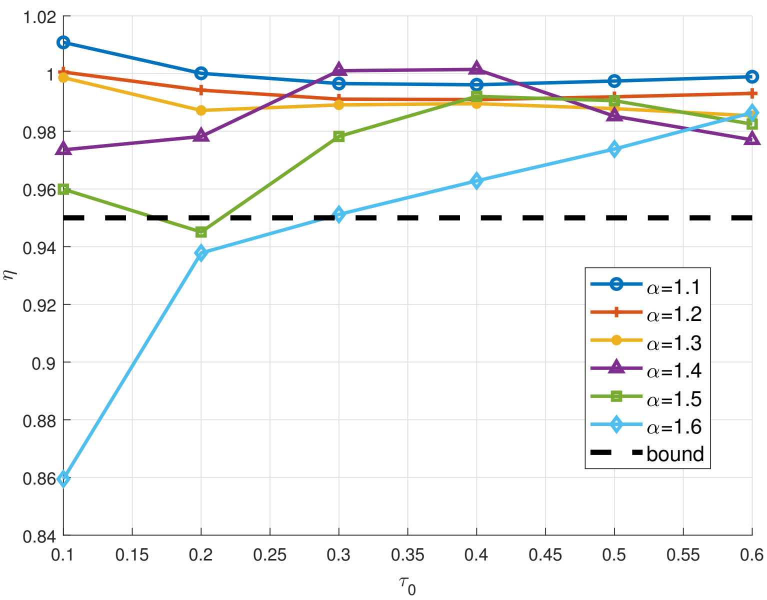

Figure 9. Index η for FOPI controllers with $M_S^t = 2.0$.

5. Simulation Examples

In order to demonstrate the effectiveness of the designed tuning rule *FOMCoRoT*, a high order process P_1 studied in [18] and shown in Equation (21), as well as a fractional order process P_2 studied in [16] and shown in Equation (22), are considered. All numerical simulations have been performed using MATLAB©.

$$P_1 = \frac{1}{(s+1)^8}, \qquad P_{m1} = \frac{e^{-4.86s}}{4.01s^{1.07}+1}, \qquad P_{m2} = \frac{e^{-4.95s}}{3.06s+1}. \tag{21}$$

Consider the process P_1. For the purpose of comparison, and in order to apply different tuning methods, two models were identified: a fractional order model P_{m1}, by using the IDFOM tool [41], and the integer order model P_{m2} that was considered in [18].

The accuracy of the model is evaluated by measuring the integral of the absolute value of the difference between the step response of the actual system P_1 and the model, P_{m1} or P_{m2} (IAE_m index). For P_{m1} the IAE_m index is equal to 0.6031 and for P_{m2} it is equal to 0.5938 and therefore, in this example, the advantages of the fractional calculus in the robustness and performance of the closed-loop system due the controller's algorithm will mainly be quantified.

Three methods were applied to tune the parameters of different structures of FOPID controllers. The technique proposed in [18] (referred here as P.&V. SP for set-point tracking or P.&V. LD for load-disturbance rejection) which considers a FOPID controller in series form and a FOPDT model, and aims at minimizing the integrated absolute error when a step change in the set-point or in the load-disturbance appears, with a constraint on the maximum sensitivity index. The second method proposed in [16], (referred here as H-F) uses series FOPID controller and a fractional order model as the one considered in this work, which aims for a trade-off performance in both set-point tracking and load-disturbance rejection, also considering the maximum sensitivity index as measure of robustness. The third approach is the *FOMCoRoT* method developed in this work. The results for performance and robustness are presented in Table 5 for $M_S^t = 1.4$ and $M_S^t = 2.0$. As was mentioned above, in this work the derivative mode was only applied to the feedback signal in order to avoid extreme changes in the controller output when a step change in the set-point value is applied. However, for the sake of fair comparison, the tracking performance when the derivative action is applied to the error signal is also evaluated. To this end, the performance indexes J_{er}^* and J_t^* are defined, similarly to JJ_{er} and J_t, respectively, but replacing $-y(t)$ with $r(t) - y(t)$.

Figure 10 shows the closed-loop responses for both tasks: a step change in the set-point value and in the load-disturbance signal. It can be noted in Table 5 that, when the constraint on robustness is given by $M_S^t = 1.4$, the best performance is achieved with the closed-loop system designed using the *FOMCoRoT* method (J_{ed} only has practically the same value in comparison with the one obtained by applying P.&V. LD). When the robustness is given by a nominal value $M_S^t = 2.0$, the best performance in the feedback system is again provided by the *FOMCoRoT* method (only J_{ed} is better with the H-F 2.0 method, but this can be attributed to a greater value of the maximum sensitivity index, $M_S = 2.8553$). If the analysis is focused on the overall performance given by the indices J_t or J_t^*, the best performance is obtained by applying the *FOMCoRoT* method.

For the fractional order process P_2 two models were identified: a fractional order model P_{m3} by applying the methodology proposed by [16] and an integer order model P_{m4} obtained by applying the *three points 123c identification method* presented in [42].

$$P_2 = \frac{1}{s^{2.6}+2.2s^{1.5}+2.9s^{1.3}+3.32s^{0.9}+1}, \quad P_{m3} = \frac{e^{-0.5171s}}{6.8601s^{1.0413}+1}, \quad P_{m4} = \frac{e^{-0.655s}}{5.406s+1}. \tag{22}$$

The model accuracy was also evaluated through the IAE_m index. For P_{m3} the IAE_m index is equal to 0.7557 and for P_{m4} it is equal to 0.2566. It can be noted that the fractional model provides a lower accuracy than the integer model. This is because, despite the increased flexibility of the fractional model, different identification methods yield different results. In any case, the fractional model is sufficiently accurate to be used effectively with the *FOMCoRoT* tuning rules.

Table 5. Comparison between tuning rules for FOPID in the control of an integer process.

Tuning Rule	M_S^t	K_p	T_i	λ	T_d	μ	M_S	J_{er}	J_{ed}	J_{er}^*	J_t	J_t^*
P.&V. SP	1.4	0.34	2.96	1	2.43	1.2	1.4026	10.7959	9.1082	9.1615	19.9041	18.2697
P.&V. LD	1.4	0.33	2.63	1	2.53	1.2	1.4283	10.8432	8.7614	8.9092	19.6046	17.6706
P.&V. SP	2.0	0.54	3.15	1	2.97	1.2	2.1195	9.4469	6.4376	6.8593	15.8845	13.2969
P.&V. LD	2.0	0.59	3.07	1	2.62	1.2	2.1156	9.467	6.4112	7.266	15.8782	13.6772
H-F	1.4	0.2352	2.43	1	4.01	1.07	1.4988	12.4727	10.4577	10.4726	22.9304	20.9302
H-F	2.0	0.5	2.43	1	4.01	1.07	2.8553	8.9615	5.5793	7.5733	14.5407	13.1526
FOMCoRoT	1.4	0.571	5.0104	1	1.9429	1.1371	1.3983	8.9024	8.769	8.771	17.6714	17.54
FOMCoRoT	2.0	0.9917	5.8036	1	1.9001	1.1353	1.9771	8.3925	5.8549	6.459	14.2473	12.3138

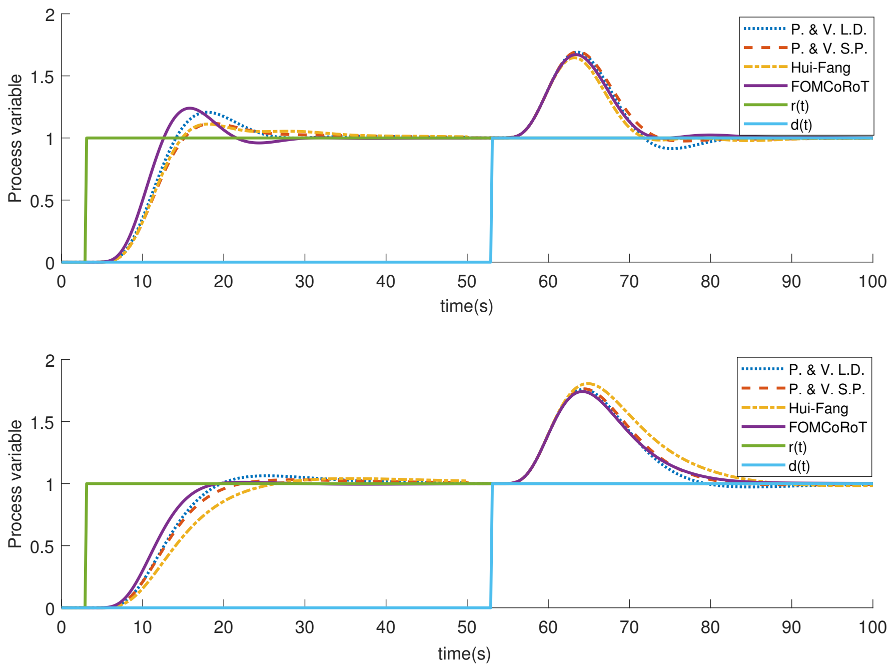

Figure 10. Closed-loop system response designed for $M_S^t = 2.0$ (at the **top**) and for $M_S^t = 1.4$ (at the **bottom**) to control P_1.

In Table 6, the performance indexes when step changes appear in the set-point and in the load-disturbance values are presented. It can be noted that the best performance for each operation mode (measured with the J_{er} or J_{ed} index) and for the overall performance (J_t index) is obtained when the controller parameters are tuned with the *FOMCoRoT* method. This can be noted also in Figure 11.

Table 6. Comparison between tuning rules for FOPID controllers in the control of a fractional process.

Tuning Rule	M_S^t	K_p	T_i	λ	T_d	μ	M_S	J_{er}	J_{ed}	J_{er}^*	J_t	J_t^*
P.&V. SP	1.4	3.8255	5.4555	1	0.2169	1.2	1.3703	1.8059	1.4266	1.6346	3.2325	3.0612
P.&V. LD	1.4	2.6563	1.2797	1	0.4811	1.1	1.3962	2.8454	0.7184	2.3704	3.5638	3.0887
P.&V. SP	2.0	5.7387	5.5095	1	0.3177	1.1	1.7828	1.4267	0.9604	1.1408	2.3872	2.1012
P.&V. LD	2.0	4.0728	0.937	1	0.4811	1	1.8299	1.9595	0.3303	1.65	2.2808	1.9803
H-F	1.4	0.2369	0.2586	1	6.8601	1.0413	1.5003	7.1647	1.1131	1.121	8.2778	2.2341
H-F	2.0	0.5036	0.2586	1	6.8601	1.0413	2.8689	6.7233	0.5238	0.5412	7.2471	1.0649
FOMCoRoT	1.4	7.6136	3.8014	1	0.1514	1.2048	1.397	1.3301	0.4993	1.1131	1.8294	1.6124
FOMCoRoT	2.0	13.9519	3.3554	1	0.1277	1.2028	2.0435	1.0017	0.2405	0.7637	1.2422	1.0042

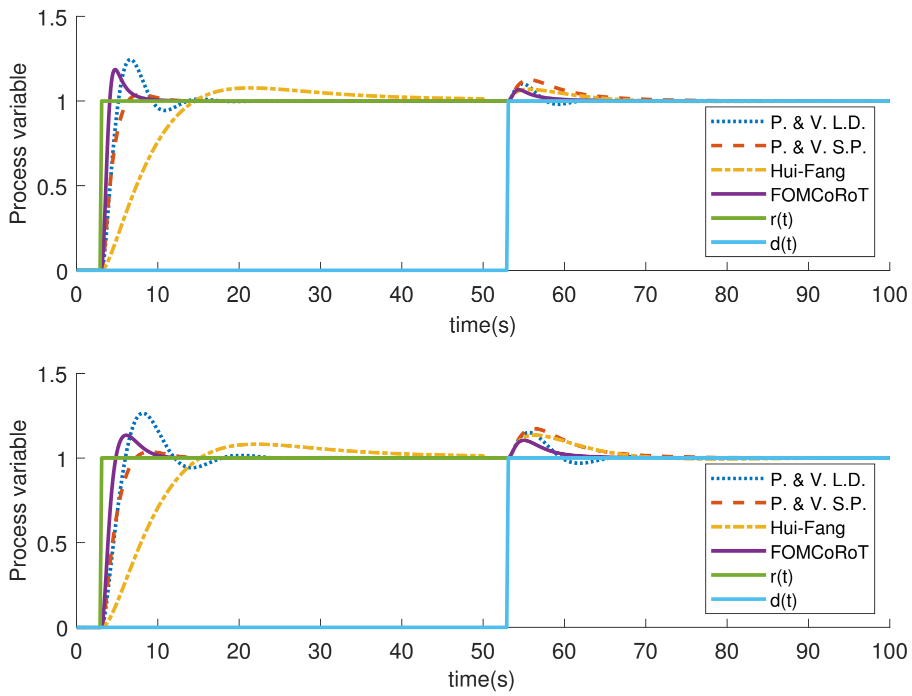

Figure 11. Closed-loop system response designed for $M_S^t = 2.0$ (at the **top**) and for $M_S^t = 1.4$ (at the **bottom**) to control P_2.

6. Conclusions

This paper deals with the design of a closed-loop control system using fractional order models and controllers. The procedure used to develop the *FOMCoRoT* method guarantees a suitable performance in both set-point tracking and load-disturbance rejection, because it has been designed to minimize a combined performance index that takes into account both operation modes. Moreover, the tuning rule designed in this work provides guaranteed stability margins due to a constraint in the control system robustness obtained by imposing suitable values for the maximum sensitivity index M_S.

Through performance analysis and simulation results, it can be concluded that FOPID and FOPI controllers provide a major impact in the closed-loop system performance when a high degree of robustness is required.

Two simulation examples have shown the effectiveness of the *FOMCoRoT* method in combination with a FFOPDT model, and this work will allow the user to have a systematic method to decide whether it is suitable to use the fractional calculus to design the control system for a given application or not.

Finally, it is worth pointing out that the proposed method deals with the design of industrial controllers, but other research lines such as controllability, observability, optimal control, robust control, etc. should be pursued in fractional order control systems in order to have a complete theoretical framework for these systems, and in particular an understanding of how the fractional order impacts the above-mentioned structural properties of a system.

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