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Classifying simply connected wandering domains

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Dedicated to Misha Lyubich on his 60th birthday

Abstract

While the dynamics of transcendental entire functions in periodic Fatou components and in multiply connected wandering domains are well understood, the dynamics in simply connected wandering domains have so far eluded classification. We give a detailed classification of the dynamics in such wandering domains in terms of the hyperbolic distances between iterates and also in terms of the behaviour of orbits in relation to the boundaries of the wandering domains. In establishing these classifications, we obtain new results of wider interest concerning non-autonomous forward dynamical systems of holomorphic self maps of the unit disk. We also develop a new general technique for constructing examples of bounded, simply connected wandering domains with prescribed internal dynamics, and a criterion to ensure that the resulting boundaries are Jordan curves. Using this technique, based on approximation theory, we show that all of the nine possible types of simply connected wandering domain resulting from our classifications are indeed realizable.

1 Introduction

We consider dynamical systems defined by the iteration of holomorphic maps $f : \mathbb{C} \to \mathbb{C}$ on the complex plane, and particularly transcendental ones, that is, those with an essential singularity at infinity. The complex plane, seen as the phase space of the system, splits into two completely invariant subsets: the *Fatou set*, or those points in a neighbourhood of which the iterates $\{f^n\}$ form a normal family, and its complement, the *Julia set*. The Fatou set is open and consists typically of infinitely many connected components called *Fatou components*.

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Fatou components map from one to another and this leads to dynamics on the set of these components.

In this setting, periodic Fatou components were completely classified a century ago by Fatou, in terms of the possible limit functions of the family of iterates; see, for example, [Ber93]. Indeed, if U is a periodic Fatou component of period $p \geq 1$, then U can only be one of the following: a domain on which the iterates $\{f^{pn}|_U\}_n$ converge to an attracting or parabolic fixed point of f^p (known as an attracting or parabolic component, respectively); or a domain on which the iterates $\{f^{pn}|_U\}_n$ converge to infinity locally uniformly (known as a *Baker domain*); or a topological disk on which f^p is conjugate to a rigid irrational rotation (known as a *Siegel disk*).

If a Fatou component U is neither periodic, nor preperiodic (that is, eventually periodic), then $f^i(U) \cap f^j(U) = \emptyset$ for all $i, j \geq 0$, $i \neq j$ and U is called a *wandering domain.* On a wandering domain all limit functions must be constant [Fat20]. Those for which the only limit function is the point at infinity are called *escaping*, while the rest are either *oscillating* (if infinity is a limit function and some other finite value also) or *dynamically bounded* (if all limit functions are points in the plane). A major open problem in transcendental dynamics is whether dynamically bounded wandering domains exist at all. We believe that any progress towards solving this problem will require a deeper knowledge of the dynamics inside (and around) wandering domains, our main motivation for the work in this paper.

An essential role in the theory of holomorphic dynamics is played by the *singular values*, that is, those points for which not all inverse branches are locally well defined. In transcendental dynamics, these can be *critical values* (images under f of zeros of f'), *asymptotic values* or accumulations thereof.

For a wide class of functions known as *finite type maps* (those maps with a finite number of singular values), every Fatou component is periodic or preperiodic. Indeed, the absence of wandering domains for polynomials (actually for rational maps) [Sul85] and for transcendental entire functions of finite type [GK86], [EL92] was a major breakthrough in the theory of complex dynamics, and meant that the possible types of dynamical behaviours of all such maps within the Fatou set was fully classified. The result about the absence of wandering domains for the class of transcendental maps of finite type was particularly striking because in the 1970's Baker [Bak76] had constructed a transcendental entire function which had a nested sequence of multiply connected Fatou components, each mapping to the next and whose orbits escaped to infinity, showing that wandering domains can indeed exist. While the wandering domains in Baker's example were multiply connected, since then a wide variety of examples of simply connected wandering domains have been given; see, for example, [Her84, p. 106], [Sul85, p. 414], [Bak84, p. 564, p. 567], [Dev90, p. 222], [EL92, Examples 1 and 2] and [FH08, Sect. 4.3.]. But it is only more recently that wandering domains have emerged as a major focus of attention, as the least understood of all the different types of Fatou components.

Indeed, several important advances have been made in recent years. For example, (oscillating) wandering domains have been constructed for functions in the Eremenko-Lyubich class $\mathcal{B}$ (those maps with a bounded sets of singular values) [Bis15, MPS20, FJL18], a landmark result because escaping wandering domains have been shown not to exist for maps in this class [EL92]. We also mention the recent construction by Bishop [Bis18] of an entire function with Julia set of Hausdorff dimension 1, solving a long standing problem in transcendental dynamics. This function has multiply connected wandering domains and also exhibits other remarkable properties; for example, all the boundary components of its wandering domains are Jordan curves.

Moreover, a detailed description of the dynamics of entire functions *within* multiply connected wandering domains was obtained in [BRS13]. Perhaps surprisingly it turns out that in these wandering domains all orbits behave in essentially the same manner, eventually landing in and remaining in a sequence of very large nested round annuli. This detailed description has proved crucial in establishing results about classes of commuting transcendental entire functions [BRS16].

Surprisingly, however, very little is known about the full range of possible behaviours of the orbits inside *simply* connected wandering domains, relative to the components themselves. This behaviour is connected to the relation between the *postsingular set* $P(f)$ (that is, the union of the forward orbits of the singular values) and the wandering domains, another major open problem in the subject. Indeed, recent results in [BFJK19, MBRG13] establish that if U is a wandering domain and U_n is the Fatou component containing $f^n(U)$ for $n \in \mathbb{N}$, then for every $z \in U$, there exists a sequence $p_n \in P(f)$ such that $\operatorname{dist}(p_n, U_n)/\operatorname{dist}(f^n(z), \partial U_n) \to 0$ as $n \to \infty$. Hence the understanding of the possible dynamics of orbits inside wandering domains may throw some light on the possible relations between the postsingular set and the wandering domains, both issues being potentially relevant in any future attempt to eliminate dynamically bounded wandering domains.

One of the challenges is that several different types of behaviour are known to exist. Let us elaborate a bit further on this observation. Consider any holomorphic self-map of $\mathbb{C} \setminus \{0\}$, or an entire map $F : \mathbb{C} \to \mathbb{C}$ for which $z = 0$ is either an omitted value or has itself as its only preimage; for example, $F_\lambda(z) = \lambda z^d \exp(z)$ with $d \in \mathbb{N}$, $\lambda \in \mathbb{C} \setminus \{0\}$. Such a map F can be lifted by the exponential map to a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ satisfying $\exp(f(z)) = F(\exp(z))$. Observe that f is not uniquely defined, since any map of the form $f_k(z) = f(z) + 2k\pi i$ for $k \in \mathbb{Z}$ will have the same property. Now notice that if F had, say, an attracting component U (not containing $z = 0$), then any logarithm of U, say $\widetilde{U}$, would be a wandering domain for f_k (for an appropriate choice of k). Nevertheless, the orbits of points in $\widetilde{U}$ would still "remember" that they were lifted from an attracting component, in the sense that the iterates of any given point would be successively closer to the orbit of $\widetilde{p} := \log p \in \widetilde{U}$, where p is the fixed point of F in U. Likewise, if U had been, for example, a Siegel disk, the iterates of points in the successive images of $\widetilde{U}$ would "rotate" around a centre point (actually orbit), again the iterates of $\widetilde{p}$. See Figure 1, and also Figure 3 in Section 3.3 for a lift of a parabolic component.

With this lifting procedure, one can construct examples of simply connected escaping wandering domains exhibiting the three different types of internal dynamics that correspond to the possible dynamics inside a periodic component: attracting, parabolic or rotation-like. Thus we already have a contrast with multiply connected wandering domains, where only one type of dynamical behaviour is possible, as noted above. These observations suggest a very natural question: How special are the three examples above in the general world of wandering domains; in other words, is there a classification of wandering domains in the spirit of Fatou's classification of periodic Fatou components or is *any* orbit behaviour realizable? Let

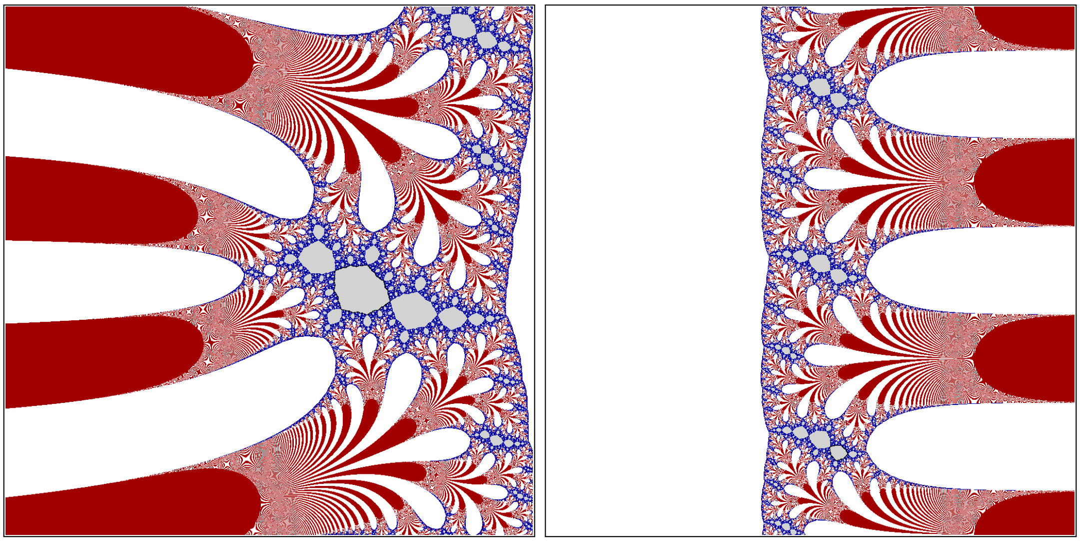

Figure 1: Left: Dynamical plane of $F(w) = \lambda w^2 e^{-w}$ with $\lambda = e^{2-\rho}/(2-\rho)$ and $\rho = e^{\pi i(1-\sqrt{5})}$. There is a (bounded) super-attracting component centred at $w = 0$ (white) and a Siegel disk centred at $w_0 = 2 - \lambda$ (gray). Right: Dynamical plane of $f(z) = 2z - e^z + \log \lambda$ satisfying $\exp(f(z)) = F(\exp(z))$. The super-attracting component lifts to a Baker domain (white), while the Siegel disk lifts to infinitely many orbits of wandering domains on which f is univalent (gray). See [Ber95a, FH08, FG03] for details. The range is $[-9, 9] \times [-9, 9]$.

us note that, due to the lack of periodicity, the dynamics of f on a sequence of wandering domains can be thought of as a non-autonomous system (at every iterate we apply a "different" map), and such systems are *a priori* difficult to study because they may exhibit a wide range of behaviours. This might be an indication that such a classification may not exist. On the other hand, the successful description of the dynamics in multiply connected wandering domains obtained in [BRS13] is encouraging.

The dynamics of points which belong to wandering domains can be seen from two perspectives. While points have to move together with the wandering domain which contains them (in the way that passengers on a cruise ship must follow the ship's trajectory), on the one hand they may or may not cluster together as they move along (as happens when lifting an attracting component but not when lifting a Siegel disk), and on the other hand orbits may stay away from the boundaries of their domains (as happens when lifting an attracting basin but not when lifting a parabolic basin).

Our results will address both of those points of view. More precisely we give a complete and precise description of the possible dynamics of orbits inside the wandering components in terms of both the contraction properties with respect to the hyperbolic metric, and the distance of orbits to the boundary of the wandering components (Theorems A, B and C). This provides a wandering version of the Fatou Classification Theorem of periodic components, a cornerstone of holomorphic dynamics. Moreover, we show that all of the possible cases exist by proving a new general and potentially very useful tool (Theorem D) to establish the existence of wandering components for a given map. Finally, we prove that under certain conditions the wandering domains have Jordan curve boundaries. Our proof includes a novel result (Theorem 7.5) on the Euclidean lengths of vertical geodesics of annuli, using a new technique involving the classical Féjer-Riesz Inequality, which is of independent interest.

Statement of results

The most natural intrinsic quantity that we have to hand – intrinsic in that it does not depend on the embedding of the wandering domains in the plane – are the hyperbolic distances between pairs of corresponding points of two orbits, and so our approach will be to evaluate how hyperbolic distances between such pairs of points evolve under iteration.

Let us recall that a domain $U \subset \mathbb{C}$ is *hyperbolic* if its boundary (in $\mathbb{C}$) contains at least two points. For a hyperbolic domain U, let $\rho_U(z)$ denote the hyperbolic density at $z \in U$ and for $z, z' \in U$ let $\text{dist}_U(z, z')$ denote the hyperbolic distance in U between z and z'. Also recall that if U, V are hyperbolic domains, and $f : U \to V$ is a holomorphic map, then the Schwarz-Pick Lemma ensures that f is a contraction for the hyperbolic distance. Hence, if $U \subset \mathbb{C}$ is a wandering domain of a transcendental entire function f and we define U_n, as above, to be the Fatou component containing $f^n(U)$, for $n \in \mathbb{N}$, we have that, given any two points $z, z' \in U$, the sequence

$$\text{dist}_{U_n}(f^n(z), f^n(z'))$$

is decreasing and therefore converges to a value that we denote by

$$c(z, z') = c_U(z, z') := \lim_{n\to\infty} \text{dist}_{U_n}(f^n(z), f^n(z')) \geq 0.$$

Our first classification result shows that whether or not $c(z, z')$ is zero does not actually depend on the chosen pair (z, z'), provided that the two points have distinct orbits. We also give a criterion to discriminate between these cases based on the concept of *hyperbolic distortion* [BM07, Sect. 5,11].

Definition 1.1 (Hyperbolic distortion). If $f : U \to V$ is a holomorphic map between two hyperbolic domains U and V, then the *hyperbolic distortion* of f at z is

$$\|Df(z)\|_U^V := \lim_{z'\to z} \frac{\text{dist}_V(f(z'), f(z))}{\text{dist}_U(z', z)},$$

and it equals the modulus of the hyperbolic derivative.

Theorem A (First classification theorem). *Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$. Define the countable set of pairs*

$$E = \{(z, z') \in U \times U : f^k(z) = f^k(z') \text{ for some } k \in \mathbb{N}\}.$$

Then, exactly one of the following holds.

(1) $\text{dist}_{U_n}(f^n(z), f^n(z')) \underset{n\to\infty}{\longrightarrow} c(z, z') = 0$ *for all $z, z' \in U$, and we say that U is* (hyperbolically) contracting*;*

(2) $\mathrm{dist}_{U_n}(f^n(z), f^n(z')) \underset{n\to\infty}{\longrightarrow} c(z,z') > 0$ *and* $\mathrm{dist}_{U_n}(f^n(z), f^n(z')) \neq c(z,z')$ *for all* $(z,z') \in (U \times U) \setminus E$*,* $n \in \mathbb{N}$*, and we say that* U *is* (hyperbolically) semi-contracting*; or*

(3) *there exists* $N > 0$ *such that for all* $n \geq N$*,* $\mathrm{dist}_{U_n}(f^n(z), f^n(z')) = c(z,z') > 0$ *for all* $(z,z') \in (U \times U) \setminus E$*, and we say that* U *is* (hyperbolically) eventually isometric.

Moreover for $z \in U$ *and* $n \in \mathbb{N}$ *let* $\lambda_n(z)$ *be the hyperbolic distortion* $\|Df(f^{n-1}(z))\|_{U_{n-1}}^{U_n}$*. Then*

- U *is contracting if and only if* $\sum_{n=1}^{\infty}(1 - \lambda_n(z)) = \infty$*;*
- U *is eventually isometric if and only if* $\lambda_n(z) = 1$*, for* n *sufficently large.*

Note that, by the Schwarz-Pick Lemma, U is eventually isometric if and only if $f : U_n \to U_{n+1}$ is univalent for large n and so a wandering domain obtained by lifting a Siegel disk is always eventually isometric. In contrast, we show that lifting an attracting or parabolic component results in a contracting wandering domain. To distinguish between these two cases, we refine the classification of contracting wandering domains according to the rate of contraction.

Definition 1.2 (Rate of contraction)**.** Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$. We say that U is *strongly contracting* if there exists $c \in (0,1)$ such that

$$\mathrm{dist}_{U_n}(f^n(z), f^n(z')) = O(c^n), \quad \text{for } z, z' \in U.$$

We say that U is *super-contracting* if it satisfies the stronger condition that

$$\lim_{n\to\infty} (\mathrm{dist}_{U_n}(f^n(z), f^n(z')))^{1/n} = 0, \quad \text{for } z, z' \in U.$$

It is easy to see that the lift of an attracting component is strongly contracting, and we prove in Section 3 that the lift of a parabolic component is contracting but not strongly contracting. We do this by a careful analysis of the behaviour of the hyperbolic distance between pairs of points in two orbits in any parabolic component; see Theorem 3.4.

A special case of super-contracting wandering domains is given by wandering domains which contain an orbit consisting of critical points. An example of such a super-contracting domain which does not arise from a lifting procedure is given in Theorem F.

Next, we give sufficient criteria for a wandering domain to be strongly contracting or super-contracting in terms of the long term average values of the hyperbolic distortion along the orbit of a point $z_0 \in U$. We also show that this quantity is independent of the point z_0.

Theorem B. *Let* U *be a simply connected wandering domain of a transcendental entire function* f *and let* U_n *be the Fatou component containing* $f^n(U)$*, for* $n \geq 0$*. Fix a point* $z_0 \in U$ *and, for* $z \in U$ *and* $n \in \mathbb{N}$*, let* $\lambda_n(z) = \|Df(f^{n-1}(z))\|_{U_{n-1}}^{U_n}$*. Then the following hold:*

(a) *If* $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z_0) < 1$*, then* U *is strongly contracting.*

(b) *If* $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z_0) = 0$*, then* U *is super-contracting.*

(c) *If* $z \in U$*, then* $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \lambda_k(z_0)$.

Once the behaviour of orbits in relation to each other within simply connected wandering domains is well understood, we turn to the question of how these orbits interact with the boundaries of the wandering domains. The concept of orbits 'approaching the boundary' is in itself delicate to define since it depends on the shape of the sets U_n, which may become highly distorted (think for example of the ratio of the diameter of the domains to their conformal radius, which may tend to infinity). There are alternative candidates for the definition of convergence to the boundary (see Section 4), but in this paper we use the following definition.

Definition 1.3 (Boundary convergence)**.** Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$. We say that the orbit of $z \in U$ *converges to the boundary* (of U_n) if and only if $\operatorname{dist}(f^n(z), \partial U_n) \to 0$ as $n \to \infty$.

We show that, with this definition, the following trichotomy holds.

Theorem C (Second classification theorem)**.** *Let* U *be a simply connected wandering domain of a transcendental entire function* f *and let* U_n *be the Fatou component containing* $f^n(U)$*, for* $n \geq 0$*. Then exactly one of the following holds:*

(a) $\liminf_{n\to\infty} \operatorname{dist}(f^n(z), \partial U_n) > 0$ *for all* $z \in U$*, that is, all orbits stay away from the boundary;*

(b) *there exists a subsequence* $n_k \to \infty$ *for which* $\operatorname{dist}(f^{n_k}(z), \partial U_{n_k}) \to 0$ *for all* $z \in U$*, while for a different subsequence* $m_k \to \infty$ *we have that*

$$\liminf_{k\to\infty} \operatorname{dist}(f^{m_k}(z), \partial U_{m_k}) > 0, \quad \text{for } z \in U;$$

(c) $\operatorname{dist}(f^n(z), \partial U_n) \to 0$ *for all* $z \in U$*, that is, all orbits converge to the boundary.*

We remark that we actually prove a stronger version of Theorem C (see Theorem 4.2), which takes into account different definitions of converging to the boundary.

Construction of examples

Theorems A and C combine to give nine different dynamical types of simply connected wandering domains. A natural question to ask is whether all of these can be realized. As far as we know, the existing examples of simply connected wandering domains in the literature belong to one of the following three cases: contracting and converging to the boundary (e.g. lifts of parabolic components); contracting and staying away from the boundary (e.g. lifts of attracting components); isometric and staying away from the boundary (e.g. lifts of Siegel disks). We will use approximation theory (see Section 5) to construct examples of each of the nine possibilities. In fact, we present a new general technique to construct *bounded* simply connected wandering domains (see Theorem 5.3) which allows us to keep good control on the internal dynamics, as well as on the degree of the resulting maps from one Fatou component to the next. As a key step we prove the following general result to show the existence of bounded simply connected wandering domains. Its statement uses the following terminology.

Definition 1.4. We say that a curve σ *surrounds* a set B if and only if B is contained in a bounded complementary component of σ. Also, for a Jordan curve η we denote by $\operatorname{int}\eta$ the bounded component of $\mathbb{C}\setminus\eta$ and by $\operatorname{ext}\eta$ the unbounded component of $\mathbb{C}\setminus\eta$.

We can now state our result.

Theorem D (Existence criteria for wandering domains)**.** *Let f be a transcendental entire function and suppose that there exist Jordan curves γ_n and Γ_n, $n\geq 0$, a bounded domain D, a subsequence $n_k\to\infty$ and compact sets L_k (associated with Γ_{n_k}) such that*

(a) *Γ_n surrounds γ_n, for $n\geq 0$;*

(b) *for every $k,n,m\geq 0$, $m\neq n$ the sets $L_k,\overline{D},\Gamma_m$ are in $\operatorname{ext}\Gamma_n$;*

(c) *γ_{n+1} surrounds $f(\gamma_n)$, for $n\geq 0$;*

(d) *$f(\Gamma_n)$ surrounds Γ_{n+1}, for $n\geq 0$;*

(e) *$f(\overline{D}\cup\bigcup_{k\geq 0}L_k)\subset D$;*

(f) $\max\{\operatorname{dist}(z,L_k):z\in\Gamma_{n_k}\}=o(\operatorname{dist}(\gamma_{n_k},\Gamma_{n_k}))$ *as $k\to\infty$.*

Then there exists an orbit of simply connected wandering domains U_n such that $\overline{\operatorname{int}\gamma_n}\subset U_n\subset\operatorname{int}\Gamma_n$, for $n\geq 0$.

Moreover, if there exists $z_n\in\operatorname{int}\gamma_n$ such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind d_n times around $f(z_n)$, then $f:U_n\to U_{n+1}$ has degree d_n, for $n\geq 0$.

We use Theorem D to construct examples of each of the nine possible types, and also to construct simply connected wandering domains that contain any prescribed (finite) number of orbits consisting of critical points. A wandering domain U will be called *k-super-attracting* if there exist critical points $z_1,\ldots,z_k\in U$, such that $f^n(z_1),\ldots,f^n(z_k)$ are critical points of f, for all $n\in\mathbb{N}$.

Theorem E (All types are realizable)**.** (a) *For each of the nine possible types of simply connected wandering domains arising from Theorems A and C, there exists a transcendental entire function with a bounded, simply connected escaping wandering domain of that type.*

(b) *For each $k\in\mathbb{N}$, there exists a transcendental entire function f having a bounded, simply connected escaping wandering domain U which is k-super-attracting.*

Note that our examples in part (b) of Theorem E are super-contracting wandering domains that are not lifts of super-attracting components.

The bounded, simply connected wandering domains, (U_n) say, constructed in Theorem E all have a shape that tends to the shape of a Euclidean disk as $n\to\infty$, but in fact the construction can easily be modified to give wandering domains with different limiting shapes. Also, in forthcoming work we combine our new ideas with techniques introduced by Eremenko and Lyubich [EL87] to construct examples of oscillating simply connected wandering domains that are bounded and have various types of internal dynamics.

Finally, we show that our methods can be adapted to construct simply connected wandering domains bounded by Jordan curves. Theorem E is proved by obtaining entire functions that approximate sequences of translates of Blaschke products associated with sequences of Jordan curves with the properties given in Theorem D, and we show that, if these Blaschke products are in a certain sense uniformly expanding and have uniformly bounded degree, then the resulting wandering domains have Jordan curve boundaries.

Structure of the paper

The first part of the paper (Sections 2, 3 and 4) is devoted to studying the possible behaviours of orbits in simply connected wandering domains, proving Theorems A, B and C. We begin in Section 2 by setting up related non-autonomous dynamical systems of self maps of the unit disk. We prove several results in this general setting which may be of wider interest. In Section 3 we use our results from Section 2 to prove Theorems A and B. We prove Theorem C in Section 4.

The second part of the paper (Sections 5, 6 and 7) is devoted to the construction of examples. In Section 5 we give the proof of Theorem D and develop a new general technique for constructing bounded wandering domains. In Section 6 we use this technique to construct examples of every possible behaviour classified in the first part of the paper, proving Theorem E. Finally, in Section 7 we show that, under certain conditions, our new construction technique gives simply connected wandering domains that are Jordan domains.

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2 Non-autonomous dynamical systems of self maps of the unit disk

In this section we prove several results in the general setting of non-autonomous forward dynamical systems of holomorphic self maps of the unit disk fixing the origin. These results may be of wider interest with applications outside holomorphic dynamics. In the next section, we apply them to the case of transcendental entire functions with simply connected wandering domains in order to prove Theorem A and Theorem B.

Our proofs are based on hyperbolic distances in the unit disk and we make frequent use of the fact that

$$\operatorname{dist}_{\mathbb{D}}(w,0) = \int_0^{|w|} \frac{2\,dt}{1-t^2} = \log\left(\frac{1+|w|}{1-|w|}\right), \quad \text{for } w \in \mathbb{D}. \tag{2.1}$$

In our first result, we characterize when the limits of such systems of holomorphic self maps of the unit disk are identically equal to zero, in terms of the values of the derivatives of the maps at 0. In particular, unless $|g_n'(0)| \to 1$ as $n \to \infty$, the limit of the maps G_n is always zero.

Theorem 2.1 (Criterion for converging to zero)**.** *For each $n \in \mathbb{N}$, let $g_n : \mathbb{D} \to \mathbb{D}$ be holomorphic with $g_n(0) = 0$ and $|g_n'(0)| = \lambda_n$, and let $G_n = g_n \circ \cdots \circ g_1$.*

(a) *If $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$, then $G_n(w) \to 0$ as $n \to \infty$, for all $w \in \mathbb{D}$.*

(b) *If $\sum_{n=1}^{\infty}(1-\lambda_n) < \infty$, then $G_n(w) \nrightarrow 0$ as $n \to \infty$, for all $w \in \mathbb{D}$ for which $G_n(w) \neq 0$ for all $n \in \mathbb{N}$.*

Proof. We begin with ideas used by Beardon and Carne [BC92]. First, it follows from the hyperbolic triangle inequality and hyperbolic contraction that, if $\psi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then for all $w \in \mathbb{D}$ we have

$$\operatorname{dist}_{\mathbb{D}}(0, \psi(w)) \leq \operatorname{dist}_{\mathbb{D}}(0, \psi(0)) + \operatorname{dist}_{\mathbb{D}}(\psi(0), \psi(w)) \leq \operatorname{dist}_{\mathbb{D}}(0, \psi(0)) + \operatorname{dist}_{\mathbb{D}}(0, w), \tag{2.2}$$

and, similarly,

$$\operatorname{dist}_{\mathbb{D}}(0, \psi(0)) \leq \operatorname{dist}_{\mathbb{D}}(0, \psi(w)) + \operatorname{dist}_{\mathbb{D}}(0, w), \text{ for all } w \in \mathbb{D}. \tag{2.3}$$

We also use the fact that

$$\sum_{n=1}^{\infty}(1-\lambda_n) = \infty \iff \lambda_{m+n} \cdots \lambda_{m+1} \to 0 \text{ as } n \to \infty, \quad \text{for all } m \in \mathbb{N}. \tag{2.4}$$

In the case when $\lambda_n \neq 0$, for all n, and the right-hand side is $\lambda_n \cdots \lambda_1 \to 0$ as $n \to \infty$, this statement is a standard property of infinite products proved by taking logarithms. Here it is possible that some or all of the terms λ_n are zero, so the right-hand side of (2.4) takes account of these possibilities.

Now take $w_0 \in \mathbb{D}$ and, for simplicity, denote $G_n(w_0)$ by w_n, for $n \in \mathbb{N}$.

To prove part (a), we assume that $\lambda_{m+n} \cdots \lambda_{m+1} \to 0$ as $n \to \infty$, for all $m \in \mathbb{N}$, and deduce that $w_n \to 0$ as $n \to \infty$. Suppose that $w_n \nrightarrow 0$ as $n \to \infty$. Since $w_n = g_n(w_{n-1})$, we deduce by Schwarz's Lemma that $|w_n| \leq |w_{n-1}|$, and hence that $|w_n|$ decreases to some $d > 0$ as $n \to \infty$.

First choose $m \in \mathbb{N}$ so large that $|w_m|$ is sufficiently close to d to ensure that

$$\operatorname{dist}_{\mathbb{D}}(0, w_{n+m}/w_m) > \operatorname{dist}_{\mathbb{D}}(0, w_m), \quad \text{for } n \in \mathbb{N}. \tag{2.5}$$

Next we fix $n \in \mathbb{N}$ and define the holomorphic map

$$\psi(w) = (g_{m+n} \circ \cdots \circ g_{m+1}(w))/w, \quad \text{for } w \in \mathbb{D} \setminus \{0\},$$

with

$$\psi(0) = (g_{m+n} \circ \cdots \circ g_{m+1})'(0) = \lambda_{m+n} \cdots \lambda_{m+1}.$$

Applying (2.2) to the function ψ at the point $w = w_m$ gives

$$\begin{aligned} \operatorname{dist}_{\mathbb{D}}(0, w_{n+m}/w_m) &= \operatorname{dist}_{\mathbb{D}}(0, \psi(w_m)) \\ &\leq \operatorname{dist}_{\mathbb{D}}(0, \psi(0)) + \operatorname{dist}_{\mathbb{D}}(0, w_m) \\ &\leq \operatorname{dist}_{\mathbb{D}}(0, \lambda_{m+n} \cdots \lambda_{m+1}) + \operatorname{dist}_{\mathbb{D}}(0, w_m). \end{aligned}$$

Since we have assumed that $\lambda_{m+n}\cdots\lambda_{m+1}\to 0$ as $n\to\infty$, it follows that $\operatorname{dist}_{\mathbb{D}}(0,w_{n+m}/w_m)\le \operatorname{dist}_{\mathbb{D}}(0,w_m)$, for m sufficiently large. This, however, contradicts (2.5), showing that $w_n\to 0$ as $n\to\infty$.

To prove part (b), we assume that, for some $m_0\in\mathbb{N}$, $\lambda_{m_0+n}\cdots\lambda_{m_0+1}\to\lambda>0$ as $n\to\infty$, and deduce that whenever $w_n\neq 0$, for all $n\in\mathbb{N}$, we have $w_n\nrightarrow 0$ as $n\to\infty$. Suppose that $w_n\to 0$ as $n\to\infty$.

First choose m so large that $m\ge m_0$ and

$$\operatorname{dist}_{\mathbb{D}}(0,w_m)<\operatorname{dist}_{\mathbb{D}}(0,\lambda), \tag{2.6}$$

and note that, for such m,

$$\lambda_{m+n}\cdots\lambda_{m+1}\ge\lambda_{m+n}\cdots\lambda_{m_0+1}=\lambda_{m_0+(m-m_0)+n}\cdots\lambda_{m_0+1}\ge\lambda,\quad\text{for } n\in\mathbb{N}. \tag{2.7}$$

Next we fix $n\in\mathbb{N}$ and apply (2.3) with ψ defined as earlier and $w=w_m$ to give

$$\operatorname{dist}_{\mathbb{D}}(0,\lambda_{m+n}\cdots\lambda_{m+1})\le\operatorname{dist}_{\mathbb{D}}(0,w_{m+n}/w_m)+\operatorname{dist}_{\mathbb{D}}(0,w_m).$$

Letting $n\to\infty$, we obtain a contradiction to (2.6) in view of (2.7) and hence to the supposition that $w_n\to 0$ as $n\to\infty$. This completes the proof. □

The following corollary to Theorem 2.1 shows that if the hyperbolic distance between two distinct orbits converges to zero, then the same occurs for every pair of orbits.

Corollary 2.2. *For $n\in\mathbb{N}$, let $g_n:\mathbb{D}\to\mathbb{D}$ be holomorphic and let $G_n=g_n\circ\cdots\circ g_1$. If there exist $w_0,w_0'\in\mathbb{D}$ such that $G_n(w_0')\neq G_n(w_0)$ for all $n\in\mathbb{N}$ and $\operatorname{dist}_{\mathbb{D}}(G_n(w_0'),G_n(w_0))\to 0$ as $n\to\infty$, then*

$$\operatorname{dist}_{\mathbb{D}}(G_n(w),G_n(w_0))\to 0\ \text{ as } n\to\infty,\quad\text{for all } w\in\mathbb{D}.$$

Proof. For each $n\in\mathbb{N}$, let $w_n=g_n(w_{n-1})$ and, for $n\ge 0$, let $M_n:\mathbb{D}\to\mathbb{D}$ be a Möbius map satisfying $M_n(w_n)=0$. Then, for each $n\in\mathbb{N}$, the map $h_n=M_n\circ g_n\circ M_{n-1}^{-1}$ is a holomorphic self map of the unit disk and $h_n(0)=0$. For $n\in\mathbb{N}$, let $H_n:=h_n\circ\cdots\circ h_1$ and notice that $H_n(0)=0$. Since Möbius maps are isometries and $H_n=M_n\circ G_n\circ M_0^{-1}$, for $n\in\mathbb{N}$, we have

$$\begin{aligned}\operatorname{dist}_{\mathbb{D}}(0,H_n(M_0(w_0')))&=\operatorname{dist}_{\mathbb{D}}(H_n(0),H_n(M_0(w_0')))\\&=\operatorname{dist}_{\mathbb{D}}(M_n\circ G_n\circ M_0^{-1}(0),M_n\circ G_n\circ M_0^{-1}\circ M_0(w_0'))\\&=\operatorname{dist}_{\mathbb{D}}(M_n\circ G_n(w_0),M_n\circ G_n(w_0'))\\&=\operatorname{dist}_{\mathbb{D}}(G_n(w_0),G_n(w_0'))\to 0\quad\text{as } n\to\infty,\end{aligned}$$

and hence $H_n(M_0(w_0'))\to 0$ as $n\to\infty$. Since $H_n(M_0(w_0'))=M_n(G_n(w_0'))\neq 0$, for each $n\in\mathbb{N}$, it follows from Theorem 2.1 that $H_n(w')\to 0$ as $n\to\infty$ for all $w'\in\mathbb{D}$. The result now follows since

$$\operatorname{dist}_{\mathbb{D}}(G_n(w),G_n(w_0))=\operatorname{dist}_{\mathbb{D}}(H_n(M_0(w)),H_n(0))=\operatorname{dist}_{\mathbb{D}}(H_n(M_0(w)),0),\quad\text{for } w\in\mathbb{D}.\ \square$$

Theorem 2.1 and Corollary 2.2 will be used in the proof of Theorem A (see Section 3.1).

We now prove several results giving estimates for the *rate* at which limits tend to zero in the case when the limit in Theorem 2.1 is identically equal to zero. The results proven in the remainder of this section will be used in Section 3.2 to prove Theorem B, that is, the subclassification of contracting wandering domains.

We use the following result which includes a generalization of Schwarz's Lemma.

Lemma 2.3 (Variation of Schwarz's Lemma)**.** *Let $\psi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then*

$$\frac{|\psi(0)| - |w|}{1 - |\psi(0)||w|} \leq |\psi(w)| \leq \frac{|\psi(0)| + |w|}{1 + |\psi(0)||w|}, \qquad \text{for } w \in \mathbb{D}.$$

Proof. The right-hand inequality arises from (2.2) and is given in [BC92, p.217]. We prove the left-hand inequality using similar methods. First note that it follows from (2.3) that

$$\operatorname{dist}_{\mathbb{D}}(0, \psi(w)) \geq \operatorname{dist}_{\mathbb{D}}(0, \psi(0)) - \operatorname{dist}_{\mathbb{D}}(0, w),$$

that is,

$$\log \frac{1 + |\psi(w)|}{1 - |\psi(w)|} \geq \log \frac{1 + |\psi(0)|}{1 - |\psi(0)|} - \log \frac{1 + |w|}{1 - |w|}.$$

By the monotonicity of the logarithm, this is equivalent to the following inequality:

$$\frac{1 + |\psi(w)|}{1 - |\psi(w)|} \geq \left(\frac{1 + |\psi(0)|}{1 - |\psi(0)|}\right)\left(\frac{1 - |w|}{1 + |w|}\right),$$

which gives

$$|\psi(w)| \geq \frac{\left(\frac{1+|\psi(0)|}{1-|\psi(0)|}\right)\left(\frac{1-|w|}{1+|w|}\right) - 1}{\left(\frac{1+|\psi(0)|}{1-|\psi(0)|}\right)\left(\frac{1-|w|}{1+|w|}\right) + 1} = \frac{|\psi(0)| - |w|}{1 - |\psi(0)||w|},$$

as claimed. □

We make frequent use of the following corollary of Lemma 2.3.

Corollary 2.4. *Let $g : \mathbb{D} \to \mathbb{D}$ be holomorphic with $g(0) = 0$ and $|g'(0)| = \lambda$. Then, for all $w \in \mathbb{D}$,*

$$|w|\left(\frac{\lambda - |w|}{1 - \lambda|w|}\right) \leq |g(w)| \leq |w|\left(\frac{\lambda + |w|}{1 + \lambda|w|}\right)$$

Proof. The result follows by applying Lemma 2.3 to the holomorphic map $\psi : \mathbb{D} \to \mathbb{D}$ defined by

$$\psi(w) = g(w)/w, \quad \text{for } w \in \mathbb{D} \setminus \{0\},$$

with $\psi(0) = g'(0)$. □

We first use Corollary 2.4 to prove the following result giving rather precise upper and lower estimates of the rate at which the sequences $|G_n(w)|$ in Theorem 2.1 decrease, expressed in terms of the derivatives $|g_n'(0)|$. This result can be used to give a more direct proof of Theorem 2.1; see the remark after the proof of Theorem 2.5.

Theorem 2.5. *For each $n \in \mathbb{N}$, let $g_n : \mathbb{D} \to \mathbb{D}$ be holomorphic with $g_n(0) = 0$ and $|g_n'(0)| = \lambda_n = 1 - \mu_n$, and let $G_n = g_n \circ \cdots \circ g_1$. If $w \in \mathbb{D}$ and $w_n = G_n(w)$, $n \in \mathbb{N}$, then*

(a)

$$|w_n| \leq |w| \prod_{k=1}^{n} (1 - c_w \mu_k), \quad \textit{where } c_w = (1 - |w|)/2; \tag{2.8}$$

(b) *if $|w| \leq \lambda_k$, for $1 \leq k \leq n$, then*

$$|w_n| \geq |w| \prod_{k=1}^{n} (1 - d_w \mu_k), \quad \textit{where } d_w = \frac{1 + |w|}{1 - |w|}. \tag{2.9}$$

Proof. Set $w_0 = w$. We begin the proof of part (a) by noting that it follows from Corollary 2.4 that, for $k \geq 0$ and $w \in \mathbb{D}$,

$$\begin{aligned} |w_{k+1}| = |g_{k+1}(w_k)| &\leq |w_k| \left(\frac{\lambda_{k+1} + |w_k|}{1 + \lambda_{k+1}|w_k|} \right) \\ &= |w_k| \left(1 - \frac{\mu_{k+1}(1 - |w_k|)}{1 + \lambda_{k+1}|w_k|} \right) \\ &\leq |w_k|(1 - \frac{\mu_{k+1}(1 - |w_k|)}{2})) \\ &\leq |w_k|(1 - c_w \mu_{k+1}), \end{aligned}$$

where the third inequality follows because $\lambda_{k+1}|G_k(w)| < 1$ and the last inequality follows because $|w_k| = |G_k(w)| \leq |w|$ by Schwarz's Lemma. The result of (2.8) now follows and this completes the proof of part (a).

We now prove part (b). Using Corollary 2.4 again,

$$|w_{k+1}| = |g_{k+1}(w_k)| \geq |w_k| \left(\frac{\lambda_{k+1} - |w_k|}{1 - \lambda_{k+1}|w_k|} \right). \tag{2.10}$$

Now we use the elementary calculus estimate that

$$\frac{\lambda - r}{1 - \lambda r} \geq 1 - \left(\frac{1 + r}{1 - r} \right) (1 - \lambda), \quad \text{for } 0 < r < \lambda \leq 1,$$

to deduce from (2.10) that, for $k \geq 0$, if $|w| \leq \lambda_{k+1}$, then

$$|w_{k+1}| \geq |w_k| \left(1 - \left(\frac{1 + |w_k|}{1 - |w_k|} \right) \mu_{k+1} \right) \geq |w_k| \left(1 - \left(\frac{1 + |w|}{1 - |w|} \right) \mu_{k+1} \right),$$

using the fact that $|w_k| = |G_k(w)| \leq |w|$ again. The result of (2.9) now follows and this completes the proof of part (b). □

Remark. Theorem 2.5 can be used to give a proof of Theorem 2.1. To do so, it is first necessary to use Hurwitz' Theorem in order to show that either $G_n(w) \to 0$ as $n \to \infty$ for all $w \in \mathbb{D}$ or $G_n(w) \to 0$ as $n \to \infty$ only for those points $w \in \mathbb{D}$ for which $G_n(w) = 0$ eventually.

We now prove another result giving upper estimates for the rate at which the sequences $|G_n(w)|$ decrease, this time expressed in terms of the average of the derivatives $|g_n'(0)|$. The proof of this result is also based on Corollary 2.4.

Theorem 2.6. *For each $n \in \mathbb{N}$, let $g_n : \mathbb{D} \to \mathbb{D}$ be holomorphic with $g_n(0) = 0$ and $|g_n'(0)| = \lambda_n = 1 - \mu_n$, and let $G_n = g_n \circ \cdots \circ g_1$. Then, for all $n \in \mathbb{N}$, if $w_0 \in \mathbb{D}$ and $w_n = G_n(w_0)$, for $n \in \mathbb{N}$,*

$$|w_n| \le \left(\frac{1}{n} \sum_{k=1}^{n} \lambda_k + \frac{1}{n} \sum_{k=0}^{n-1} |w_k| \right)^n . \tag{2.11}$$

Hence

(a) *if*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \lambda_k = a < 1,$$

then

$$|G_n(w)| = O(c^n) \quad \text{as } n \to \infty, \quad \text{for } w \in \mathbb{D}, \text{ where } c \in (a, 1);$$

(b) *if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \lambda_k = 0,$$

then

$$|G_n(w)|^{1/n} \to 0 \quad \text{as } n \to \infty, \quad \text{for } w \in \mathbb{D}.$$

Proof. By using Corollary 2.4, and then applying the fact that the geometric mean of n positive numbers is at most equal to their arithmetic mean, we see that, for $w_0 \in \mathbb{D}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
|w_n| &\le |w_0| \left(\frac{\lambda_n + |w_{n-1}|}{1 + \lambda_n |w_{n-1}|} \right) \cdots \left(\frac{\lambda_1 + |w_0|}{1 + \lambda_1 |w_0|} \right) \\
&\le |w_0| ((\lambda_n + |w_{n-1}|) \dots (\lambda_1 + |w_0|)) \\
&\le |w_0| \left(\frac{1}{n} ((\lambda_n + |w_{n-1}|) + \cdots + (\lambda_1 + |w_0|)) \right)^n \\
&= |w_0| \left(\frac{1}{n} \sum_{k=1}^{n} \lambda_k + \frac{1}{n} \sum_{k=0}^{n-1} |w_k| \right)^n .
\end{aligned}$$

This proves (2.11).

Next, if $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \lambda_k = a < 1$, then $\sum_{k=1}^{\infty} (1 - \lambda_k) = \infty$ and so it follows from Theorem 2.1 that $w_n \to 0$ as $n \to \infty$ and hence that $\frac{1}{n} \sum_{k=0}^{n-1} |w_k| \to 0$ as $n \to \infty$. So, in this case, it follows from (2.11) that

$$|G_n(w_0)|^{1/n} = |w_n|^{1/n} \le a + o(1) \quad \text{as } n \to \infty.$$

The results of parts (a) and (b) now follow. $\square$

Theorem 2.5 (a) and Theorem 2.6 give uniform upper estimates on the rate that $|G_n(w)|$ tends to 0, in the situation where $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. It is natural to ask whether we can demonstrate such a uniform rate if we know the rate at which $|G_n(w)|$ tends to 0 on some subset of $\mathbb{D}$. It is clear that we cannot deduce any uniform rate at which $G_n(w) \to 0$ from information about the behaviour of G_n at a *single* point $w_0 \in \mathbb{D}$. However, if we have an upper bound for $|G_n(w)|$ on some circle $\{w : |w| = r_0\}$, where $0 < r_0 < 1$, then we can obtain an upper estimate for $|G_n(w)|$ for all $w \in \mathbb{D}$ by applying the following simple proposition.

Proposition 2.7 (Hadamard convexity)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and satisfy*

$$|f(w)| \le a, \quad \textit{for } |w| \le r_0,$$

where $0 < a \le r_0 < 1$. Then,

$$|f(w)| \le a^{\frac{\log r}{\log r_0}} \quad \textit{for } |w| \le r,$$

for all r such that $r_0 \le r < 1$.

Proof. For $0 \le r < 1$, let

$$M(r) = M(r, f) := \sup_{|z|=r} |f(z)|$$

denote the maximum modulus function and put

$$\varphi(t) = \log M(e^t), \quad \text{for } -\infty < t < 0.$$

Then φ is convex by Hadamard's Three Circles Theorem [Tit39, page 172], negative and increasing, and by hypothesis $\varphi(\log r_0) \le \log a$. Hence

$$\varphi(t) \le \left(\frac{\log a}{\log r_0}\right) t, \quad \text{for } -\infty < t < 0;$$

that is,

$$\log M(r) \le \left(\frac{\log a}{\log r_0}\right) \log r, \quad \text{for } r_0 \le r < 1,$$

and hence

$$M(r) \le a^{\frac{\log r}{\log r_0}}, \quad \text{for } r_0 \le r < 1,$$

as required. □

Remark. In Proposition 2.7, the circle $\{w : |w| = r_0\}$ can be replaced by any subset of $\mathbb{D}$ of positive logarithmic capacity, using a more delicate argument involving Green potentials in $\mathbb{D}$. We omit the details.

3 Contraction trichotomy: Proof of Theorems A and B

This section is devoted to a classification of simply connected wandering domains based on hyperbolic distances between orbits of points. More precisely we prove Theorems A and B and we also show that lifts of parabolic components are contracting yet not strongly contracting; (see Theorem 3.4).

The proofs are based on the results from Section 2 concerning self maps of the unit disk. We first show how the hyperbolic distances between orbits of points in the wandering domain compare with the distances between related orbits of points in the unit disk. We also compare the hyperbolic distortion along an orbit of a point in the wandering domain with the derivatives of the related maps of the unit disk.

Let f be a transcendental entire function with a simply connected wandering domain U and let U_n be the Fatou component containing $f^n(U_0)$, for $n \geq 0$. Note that each of the domains U_n is simply connected; indeed, if some U_n is multiply connected, then by [Bak84, Theorem 3.1], all the Fatou components are bounded, so f is a proper map between Fatou components and the claim follows from the Riemann–Hurwitz formula. Although $U_n = f^n(U_0)$ if U_0 is bounded, this is not necessarily true in the case that U_0 is unbounded when $U_n \setminus f^n(U)$ may contain one point; see for example [Her98].

We prove Theorem A and Theorem B by considering a sequence (g_n) of holomorphic self maps of the unit disk associated to f and U_n in the following way. Fix a point $z_0 \in U_0$ and, for each $n \geq 0$, choose $\varphi_n : U_n \to \mathbb{D}$ to be a Riemann map such that $\varphi_n(f^n(z_0)) = 0$. Then, for $n \in \mathbb{N}$, consider the holomorphic maps $g_n : \mathbb{D} \to \mathbb{D}$ defined as

$$g_n = \varphi_n \circ f \circ \varphi_{n-1}^{-1},$$

and the composite maps $G_n : \mathbb{D} \to \mathbb{D}$ defined as

$$G_n = g_n \circ \cdots \circ g_1 = \varphi_n \circ f^n \circ \varphi_0^{-1}.$$

Because of the choice of normalization for the Riemann maps we have that $g_n(0) = G_n(0) = 0$. This set up is illustrated in Figure 3. Each of the maps g_n and G_n is an inner function, but we do not use this fact in this paper.

Before stating the next theorem, we recall that if $f : U \to V$ is a holomorphic map between two hyperbolic domains U and V, then the hyperbolic distortion of f at z is defined to be

$$\|Df(z)\|_U^V := \lim_{z' \to z} \frac{\text{dist}_V(f(z'), f(z))}{\text{dist}_U(z', z)}.$$

Lemma 3.1. *Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$. Let $z_0 \in U_0$ and let g_n, G_n be as defined above.*

(a) *If $z \in U$ and $\varphi_0(z) = w$, then*

$$\text{dist}_{U_n}(f^n(z), f^n(z_0)) = \log\left(\frac{1 + |G_n(w)|}{1 - |G_n(w)|}\right), \quad \textit{for } n \in \mathbb{N}.$$

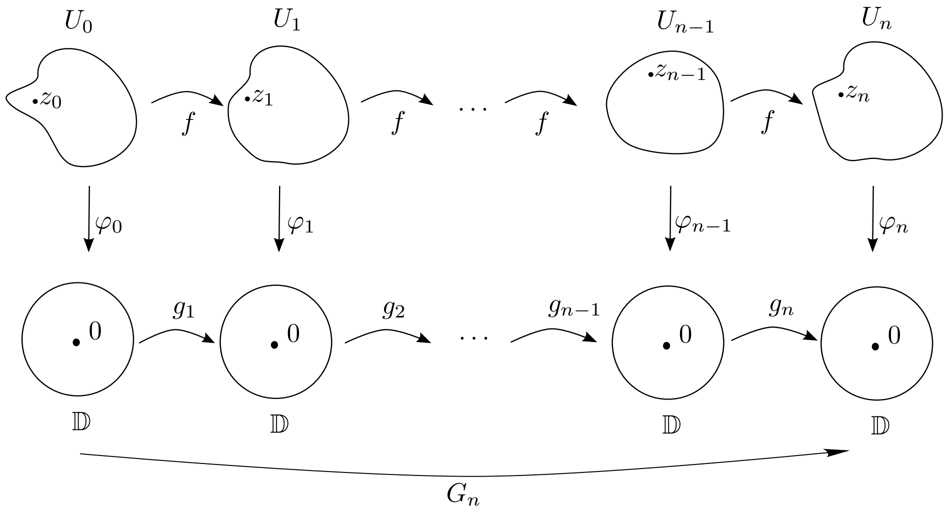

Figure 2: Self maps of the unit disk arising from an orbit of wandering domains

(b) *For each* $n \in \mathbb{N}$,

$$|g_n'(0)| = \lambda_n(z_0) := \|Df(f^{n-1}(z_0))\|_{U_{n-1}}^{U_n}.$$

Proof. (a) Let $n \in \mathbb{N}$. Since $G_n = \varphi_n \circ f^n \circ \varphi_0^{-1}$ and φ_n is conformal, if $z \in U$ and $\varphi_0(z) = w$ then

$$\mathrm{dist}_{U_n}(f^n(z), f^n(z_0)) = \mathrm{dist}_{\mathbb{D}}(G_n(w), G_n(0)) = \mathrm{dist}_{\mathbb{D}}(G_n(w), 0) = \log\left(\frac{1+|G_n(w)|}{1-|G_n(w)|}\right),$$

where the last equality follows from (2.1).

(b) Let $n \in \mathbb{N}$. Since $g_n = \varphi_n \circ f \circ \varphi_{n-1}^{-1}$ and φ_n is conformal we have

$$\|Dg_n(0)\|_{\mathbb{D}}^{\mathbb{D}} = \|Df(f^{n-1}(z_0))\|_{U_{n-1}}^{U_n} = \lambda_n(z_0).$$

Since $g_n(0) = 0$, it follows from (2.1) that

$$\begin{aligned} \|Dg_n(0)\|_{\mathbb{D}}^{\mathbb{D}} &= \lim_{w\to 0} \frac{\mathrm{dist}_{\mathbb{D}}(g_n(w), g_n(0))}{\mathrm{dist}_{\mathbb{D}}(w,0)} = \lim_{w\to 0} \frac{\mathrm{dist}_{\mathbb{D}}(g_n(w), 0)}{\mathrm{dist}_{\mathbb{D}}(w,0)} \\ &= \lim_{w\to 0} \frac{\log\left(\frac{1+|g_n(w)|}{1-|g_n(w)|}\right)}{\log\left(\frac{1+|w|}{1-|w|}\right)} \\ &= \lim_{w\to 0} \frac{2|g_n(w)|}{2|w|} = |g_n'(0)|, \end{aligned}$$

by using the Taylor expansion for the logarithm. □

3.1 Proof of Theorem A

We now use the results of Section 2 together with Lemma 3.1 to prove Theorem A, that is, the classification of simply connected wandering domains according to the behaviour of the hyperbolic distances between orbits of points.

Let U be a simply connected wandering domain of a transcendental entire function f and let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$. Also, let

$$E = \{(z, z') \in U \times U : f^k(z) = f^k(z') \text{ for some } k \in \mathbb{N}\}.$$

Let $z_0 \in U_0$ and let φ_n, g_n, G_n be as defined at the beginning of this section.

First we suppose that there exists $z_0' \in U_0$ with

$$(z_0', z_0) \notin E \quad \text{and } \operatorname{dist}_{U_n}(f^n(z_0'), f^n(z_0)) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.1}$$

Let $w_0' = \varphi_0(z_0')$. By (3.1) and Lemma 3.1 (a), we have that $G_n(w_0') \underset{n\to\infty}{\longrightarrow} 0$ and $G_n(w_0') \neq 0$ for all $n \in \mathbb{N}$. Hence by Theorem 2.1 (b) we have that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$, where $\lambda_n = |g_n'(0)|$, and therefore that $G_n(w) \underset{n\to\infty}{\longrightarrow} 0$, for all $w \in \mathbb{D}$, by Theorem 2.1 (a). By Lemma 3.1 (a) again, $\operatorname{dist}_{U_n}(f^n(z), f^n(z_0)) \underset{n\to\infty}{\longrightarrow} 0$, for all $z \in U_0$. We conclude that $\operatorname{dist}_{U_n}(f^n(z), f^n(z')) \underset{n\to\infty}{\longrightarrow} 0$, for all $z, z' \in U_0$, by the triangle inequality, which is case (1).

We have shown that (3.1) implies that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ and that this implies that U_0 is contracting. Thus U_0 is contracting if and only if $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$, where

$$\lambda_n = |g_n'(0)| = \|Df(f^{n-1}(z_0))\|_{U_{n-1}}^{U_n} = \lambda_n(z_0), \quad \text{for } n \in \mathbb{N},$$

by Lemma 3.1 (b).

Now suppose that there exist $z, z' \in U_0$ and $N \in \mathbb{N}$ with

$$\operatorname{dist}_{U_n}(f^n(z), f^n(z')) = c(z, z') > 0, \qquad \text{for all } n \geq N. \tag{3.2}$$

Then, by the Schwarz-Pick Lemma, $f : U_n \to U_{n+1}$ is an isometry, for all $n \geq N$, and so for every pair $z, z' \in U_0$ we have that

$$\operatorname{dist}_{U_n}(f^n(z), f^n(z')) = \operatorname{dist}_{U_N}(f^N(z), f^N(z')), \quad \text{for } n \geq N.$$

Thus, if $(z, z') \in (U \times U) \setminus E$ we have that $\operatorname{dist}_{U_n}(f^n(z), f^n(z')) = c(z, z') > 0$ for all $n \geq N$ and that U_0 is eventually isometric, which is case (3). In this case, $\lambda_n(z) = 1$ for all $z \in U_0$ and for $n \geq N$, by the Schwarz-Pick Lemma, as required.

Finally, we show that case (2) is the only other possibility. It follows from the above proof that, if there exists $z_0' \in U_0$ for which neither (3.1) nor (3.2) holds, then the only possibility is that neither of these conditions hold for *any* $z \in U_0$; that is, U_0 is semi-contracting, which is case (2). This completes the proof of Theorem A.

3.2 Subclassification of contracting wandering domains: Proof of Theorem B

In this subsection we prove Theorem B, which gives sufficient conditions for a simply connected wandering domain to be strongly contracting or super-contracting. We prove parts (a) and (b) by using the results of Section 2 together with Lemma 3.1.

Let U be a simply connected wandering domain of a transcendental entire function f and let U_n denote the Fatou component containing $f^n(U)$, for $n \geq 0$. Let $z_0 \in U_0$ and let g_n, G_n be as defined in the beginning of this section.

Also, for $n \in \mathbb{N}$, we let $\lambda_n = \lambda_n(z_0) = \|Df(f^{n-1}(z_0))\|_{U_{n-1}}^{U_n}$ and note from Lemma 3.1 (b) that $\lambda_n = |g_n'(0)|$.

To prove part (a), observe that if $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k = a < 1$, then it follows from Theorem 2.6 (a) that

$$|G_n(w)| = O(c^n) \text{ as } n \to \infty, \quad \text{for } w \in \mathbb{D},\ c \in (a,1).$$

So, by Lemma 3.1 (a), if we take $z \in U_0$ and put $w = \varphi_0(z)$, then

$$\operatorname{dist}_{U_n}(f^n(z), f^n(z_0)) = O(c^n) \text{ as } n \to \infty, \quad \text{for } c \in (a,1).$$

This proves part (a) of Theorem B.

To prove part (b), we note that, if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k = 0$, then, from Theorem 2.6 (b),

$$(\operatorname{dist}_{U_n}(f^n(z), f^n(z_0)))^{1/n} \to 0 \quad \text{as } n \to \infty,$$

and hence U_0 is super-contracting.

To prove part (c) we need to show that, for $n \in \mathbb{N}$, $z \in U_0$,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k(z) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k, \quad \text{for } z \in U_0. \tag{3.3}$$

(Recall that $\lambda_k = \lambda_k(z_0)$.) We begin by supposing that $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k = a < 1$ and fix $c \in (a,1)$ and $z \in U_0$. From part (a) above, there exists $C > 0$ such that

$$\operatorname{dist}_{U_k}(f^k(z), f^k(z_0)) \leq Cc^k, \quad \text{for } k \in \mathbb{N}. \tag{3.4}$$

We now use the following result of Beardon and Minda to obtain a bound on the difference between $\lambda_k(z)$ and λ_k.

Lemma 3.2 ([BM07, Theorem 11.2])**.** *Let U, V be hyperbolic domains and let $f : U \to V$ be holomorphic. Then*

$$\operatorname{dist}_{\mathbb{D}}(\|Df(z)\|_U^V, \|Df(w)\|_U^V) \leq 2\operatorname{dist}_U(z,w), \quad \textit{for all } z, w \in U.$$

It follows from Lemma 3.2 together with (3.4) that, under our supposition,

$$\operatorname{dist}_{\mathbb{D}}(\lambda_k(z), \lambda_k) \leq 2Cc^k, \quad \text{for } k \in \mathbb{N}.$$

Since

$$\operatorname{dist}_{\mathbb{D}}(\lambda_k(z), \lambda_k) = \left|\int_{\lambda_k}^{\lambda_k(z)} \frac{2\,dt}{1-t^2}\right| \geq \left|\int_{\lambda_k}^{\lambda_k(z)} 2\,dt\right| = 2|\lambda_k(z) - \lambda_k|,$$

it follows that

$$|\lambda_k(z) - \lambda_k| \leq Cc^k, \quad \text{for } k \in \mathbb{N}.$$

So, if $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k = a < 1$, then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k(z) &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k + \limsup_{n\to\infty} \frac{C}{n}\sum_{k=1}^n c^k\\ &= \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k = a.\end{aligned}$$

Since the roles of z_0 and z are interchangeable, we have shown that (3.3) holds whenever $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k < 1$. The only remaining case is that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k(z) = 1 = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \lambda_k.$$

This completes the proof of Theorem B.

3.3 Rate of contraction in parabolic components

It is clear that if a wandering domain U is the lift of an attracting component V, then U is strongly contracting and, if V is super-attracting, then U is super-contracting. We end this section by showing that if a wandering domain U occurs as a lift of a parabolic component, then U is contracting but not strongly contracting. We need the following lemma; see [Sha93, p. 157], for example.

Lemma 3.3. *If G is a simply connected domain, not the whole complex plane, then for $z, w \in G$,*

$$\operatorname{dist}_G(z,w) \ge \frac{1}{2}\log\left(1 + \frac{|z-w|}{\min\{\operatorname{dist}(z,\partial G), \operatorname{dist}(w,\partial G)\}}\right).$$

We have the following general result about the contraction rate in a parabolic component. The estimates in this result may be known but we are not aware of a reference.

Theorem 3.4. *Let V be an invariant parabolic component of a transcendental entire function f. Then, for all $z_0, z_0' \in V$, either $f^m(z_0) = f^m(z_0')$ for some $m \in \mathbb{N}$ or there exist positive constants k and K depending on z_0, z_0' and p, the number of petals, such that*

$$\frac{k}{n} \le \operatorname{dist}_V(f^n(z_0), f^n(z_0')) \le \frac{K}{n}, \quad \textit{for } n \in \mathbb{N}. \tag{3.5}$$

Proof. Without loss of generality we assume that 0 is the parabolic fixed point in ∂V and let p be the number of petals of f at 0. The following proof uses detailed estimates from the discussion of Abel's functional equation in [Bea91, pages 110–122] and we start by summarising this discussion, mainly using the notation from [Bea91].

First, the function f is conformally conjugate near 0 to a function of the form

$$F(z) = z - z^{p+1} + O(z^{2p+1}) \quad \text{as } z \to 0.$$

Substituting $w = z^{-p}$, $z = w^{-1/p}$, where $w^{-1/p}$ denotes the principal root, we obtain

$$g(w) = 1/\left(F(w^{-1/p})\right)^p = w + p + A/w + O(1/w^{1+1/p}) \quad \text{as } w \to \infty,$$

for some constant A, from which it follows that there exists a parabola-shaped domain of the form $\Pi = \{u + iv : v^2 > 4K(K - u)\}$, $K > 0$, that is forward invariant under g. For $w \in \Pi$, we have

$$g^n(w) = np + \frac{A}{p} \log n + u_n(w), \quad \text{for } n \in \mathbb{N}, \tag{3.6}$$

where the functions u_n are holomorphic in Π and converge locally uniformly on Π to a univalent function u, which satisfies $u(g(w)) = u(w)+p$, a form of Abel's functional equation.

Now suppose that $w_0, w_0' \in \Pi$ are distinct points and put $z_0 = w_0^{-1/p}$ and $z_0' = (w_0')^{-1/p}$. Then z_0 and z_0' lie in the invariant petal-shaped domain for F at 0, which corresponds to Π under the mapping $w \mapsto w^{-1/p}$ and which is symmetric with respect to the positive real axis, subtending an angle of $2\pi/p$ at 0. It follows from the above properties of the functions u_n and the univalence of u that

$$g^n(w_0') - g^n(w_0) = u_n(w_0') - u_n(w_0) \to u(w_0') - u(w_0) \neq 0 \quad \text{as } n \to \infty. \tag{3.7}$$

On substituting $z = w^{-1/p}$ we find that $z_n = F^n(z_0)$ and $z_n' = F^n(z_0')$ both approach 0 tangentially to the positive real axis through the petal-shaped domain mentioned above. Moreover, by (3.6),

$$|z_n| = |g^n(w_0)|^{-1/p} = \frac{1}{|np + \frac{A}{p}\log n + u_n(w_0)|^{1/p}} \sim \frac{1}{(np)^{1/p}} \quad \text{as } n \to \infty, \tag{3.8}$$

and

$$|z_n'| = |g^n(w_0')|^{-1/p} = \frac{1}{|np + \frac{A}{p}\log n + u_n(w_0')|^{1/p}} \sim \frac{1}{(np)^{1/p}} \quad \text{as } n \to \infty. \tag{3.9}$$

Also, by (3.6), (3.7) and a short calculation,

$$|z_n - z_n'| = |g^n(w_0)^{-1/p} - g^n(w_0')^{-1/p}| \sim \frac{|u(w_0) - u(w_0')|}{p(np)^{1+1/p}} \quad \text{as } n \to \infty. \tag{3.10}$$

Since F is conformally conjugate to f near 0, we deduce that these estimates for z_n and z_n' hold if $z_n = f^n(z_0)$ and $z_n' = f^n(z_0')$, for $n \in \mathbb{N}$, where z_0 and z_0' are redefined to be the corresponding points in the invariant parabolic component V for f. Also, note that V is one of p distinct invariant parabolic components for f at 0, each containing an invariant petal-shaped domain subtending an angle of $2\pi/p$ at 0.

Therefore, by Lemma 3.3, (3.8), (3.9) and (3.10), together with the fact that $\operatorname{dist}(z_n, \partial V) \leq |z_n|$, we have

$$\begin{aligned} \operatorname{dist}_V(z_n, z_n') &\geq \frac{1}{2}\log\left(1 + \frac{|z_n - z_n'|}{\min\{\operatorname{dist}(z_n, \partial V), \operatorname{dist}(z_n', \partial V)\}}\right) \\ &\geq k\frac{1/n^{1+1/p}}{1/n^{1/p}} = \frac{k}{n}, \quad \text{for } n \in \mathbb{N}, \end{aligned}$$

for some positive constant k depending on z_0, z_0' and p.

Finally, for $n \in \mathbb{N}$, let γ_n denote the line segment joining z_n to z_n'. Then, in view of the fact that z_n and z_n' approach 0 tangentially to the positive real axis, the line segment γ_n, for n sufficiently large, lies in the invariant petal-shaped domain in V, which is conformally equivalent near 0 to the petal-shaped domain obtained from Π. Also, for n sufficiently large, we have

$$\min\{\mathrm{dist}(z_n, \partial V), \mathrm{dist}(z_n', \partial V)\} \geq \frac{1}{2}\sin(\pi/p)|z_n|.$$

Therefore, by the standard hyperbolic density estimate in a simply connected domain (see, for example, [CG93, page 13]), (3.8), (3.9), and the triangle inequality, we have

$$\begin{aligned}
\mathrm{dist}_V(z_n, z_n') &\leq \int_{\gamma_n} \rho_V(z)\,|dz| \\
&\leq \int_{\gamma_n} \frac{2}{\mathrm{dist}(z, \partial V)}\,|dz| \\
&\leq \frac{2|z_n - z_n'|}{\min\{\mathrm{dist}(z_n, \partial V), \mathrm{dist}(z_n', \partial V)\} - \frac{1}{2}|z_n - z_n'|} \\
&\leq K\frac{1/n^{1+1/p}}{1/n^{1/p}} = \frac{K}{n},
\end{aligned}$$

for some positive constant K depending on z_0, z_0' and p, and n sufficiently large.

Finally, we note that, for all pairs of points $z_0, z_0' \in V$ with disjoint orbits, we have $f^n(z_0), f^n(z_0') \in V$ for n sufficiently large. This completes the proof. □

Remark. Using a more careful analysis of the size of the hyperbolic density in V near the points z_n and z_n' we can show that the estimate (3.5) in Theorem 3.4 can be replaced by

$$\mathrm{dist}_V(f^n(z_0), f^n(z_0')) \sim \frac{c}{n} \quad \text{as } n \to \infty,$$

for some positive constant c depending on z_0, z_0' and p. The proof uses results about the behaviour of a Riemann map from a sector of angle $2\pi/p$ onto V which maps 0 to 0, justified by using standard results about angular derivatives of conformal mappings at boundary points.

By conformality and Definition 1.2, we have the following corollary of Theorem 3.4.

Corollary 3.5. *Let U be a simply connected wandering domain that is the lift of an invariant parabolic component V and let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$. Then, for all $z_0, z_0' \in U$, either $f^m(z_0) = f^m(z_0')$ for some $m \in \mathbb{N}$ or there exist positive constants k and K depending on z_0 and z_0' such that*

$$\frac{k}{n} \leq \mathrm{dist}_{U_n}(z_n, z_n') \leq \frac{K}{n}, \quad \textit{for } n \in \mathbb{N}.$$

In particular, U is contracting but not strongly contracting.

Here are two examples of simply connected wandering domains, obtained by lifting parabolic components, which are contracting but not strongly contracting.

Example 1. Consider the entire functions

$$f(z) = z + e^{-z} + 2\pi i, \quad g(z) = z + e^{-z} \quad \text{and} \quad F(w) = we^{-w}.$$

Then both f and g are obtained by lifting F under the exponential function $w = e^{-z}$. Since F has an invariant parabolic component associated with the fixed point at 0, the function g has congruent unbounded invariant Baker domains U_n, $n \in \mathbb{Z}$, such that $U_n \subset \{z : (2n-1)\pi < \text{Im}(z) < (2n+1)\pi\}$; see [Dom98, FH06]. Since $J(f) = J(g)$, by [Ber95b], the components U_n form a sequence of simply connected wandering domains which, by Corollary 3.5, are contracting but not strongly contracting.

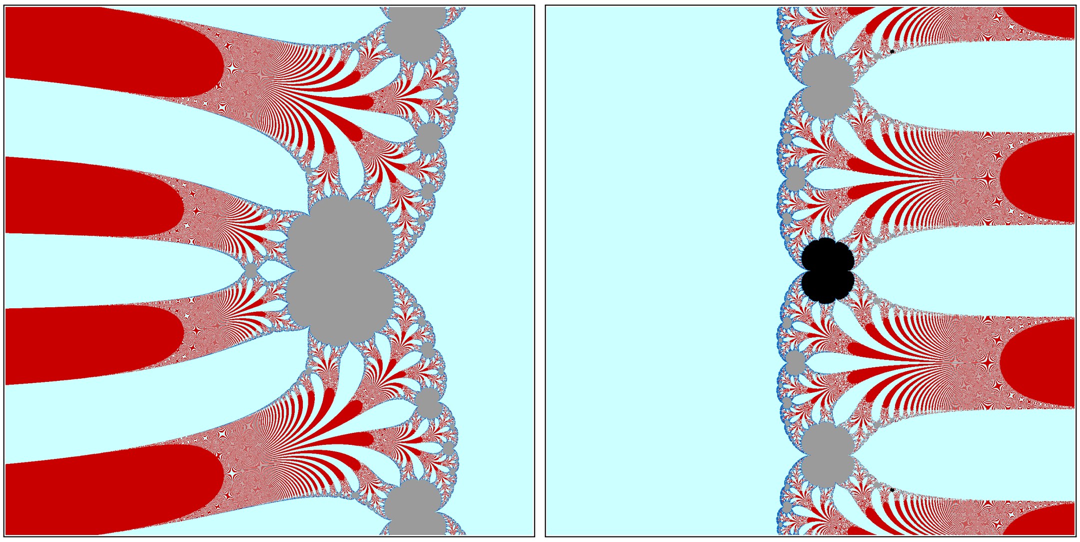

Figure 3: Left: Dynamical plane of F in Example 2. The super-attracting basin of $w = 0$ is shown in light blue, while in gray we see the parabolic basin of $w = 1$. Right: Dynamical plane of f. In blue the Baker domain (lift of the superattracting basin). In black the parabolic invariant basin at $z = 0$. In gray the wandering domains. The range is $[-9, 9] \times [-9, 9]$.

Example 2. As another example, consider

$$f(z) = 2z + 1 - e^z \quad \text{and} \quad F(w) = e\,w^2 e^{-w},$$

which belongs to the same family as the Example in Figure 1, both closely related to an example of Bergweiler [Ber95a]. In this case, f is a lift of F under $w = e^z$, and F has an invariant parabolic component associated with the fixed point at 1 which lifts to congruent, bounded, simply connected Fatou components, V_n, $n \in \mathbb{Z}$, say, of f such that $0 \in \partial V_0$ and

$$V_n = V_0 + 2n\pi i, \quad \text{for } n \in \mathbb{Z}, \quad \text{and} \quad f(V_n) = V_{2n}, \quad \text{for } n \in \mathbb{Z}.$$

From this it follows that V_{2n}, $n \geq 1$, is a sequence of bounded, escaping, simply connected wandering domains which, by Corollary 3.5, are contracting but not strongly contracting.

4 Convergence to the boundary: Proof of Theorem C

In this section we give the proof of Theorem C, the classification of simply connected wandering domains in terms of whether orbits of points converge to the boundary. Recall that the

Euclidean distance of a point z from the boundary of a hyperbolic domain U is closely related to the hyperbolic density $\rho_U(z)$ of the point in the domain. Indeed, if U is a simply connected wandering domain of a transcendental entire function f, U_n is the Fatou component containing $f^n(U)$, for $n \geq 0$, and $z \in U_0$, then by standard estimates [CG93, page 13]

$$\text{dist}_{\text{Eucl}}(f^n(z), \partial U_n) \to 0 \iff \rho_{U_n}(f^n(z)) \to \infty.$$

We prove Theorem C by considering the hyperbolic densities $\rho_{U_n}(f^n(z))$. In fact, we show that a trichotomy as in Theorem C occurs if we consider the quantities $a_n \rho_{U_n}(f^n(z))$, for *any* sequence a_n and not just for $a_n = 1$. As we mentioned in the introduction, the issue of convergence to the boundary is somehow delicate in that it is tightly connected to the shape of the wandering domains, and there may be situations where it is more appropriate to use an alternative definition involving different sequences a_n. For example, if the domains U_n are shrinking then it may make sense to say that z_n converges to the boundary if $a_n \rho_{U_n}(f^n(z)) \to \infty$ as $n \to \infty$ where

$$a_n = \sup_D \{\text{diam}\, D : D \text{ is a disk contained in } U_n\}.$$

In order to prove Theorem C we need the following lemma, which can be thought of as a Harnack inequality for hyperbolic density in a simply connected domain; see [BC08, Lemma 6.2] for a similar type of result (with a different proof) for hyperbolic density in the unit disk.

Lemma 4.1 (Estimate of hyperbolic quantities)**.** *Let $U \subset \mathbb{C}$ be a simply connected domain. Then, for all $z, z' \in U$,*

$$\exp(-2\,\text{dist}_U(z, z')) \leq \frac{\rho_U(z')}{\rho_U(z)} \leq \exp(2\,\text{dist}_U(z, z')).$$

Proof. Let $z, z' \in U$ and let $\varphi : \mathbb{D} \to U$ be a Riemann map with $\varphi(0) = z$ and $\varphi(r) = z'$, for some $r \in [0, 1)$. By conformal invariance of the hyperbolic metric, together with (2.1),

$$\text{dist}_U(z, z') = \text{dist}_{\mathbb{D}}(0, r) = \log \frac{1 + r}{1 - r},$$

and, by the definition of the hyperbolic density on U,

$$\begin{aligned} \rho_U(z) &= \rho_{\mathbb{D}}(0)/|\varphi'(0)| = 2/|\varphi'(0)| \\ \rho_U(z') &= \rho_{\mathbb{D}}(r)/|\varphi'(r)| = \frac{2}{1 - r^2} \,/|\varphi'(r)|\,. \end{aligned}$$

Also, by a standard distortion theorem for conformal maps [Pom92, p. 9],

$$\frac{1 - r}{(1 + r)^3} \leq \frac{|\varphi'(r)|}{|\varphi'(0)|} \leq \frac{1 + r}{(1 - r)^3}.$$

Putting everything together we obtain the lower bound,

$$\frac{\rho_U(z')}{\rho_U(z)} = \frac{1}{1 - r^2} \frac{|\varphi'(0)|}{|\varphi'(r)|} \geq \frac{(1 - r)^2}{(1 + r)^2} = \exp(-2\,\text{dist}_U(z, z')),$$

and the upper bound follows by symmetry. □

Remark It is easy to check that the inequalities in Lemma 4.1 are sharp in the case when U is $\mathbb{C} \setminus (-\infty, 0]$ and the points z, z' lie on the positive real axis.

We now prove the main result of this section.

Theorem 4.2. *Let U be a simply connected wandering domain of a transcendental entire function f, let U_n be the Fatou component containing $f^n(U)$, for $n \geq 0$, and let (a_n) be a real positive sequence.*

(a) *If there is a subsequence $n_k \to \infty$ and a point $z \in U_0$ such that $a_{n_k}\rho_{U_{n_k}}(f^{n_k}(z)) \to \infty$, then the same is true for all other points in U_0.*

(b) *If there is a subsequence $m_k \to \infty$ and a point $z \in U_0$ such that $a_{m_k}\rho_{U_{m_k}}(f^{m_k}(z))$ is bounded, then the same is true for all other points in U_0.*

Proof. (a) Suppose that $a_{n_k}\rho_{U_{n_k}}(f^{n_k}(z)) \to \infty$ as $k \to \infty$ and let $z' \in U_0$ with $z' \neq z$. By the contraction property of the hyperbolic metric, we have that

$$\mathrm{dist}_{U_n}(f^n(z), f^n(z')) \leq \mathrm{dist}_{U_0}(z, z') =: C, \qquad \text{for } n \in \mathbb{N}.$$

By Lemma 4.1, $\rho_{U_n}(f^n(z')) \geq e^{-2C}\rho_{U_n}(f^n(z))$, for $n \in \mathbb{N}$. Hence

$$a_{n_k}\rho_{U_{n_k}}(f^{n_k}(z')) \geq e^{-2C}a_{n_k}\rho_{U_{n_k}}(f^{n_k}(z)) \to \infty \quad \text{as } k \to \infty$$

(b) Now suppose that $a_{m_k}\rho_{U_{m_k}}(f^{m_k}(z)) \leq M$, for $k \in \mathbb{N}$, and let $z' \in U_0$ with $z' \neq z$. Again, by the contraction property of the hyperbolic metric, we have that

$$\mathrm{dist}_{U_{m_k}}(f^{m_k}(z), f^{m_k}(z')) \leq \mathrm{dist}_{U_0}(z, z') =: C.$$

Now, applying Lemma 4.1 and interchanging z and z', we obtain that

$$\rho_{U_{m_k}}(f^{m_k}(z')) \leq e^{2C}\rho_{U_{m_k}}(f^{m_k}(z)),$$

which implies that

$$a_{m_k}\rho_{U_{m_k}}(f^{m_k}(z')) \leq e^{2C}a_{m_k}\rho_{U_{m_k}}(f^{m_k}(z)) \leq Me^{2C},$$

so $a_{m_k}\rho_{U_{m_k}}(f^{m_k}(z'))$ is bounded, for $k \in \mathbb{N}$.

□

The result of Theorem C follows from Theorem 4.2, by taking $a_n = 1$, for $n \geq 0$.

5 Constructing wandering domains

We begin this section with the proof of Theorem D, which we then use together with an extension of Runge's Approximation Theorem to prove Theorem 5.3. This result enables us to construct bounded simply connected wandering domains in which various different dynamical behaviours can be specified and is the main tool that we use to construct examples in Section 6.

5.1 Proof of Theorem D

Let f be a transcendental entire function and let γ_n, Γ_n, n_k, L_k, and D be as in Theorem D; see Figure 4. It follows from properties (a) and (b) of Theorem D that for each $n, m \in \mathbb{N}$ with $n \neq m$ the curve γ_n is in $\operatorname{ext}\gamma_m$ and so, by property (c) and Montel's theorem, there exist Fatou components U_n such that

$$\overline{\operatorname{int}\gamma_n} \subset U_n, \text{ for } n \geq 0. \tag{5.1}$$

Notice that, a priori, the components U_n need not be different from each other. One of our goals is to show that they are indeed different, by proving that $U_n \subset \operatorname{int}\Gamma_n$, for $n \geq 0$.

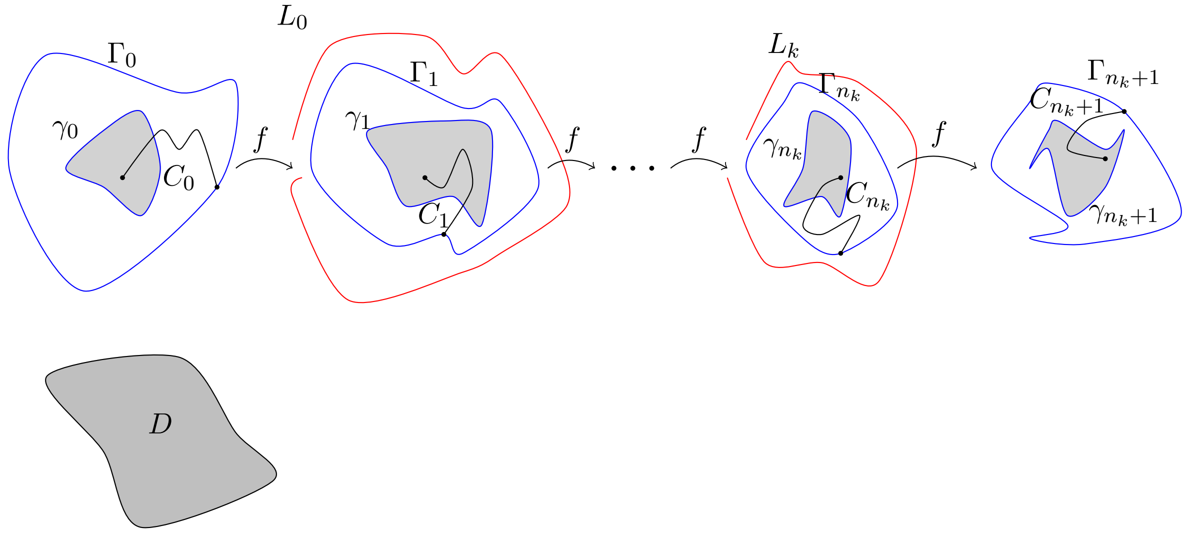

Figure 4: Sketch of the setup of the proof of Theorem D.

By property (e), the domain D must contain an attracting fixed point and so it is contained in an attracting Fatou component, say V. It then follows by property (e) that for all $k \geq 0$ the set L_k is contained in a union of Fatou components, V_k say, that maps into V. As above, notice that the V_k's may all be part of the same component. Since for every n we have that $\overline{D} \subset \operatorname{ext}\Gamma_n$ while $\gamma_n \subset \operatorname{int}\Gamma_n$ we deduce that U_0 is not in the grand orbit of V and hence that $\bigcup_{n\geq 0} U_n \cap \bigcup_{k\geq 0} V_k = \emptyset$. Therefore

$$\operatorname{dist}(z', \partial U_k) < \delta_k := \max\{\operatorname{dist}(z, L_k) : z \in \Gamma_{n_k}\}, \quad \text{for all } z' \in \Gamma_{n_k} \cap U_k. \tag{5.2}$$

Note that U_n is simply connected for $n \geq 0$. Indeed, if U_n is multiply connected for some $n \geq 0$, then it is a wandering domain and by [BRS13, Theorem 1.2] there exists $N > 0$ such that $f^k(\operatorname{int}\gamma_n)$ contains an annulus $A(r_k, R_k)$ for all $k \geq N$ with $R_k/r_k \to \infty$ as $k \to \infty$. It follows by property (c) that $A(r_k, R_k)$ is contained in $\operatorname{int}\gamma_{n+k}$ and this contradicts property (b). So U_n must be simply connected for $n \geq 0$.

We now show that $U_n \subset \operatorname{int}\Gamma_n$, for $n \geq 0$, using proof by contradiction. If there exists $m \geq 0$ for which U_m is not a subset of int Γ_m, then it follows from (5.1) and property (a) that $U_m \cap \Gamma_m \neq \emptyset$ and so we can take $z_m \in \operatorname{int} \gamma_m$ and $z'_m \in U_m \cap \Gamma_m$, and join them by a compact curve $C_m \subset (U_m \cap \operatorname{int}\ \Gamma_m)$.

Then, by properties (c) and (d), we can choose simple curves C_n, $n \geq m$, such that $C_n \subset f^{n-m}(C_m) \subset (U_n \cap \operatorname{int}\ \Gamma_n)$ and also C_n joins $z_n := f^{n-m}(z_m) \in \operatorname{int} \gamma_n$ to a point $z'_n \in \Gamma_n \cap f^{n-m}(C_m) \subset U_n$, while C_n lies in $\overline{\operatorname{int} \Gamma_n}$. Such a curve C_n must also intersect γ_n. Then, on the one hand, since $C_n \subset f^{n-m}(C_m)$ and $f^{n-m} : U_m \to U_n$ is a hyperbolic contraction, we have that

$$\operatorname{length}_{U_n} C_n \leq \operatorname{length}_{U_n} f^{n-m}(C_m) \leq \operatorname{length}_{U_m} C_m < \infty, \tag{5.3}$$

for all $n \geq m$. On the other hand, by Lemma 3.3 and (5.2), for $n_k \geq m$, we have

$$\begin{aligned} \operatorname{length}_{U_{n_k}} C_{n_k} &\geq \operatorname{dist}_{U_{n_k}}(z_{n_k}, z'_{n_k}) \\ &\geq \frac{1}{2} \log \left(1 + \frac{|z_{n_k} - z'_{n_k}|}{\min\{\operatorname{dist}(z_{n_k}, \partial U_{n_k}), \operatorname{dist}(z'_{n_k}, \partial U_{n_k})\}} \right) \\ &\geq \frac{1}{2} \log \left(1 + \frac{|z_{n_k} - z'_{n_k}|}{\operatorname{dist}(z'_{n_k}, \partial U_{n_k})} \right) \\ &\geq \frac{1}{2} \log \left(1 + \frac{\operatorname{dist}(\gamma_{n_k}, \Gamma_{n_k})}{\delta_k} \right). \end{aligned}$$

By property (f), this quantity tends to infinity as $k \to \infty$, which contradicts (5.3), so $U_m \subset \operatorname{int} \Gamma_m$ and hence U_m is a bounded wandering domain by property (b).

Finally, suppose that, for some $n \geq 0$, there exists $z_n \in \operatorname{int} \gamma_n$ such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind d_n times round $f(z_n)$. Since $f(\Gamma_n)$ winds d_n times around $f(z_n)$, we deduce that f takes the value $f(z_n)$ exactly d_n times in $\operatorname{int} \Gamma_n$. Similarly, f takes the value $f(z_n)$ exactly d_n times in $\operatorname{int} \gamma_n$. Hence f takes the value $f(z_n)$ exactly d_n times in U_n. Since U_n is a bounded Fatou component, $f : U_n \to U_{n+1}$ is a proper map; since the above argument holds for a neighbourhood of $f(z_n)$, we deduce that the degree of f on U_n is equal to d_n.

5.2 Main construction result

In the proof of our main construction result, Theorem 5.3 below, we use the following extension of the main lemma in [EL87], which is a strong version of the well-known Runge's Approximation Theorem.

Lemma 5.1 (Approximating on infinitely many compact sets)**.** *Let (E_n) be a sequence of compact subsets of $\mathbb{C}$ with the following properties:*

(i) $\mathbb{C} \setminus E_n$ *is connected, for* $n \geq 0$*;*

(ii) $E_n \cap E_m = \emptyset$*, for* $n \neq m$*;*

(iii) $\min\{|z| : z \in E_n\} \to \infty$ *as* $n \to \infty$*.*

Suppose ψ is holomorphic on $E = \bigcup_{n=0}^{\infty} E_n$ and $j \in \mathbb{N}$. For $n \geq 0$, let $\varepsilon_n > 0$ and let $z_{n,i} \in E_n$, $1 \leq i \leq j$. Then there exists an entire function f satisfying, for $n \geq 0$,

$$|f(z) - \psi(z)| < \varepsilon_n, \quad \text{for } z \in E_n; \tag{5.4}$$

$$f(z_{n,i}) = \psi(z_{n,i}), \quad f'(z_{n,i}) = \psi'(z_{n,i}), \quad \text{for } 1 \leq i \leq j. \tag{5.5}$$

The main lemma in [EL87] allows for one point z_n in every compact set at which f and f' can be specified, but its proof can easily be modified to hold for finitely many points in every E_n, as stated above.

The following lemma will also be used in the proof of Theorem 5.3.

Lemma 5.2 (Hyperbolic distance on disks)**.** *Suppose that* $0 < s < r < 1 < R$ *and set*

$$c(s,R) = \frac{1-s^2}{R-s^2/R}, \quad D_r = D(0,r) \quad \textit{and} \quad D_R = D(0,R).$$

If $|z|, |w| \leq s$, *then*

$$\text{dist}_{D_R}(z,w) = \text{dist}_{\mathbb{D}}(z/R, w/R) \geq c(s,R)\,\text{dist}_{\mathbb{D}}(z,w), \tag{5.6}$$

and

$$\text{dist}_{D_r}(z,w) = \text{dist}_{\mathbb{D}}(z/r, w/r) \leq \frac{1}{c(s/r,1/r)}\,\text{dist}_{\mathbb{D}}(z,w). \tag{5.7}$$

Also, $0 < c(s,R) < 1$ *and if the variables* s, r *and* R *satisfy in addition*

$$1-r = o(1-s) \ \ \textit{as } s \to 1 \quad \textit{and} \quad R-1 = O(1-r) \ \ \textit{as } r \to 1, \tag{5.8}$$

then

$$c(s,R) \to 1 \ \ \textit{as } s \to 1, \tag{5.9}$$

and

$$c\,(s/r, 1/r) \to 1 \ \ \textit{as } s \to 1. \tag{5.10}$$

Proof. Suppose that $0 < s < r < 1 < R$ and take $z, w \in \mathbb{D}$ with $|z|, |w| \leq s$. Let γ be the hyperbolic geodesic in $\mathbb{D}$ joining z/R to w/R. Then

$$\text{dist}_{\mathbb{D}}(z/R, w/R) = \int_\gamma \frac{2\,|dt|}{1-|t|^2}.$$

Now substitute $\zeta = Rt$, $t \in \mathbb{D}$, so $|d\zeta| = R|dt|$. Also let $R\gamma := \{Rz : z \in \gamma\}$. Since $R > 1$, we have

$$\text{dist}_{\mathbb{D}}(z/R, w/R) = \frac{1}{R}\int_{R\gamma} \frac{2\,|d\zeta|}{1-|\zeta|^2/R^2} = R\int_{R\gamma} \frac{2\,|d\zeta|}{R^2-|\zeta|^2}.$$

Now for $\zeta \in R\gamma$ we have $|\zeta| \leq s$, so

$$\frac{R^2-|\zeta|^2}{1-|\zeta|^2} \leq \frac{R^2-s^2}{1-s^2}, \quad \text{for } \zeta \in R\gamma.$$

Hence

$$\text{dist}_{\mathbb{D}}(z/R, w/R) = R\int_{R\gamma} \frac{2\,|d\zeta|}{R^2-|\zeta|^2} \geq \frac{R(1-s^2)}{R^2-s^2}\int_{R\gamma} \frac{2\,|d\zeta|}{1-|\zeta|^2} \geq c(s,R)\,\text{dist}_{\mathbb{D}}(z,w),$$

since

$$\text{dist}_{\mathbb{D}}(z,w) = \min\left\{\int_{\gamma'} \frac{2\,|d\zeta|}{1-|\zeta|^2} : \text{ for all paths } \gamma' \text{ joining } z \text{ to } w \text{ in } \mathbb{D}\right\}.$$

This proves (5.6).

Next,

$$\mathrm{dist}_{D_r}(z,w) = \mathrm{dist}_{\mathbb{D}}(z/r, w/r) \ \text{ and } \ \left|\frac{z}{r}\right|, \left|\frac{w}{r}\right| \le \frac{s}{r} < 1.$$

Hence, by (5.6), with r and R replaced by s/r and $1/r$, and z, w replaced by z/r and w/r, we obtain

$$\mathrm{dist}_{\mathbb{D}}(z/r, w/r) \le \frac{1}{c(s/r, 1/r)} \mathrm{dist}_{D_{1/r}}(z/r, w/r) = \frac{1}{c(s/r, 1/r)} \mathrm{dist}_{\mathbb{D}}(z, w).$$

This proves (5.7).

It is clear that $0 < c(s,R) < 1$ since $0 < s < 1 < R$. Finally, suppose that (5.8) holds. Then $R - 1 = O(1-r) = o(1-s)$ as $s \to 1$ and hence

$$c(s,R) = \frac{R(1-s)(1+s)}{(R-s)(R+s)} = \frac{R(1-s)}{1-s+o(1-s)} \frac{1+s}{R+s} \to 1 \ \text{ as } s \to 1,$$

and

$$c(s/r, 1/r) = \frac{(r-s)(1+s/r)}{(1-s)(1+s)} = \frac{1-s+o(1-s)}{1-s} \frac{1+s/r}{1+s} \to 1 \ \text{ as } s \to 1,$$

which give (5.9) and (5.10). □

We now give our main construction result, which we use in Section 6 to construct examples. In these examples, we shall prescribe the orbits of at most two points $z_1, z_2 \in D(0, r_0)$, although the result below allows us to prescribe the orbits of any finite number of points in $D(0, r_0)$.

Theorem 5.3 (Main construction). *Let b_n, $n \in \mathbb{N}$, be a sequence of Blaschke products of corresponding degrees $d_n \ge 1$ and let T_n, $n \ge 0$, be the sequence of translations $z \mapsto z + 4n$ and D_n, $n \ge 0$, be the sequence of disks $D_n = \{z : |z - 4n| < 1\}$. Suppose also that $j \in \mathbb{N}$ and $z_i \in D_0$, $1 \le i \le j$. Then there exists a transcendental entire function f having an orbit of bounded, simply connected, escaping, wandering domains U_n such that, for $n \ge 0$,*

(i) $\overline{\Delta_n'} := \overline{D(4n, r_n)} \subset U_n \subset D(4n, R_n) := \Delta_n$, where $0 < r_n < 1 < R_n$ and $r_n, R_n \to 1$ as $n \to \infty$;

(ii) $f_{n+1} := T_{n+1} \circ b_{n+1} \circ T_n^{-1}$ is holomorphic on $\overline{\Delta_n}$, and $|f(z) - f_{n+1}(z)| \to 0$ uniformly on $\overline{\Delta_n}$ as $n \to \infty$;

(iii) $f^n(z_i) = F_n(z_i)$ and $f'((f^n)(z_i)) = f'_{n+1}(F_n(z_i))$, $1 \le i \le j$, where $F_n = f_n \circ \cdots \circ f_1$;

(iv) $f : U_n \to U_{n+1}$ has degree d_{n+1}.

Finally, if $z, z' \in \overline{D(0, r_0)}$, then we have

$$k_n \, \mathrm{dist}_{D_n}(f^n(z), f^n(z')) \le \mathrm{dist}_{U_n}(f^n(z), f^n(z')) \le K_n \, \mathrm{dist}_{D_n}(f^n(z), f^n(z')), \tag{5.11}$$

where $0 < k_n < 1 < K_n$ and $k_n, K_n \to 1$ as $n \to \infty$.

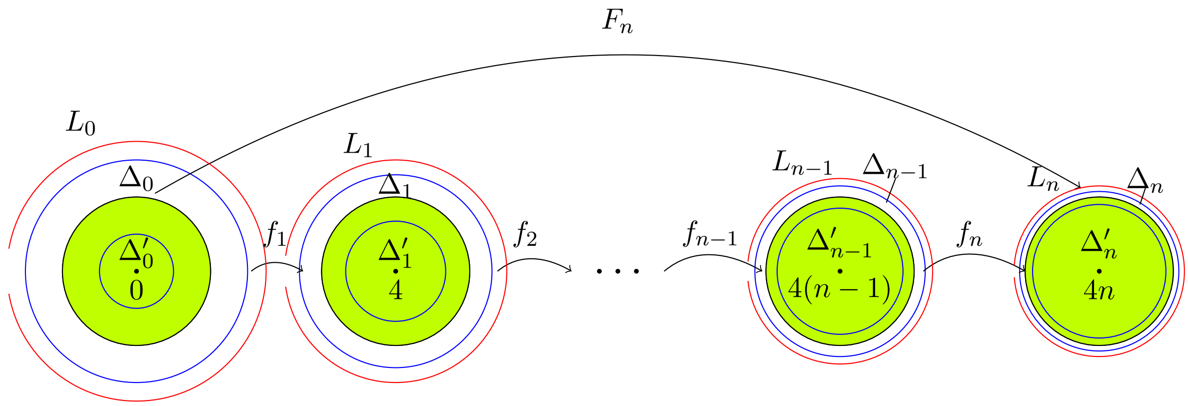

Figure 5: Sketch of the setup of Theorem 5.3. In green, the disks D_n centred at $4n$. In blue, the boundaries of the disks of radii r_n and R_n in between which lie the boundaries of the wandering domains. In red, the curves L_n introduced in the proof.

Proof. For $n \geq 0$, let

$$b_n(z) = e^{i\theta_n} \prod_{j=1}^{d_n} \frac{z + a_{n,j}}{1 + \overline{a_{n,j}} z},$$

where $a_{n,j} \in \mathbb{D}$ are not necessarily different from each other, and $\theta_n \in [0, 2\pi)$.

We first define the increasing sequence (r_n) and the decreasing sequence (R_n) inductively. These sequences determine the following circles which play a key role in the proof (see Figure 5):

$$\gamma_n = \{z : |z - 4n| = r_n\} \quad \text{and} \quad \Gamma_n = \{z : |z - 4n| = R_n\}. \tag{5.12}$$

First, take $R_0 \in (1, 3/2)$ such that $R_0 < 1/\max_j\{|a_{1,j}|\}$, which ensures that b_1 is holomorphic inside and in a neighborhood of Γ_0, and take $r_0 \in (1/2, 1)$ such that $r_0 > \max_i |z_i|$ and also such that $b_1(z) = w$ has exactly d_1 solutions in $D(0, r_0)$ for $w \in D(0, 1/2)$. Now assume that r_k, R_k have been chosen for $k = 0, \ldots, n-1$, for some $n \in \mathbb{N}$. We choose r_n and R_n so that the following statements all hold:

$$0 < 1 - r_n \leq \min\left\{\frac{1 - r_{n-1}}{2}, \operatorname{dist}(f_n(\gamma_{n-1}), \partial D_n)^2\right\}; \tag{5.13}$$

$$f_{n+1}(\gamma_n) \ \text{ winds exactly } d_{n+1} \text{ times round } D(4n, 1/2); \tag{5.14}$$

$$0 < R_n - 1 \leq \min\left\{\frac{R_{n-1} - 1}{2}, 1 - r_n, \frac{1}{2}\operatorname{dist}(f_n(\Gamma_{n-1}), \partial D_n), \frac{1}{\max_j\{|a_{n+1,j}|\}} - 1\right\}. \tag{5.15}$$

These properties prescribe the values r_n and R_n, and hence the circles γ_n and Γ_n. In particular, by (5.13) and (5.15), the sequence (r_n) increases to 1 and the sequence (R_n) decreases to 1, and the maps f_{n+1}, $n \geq 0$, defined in property (ii), satisfy

$$\gamma_{n+1} \ \text{ surrounds } f_{n+1}(\gamma_n), \tag{5.16}$$

$$f_{n+1}(\Gamma_n) \ \text{ surrounds } \Gamma_{n+1}. \tag{5.17}$$

Our aim is to use Lemma 5.1 to approximate all the maps f_n by a single entire function f such that, for $n \geq 0$, γ_{n+1} surrounds $f(\gamma_n)$ and $f(\Gamma_n)$ surrounds Γ_{n+1}.

We first define

$$\delta_n = R_n - r_n, \quad \text{for } n \geq 0, \tag{5.18}$$

and observe that $\delta_n \to 0$ as $n \to \infty$.

We then define L_n, $n \geq 0$, to be the curve

$$L_n := \{z : |z - 4n| = R_n + \delta_n^2/2,\ |\arg(z)| \leq \pi - \delta_n^2\}, \tag{5.19}$$

so

$$\max\{\text{dist}(z, L_n) : z \in \Gamma_n\} \leq 2\delta_n^2, \quad \text{for } n \geq 0, \tag{5.20}$$

and define the error quantities

$$\varepsilon_n = \min\left\{\frac{1}{4}\,\text{dist}(f_n(\gamma_{n-1}), \partial D_n), \frac{1}{4}\,\text{dist}(f_n(\Gamma_{n-1}), \partial D_n), \delta_n/4\right\} > 0, \tag{5.21}$$

for $n \geq 1$. Since $0 < \varepsilon_n \leq \delta_n/4$, we have that $\varepsilon_n < \delta_0/4 < 1/4$, $n \geq 1$, and $\varepsilon_n \to 0$ as $n \to \infty$.

We now apply Lemma 5.1 to the sets $E_0 = \overline{D(-4,1)}$ and $E_{2k+1} = L_k$, $E_{2k+2} = \overline{\Delta_k}$, for $k \geq 0$, with the function ψ defined by

$$\psi(z) = \begin{cases} -4, & \text{if } z \in \overline{D(-4,1)}, \\ -4, & \text{if } z \in L_n,\ n \geq 0, \\ f_{n+1}(z), & \text{if } z \in \overline{\Delta_n},\ n \geq 0. \end{cases}$$

Lemma 5.1 allows us to choose finitely many points $z_{n,i}$, $1 \leq i \leq j$, in each set E_n where we do the approximation. The choice of these points in $\overline{D(-4,1)} \cup \bigcup_{k=0}^{\infty} L_k$ plays no role in our argument. In $E_2 = \overline{\Delta_0}$ we choose $z_{2,i} = z_i \in D_0$, $1 \leq i \leq j$, and in $E_{2k+2} = \overline{\Delta_k}$, $k \geq 1$, we choose $z_{2k+2,i} = F_k(z_i)$, $1 \leq i \leq j$, where $F_k = f_k \circ \cdots \circ f_1$.

It then follows from Lemma 5.1 that there exists an entire function f such that, for $n \geq 0$,

$$|f(z) - f_{n+1}(z)| < \varepsilon_{n+1}, \quad \text{for } z \in \overline{\Delta_n}; \tag{5.22}$$

$$|f(z) + 4| \leq 1/2, \quad \text{for } z \in L_n; \tag{5.23}$$

$$|f(z) + 4| \leq 1/2, \quad \text{for } z \in \overline{D(-4,1)}; \tag{5.24}$$

$$f^n(z_i) = F_n(z_i), \quad \text{for } 1 \leq i \leq j; \tag{5.25}$$

$$f'((f^n)(z_i)) = f'(F_n(z_i)) = f'_{n+1}(F_n(z_i)), \quad \text{for } 1 \leq i \leq j. \tag{5.26}$$

It follows from (5.13), (5.15), (5.22) and (5.21) that, for $n \geq 0$,

$$\gamma_{n+1} \text{ surrounds } f(\gamma_n) \text{ (which surrounds the point } 4(n+1)); \tag{5.27}$$

$$f(\Gamma_n) \text{ surrounds } \Gamma_{n+1}. \tag{5.28}$$

We now apply Theorem D to the Jordan curves γ_n, Γ_n, $n \geq 0$, the compact curves L_n, $n \geq 0$, and the bounded domain $D = D(-4,1) \subset E_0$, noting that these sets satisfy the required hypotheses. Indeed, the hypotheses (a) and (b) are clearly true, (c) follows from (5.27), (d) follows from (5.28), (e) holds by (5.23) and (5.24), and (f) holds by (5.18) and (5.20).

Part (i) of our result now follows from Theorem D, part (ii) is true by construction, and part (iii) follows from (5.25) and (5.26). We now show that part (iv) holds.

By (5.21) and (5.22), we can write $f(z) = f_{n+1}(z) + e_{n+1}(z)$ for some holomorphic map $e_n(z)$ which satisfies $|e_{n+1}(z)| < 1/4$, for $z \in \overline{\Delta_n}$.

By (5.14), we have

$$|f_{n+1}(z) - 4(n+1)| \geq 1/2, \quad \text{for } z \in \gamma_n = \partial\Delta'_n.$$

It follows from this together with the fact that $|e_n(z)| < 1/4$, for $z \in \overline{\Delta_n}$ and (5.14) that

$$|f(z) - 4(n+1)| \geq 1/4, \quad \text{for } z \in \gamma_n, \tag{5.29}$$

and $f(\gamma_n)$ winds exactly d_{n+1} times around $4(n+1)$, so f takes the value $4(n+1)$ exactly d_{n+1} times in Δ'_n. Similarly, by (5.14), (5.21) and (5.22), $f(\Gamma_n)$ winds exactly d_{n+1} times around $4(n+1)$, so f takes the value $4(n+1)$ exactly d_{n+1} times in Δ_n. Therefore, by the final statement of Theorem D, $f : U_n \to U_{n+1}$ has degree d_{n+1}.

It remains to prove the double inequality (5.11), which compares the hyperbolic distances in U_n between points of two orbits under f with the corresponding hyperbolic distances in the disks D_n. To do this, we let $s_n := 1 - \frac{3}{4}\operatorname{dist}(f_n(\gamma_{n-1}), \partial D_n)$, for $n \geq 1$, and note that, if $z, z' \in \overline{D(0, r_0)}$, then

$$f^n(z), f^n(z') \in \overline{D(4n, s_n)} \subset \Delta'_n, \quad \text{for } n \in \mathbb{N},$$

by (5.16), (5.21) and (5.22).

Now $1 - r_n = o(1 - s_n)$ as $n \to \infty$, by (5.13), and $R_n - 1 \leq 1 - r_n$, by (5.15), so the properties (5.8) hold for the sequences (s_n), (r_n) and (R_n). Also,

$$\operatorname{dist}_{\Delta_n}(f^n(z), f^n(z')) \leq \operatorname{dist}_{U_n}(f^n(z), f^n(z')) \leq \operatorname{dist}_{\Delta'_n}(f^n(z), f^n(z')),$$

since $\Delta'_n \subset D_n \subset \Delta_n$. Therefore, we deduce from Lemma 5.2 (translated to the disks D_n) that

$$c(s_n, R_n)\operatorname{dist}_{D_n}(f^n(z), f^n(z')) \leq \operatorname{dist}_{U_n}(f^n(z), f^n(z')) \leq \frac{1}{c(s_n/r_n, 1/r_n)}\operatorname{dist}_{D_n}(f^n(z), f^n(z'))$$

and

$$c(s_n, R_n) \to 1 \ \text{ as } n \to \infty, \quad c(s_n/r_n, 1/r_n) \to 1 \ \text{ as } n \to \infty,$$

which gives (5.11). □

6 Examples: Proof of Theorem E

In this section we construct the examples described in Theorem E. In every case we use Theorem 5.3 and the notation there. Hence (b_n) denotes the sequence of Blaschke products of degree $d_n \geq 1$; (T_n) the sequence of real translations $z \mapsto z + 4n$; and (D_n) the sequence of disks $D_n = \{z : |z - 4n| < 1\}$, $n \geq 0$. Moreover, for $n \in \mathbb{N}$, we set $B_n = b_n \circ \cdots \circ b_1$, $f_n = T_n \circ b_n \circ T_{n-1}^{-1}$, and $F_n = f_n \circ \cdots \circ f_1$, so $F_n = T_n \circ B_n$; see Figure 6.

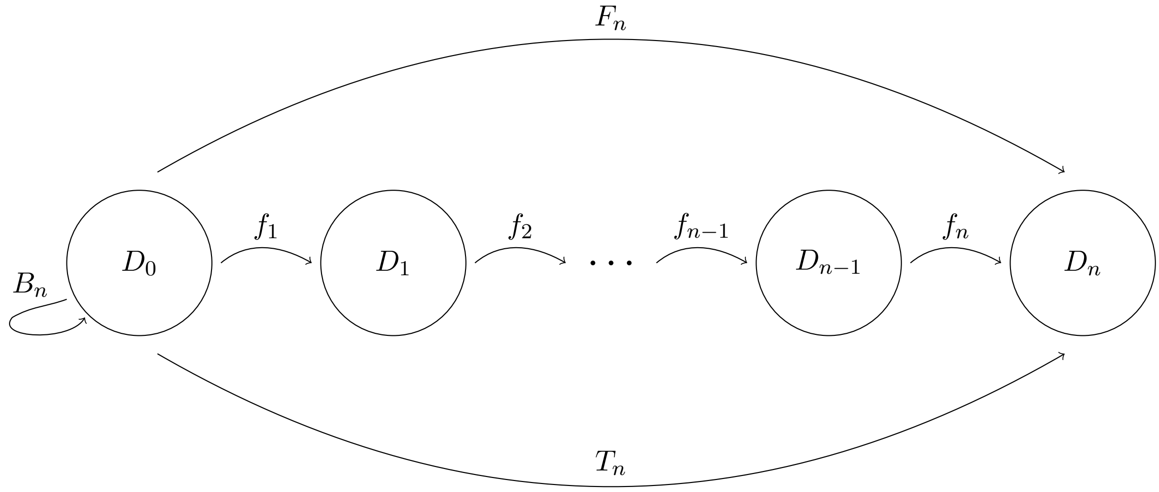

Figure 6: The maps f_n, B_n, T_n

6.1 Preliminary lemmas

We first prove two lemmas that will be used in the constructions.

Lemma 6.1. *Let f be a transcendental entire function with an orbit of bounded, simply connected, wandering domains U_n, $n \geq 0$, arising from Theorem 5.3, with Blaschke products b_n and associated functions B_n and F_n such that $f^n(0) = F_n(0)$, for $n \in \mathbb{N}$. Then, we have the following cases.*

(a) *If $B_n(0) \to 0$ as $n \to \infty$, then, for all $z \in U_0$,*

$$\liminf_{n\to\infty} \operatorname{dist}(f^n(z), \partial U_n) > 0,$$

that is, all orbits stay away from the boundary.

(b) *If there exists a subsequence $n_k \to \infty$ with $B_{n_k}(0) \to 1$ and a different subsequence $m_k \to \infty$ with $B_{m_k}(0) \to 0$, then $\operatorname{dist}(f^{n_k}(z), \partial U_{n_k}) \to 0$ for all $z \in U_0$, while*

$$\liminf_{k\to\infty} \operatorname{dist}(f^{m_k}(z), \partial U_{m_k}) > 0, \quad \text{for all } z \in U_0.$$

(c) *If $B_n(0) \to 1$ as $n \to \infty$, then $\operatorname{dist}(f^n(z), \partial U_n) \to 0$ for all $z \in U_0$, that is, all orbits converge to the boundary.*

Proof. It follows from Theorem C that all points in a simply connected wandering domain have the same limiting behaviour in relation to the boundary and so, in each case, it is sufficient to find just one point whose orbit behaves as required. We choose this point to be $0 \in U_0$.

If $B_n(0) \to 0$ as $n \to \infty$ then

$$f^n(0) - 4n = F_n(0) - 4n \to 0 \quad \text{as } n \to \infty$$

and so, by Theorem 5.3 part (i), we have

$$\liminf_{n\to\infty} \operatorname{dist}(f^n(0), \partial U_n) = 1 > 0,$$

which is sufficient to prove part (a).

If $B_n(0) \to 1$ as $n \to \infty$ then

$$f^n(0) - (4n+1) = F_n(0) - (4n+1) \to 0 \quad \text{as } n \to \infty$$

and so, by Theorem 5.3 part (i), we have

$$\operatorname{dist}(f^n(0), \partial U_n) \to 0 \quad \text{as } n \to \infty,$$

which is sufficient to prove part (c).

The proof of part (b) follows in a similar way. □

In some of our constructions we use the following properties about a specific family of Blaschke products of degree 2.

Lemma 6.2. *Let* $b(z) = \left(\frac{z+a}{1+az}\right)^2$, *where* $1/3 \le a < 1$, *and let* $0 < r < s < 1$. *Then*

(a) the function b *has a fixed point at 1, which is attracting if* $a > 1/3$ *and parabolic if* $a = 1/3$, *and* $b^n(r) \to 1$ *as* $n \to \infty$ *for all* $a \ge 1/3$*;*

(b) $\operatorname{dist}_{\mathbb{D}}(b^n(r), b^n(s)) \not\to 0$ *as* $n \to \infty$ *if* $a > 1/3$*;*

(c) $\operatorname{dist}_{\mathbb{D}}(b^n(r), b^{n+1}(r)) = O(1/n)$ *as* $n \to \infty$ *if* $a = 1/3$.

Proof. The proof of part (a) is straightforward.

For part (b) note first that

$$\operatorname{dist}_{\mathbb{D}}(b^n(r), b^n(s)) = \int_{b^n(r)}^{b^n(s)} \frac{2\,dt}{1-t^2} \ge \int_{b^n(r)}^{b^n(s)} \frac{dt}{1-t} = \log \frac{1-b^n(r)}{1-b^n(s)}. \tag{6.1}$$

Also, since 1 is an attracting fixed point of b when $a > 1/3$, there exist $\lambda \in (0,1)$ and $d > c > 0$ such that $1 - b^n(r) \sim d\lambda^n$ and $1 - b^n(s) \sim c\lambda^n$ as $n \to \infty$. Hence, by (6.1),

$$\lim_{n\to\infty} \operatorname{dist}_{\mathbb{D}}((b^n(r), b^n(s)) \ge \log \frac{d}{c} > 0.$$

For part (c) we use a similar approach. First note that

$$\operatorname{dist}_{\mathbb{D}}(b^n(r), b^{n+1}(r)) \le \int_{b^n(r)}^{b^{n+1}(r)} \frac{2\,dt}{1-t} = 2\log \frac{1-b^n(r)}{1-b^{n+1}(r)}. \tag{6.2}$$

When $a = 1/3$, we have $b(1) = 1$, $b'(1) = 1$, $b''(1) = 0$ and $b'''(1) \neq 0$, so $1 - b^n(r) \sim c/n^{1/2}$ as $n \to \infty$, where $c > 0$; see (3.8). We deduce that

$$\frac{1-b^n(r)}{1-b^{n+1}(r)} \sim \frac{(n+1)^{1/2}}{n^{1/2}} = \left(1 + \frac{1}{n}\right)^{1/2} = 1 + \frac{1}{2n} + O(1/n^2) \quad \text{as } n \to \infty.$$

The result now follows by putting this estimate into (6.2). □

6.2 The nine types of simply connected wandering domains

We now prove part (a) of Theorem E by constructing examples corresponding to each of the nine cases given in Theorems A and C. The following maps will play key roles in the constructions:

$$b(z) = \left(\frac{z+a}{1+az}\right)^2, \quad \text{for } 1/3 \le a < 1,$$

$$\mu_n(z) = \frac{z+a_n}{1+a_n z} \quad \text{and} \quad \widetilde{\mu}_n(z) = \frac{z-a_n^2}{1-a_n^2 z}, \quad \text{for } n \in \mathbb{N}, \tag{6.3}$$

where $a_n \in (0,1)$ is an arbitrary sequence satisfying $a_n \to 1$ as $n \to \infty$.

Examples 1, 2 and 3, which follow, correspond to the three cases of Theorem A. Within each of them we give three functions, corresponding to the three cases of Theorem C.

Example 1 (Three contracting wandering domains). For each of the cases (a), (b) and (c) of Theorem C, there exists a transcendental entire function f having an orbit of bounded, simply connected, escaping contracting wandering domains U_n, $n \ge 0$, with the stated behaviour:

(a) for all $z \in U_0$,

$$\liminf_{n\to\infty} \operatorname{dist}(f^n(z), \partial U_n) > 0,$$

that is, all orbits stay away from the boundary;

(b) there exists a subsequence $n_k \to \infty$ for which $\operatorname{dist}(f^{n_k}(z), \partial U_{n_k}) \to 0$ for all $z \in U$, while for a different subsequence $m_k \to \infty$ we have that

$$\liminf_{k\to\infty} \operatorname{dist}(f^{m_k}(z), \partial U_{m_k}) > 0, \quad \text{for } z \in U_0;$$

(c) $\operatorname{dist}(f^n(z), \partial U_n) \to 0$ for all $z \in U_0$, that is, all orbits converge to the boundary.

Proof. (a) Let $b_n(z) = z^2$, for $n \in \mathbb{N}$, and apply Theorem 5.3 with the points $z_1 = 0$ and $z_2 = 1/2$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \operatorname{dist}_{D_n}(f^n(0), f^n(1/2)) &= \operatorname{dist}_{\mathbb{D}}(F_n(0), F_n(1/2)) \\ &= \operatorname{dist}_{\mathbb{D}}(B_n(0), B_n(1/2)) \\ &= \operatorname{dist}_{\mathbb{D}}(0, 1/2^{2^n}) \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

It follows from (5.11) that

$$\operatorname{dist}_{U_n}(f^n(0), f^n(1/2)) \to 0 \text{ as } n \to \infty.$$

By Theorem A, this is sufficient to show that U_0 is contracting. Since $B_n(0) = 0$, for $n \in \mathbb{N}$, the result now follows from case (a) of Lemma 6.1.

(b) In this case, for $n \in \mathbb{N}$, we let

$$b_n(z) = \begin{cases} z^2 & \text{if } n = 3k-2,\ k \geq 1, \\ \mu_k(z), & \text{if } n = 3k-1,\ k \geq 1, \\ \mu_k^{-1}(z), & \text{if } n = 3k,\ k \geq 1, \end{cases}$$

where μ_k is as defined in (6.3). As in case (a), we apply Theorem 5.3 with $z_1 = 0$ and $z_2 = 1/2$. For $k \in \mathbb{N}$, we have

$$\begin{aligned} \operatorname{dist}_{D_{3k}}(f^{3k}(0), f^{3k}(1/2)) &= \operatorname{dist}_{\mathbb{D}}(F_{3k}(0), F_{3k}(1/2)) \\ &= \operatorname{dist}_{\mathbb{D}}(B_{3k}(0), B_{3k}(1/2)) \\ &= \operatorname{dist}_{\mathbb{D}}(0, 1/2^{2^k}) \to 0 \ \text{ as } k \to \infty. \end{aligned}$$

As in case (a), this is sufficient to show that U_0 is contracting. Since $B_{3k}(0) = 0$, for $k \in \mathbb{N}$, and $B_{3k-1}(0) = a_{3k-1} \to 1$ as $k \to \infty$, the conclusion now follows from case (b) of Lemma 6.1.

(c) In this case we let $b_n(z) = b(z) = \left(\frac{z+1/3}{1+z/3}\right)^2$, for $n \in \mathbb{N}$, and we apply Theorem 5.3 with $z_1 = 0$ and $z_2 = b(0)$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \operatorname{dist}_{D_n}(f^n(0), f^n(b(0))) &= \operatorname{dist}_{\mathbb{D}}(F_n(0), F_n(b(0))) \\ &= \operatorname{dist}_{\mathbb{D}}(B_n(0), B_n(b(0))) \\ &= \operatorname{dist}_{\mathbb{D}}(b^n(0), b^{n+1}(0)) \to 0 \text{ as } n \to \infty, \end{aligned}$$

by Lemma 6.2(c). As before, this is sufficient to show that U_0 is contracting. It also follows from Lemma 6.2(a) that $B_n(0) = b^n(0) \to 1$ as $n \to \infty$ and the result now follows from case (c) of Lemma 6.1 □

Remark. All three cases of Example 1 are in fact super-contracting (see Definition 1.2).

Example 2 (Three semi-contracting wandering domains). For each of the cases (a), (b) and (c) of Theorem C, there exists a transcendental entire function f having an orbit of bounded, simply connected, escaping, semi-contracting, wandering domains U_n, $n \geq 0$, with the stated behaviour.

Proof. (a) In this case we let $b_n(z) = \widetilde{\mu}_n((\mu_n(z))^2)$, for $n \in \mathbb{N}$, where μ_n and $\widetilde{\mu}_n$ are as defined in (6.3). We apply Theorem 5.3 with the points $z_1 = 0$ and $z_2 = 1/2$

A calculation shows that, for $n \in \mathbb{N}$, we have $b_n(0) = 0$ and $b_n'(0) = \frac{2a_n}{1+a_n^2} \to 1$ as $n \to \infty$. Hence we can choose (a_n) so that, in addition,

$$\sum_{n=1}^{\infty} (1 - b_n'(0)) < \infty. \tag{6.4}$$

It follows from Theorem 2.1(b) that $B_n(1/2) \not\to 0$ as $n \to \infty$. Thus

$$\begin{aligned} \operatorname{dist}_{D_n}(f^n(0), f^n(1/2)) &= \operatorname{dist}_{\mathbb{D}}(F_n(0), F_n(1/2)) \\ &= \operatorname{dist}_{\mathbb{D}}(B_n(0), B_n(1/2)) \\ &= \operatorname{dist}_{\mathbb{D}}(0, B_n(1/2)) \not\to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows from (5.11) that

$$\operatorname{dist}_{U_n}(f^n(0), f^n(1/2)) \not\to 0 \quad \text{as } n \to \infty,$$

and so U_0 is not contracting. Also, for $n \in \mathbb{N}$, the Blaschke product b_n has degree 2 and so, by Theorem 5.3 part (iv), $f : U_{n-1} \to U_n$ has degree 2. Thus U_0 is not eventually isometric and so it follows from Theorem A that U_0 is semi-contracting.

Since $B_n(0) = 0$, for $n \in \mathbb{N}$, the result now follows from case (a) of Lemma 6.1.

(b) In this case, for $n \in \mathbb{N}$, we let

$$b_n(z) = \begin{cases} \mu_k(z), & \text{if } n = 3k-2,\ k \geq 1, \\ z^2, & \text{if } n = 3k-1,\ k \geq 1, \\ \widetilde{\mu}_k(z), & \text{if } n = 3k,\ k \geq 1, \end{cases}$$

where μ_k and $\widetilde{\mu}_k$ are as defined in (6.3). Note the similarity to case (a), where each Blaschke product b_n was defined to be the composite of the three maps above.

As in case (a), we apply Theorem 5.3 with $z_1 = 0$ and $z_2 = 1/2$. Using similar arguments to those used in part (a), we can choose (a_n) such that

$$\operatorname{dist}_{U_{3k}}(f^{3k}(0), f^{3k}(1/2)) \not\to 0 \quad \text{as } k \to \infty,$$

and so U_0 is not contracting. Also, for $k \in \mathbb{N}$, the Blaschke product b_{3k-1} has degree 2 and so, by Theorem 5.3 part (iv), $f : U_{3k-2} \to U_{3k-1}$ has degree 2. Thus U_0 is not eventually isometric and so it follows from Theorem A that U_0 is semi-contracting.

Since $B_{3k}(0) = 0$, for $k \in \mathbb{N}$, and $B_{3k-2}(0) = a_{3k-2} \to 1$ as $k \to \infty$, the conclusion now follows from case (b) of Lemma 6.1.

(c) In this case we choose $a > 1/3$ and let $b_n(z) = b(z) = \left(\frac{z+a}{1+az}\right)^2$, for $n \in \mathbb{N}$. As in Example 1(c), we apply Theorem 5.3 with $z_1 = 0$ and $z_2 = b(0)$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \operatorname{dist}_{D_n}(f^n(0), f^n(b(0))) &= \operatorname{dist}_{\mathbb{D}}(F_n(0), F_n(b(0))) \\ &= \operatorname{dist}_{\mathbb{D}}(B_n(0), B_n(b(0))) \\ &= \operatorname{dist}_{\mathbb{D}}(b^n(0), b^{n+1}(0)) \not\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

by Lemma 6.2(b). Arguing as above, this is sufficient to show that U_0 is not contracting. Also, for $n \in \mathbb{N}$, the Blaschke product b_n has degree 2 and so, as above, it follows that U_0 is not eventually isometric. Hence, by Theorem A, it is semi-contracting.

It also follows from Lemma 6.2(a) that $B_n(0) = b^n(0) \to 1$ as $n \to \infty$ and the result now follows from case (c) of Lemma 6.1 □

Example 3 (Three eventually isometric wandering domains). For each of the cases (a), (b) and (c) of Theorem C, there exists a transcendental entire function f having an orbit of bounded, simply connected, escaping, eventually isometric, wandering domains U_n, $n \geq 0$, with the stated behaviour.

Proof. (a) In this case we let $b_n(z) = z$, for $n \in \mathbb{N}$, and apply Theorem 5.3 with $z_1 = 0$. For $n \in \mathbb{N}$, the map b_n is univalent and so, by Theorem 5.3 part (iv), $f : U_{n-1} \to U_n$ is also univalent. Thus U_0 is eventually isometric.

Since $B_n(0) = 0$, for $n \in \mathbb{N}$, the result now follows from case (a) of Lemma 6.1.

(b) In this case, for $n \in \mathbb{N}$, we let

$$b_n(z) = \begin{cases} \mu_k(z), & \text{if } n = 2k-1,\ k \geq 1, \\ \mu_k^{-1}(z), & \text{if } n = 2k,\ k \geq 1, \end{cases}$$

where μ_k is as defined in (6.3). We apply Theorem 5.3 with $z_1 = 0$. For $n \in \mathbb{N}$, the map b_n is univalent and so, as in case (a), U_0 is eventually isometric.

Since $B_{2k}(0) = 0$, for $k \in \mathbb{N}$, and $B_{2k-1}(0) = a_{2k-1} \to 1$ as $k \to \infty$, the conclusion now follows from case (b) of Lemma 6.1.

(c) In this case, for $n \in \mathbb{N}$, we let $b_n(z) = \frac{z+1/2}{1+z/2}$. As in cases (a) and (b), we apply Theorem 5.3 with $z_1 = 0$ and, since the map b_n is univalent, for $n \in \mathbb{N}$, we deduce that U_0 is eventually isometric. Since $b_n(x) > x$, for $x \in [0, 1)$, we deduce that $B_n(0) \to 1$ as $n \to \infty$. The conclusion now follows from case (c) of Lemma 6.1. □

6.3 2−super–attracting wandering domains

We prove part (b) of Theorem E by giving an example of a transcendental entire function with an orbit of wandering domains (U_n) containing two orbits consisting of critical points. The case of finitely many critical orbits is completely analogous.

We choose the sequence of Blaschke products (b_n). We let

$$b_1(z) = z^2 \frac{z + a_1}{1 + a_1 z},$$

with $a_1 < 0$ chosen so that $1/2$ is a critical point, and, for $n \in \mathbb{N}$ define b_n inductively by setting

$$b_{n+1}(z) = z^2 \frac{z + a_{n+1}}{1 + a_{n+1} z},$$

with $a_{n+1} \in (-1, 1)$ chosen so that $b_n \circ b_{n-1} \circ \cdots \circ b_1(1/2)$ is a critical point of b_{n+1}. In this way we construct a sequence of Blaschke products (b_n) of degree 3 such that, for $n \in \mathbb{N}$, we have $b_n(0) = 0$ and the two critical points of b_n are $0, B_{n-1}(1/2)$.

We now apply Theorem 5.3 with $z_1 = 0$ and $z_2 = 1/2$. We deduce that there exists a transcendental entire function f which has a sequence of bounded, simply connected, escaping, wandering domains (U_n) such that, for $n \geq 0$, $f^n(0) = F_n(0) = 4n$, $f^n(1/2) = F_n(1/2)$ and $f'(f^n(0)) = f'(f^n(1/2)) = 0$. Hence, there are two points in U_0, namely 0 and $1/2$, whose orbits under f consist of critical points of f.

Remark. It follows from Theorem 5.3 that $\text{dist}_{U_n}(f^n(0), f^n(1/2)) \to 0$ as $n \to \infty$ and, in fact, one can check that these wandering domains are super-contracting.

7 Wandering domains whose boundaries are Jordan curves

In this section we prove that, if the Blaschke products in Theorem 5.3 satisfy certain conditions, then the boundaries of the resulting wandering domains are Jordan curves. Apart from wandering domains arising from lifting constructions, as far as we are aware these are the first examples of simply connected wandering domains for which it is possible to obtain information concerning the boundary. Examples of *multiply* connected wandering domains for which it is known that connected components of the boundary are Jordan curves can be found in [Bis18] and in [Bau15].

In order for the boundaries of the resulting wandering domains to be Jordan curves, we need the Blaschke products in Theorem 5.3 to be uniformly expanding in the following precise sense.

Definition 7.1 (Uniformly expanding Blaschke products)**.** Let b_n, $n \in \mathbb{N}$, be a sequence of Blaschke products. We say that the Blaschke products in the sequence (b_n) are *uniformly expanding* if there exists $\xi > 1$ and an ε-neighborhood U_ε of the unit circle such that

1. each b_n is holomorphic in U_ε, that is, b_n has no poles in U_ε;
2. $|b_n'| \geq \xi$ on U_ε.

Note that the second condition implies that the b_n have no critical points in U_ε.

Our aim for this section is to prove the following theorem.

Theorem 7.2. *Let b_n, $n \in \mathbb{N}$, be a sequence of uniformly expanding Blaschke products such that $\max_n\{\deg b_n\} < \infty$ and let U_n, $n \geq 0$, be the resulting orbit of wandering domains given by Theorem 5.3. Then, for $n \geq 0$, the boundary of the wandering domain U_n is a Jordan curve.*

The proof of Theorem 7.2 follows in outline the proof that the Julia sets of certain quadratic polynomials are Jordan curves (see [Bea91, Section 9.9], for example) but with very significant changes and novel arguments due to the fact that we are dealing with a lack of uniformity arising from the associated non-autonomous system of maps.

The proof has three steps:

1. For each $n \in \mathbb{N}$, we let A_n be the annulus bounded by the circles γ_n and Γ_n, which were defined in (5.12) and played a key role in the proof of Theorem 5.3. We consider the annulus $\breve{A}_n$ lying in A_n between Γ_n and a component of $f^{-1}(A_{n+1})$ and show that the vertical geodesics of $\breve{A}_n$ have uniformly bounded Euclidean length.
2. We then use pullbacks under f^{-n} of these vertical geodesics together with the uniformly expanding property of the functions b_n (and hence of f on the annuli A_n) to induce a continuous map Σ from Γ_0 to a closed curve $\Sigma(\Gamma_0)$, and a continuous map σ from γ_0 to a closed curve $\sigma(\gamma_0)$.
3. Finally, we show that ∂U_0, which is squeezed between $\Sigma(\Gamma_0)$ and $\sigma(\gamma_0)$, is a Jordan curve.

The first step in the proof relies on a general geometric result of independent interest about the Euclidean lengths of vertical geodesics of annuli, which we prove using the Fejér-Riesz Inequality. We prove this and other preliminary geometric results in Section 7.1, and then give the proof of Theorem 7.2 in Section 7.2.

7.1 Preliminary results

We begin with a result about pre-images of annuli bounded by Jordan curves. Related results appear in [Bis18, Lemma 11.1] and [RS11, Lemma 5].

In this lemma, we denote the inner boundary component of a topological annulus A by ∂A_{inn} and the outer boundary component by ∂A_{out}. As usual, when we say that f is holomorphic on $\overline{A}$ we mean that f is holomorphic on a neighborhood of $\overline{A}$.

Lemma 7.3. *Let A and B be annuli with Jordan curve boundary components, both surrounding 0, and let f be holomorphic on $\overline{A}$, with f and f' non-zero on $\overline{A}$. Suppose that*

$$\partial B_{\text{inn}} \text{ surrounds } f(\partial A_{\text{inn}}) \quad \text{and} \quad f(\partial A_{\text{out}}) \text{ surrounds } \partial B_{\text{out}}. \tag{7.1}$$

Then A contains a unique component $\widehat{A}$ of $f^{-1}(B)$, which is an annulus that surrounds 0 with Jordan curve boundary components that satisfy

$$f(\partial \widehat{A}_{\text{inn}}) = \partial B_{\text{inn}} \quad \text{and} \quad f(\partial \widehat{A}_{\text{out}}) = \partial B_{\text{out}}. \tag{7.2}$$

Proof. Since $f(\partial A) \cap B = \emptyset$ we deduce that every connected component H of $f^{-1}(B)$ which intersects A is in fact contained in A. Since $f(\partial A)$ intersects both components of B^c, we deduce that $f(A) \supset B$ and that there exists at least one component $\hat{A}$ of $f^{-1}(B)$ that is contained in A. We now claim that $\hat{A}$ is doubly connected, surrounds 0, and that there are no other preimage components of B in A.

Let H be any preimage component of B which is contained in A. By the Riemann–Hurwitz formula H is at least doubly connected. Let X be a bounded complementary component of H. Since $f : H \to B$ is proper, $f(X)$ is the bounded complementary component of B, hence $0 \in f(X)$. If X does not contain 0, then $X \subset A$, on which f cannot take the value 0 by assumption, giving a contradiction. Hence every preimage component H of B in A is doubly connected and surrounds 0. Now suppose that there are two such components. Then one of them is contained in the bounded complementary component of the other, and hence maps to the bounded complementary component of B, again a contradiction since it is a preimage component of B.

Since the boundary components of B are Jordan curves, the fact that $f : \hat{A} \to B$ is proper, together with the fact that $f' \neq 0$ on $\overline{A}$, implies that the boundaries of $\hat{A}$ are also Jordan curves, and also implies (7.2). □

The main result in this subsection concerns the notion of a *vertical foliation* of an annulus which we now define.

Definition 7.4 (Vertical foliations)**.** Let A be an open annulus and consider the straight annulus $\mathbb{A}_\rho = \{z : \rho < |z| < 1\}$ such that $\varphi : \mathbb{A}_\rho \to A$ is a biholomorphism. The *vertical foliation* $\mathcal{F}_A$ of A consists of the image curves under φ of the radial segments connecting the two circles which form the boundary of $\mathbb{A}_\rho$. Each of these image curves is a hyperbolic geodesic which we refer to as a *vertical geodesic*.

We can now state the main result of this subsection, which concerns the Euclidean lengths of geodesics in vertical foliations of annuli. The proof uses the Fejér-Riesz Inequality (stated below); similar reasoning using instead the Gehring-Hayman Theorem (see [GH62] or [Pom92, Section 4]) is possible.

Theorem 7.5. *Let A be an annulus for which both boundary components are analytic Jordan curves with length at most S, and such that the bounded component of $\mathbb{C} \setminus A$ contains a disk of radius $r > 0$. Then there exists $M = M(S, r) > 0$ such that*

$$\ell_{\mathrm{Eucl}}(\gamma) \leq M, \quad \textit{for all } \gamma \in \mathcal{F}_A.$$

We prove Theorem 7.5 using the following technical lemma.

Lemma 7.6. *Let A be an annulus for which both boundary components are analytic Jordan curves with length at most S, and consider the straight annulus $\mathbb{A}_\rho = \{z : \rho < |z| < 1\}$ such that $\varphi : \mathbb{A}_\rho \to A$ is a biholomorphism. For $\theta \in [0, 2\pi]$, let $\sigma_\theta := \varphi(\{re^{i\theta} : \rho < r < 1\})$ and $\ell(\theta) := \ell_{\mathrm{Eucl}}(\sigma_\theta)$.*

Then the Lebesgue measure of the set of θ such that $\ell(\theta) < \frac{2S}{\rho}(1 - \rho)$ is at least $2\pi - \frac{1}{2}$.

Proof. Consider the integral

$$I = \int_\rho^1 \int_0^{2\pi} |\varphi'(re^{i\theta})| \, dr d\theta = \int_\rho^1 dr \int_0^{2\pi} |\varphi'(re^{i\theta})| \, d\theta.$$

The function

$$I(r) := \int_0^{2\pi} |\varphi'(re^{i\theta})| \, d\theta, \quad \rho < r < 1,$$

is a convex function of $\log r$ since $|\varphi'|$ can be extended to be subharmonic in a neighbourhood of $\overline{\mathbb{A}_\rho}$, so $I(r) \leq \max\{I(\rho), I(1)\}$, for $\rho \leq r \leq 1$. Then

$$\rho I(\rho) = \int_0^{2\pi} \rho |\varphi'(\rho e^{i\theta})| \, d\theta = \ell_{\mathrm{Eucl}}(\partial A_{\mathrm{in}}) \leq S,$$
$$I(1) = \int_0^{2\pi} |\varphi'(e^{i\theta})| \, d\theta = \ell_{\mathrm{Eucl}}(\partial A_{\mathrm{out}}) \leq S.$$

Hence $I(r) \leq S/\rho$, for $\rho \leq r \leq 1$, so

$$I = \int_\rho^1 I(r) \, dr \leq \frac{S}{\rho}(1 - \rho).$$

Changing the order of integration we obtain

$$I = \int_0^{2\pi} \left(\int_\rho^1 |\varphi'(re^{i\theta}| \, dr \right) d\theta = \int_0^{2\pi} \ell(\theta) \, d\theta \leq \frac{S}{\rho}(1 - \rho).$$

Hence the Lebesgue measure of the set $\{\theta : \ell(\theta) > \frac{2S}{\rho}(1 - \rho)\}$ is at most 1/2. □

In particular, Lemma 7.6 shows that the annulus A has many vertical geodesics whose Euclidean length is at most $2S(1-\rho)/\rho$.

Next, we state the following classical result (see[FR21] and [Dur70, Theorem 3.13]) about the space H^p, $p>0$, of functions g holomorphic in $\mathbb{D}$ such that

$$\sup_{0\le r<1}\left\{\int_0^{2\pi}|g(re^{i\theta})|^p\,d\theta\right\}<\infty.$$

Lemma 7.7 (Fejér-Riesz Inequality)**.** *If $g\in H^p$, then*

$$\int_{-1}^{1}|g(x)|^p\,dx\le\frac{1}{2}\int_0^{2\pi}|g(e^{i\theta})|^p\,d\theta.$$

We can now give a proof of Theorem 7.5.

Proof of Theorem 7.5. Since the result remains true under a translation, we can assume that $\mathbb{C}\setminus A$ contains a disk of radius r centered at 0.

We first claim that there exists $L=L(S,r)$ and a vertical geodesic $\sigma\in\mathcal{F}_A$ such that $\ell_{\mathrm{Eucl}}(\sigma)\le L$. Since $\mathbb{C}\setminus A$ contains a disk of radius r centered at 0, and the outer boundary has length at most S, the modulus of A is bounded from above by a constant depending only on S and r. The claim then follows by Lemma 7.6, since ρ is bounded from below by a positive constant depending only on S and r.

Now let $\log A$ be a lift of $A\setminus\sigma$ under the exponential map, using a suitable branch of the logarithm. Observe that $\log A$ is simply connected and that vertical geodesics in A lift to geodesic cross cuts in $\log A$. For any vertical geodesic $\gamma\in\mathcal{F}_A$ consider its lift $\log\gamma$ in $\log A$. Let $\psi:\mathbb{D}\to\log A$ be a biholomorphism such that $\psi(\{z:-1<\operatorname{Re}z<1\})=\log\gamma$. This can be done by mapping 0 to a point in $\log\gamma$, observing that geodesics are mapped to geodesics, and pre-composing with a rotation if necessary. By applying the Fejér-Riesz Inequality (Lemma 7.7) with $p=1$ and $g=\psi'$, we obtain

$$\ell_{\mathrm{Eucl}}(\log\gamma)\le\frac{1}{2}\ell_{\mathrm{Eucl}}(\partial\log A).\tag{7.3}$$

So it remains to show that $\ell_{\mathrm{Eucl}}(\partial\log A)$ is bounded by a uniform constant and that the resulting bound on $\ell_{\mathrm{Eucl}}(\log\gamma)$ can be translated into a bound for $\ell_{\mathrm{Eucl}}(\gamma)$. We do this by studying the distortion of lengths of curves under the lift via the exponential. Let $t\mapsto z(t)$ for $t\in[0,1]$ be a parametrization of a curve C in $\overline{A}$ and let $\log C$ be its lift in $\overline{\log A}$. Then $t\mapsto\log(z(t))$ for $t\in[0,1]$ is a parametrization of the curve $\log C$, so

$$\ell_{\mathrm{Eucl}}(C)=\int_0^1|z'(t)|\,dt\ \text{ and }\ \ell_{\mathrm{Eucl}}(\log C)=\int_0^1\left|\frac{z'(t)}{z(t)}\right|dt.$$

Since A is contained in the straight annulus $\mathbb{A}(r,S/2)$ (this follows from considering the extremal case for ∂A_{out}), we have that $r\le|z(t)|\le S/2$, so

$$\frac{2}{S}\ell_{\mathrm{Eucl}}(C)\le\ell_{\mathrm{Eucl}}(\log C)\le\frac{1}{r}\ell_{\mathrm{Eucl}}(C).\tag{7.4}$$

It follows that $\ell_{\text{Eucl}}(\log \sigma) \leq \frac{1}{r}\ell_{\text{Eucl}}(\sigma) \leq \frac{L}{r}$ and that if $\alpha = \partial A_{\text{inn}}$ and $\beta = \partial A_{\text{out}}$ are the inner and outer boundary components, respectively, of A, then we have

$$\ell_{\text{Eucl}}(\log \alpha) + \ell_{\text{Eucl}}(\log \beta) \leq \frac{1}{r}(\ell_{\text{Eucl}}(\alpha) + \ell_{\text{Eucl}}(\beta)) \leq \frac{2S}{r}.$$

So

$$\ell_{\text{Eucl}}(\partial \log A) = 2\ell_{\text{Eucl}}(\log \sigma) + \ell_{\text{Eucl}}(\log \alpha) + \ell_{\text{Eucl}}(\log \beta) \leq \frac{2L}{r} + \frac{2S}{r}.$$

It now follows from (7.3) and (7.4) that, for any vertical geodesic $\gamma \in \mathcal{F}_A$, we have

$$\ell_{\text{Eucl}}(\gamma) \leq \frac{S}{2}\ell_{\text{Eucl}}(\log \gamma) \leq \frac{S(L+S)}{2r}.$$

This concludes the proof of Theorem 7.5. □

7.2 Proof of Theorem 7.2

Let b_n, $n \in \mathbb{N}$, be a sequence of uniformly expanding Blaschke products of degree at most d and let f be the transcendental entire function with an associated orbit of wandering domains (U_n) arising from Theorem 5.3. We will show that the boundary of U_0 is a Jordan curve.

For each $n \geq 0$, we let A_n be the annulus bounded by the circles γ_n and Γ_n which were defined in (5.12) in the proof of Theorem 5.3. By the uniform expansivity condition on the functions b_n and the fact that $\max\{|f(z) - f_n(z)| : z \in A_n\} \to 0$ as $n \to \infty$ (see (5.22)), we deduce using Cauchy's estimate that there exists $\eta > 1$ such that, for sufficiently large $n \in \mathbb{N}$,

$$|f'| \geq \eta > 1 \text{ on a neighborhood of } A_n; \tag{7.5}$$

in particular, f has no critical points in a neighborhood of A_n for such n. Relabeling the U_n if necessary, we can assume that the above conditions hold for any $n \geq 0$.

Step 1 For each $n \geq 0$, we let $\widehat{A}_n$ denote the pre-image component of A_{n+1} under f in A_n, given by Lemma 7.3, with inner and outer boundary components $\widehat{\gamma}_n$ and $\widehat{\Gamma}_n$, say, respectively. Then let $\breve{A}_n$ denote the annulus lying between $\widehat{A}_n$ and Γ_n (see Figure 7). Our first claim is that there exists $M = M(\eta, d) > 0$ such that, for $n \geq 0$, each geodesic in the vertical foliation $\mathcal{F}_n$ of the annulus $\breve{A}_n$ has Euclidean length at most M.

We start by showing that each Jordan curve $\widehat{\Gamma}_n$ has length which is uniformly bounded by $3\pi d/\eta$. Indeed, we can parametrize $\widehat{\Gamma}_n : [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{d-1}, t_d] \to \mathbb{C}$, where $t_0 < t_1 < \cdots < t_d$, with $\widehat{\Gamma}_n(t_0) = \widehat{\Gamma}_n(t_d)$, in such a way that f is univalent on $\widehat{\Gamma}_n(t_i, t_{i+1})$, $0 \leq i < d-1$. This can be done because the degree of b_n (and hence of f on $\breve{A}_n$, for n sufficiently large) is bounded above by d, and $\widehat{\Gamma}_n$ is a Jordan curve. Notice that $f(\widehat{\Gamma}_n[t_i, t_{i+1}]) \subseteq \Gamma_{n+1}$. For n large, we have, by (7.5),

$$\begin{aligned} 3\pi \geq \ell_{\text{Eucl}}(\Gamma_{n+1}) &\geq \frac{1}{d}\left(\int_{t_0}^{t_1} |f'(\widehat{\Gamma}_n(t))||\widehat{\Gamma}'_n(t)|\,dt + \cdots + \int_{t_{d-1}}^{t_d} |f'(\widehat{\Gamma}_n(t))||\widehat{\Gamma}'_n(t)|\,dt\right) \\ &\geq \frac{\eta}{d}\int_{t_0}^{t_d} |\widehat{\Gamma}'_n(t)|\,dt = \frac{\eta}{d}\ell_{\text{Eucl}}(\widehat{\Gamma}_n). \end{aligned}$$

(The second inequality becomes an equality if $f : \widehat{\Gamma}_n[t_i, t_{i+1}] \to \Gamma_{n+1}$ is surjective for every i.) Since, by construction, the bounded component of $\mathbb{C} \setminus \widehat{A}_n$ contains the circle $\gamma_n = \{z : |z| = r_n\}$ and $r_n \geq 1/2$ for $n \geq 0$, the annulus $\widehat{A}_n$ satisfies the hypotheses of Theorem 7.5, giving the claim.

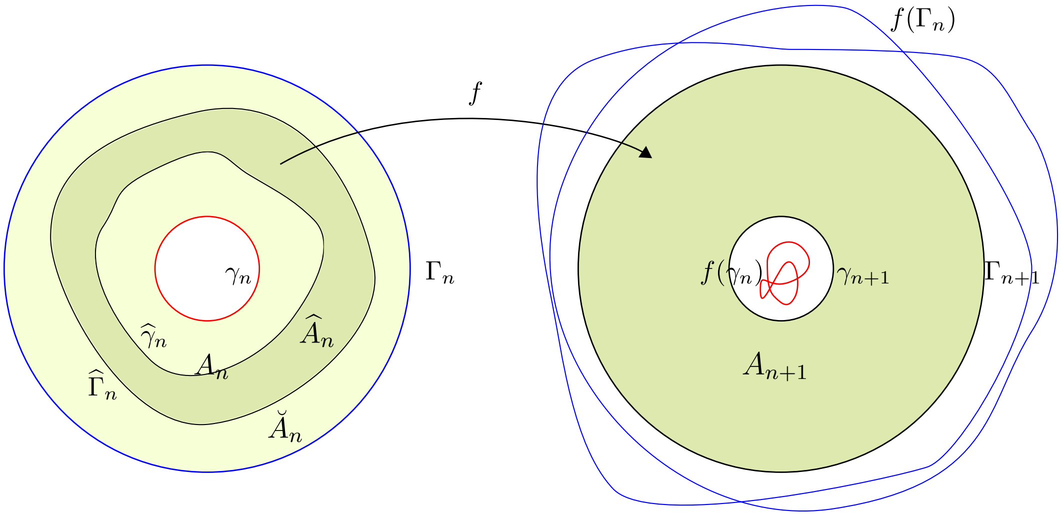

Figure 7: Sketch of the setup in Step 1.

Step 2 For $n \in \mathbb{N}$, let $\widetilde{\Gamma}_n$, $\widetilde{\gamma}_n$ be those pre-images under f^n of Γ_n, γ_n, respectively, that are contained in A_0 and are such that $\widetilde{\Gamma}_n$ surrounds $\widetilde{\Gamma}_{n+1}$ for every n, while $\widetilde{\gamma}_n$ is surrounded by $\widetilde{\gamma}_{n+1}$ for every n. The existence of $\widetilde{\gamma}_n$ and $\widetilde{\Gamma}_n$, and the fact that they are Jordan curves, follows by applying inductively Lemma 7.3, since there are no critical points in A_n for any n by the uniform expansivity condition.

We now concentrate on the family of Jordan curves $\widetilde{\Gamma}_n$ and construct a continuous map Σ from Γ_0 to the limit of the $\widetilde{\Gamma}_n$, defined in an appropriate way.

Fix $z_0 \in \Gamma_0$. Let $\Sigma_0(z_0)$ be the (unique) geodesic in $\mathcal{F}_0$ which connects z_0 to some point $z_1 \in \widetilde{\Gamma}_1$, and let us parametrize it as a curve $\Sigma_0(z_0, t)$, $t \in [0, 1]$, with $\Sigma_0(z_0, 0) = z_0$, $\Sigma_0(z_0, 1) = z_1$. Consider $f(z_1) \in \Gamma_1$, and the geodesic $\omega_1 \in \mathcal{F}_1$ which connects $f(z_1)$ to some point z_2' say in $f^{-1}(\Gamma_2)$. Notice that the definition of $\mathcal{F}_1$ automatically specifies the connected component of $f^{-1}(\Gamma_2)$ to which z_2' belongs. The preimage of ω_1 under f which contains z_1 is an arc that can be parametrized as $\Sigma_1(z_0, t)$, $t \in [1, 2]$, connecting z_1 to some point $z_2 \in f^{-1}(z_2') \cap \widetilde{\Gamma}_2$. Proceeding in this way, for each n we can construct a point z_n and a curve $\Sigma_n(z_0, t)$, $t \in [n, n+1]$, connecting z_n to z_{n+1}. This can be repeated for any starting point $z \in \Gamma_0$ to construct a continuous injective curve

$$\Sigma(z, t) : \Gamma_0 \times [0, \infty] \to A_0,$$

such that $\Sigma(z, n) \in \widetilde{\Gamma}_n$ and, for each $z \in \Gamma_0$ and each $j \leq n$, $n \in \mathbb{N}$, we have $f^j(\Sigma(z, [n, n+1])) \subset A_j$ and $f^n(\Sigma(z, [n, n+1]))$ is a geodesic in $\mathcal{F}_n$.

Recalling that the Euclidean length of elements in $\mathcal{F}_n$ is bounded uniformly in n by a constant $M = M(\eta, d) > 0$, and using the expansivity estimate (7.5) on f, we obtain

$$\ell_{\mathrm{Eucl}}(\Sigma(z, [n, n+1])) \leq \frac{1}{\eta^n} M, \quad \text{for } z \in \Gamma_0, n \in \mathbb{N}.$$

It follows that for each $z \in \Gamma_0$ the curve $\Sigma(z,t)$ converges to $\Sigma(z)$, say, as $t \to \infty$, and moreover that the map $\Sigma : \Gamma_0 \to \Sigma(\Gamma_0)$ is continuous in z, so $\Sigma(\Gamma_0)$ is a closed curve. Note, however, that we have not shown that Σ is a Jordan curve, since the map $z \mapsto \Sigma(z)$ has not been shown to be injective.

We can construct an analogous map $\sigma : \gamma_0 \to \sigma(\gamma_0)$ and obtain a closed curve $\sigma(\gamma_0)$ as a uniform limit using the Jordan curves $\widetilde{\gamma}_n$ in a similar manner.

Step 3 We now do the final step of showing that ∂U_0 is indeed a Jordan curve. It is sufficient (see [New61, Chapter VI, Theorem 16.1]) to show that each point of ∂U_0 is accessible from both complementary components. By the construction in Step 2, it is enough to show that

$$\partial U_0 \subset \sigma(\gamma_0) \cap \Sigma(\Gamma_0).$$

Let C_n, c_n denote the bounded complementary components of the Jordan curves $\widetilde{\Gamma_n}$, $\widetilde{\gamma_n}$, respectively. Then (C_n) and (c_n) each form a sequence of nested topological disks which are respectively decreasing and increasing, because for each n we have that $f(\Gamma_n)$ surrounds Γ_{n+1} and $f(\gamma_n)$ is surrounded by γ_{n+1}. Notice that for each n, the annulus $\widetilde{A}_n := C_n \setminus \overline{c_n}$ contains ∂U_0. This is because, for $n \geq 0$, we have Γ_n surrounds U_n and $\gamma_n \subset U_n$.

Now suppose that $\partial U_0 \not\subset \Sigma(\Gamma_0)$, and let $\zeta \in \partial U_0 \setminus \Sigma(\Gamma_0)$. Then for some $r > 0$ the disk $D(\zeta, r)$ does not meet the closed curve $\Sigma(\Gamma_0)$, so there is some open disk $D(\zeta', r') \subset D(\zeta, r)$ that lies in both the exterior of $\overline{U_0}$ and in the bounded complementary component of $\Sigma(\Gamma_0)$ which contains U_0. Hence $D(\zeta', r') \subset \widetilde{A}_n$, for all $n \in \mathbb{N}$, and by construction $f^n(D(\zeta', r')) \subset A_n$ for every n. Now, the maximal radii of the Euclidean disks contained in the annuli A_n are bounded, and indeed converge to 0 as $n \to \infty$. For large n, this contradicts the fact that, since $|(f^n)'(\zeta')| \geq \eta^n$, the image $f^n(D(\zeta', r'))$ contains a disk of radius at least $Br'\eta^n$, where $B > 0$ is Bloch's constant. Hence $\partial U_0 \subset \Sigma(\Gamma_0)$.

A similar argument shows that $\partial U_0 \subset \sigma(\gamma_0)$, which completes the proof.

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