This is a preprint of: "Non–equivalence between the Melnikov and the averaging methods for nonsmooth differential systems", Zhifei Guo, Jaume Llibre, *Qual. Theory Dyn. Syst.*, vol. 21, 114:1–13, 2022.
DOI: [10.1007/s12346-022-00643-5]

NON–EQUIVALENCE BETWEEN THE MELNIKOV AND THE AVERAGING METHODS FOR NONSMOOTH DIFFERENTIAL SYSTEMS

ZHIFEI GUO[1] AND JAUME LLIBRE[2]

ABSTRACT. It is known that for smooth differential systems in the plane $\mathbb{R}^2$ the Melnikov and the averaging methods for studying the limit cycles produce the same results. Here we prove that this is not the case for nonsmooth differential systems in the plane.

More precisely, we prove that the linear center $\dot{x} = y$, $\dot{y} = -x$, can produce at most 5 limit cycles using the first order averaged function and also produce at most 5 limit cycles using the second order averaged function, when it is perturbed by a discontinuous piecewise differential systems of two pieces separated by the cubic curve $y = x^3$, and having in each piece a quadratic polynomial differential system. While using the Melnikov method up to order two these discontinuous piecewise differential systems already produce 7 limit cycles having in each piece a linear polynomial differential system.

1. Introduction and statement of the main results

In the past few decades, nonsmooth differential systems with discontinuous right-hand sides have received a lot of attentions because they play a crucial role in different fields, such as control systems, economy, electrical circuits and mechanics (see the books [3, 29] and the survey [27]). The vector fields at each point of the phase space are unique for a smooth differential system. A nonsmooth differential system with discontinuous right-hand sides is multivalued in its discontinuous boundary. For a nonsmooth differential system we use the Filippov's convention to define the vector fields at points of its discontinuous boundary (see [11]).

In recent years many authors focus on the number of limit cycles or the cyclicity of the Hopf bifurcation for a nonsmooth differential system (see [8–10, 16, 23, 25]). The method of averaging and the Melnikov method are two classical and mature tools to study the dynamics of nonlinear differential systems (see [5, 12, 21, 28] for smooth differential systems and [13, 15, 20, 22, 30] for nonsmooth differential systems).

It is well-known that planar smooth linear differential systems may have periodic orbits but no limit cycles. A planar polynomial differential system having a linear center is equivalent to the linear system $(\dot{x}, \dot{y})^T = (y, -x)^T$. The linear differential system perturbed by discontinuous piecewise polynomials of two zones $\Sigma_\pm = \mathbb{R}^2 \setminus \Sigma$,

---

2010 *Mathematics Subject Classification.* 34C29, 34C25, 34A36.
Key words and phrases. Limit cycles, the method of averaging, discontinuous piecewise differential systems.

where Σ is the switching boundary, takes the form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y+\Sigma_{i=1}^{m}\varepsilon^i f_i^{\pm}(x,y)\\ -x+\Sigma_{i=1}^{m}\varepsilon^i g_i^{\pm}(x,y)\end{pmatrix} \text{ if } (x,y)\in\Sigma_{\pm}, \tag{1}$$

where $f_i^{\pm}$ and $g_i^{\pm}$ $(i = 1, \ldots, m)$ are real polynomials. The perturbed system (1) can have limit cycles even if $f_i^{\pm}$ and $g_i^{\pm}$ $(i = 1, \ldots, m)$ are real polynomials of degree 1.

Attentions were made to investigate limit cycles of system (1) using the method of averaging or the Melnikov method (see [1, 2, 6, 7, 19, 20, 24]). When the switching boundary Σ is the x-axis, i.e. $\Sigma = \{y = 0\}$, Buzzi, Pessoa and Torregrosa [6] used the Melnikov functions up to order 7 to prove that system (1) with linear perturbations has at most 3 limit cycles. For the same switching boundary Σ as [6], Llibre and Tang [24] applied the method of averaging up to order 5 to obtain that system (1) has at most 8 crossing limit cycles for quadratic perturbations and 13 crossing limit cycles for cubic perturbations, respectively.

When the switching boundary $\Sigma = \Sigma_\alpha$ is nonregular, for example, Σ is the nonnegative x-axis and the ray $x = y \cot\alpha$ with $\alpha \in (0, \pi)$ and $y > 0$, Cardin and Torregrosa [7] used the Melnikov functions up to order 6 to obtain that system (1) with linear perturbations has at most 5 limit cycles. Moreover they proved again using the Melnikov functions up to order 6 that the maximum number of crossing limit cycles is 2 for discontinuous piecewise linear Liénard system (1), where $g_{\pm} = 0$ and $f_{\pm}$ are only functions of x. Later on Li and Llibre [19] used the averaging method up to any order, which was developed in [17], to provide an upper bound for the maximum number of limit cycles of system (1).

Recently, Llibre, Mereu and Novaes [20] discussed crossing limit cycles of system (1) with quadratic perturbations for the nonlinear switching boundary $\Sigma = \{y = x^2\}$, and obtained that the maximum number of crossing limit cycles is 6 using the method of averaging up to order 2. Bastos, Buzzi, Llibre and Novaes [2] proved that the maximum number of limit cycles is 7 for system (1) with linear perturbations and $\Sigma = \{y = x^3\}$ using the Melnikov functions up to order 2. This last result together with our results will prove that the Melnikov and the averaging methods do not produce the same result on the number of limit cycles when the differential systems are nonsmooth. Later on Andrade, Cespedes, Cruz and Novaes [1] considered the case $\Sigma = \{y = x^n\}$ for system (1) with linear perturbations using the higher order Melnikov methods, and obtained that $H(2) \geq 4, H(3) \geq 8, H(n) \geq 7$, for $n \geq 4$ even, and $H(n) \geq 9$, for $n \geq 5$ odd, where $H(n)$ denotes an upper bound of the maximum number of limit cycles for system (1) with linear perturbations and $\Sigma = \{y = x^n\}$.

In this paper stimulated by the references [2, 20], we use the method of averaging to investigate the number of crossing limit cycles for system (1) in the case that $\Sigma = \{y = x^3\}$ but with perturbations of order two in a small parameter ε, i.e. the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon A_1(x,y) + \varepsilon^2 A_2(x,y) \\ -x + \varepsilon B_1(x,y) + \varepsilon^2 B_2(x,y) \end{pmatrix} \text{if } h(x,y) \geq 0, \\ \begin{pmatrix} y + \varepsilon C_1(x,y) + \varepsilon^2 C_2(x,y) \\ -x + \varepsilon D_1(x,y) + \varepsilon^2 D_2(x,y) \end{pmatrix} \text{if } h(x,y) \leq 0, \end{cases} \tag{2}$$

where $h(x,y) := y - x^3$ and for $i = 1, 2$,

$$A_i(x,y) := \sum_{j+k=0}^{2} a_{ijk} x^j y^k, \qquad B_i(x,y) := \sum_{j+k=0}^{2} b_{ijk} x^j y^k,$$
$$C_i(x,y) := \sum_{j+k=0}^{2} c_{ijk} x^j y^k, \qquad D_i(x,y) := \sum_{j+k=0}^{2} d_{ijk} x^j y^k.$$

Let $\mathcal{Q}_1$ be the set of the following conditions

$$\begin{array}{l} a_{110} = -b_{101} - c_{110} - d_{101}, \quad a_{120} = c_{120}, \\ a_{111} = 3a_{100} - b_{120} - 3c_{100} + c_{111} + d_{120}, \\ a_{102} = -b_{111} + c_{102} + d_{111}, \quad b_{100} = d_{100}, \quad b_{102} = d_{102}, \end{array} \tag{3}$$

and $\mathcal{Q}_2$ the set of the following conditions

$$\begin{array}{l} a_{100} = a_{120} = b_{100} = b_{102} = c_{100} = c_{110} = c_{120} = d_{100} = d_{101} = d_{102} = 0, \\ a_{102} = -b_{111}, \quad a_{110} = -b_{101}, \quad a_{111} = -b_{120}, \quad c_{101} = -d_{110}, \\ c_{111} = -d_{120}, \quad c_{102} = -d_{111}, \end{array} \tag{4}$$

where $\mathcal{Q}_2$ is a subset of $\mathcal{Q}_1$. Our main results are the following.

Theorem 1. *For $|\varepsilon|$ sufficiently small system* (2) *using the averaging theory of first order has at most 5 crossing limit cycles when the condition $\mathcal{Q}_1$ does not hold. Moreover we can choose parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} such that system* (2) *has exactly* $0, 1, 2, 3, 4$ *or* 5 *limit cycles.*

Theorem 2. *For $|\varepsilon|$ sufficiently small system* (2) *using the averaging theory of second order has at most 5 crossing limit cycles when the condition $\mathcal{Q}_2$ holds. Moreover we can choose parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} such that system* (2) *has exactly* $0, 1, 2, 3, 4$ *or* 5 *limit cycles.*

Theorems 1 and 2 are proven in Section 3.

In this paper we study the maximum number of crossing limit cycles for system (2) using the method of averaging up to order 2. We prove that system (2) produces at most 5 crossing limit cycles using the first order averaging theory. Here the second order averaging theory produces the same result as the first order averaging theory. However, Bastos, Buzzi, Llibre and Novaes [2] using the Melnikov functions up to order 2 proved that system (1) with linear perturbations and $\Sigma = \{y = x^3\}$ has at most 7 limit cycles. Their results together with our results show that the Melnikov and the averaging methods do not produce the same result on the number of limit cycles when the differential systems are nonsmooth, while the number of limit cycles obtained by these two methods coincide for smooth systems (see [14]).

2. Preliminaries

In order to prove our results we state some necessary elements. Let A and D be open subsets of $\mathbb{S}^1 \times \mathbb{R}^d$ and $\mathbb{S}^1 = R/T$ be the circle with period T. We denote the characteristic function by $\chi_A(t,x) = 1$ (resp. 0) if $(t,x) \in A$ (resp. $\notin A$). Given the following discontinuous piecewise differential system

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{5}$$

where $F_i := \Sigma_{j=1}^M \chi_{\bar{S}_j}(t,x) F_i^j(t,x)$ for $i = 1,2$, $R := \Sigma_{j=1}^M \chi_{\bar{S}_j}(t,x) R_i^j(t,x,\varepsilon)$, and $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}^d$, $R_i^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ with $\varepsilon_0 > 0$, $i = 1,2$, $j = 1,\dots,M$, are all continuous functions and are all T-periodic in the variable t. From [20] the averaged functions of orders 1 and 2 for system (5) are

$$f_1(z) = \int_0^T F_1(t,z)dt, \ \ f_2(z) = \int_0^T \left(\frac{\partial F_1(t,z)}{\partial x} y_1(t,z) + F_2(t,z) \right) dt, \tag{6}$$

respectively, where $y_1(t,z) = \displaystyle\int_0^t F_1(s,z)ds$.

A point $p \in \Sigma$ is called a generic point of discontinuity if there exists a neighborhood U of p such that $S_p = U \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface.

The crossing hypothesis (HC) given in [20] for system (5) is

(HC) There exists an open bounded set $C \subset D$ such that for each $z \in \overline{C}$ the curve $\{(t,z) : t \in \mathbb{S}^1\}$ reaches transversally the set Σ and only at generic points of discontinuity.

In what follows we state the averaging theory for computing periodic orbits up to order one and two for discontinuous piecewise differential systems that we need for studying system (5), where the notation $d_B(f_i, U, 0)$ $(i = 1,2)$ denote the Brouwer degree of the function f_i in the neighborhood U of zero (see [4] or the Appendix A of [20]).

Theorem 3 ([20, Theorem A])**.** *In addition to the crossing hypothesis (HC) assume the following conditions.*

(Ha1) For $i = 1,2$ and $j = 1,2,\dots,M$, the continuous functions F_i^j and R_i^j are locally Lipschitz with respect to x, and T-periodic with respect to the time t. Furthermore, for $j = 1,2,\dots,M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.

(Ha2) For $a^* \in C$ with $f_1(a^*) = 0$, there exists a neighborhood $U \subset C$ of a^* such that $f_1(z) \neq 0$ for all $z \in \overline{U} \backslash \{a^*\}$ and $d_B(f_1, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ sufficiently small, there exists a T-periodic solution $x(t,\varepsilon)$ of system (5) such that $x(0,\varepsilon) \to a^*$ as $\varepsilon \to 0$.

Theorem 4 ([20, Theorem B])**.** *Suppose that $f_1(z) \equiv 0$. In addition to the crossing hypothesis (HC) assume the following conditions.*

(Hb1) For $j = 1,2,\dots,M$, the functions $F_1^j(t,\cdot)$ are of class $\mathcal{C}^1$ for all $t \in \mathbb{R}$; for $j = 1,2,\dots,M$, the functions $D_x F_1^j, F_2^j$ and R are locally Lipschitz with respect to x. Furthermore, for $j = 1,2,\dots,M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.

(Hb2) *If* $(t,z)\in\Sigma$ *then* $(0,y_1(t,z))\in T_{(t,z)}\Sigma$.

(Hb3) *For* $a^*\in C$ *with* $f_2(a^*)=0$, *there exists a neighborhood* $U\subset C$ *of* a^* *such that* $f_2(z)\neq 0$ *for all* $z\in\overline{U}\backslash\{a^*\}$ *and* $d_B(f_2,U,0)\neq 0$.

Then for $|\varepsilon|\neq 0$ *sufficiently small, there exists a* T*-periodic solution* $x(t,\varepsilon)$ *of system* (5) *such that* $x(0,\varepsilon)\to a^*$ *as* $\varepsilon\to 0$.

It is known that if a function f is $\mathcal{C}^1$ then it is sufficient to check that the determinant of the Jacobian matrix $D(f)$ is non-zero in order to have that the Brouwer degree $d_B(f,U,0)\neq 0$, for more details see [26].

Besides we also resort to the ECT-system that we shall use in the proof of our results. Let I denote a proper real interval of $\mathbb{R}$. An ordered set of complex-valued functions $F=(f_0,f_1,\ldots,f_n)$ defined on I is called an Extended Chebyshev system or ET-system on I if and only if any nontrivial linear combination of f_i $(i=0,1,\ldots,n)$ has at most n zeros counting multiplicities. Furthermore F becomes an Extended Complete Chebyshev system or an ECT-system on I if and only if for any $0\le k\le n$, $(f_0,f_1,\ldots,f_k)$ is an ET-system. We can see the monograph [18] for more details. The set F is an ECT-system on I if and only if $W(f_0,f_1,\ldots,f_k)(t)\neq 0$ on I for $0\le k\le n$, where $W(f_0,f_1,\ldots,f_k)(t)$ denotes the Wronskian of the functions $(f_0,f_1,\ldots,f_k)$ with respect to t, i.e.

$$W(f_0,f_1,\ldots,f_k)(t)=\begin{vmatrix} f_0(t) & f_1(t) & \cdots & f_k(t)\\ f_0'(t) & f_1'(t) & \cdots & f_k'(t)\\ \vdots & \vdots & \ddots & \vdots\\ f_0^{(k)}(t) & f_1^{(k)}(t) & \cdots & f_k^{(k)}(t)\end{vmatrix}.$$

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. In the polar coordinates $x=r\cos\theta$ and $x=r\sin\theta$ system (2) takes the form

$$(7)\qquad \begin{pmatrix}\dot r\\ \dot\theta\end{pmatrix}=\begin{cases}\begin{pmatrix}\varepsilon\big(A_1(x,y)\cos\theta+B_1(x,y)\sin\theta\big)\\ +\varepsilon^2\big(A_2(x,y)\cos\theta+B_2(x,y)\sin\theta\big),\\ -1+\varepsilon\big(B_1(x,y)\cos\theta-A_1(x,y)\sin\theta\big)/r\\ +\varepsilon^2\big(B_2(x,y)\cos\theta-A_2(x,y)\sin\theta\big)/r\end{pmatrix} & \text{if } \tilde h(\theta,r)\ge 0,\\ \begin{pmatrix}\varepsilon\big(C_1(x,y)\cos\theta+D_1(x,y)\sin\theta\big)\\ +\varepsilon^2\big(C_2(x,y)\cos\theta+D_2(x,y)\sin\theta\big),\\ -1+\varepsilon\big(D_1(x,y)\cos\theta-C_1(x,y)\sin\theta\big)/r\\ +\varepsilon^2\big(D_2(x,y)\cos\theta-C_2(x,y)\sin\theta\big)/r\end{pmatrix} & \text{if } \tilde h(\theta,r)\le 0,\end{cases}$$

where $x=r\cos\theta$, $y=r\sin\theta$ and $\tilde h(\theta,r):=r\sin\theta-r^3\cos^3\theta$. Taking θ as the new time variable and expanding it as a power series of ε, system (7) is equivalent to the following

$$(8)\qquad \dot r=\begin{cases}P(\theta,r):=\varepsilon P_1(\theta,r)+\varepsilon^2P_2(\theta,r)+\mathcal{O}(\varepsilon^3) \text{ if } \tilde h(\theta,r)\ge 0,\\ Q(\theta,r):=\varepsilon Q_1(\theta,r)+\varepsilon^2Q_2(\theta,r)+\mathcal{O}(\varepsilon^3) \text{ if } \tilde h(\theta,r)\le 0,\end{cases}$$

where now $\dot{r}$ denotes the derivative with respect to the variable θ, $Q_i(\theta,r) := P_i(\theta,r)|_{a_{ijk}=c_{ijk},\, b_{ijk}=d_{ijk}}$ for $i=1,2,\ j,k=0,1,2,\ 0\le j+k\le 2$, and

$$
\begin{aligned}
P_1(\theta,r) :=& -a_{120}r^2\cos^3\theta-(r^2(a_{111}+b_{120})\sin\theta+a_{110}r)\cos^2\theta\\
&-(r^2(a_{102}+b_{111})\sin^2\theta+r(a_{101}+b_{110})\sin\theta+a_{100})\cos\theta\\
&-b_{102}r^2\sin^3\theta-b_{101}r\sin^2\theta-b_{100}\sin\theta,\\
P_2(\theta,r) :=& \{-a_{120}b_{120}r^4\cos^6\theta-(r^4(a_{111}b_{120}-a_{120}^2+a_{120}b_{111}+b_{120}^2)\sin\theta\\
&+r^3(a_{110}b_{120}+a_{120}b_{110}+a_{220}))\cos^5\theta-(r^4(a_{102}b_{120}-2a_{111}a_{120}\\
&+a_{111}b_{111}+a_{120}b_{102}-a_{120}b_{120}+2b_{111}b_{120})\sin^2\theta+r^3(a_{101}b_{120}\\
&-2a_{110}a_{120}+a_{110}b_{111}+a_{111}b_{110}+a_{120}b_{101}+2b_{110}b_{120}+a_{211}\\
&+b_{220})\sin\theta+r^2(a_{100}b_{120}+a_{110}b_{110}+a_{120}b_{100}+a_{210}))\cos^4\theta\\
&+(r^4(2a_{102}a_{120}-a_{102}b_{111}+a_{111}^2-a_{111}b_{102}+a_{111}b_{120}+a_{120}b_{111}\\
&-2b_{102}b_{120}-b_{111}^2)\sin^3\theta+r^3(2a_{101}a_{120}-a_{101}b_{111}-a_{102}b_{110}\\
&+2a_{110}a_{111}-a_{110}b_{102}+a_{110}b_{120}-a_{111}b_{101}+a_{120}b_{110}-2b_{101}b_{120}\\
&-2b_{110}b_{111}-a_{202}-a_{220}-b_{211})\sin^2\theta+r^2(2a_{100}a_{120}-a_{100}b_{111}\\
&-a_{101}b_{110}+a_{110}^2-a_{110}b_{101}-a_{111}b_{100}-2b_{100}b_{120}-b_{110}^2-a_{201}\\
&-b_{210})\sin\theta-r(a_{100}b_{110}+a_{110}b_{100}+a_{200}))\cos^3\theta+(r^4(2a_{102}a_{111}\\
&-a_{102}b_{102}+a_{102}b_{120}+a_{111}b_{111}+a_{120}b_{102}-2b_{102}b_{111})\sin^4\theta\\
&+r^3(2a_{101}a_{111}-a_{101}b_{102}+a_{101}b_{120}+2a_{102}a_{110}-a_{102}b_{101}+a_{110}b_{111}\\
&+a_{111}b_{110}+a_{120}b_{101}-2b_{101}b_{111}-2b_{102}b_{110}-a_{211}-b_{202}-b_{220})\sin^3\theta\\
&+r^2(2a_{100}a_{111}-a_{100}b_{102}+a_{100}b_{120}+2a_{101}a_{110}-a_{101}b_{101}-a_{102}b_{100}\\
&+a_{110}b_{110}+a_{120}b_{100}-2b_{100}b_{111}-2b_{101}b_{110}-a_{210}-b_{201})\sin^2\theta\\
&+r(2a_{100}a_{110}-a_{100}b_{101}-a_{101}b_{100}-2b_{100}b_{110}-b_{200})\sin\theta\\
&-b_{100}a_{100})\cos^2\theta+(r^4(a_{102}^2+a_{102}b_{111}+a_{111}b_{102}-b_{102}^2)\sin^5\theta\\
&+r^3(2a_{101}a_{102}+a_{101}b_{111}+a_{102}b_{110}+a_{110}b_{102}+a_{111}b_{101}-2b_{101}b_{102}\\
&-a_{202}-b_{211})\sin^4\theta+r^2(2a_{100}a_{102}+a_{100}b_{111}+a_{101}^2+a_{101}b_{110}\\
&+a_{110}b_{101}+a_{111}b_{100}-2b_{100}b_{102}-b_{101}^2-a_{201}-b_{210})\sin^3\theta\\
&+r(2a_{100}a_{101}+a_{100}b_{110}+a_{110}b_{100}-2b_{100}b_{101}-a_{200})\sin^2\theta\\
&+(a_{100}-b_{100})(a_{100}+b_{100})\sin\theta)\cos\theta+a_{102}b_{102}r^4\sin^6\theta\\
&+r^3(a_{101}b_{102}+a_{102}b_{101}-b_{202})\sin^5\theta+r^2(a_{100}b_{102}+a_{101}b_{101}\\
&+a_{102}b_{100}-b_{201})\sin^4\theta+r(a_{100}b_{101}+a_{101}b_{100}-b_{200})\sin^3\theta\\
&+a_{100}b_{100}\sin^2\theta\}/r.
\end{aligned}
$$

We write

$$
\tilde{h}(\theta,r)=r\sin\theta-r^3\cos^3\theta=\begin{cases} r \text{ if } \theta=\pi/2 \text{ or } -r \text{ if } \theta=3\pi/2,\\ \dfrac{\left(\tan^3\theta+\tan\theta-r^2\right)r\cos\theta}{1+\tan^2\theta} \text{ if } \theta\neq\pi/2,\ 3\pi/2.\end{cases}
$$

Solving $\tilde{h}(\theta,r)=0$ (i.e. $\tan^3\theta+\tan\theta-r^2=0$) yields $\theta=\theta_1(r)$ or $\theta=\theta_2(r)$, where $\theta_1\in[0,\pi/2)$ and

$$\theta_2:=\theta_1+\pi,\quad \theta_1:=\arctan(\varpi(r)),$$
$$\varpi:=\sqrt[3]{\frac{r^2}{2}+\sqrt{\frac{r^4}{4}+\frac{1}{27}}}-\sqrt[3]{-\frac{r^2}{2}+\sqrt{\frac{r^4}{4}+\frac{1}{27}}}. \tag{9}$$

Then the discontinuous set of system (8) is given by $\widetilde{\Sigma}:=\{(\theta_1(r),r):r>0\}\cup\{(\theta_2(r),r):r>0\}$. Moreover one can check that $\tilde{h}(r,\theta)>0$ (resp. <0) if $\theta\in(\theta_1,\theta_2)$ (resp. $\theta\in[0,\theta_1)\cup(\theta_2,2\pi]$). Computation shows that

$$\begin{aligned}
&\left\langle\nabla\tilde{h}(\theta_1(r),r),(1,P(\theta_1(r),r))\right\rangle\left\langle\nabla\tilde{h}(\theta_1(r),r),(1,Q(\theta_1(r),r))\right\rangle\\
&=\frac{r^2(3\varpi r^2+\varpi^2+1)^2}{(\varpi^2+1)^3}+\mathcal{O}(\varepsilon)>0,\\
&\left\langle\nabla\tilde{h}(\theta_2(r),r),(1,P(\theta_2(r),r))\right\rangle\left\langle\nabla\tilde{h}(\theta_2(r),r),(1,Q(\theta_2(r),r))\right\rangle\\
&=\frac{r^2(3\varpi r^2+\varpi^2+1)^2}{(\varpi^2+1)^3}+\mathcal{O}(\varepsilon)>0.
\end{aligned}$$

Hence the hypothesis (HC) below (6) holds for system (8). From (6) the averaged function of order 1 is

$$\begin{aligned}
f_1(r)&=\int_0^{2\pi}\frac{d\dot{r}}{d\varepsilon}\Big|_{\varepsilon=0}d\theta\\
&=\int_0^{\theta_1}Q_1(\theta,r)d\theta+\int_{\theta_1}^{\theta_2}P_1(\theta,r)d\theta+\int_{\theta_2}^{2\pi}Q_1(\theta,r)d\theta\\
&=-\{-4(2a_{120}+b_{111}+a_{102}-2c_{120}-c_{102}-d_{111})r^2\varpi^3\\
&\quad+12(b_{102}-d_{102})r^2\varpi^2-12(a_{120}-c_{120})r^2\varpi+4(a_{111}+b_{120}+2b_{102}\\
&\quad-c_{111}-d_{120}-2d_{102})r^2+3\pi(a_{110}+b_{101}+c_{110}+d_{101})(\varpi^2+1)^{3/2}r\\
&\quad-12(a_{100}-c_{100})\varpi^3+12(b_{100}-d_{100})\varpi^2-12(a_{100}-c_{100})\varpi\\
&\quad+12(b_{100}-d_{100})\}/\{6(\varpi^2+1)^{3/2}\},
\end{aligned}$$

where θ_i $(i=1,2)$ are given in (9). Introducing new variable $u=\sqrt{\varpi(r)}=r-(1/2)r^5+(11/8)r^9+O(|r|^{13})$, equivalently $r=u\sqrt{1+u^4}=u+(1/2)u^5-(1/8)u^9+O(|u|^{13})$ we reduce $f_1(r)$ to

$$\tilde{f}_1(u):=f_1(r(u))=\frac{k_1+k_2u+k_3u^2+k_4u^4+k_5u^5+k_6u^6+k_7u^8}{6\sqrt{1+u^4}},$$

where $k_5:=k_2$ and

$$\begin{aligned}
&k_1:=-12(b_{100}-d_{100}),\quad k_2:=-3\pi(c_{110}+d_{101}+a_{110}+b_{101}),\\
&k_3:=4(3a_{100}-3c_{100}-2b_{102}-a_{111}-b_{120}+2d_{102}+d_{120}+c_{111}),\\
&k_4:=12(a_{120}-c_{120}),\qquad k_6:=-12(b_{102}-d_{102}),\\
&k_7:=4(a_{102}+2a_{120}+b_{111}-c_{102}-2c_{120}-d_{111}).
\end{aligned}$$

Obviously the zeros of $\tilde{f}_1(u)$ are determined by its numerator, which is a polynomial of degree at most 8. From Theorem 3 we know that every simple zero of $\tilde{f}_1(u)$ on the interval $(0, +\infty)$ corresponds to a crossing limit cycle of system (2). We take

$$g_1 := 1,\ g_2 := u^2,\ g_3 := u^4,\ g_4 := u^6,\ g_5 := u^8,\ g_6 := u + u^5.$$

One can compute the six Wronskians

$$W_1(g_1) := 1,\ W_2(g_1, g_2) := 2u,\ W_3(g_1, g_2, g_3) := 16u^3,\ W_4(g_1, g_2, g_3, g_4) := 768u^6,$$
$$W_5(g_1, g_2, g_3, g_4, g_5) := 294912u^{10},\ W_6(g_1, g_2, g_3, g_4, g_5, g_6) := 4423680u^6(3u^4 + 7).$$

Note that none of the six Wronskians is zero for $u > 0$. It implies that the set of functions $\{g_1, g_2, g_3, g_4, g_5, g_6\}$ is an ECT-system on the interval $(0, +\infty)$. Then $\tilde{f}_1(u)$ has at most 5 simple zeros on the interval $(0, +\infty)$. Hence system (2) has at most 5 crossing limit cycles bifurcated from the center annulus. Furthermore one can compute that

$$\det \frac{\partial(k_1, k_2, k_3, k_4, k_6, k_7)}{\partial(a_{110}, a_{120}, a_{111}, a_{102}, b_{100}, b_{102})} = \det \begin{pmatrix} 0 & 0 & 0 & 0 & -12 & 0 \\ -3\pi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & -8 \\ 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 8 & 0 & 4 & 0 & 0 \end{pmatrix}$$
$$= 82944\pi \neq 0,$$

which implies that k_1, k_2, k_3, k_4, k_6 and k_7 are linearly independent with respect to $a_{110}, a_{120}, a_{111}, a_{102}, b_{100}$ and b_{102}. Thus we can choose parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} $(i = 1, 2$ and $0 \leq j + k \leq 2)$ such that system (2) has exactly $0, 1, 2, 3, 4$ or 5 crossing limit cycles. This completes the proof of Theorem 1.

Proof of Theorem 2. We compute the averaged function of order 2. Solving $\tilde{f}_1(u) \equiv 0$, equivalently $k_1 = \ldots = k_7 = 0$, yields the condition $\mathcal{Q}_1$ given in (3). Let

$$\Phi_1(r) := \langle \nabla h(\theta_1(r), r), (s, y_1(\theta_1(r), r)) \rangle, \quad \Phi_2(r) := \langle \nabla h(\theta_2(r), r), (s, y_1(\theta_2(r), r)) \rangle,$$

where θ_i $(i = 1, 2)$ are given in (9) and

$$y_1(\theta, r) := \int_0^\theta \frac{d\dot{r}}{d\varepsilon}\Big|_{(\varepsilon,\theta)=(0,\varphi)} d\varphi$$

with $\dot{r}$ in (8). Computations show that

$$\begin{aligned} \Phi_1(r) &= \Big\langle \nabla h\big(\theta_1(r), r\big), \Big(s, \int_0^{\theta_1} Q_1(t, r)dt\Big) \Big\rangle \\ &= s(3r^3 \sin\theta_1 \cos^2\theta_1 + r\cos\theta_1) + \frac{(\sin\theta_1 - 3r^2\cos^3\theta_1)\Psi_1(r)}{12}, \\ \Phi_2(r) &= \Big\langle \nabla h\big(\pi + \theta_1, r\big), \Big(s, \int_0^{\theta_1} Q_1(t, r)dt + \int_{\theta_1}^{\pi+\theta_1} P_1(t, r)dt\Big) \Big\rangle \\ &= -s(3r^3 \sin\theta_1 \cos^2\theta_1 + r\cos\theta_1) - \frac{(\sin\theta_1 - 3r^2\cos^3\theta_1)\big(\Psi_1(r) + \Psi_2(r)\big)}{12}, \end{aligned}$$

where

$$\begin{aligned} \Psi_1(r) := &- 12d_{100} - 3(c_{101} + d_{110})r - 12c_{100}\sin\theta_1 + 3(c_{101} + d_{110})r\cos(2\theta_1) \\ &- 3(c_{102} + 3c_{120} + d_{111})r^2\sin\theta_1 + (c_{102} - c_{120} + d_{111})r^2\sin(3\theta_1) \\ &- 6(c_{110} + d_{101})r\theta_1 - 3(c_{110} - d_{101})r\sin(2\theta_1) + 12d_{100}\cos\theta_1 \end{aligned}$$

$$\begin{aligned}
&+3(c_{111}+3d_{102}+d_{120})r^2\cos\theta_1+(c_{111}-d_{102}+d_{120})r^2\cos(3\theta_1)\\
&-4(c_{111}+2d_{102}+d_{120})r^2,\\
\Psi_2(r):=&-6\pi(a_{110}+b_{101})r+24a_{100}\sin\theta_1+6(a_{102}+3a_{120}+b_{111})r^2\sin\theta_1\\
&-2(a_{102}-a_{120}+b_{111})r^2\sin(3\theta_1)-6(a_{111}+3b_{102}+b_{120})r^2\cos\theta_1\\
&-2(a_{111}-b_{102}+b_{120})r^2\cos(3\theta_1)-24b_{100}\cos\theta_1.
\end{aligned}$$

In order to ensure that $\Phi_i=0$ if and only if $s=0$, we need to eliminate Ψ_i for $i=1,2$, equivalently that all coefficients of Ψ_i are equal to zero, from which we obtain

$$\begin{aligned}
&a_{100}=a_{120}=b_{100}=b_{102}=c_{100}=c_{110}=c_{120}=d_{100}=d_{101}=d_{102}=0,\\
&a_{102}=-b_{111},\quad a_{110}=-b_{101},\quad a_{111}=-b_{120},\\
&c_{101}=-d_{110},\quad c_{111}=-d_{120},\quad c_{102}=-d_{111}.
\end{aligned}\tag{10}$$

Under (10) the assumption $(Hb2)$ in Theorem 4 holds for system (8). Taking the intersection between $\mathcal{Q}_1$ and (10) yields the condition $\mathcal{Q}_2$ given in (4), where $\mathcal{Q}_2$ is actually (10) because (10) is a subset of $\mathcal{Q}_1$. From (6) the averaged function of order 2 is

$$\begin{aligned}
f_2(r)=&\int_0^{2\pi}\left\{\left(\frac{\partial^2\dot r}{\partial r\partial\varepsilon}\Big|_{\varepsilon=0}\int_0^{\theta}\frac{\partial\dot r}{\partial\varepsilon}\Big|_{(\varepsilon,\theta)=(0,\varphi)}d\varphi\right)+\frac{1}{2}\frac{\partial^2\dot r}{\partial\varepsilon^2}\Big|_{\varepsilon=0}\right\}d\theta\\
=&\int_0^{\theta_1}\left\{\frac{\partial Q_1(\theta,r)}{\partial r}\int_0^{\theta}Q_1(\varphi,r)d\varphi+Q_2(\theta,r)\right\}d\theta\\
&+\int_{\theta_1}^{\theta_2}\left\{\frac{\partial P_1(\theta,r)}{\partial r}\left(\int_0^{\theta_1}Q_1(\varphi,r)d\varphi+\int_{\theta_1}^{\theta}P_1(\varphi,r)d\varphi\right)+P_2(\theta,r)\right\}d\theta\\
&+\int_{\theta_2}^{2\pi}\left\{\frac{\partial Q_1(\theta,r)}{\partial r}\left(\int_0^{\theta_1}Q_1(\varphi,r)d\varphi+\int_{\theta_1}^{\theta_2}P_1(\varphi,r)d\varphi+\int_{\theta_2}^{\theta}Q_1(\varphi,r)d\varphi\right)\right.\\
&\left.\qquad+Q_2(\theta,r)\right\}d\theta.
\end{aligned}$$

Using the change $u=\sqrt{\varpi(r)}$ (i.e. $r=u\sqrt{1+u^4}$) we reduce $f_2(r)$ to

$$\tilde f_2(u):=f_2(r(u))=\frac{\tilde k_1+\tilde k_2u^2+\tilde k_3u^4+\tilde k_4u^6+\tilde k_5u^8+\tilde k_6(u+u^5)}{6\sqrt{1+u^4}},$$

where

$$\begin{aligned}
&\tilde k_1:=12(d_{200}-b_{200}),\\
&\tilde k_2:=4(3a_{200}-a_{211}-b_{101}b_{111}-a_{101}b_{120}-b_{110}b_{120}-2b_{202}-b_{220}-3c_{200}\\
&\qquad+c_{211}+2d_{202}+d_{220}),\\
&\tilde k_3:=12(a_{220}-b_{101}b_{120}-c_{220}),\ \tilde k_4:=-12(b_{101}b_{111}+b_{202}-d_{202}),\\
&\tilde k_5:=4(a_{202}+2a_{220}+a_{101}b_{111}+b_{110}b_{111}-b_{101}b_{120}+b_{211}-c_{202}-2c_{220}-d_{211}),\\
&\tilde k_2:=-3\pi(a_{210}+b_{201}+c_{210}+d_{201}).
\end{aligned}$$

Note that $\tilde f_2(u)$ has the same form as $\tilde f_1(u)$. Combining that $\{g_1,\dots,g_6\}$ is an ECT-system, which is proven in the proof of Theorem 1, we obtain $\tilde f_2(u)$ has at most 5 simple zeros. Furthermore one can compute the determinant

$$\det \frac{\partial(\tilde{k}_1,\tilde{k}_2,\tilde{k}_3,\tilde{k}_4,\tilde{k}_5,\tilde{k}_6)}{\partial(a_{200},a_{210},a_{202},a_{220},d_{200},d_{202})} = \det \begin{pmatrix} 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & -3\pi & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 4 & 8 & 0 & 0 \end{pmatrix} = 248832\pi \neq 0,$$

implying that $\tilde{k}_i$ $(i = 1, \ldots, 6)$ are linearly independent. Hence there exist parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} $(i = 1, 2$ and $0 \le j + k \le 2)$ such that system (2) has exactly $0, 1, 2, 3, 4$ or 5 limit cycles. This completes the proof of Theorem 2.

Acknowledgements

The first author is partially supported by the grant CSC #202106240110 from the P. R. China. The second author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

Data availability

This paper has no data.

References

[1] K. d. S. Andrade, O. A. R. Cespedes, D. R. Cruz, D. D. Novaes, Higher order Melnikov analysis for planar piecewise linear vector fields with nonlinear switching curve, *J. Differential Equations* **287** (2021), 1–36.

[2] J. R. Bastos, C. A. Buzzi, J. Llibre, D. D. Novaes, Melnikov analysis in nonsmooth differential systems with nonlinear switching manifold, *J. Differential Equations* **267** (2019), 3748–3767.

[3] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, *Piecewise-Smooth Dynamical Systems, Theory and Applications*, Springer-Verlag, London, 2008.

[4] F. Browder, Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc.* **9** (1983), 1–39.

[5] A. Buică, L. Llibre, Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **128** (2004), 7–22.

[6] C. A. Buzzi, C. Pessoa, J. Torregrosa, Piecewise linear perturbations of a linear center, *Discrete Contin. Dyn. Syst.* **9** (2013), 3915–3936.

[7] P. T. Cardin, J. Torregrosa, Limit cycles in planar piecewise linear differential systems with nonregular separation line, *Physica D* **337** (2016), 67–82.

[8] X. Chen, J. Llibre, W. Zhang, Averaging approach to cyclicity of Hopf bifurcation in planar linear-quadratic polynomial discontinuous differential systems, *Discrete Contin. Dyn. Syst. Ser. B* **22** (2017), 3953–3965.

[9] X. Chen, J. Llibre, W. Zhang, Cyclicity of $(1, 3)$-switching FF type equilibria, *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019), 6541–6552.

[10] X. Chen, V. G. Romanovski, W. Zhang, Degenerate Hopf bifurcations in a family of FF-type switching systems, *J. Math. Anal. Appl.* **432** (2015), 1058–1076.

[11] A. F. Filippov, *Differential Equation with Discontinuous Right-Hand Sides*, Kluwer Academic, Amsterdam, 1988.

[12] J. Giné, M. Grau, J. Llibre, Averaging theory at any order for computing periodic orbits, *Phys. D* **250** (2013), 58–65.

[13] M. Han, On the maximum number of periodic solutions of piecewise smooth periodic equations by average method, *J. Appl. Anal. Comput.* **7** (2017), 788–794.

[14] M. Han, V. G. Romanovski, X. Zhang, Equivalence of the Melnikov function method and the averaging method, *Qual. Theory Dyn. Syst.* **15** (2016), 471–479.

[15] M. Han, L. Sheng, Bifurcation of limit cycles in piecewise smooth systems via Melnikov function, *J. Appl. Anal. Comput.* **5** (2015), 809–815.

[16] M. Han, W. Zhang, On Hopf bifurcation in non-smooth planar systems, *J. Differential Equations* **248** (2010), 2399–2416.

[17] J. Itikawa, J. Llibre, D. D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Rev. Mat. Iberoam.* **33** (2017), 1247–1265

[18] S. J. Karlin, W. J. Studden, *Tchebycheff systems: With Applications in Analysis and Statistics*, Pure Appl. Math., Interscience Publishers, New York, London, Sidney, 1966.

[19] T. Li, J. Llibre, Limit cycles in piecewise polynomial systems allowing a non-regular switching boundary, *Physica D* **419** (2021), 132855.

[20] J. Llibre, A. C. Mereu, D. D. Novaes, Averaging theory for discontinuous piecewise differential systems, *J. Differential Equations* **258** (2015), 4007–4032.

[21] J. Llibre, D. D. Novaes, M. A. Teixeira, Higher order averaging theory for finding periodic solutions via Brouwer degree, *Nonlinearity* **27** (2014), 563–583.

[22] J. Llibre, D. D. Novaes, M. A. Teixeira, On the birth of limit cycles for non-smooth dynamical systems, *Bull. Sci. math.* **139** (2015), 229–244.

[23] J. Llibre, E. Ponce, Three nested limit cycles in discontinuous piecewise linear differential systems with two zones, *Dynam. Contin. Discrete Impuls. Systems. Ser. B Appl. Algorithms* **19** (2012), 325–335.

[24] J. Llibre, Y. Tang, Limit cycles of discontinuous piecewise quadratic and cubic polynomial perturbations of a linear center, *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019), 1769–1784.

[25] J. Llibre, X. Zhang, Limit cycles for discontinuous planar piecewise linear differential systems separated by one straight line and having a center, *J. Math. Anal. Appl.* **467** (2018), 537–549.

[26] N. G. Lloyd, *Degree Theory*, Cambridge Tracts in Mathematics, vol. **73**, Cambridge University Press, Cambridge-New York-Melbourne, 1978.

[27] O. Makarenkov, J. S. W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey, *Phys. D* **241** (2012), 1826–1844.

[28] J. A. Sanders, F. Verhulst, J. Murdock, *Averaging Method in Nonlinear Dynamical Systems*, Appl. Math. Sci., vol. **59**, Springer, New York, 2007.

[29] D. J. W. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems*, World Scientific Series on Nonlinear Science A, vol **69**, World Scientific, Singapore, 2010.

[30] L. Wei, X. Zhang, Averaging theory of arbitrary order for piecewise smooth differential systems and its application, *J. Dynam. Differential Equations* **30** (2018), 55–79.

[1] School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China

Email address: `mathgzf@gmail.com`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra 08193, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat