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FLOW CURVATURE MANIFOLD AND ENERGY OF GENERALIZED LIÉNARD SYSTEMS

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Abstract. In his famous book entitled *Theory of Oscillations*, Nicolas Minorsky wrote: "*each time the system absorbs energy the curvature of its trajectory decreases* and *vice versa*". By using the *Flow Curvature Method*, we establish that, in the ε-vicinity of the *slow invariant manifold* of generalized Liénard systems, the *curvature of trajectory curve* increases while the *energy* of such systems decreases. Hence, we prove Minorsky's statement for the generalized Liénard systems. These results are then illustrated with the classical Van der Pol and generalized Liénard *singularly perturbed systems.*

1. Introduction

At the end of the 1930s, a general equation of *self-sustained oscillations* (1) was stated by the French engineer Alfred Liénard [25]. It encompassed the prototypical equation of the Dutch physicist Balthasar Van der Pol [32] modelling the so-called *relaxation oscillations*[1].

$$\frac{d^2x}{dt^2} + \omega f(x) \frac{dx}{dt} + \omega^2 x = 0. \tag{1}$$

Less than fifteen years later, a more general form was provided by the American mathematicians Norman Levinson and his former student Oliver K. Smith [23]:

$$\frac{d^2x}{dt^2} + \mu f(x) \frac{dx}{dt} + g(x) = 0. \tag{2}$$

At that time, the classical geometric theory of differential equations developed originally by Andronov [1], Tikhonov [31] and Levinson [24] stated that such *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly, and toward which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. So, these manifolds have been called asymptotically stable (or unstable) *slow invariant manifolds.* Their *local invariance* has then been stated according to Fenichel [4, 5, 6, 7] theory[2] for the *persistence of normally hyperbolic invariant manifolds.*

Key words and phrases. Generalized Liénard systems, singularly perturbed systems, Flow Curvature Method.

[1]For more details see J.-M. Ginoux [12].

[2]The theory of invariant manifolds for an ordinary differential equation is based on the work of Hirsch, *et al.* [15]

During the last century, various methods have been developed to approximate the *slow invariant manifold* equation in the form of an asymptotic expansion in power of ε. The seminal works of Wasow [33], Cole [3], O'Malley [27, 28] and Fenichel [4, 5, 6, 7] to name but a few, gave rise to the so-called *Geometric Singular Perturbation Method*. Fifteen years ago, a new approach of n-dimensional singularly perturbed dynamical systems of ordinary differential equations with two time scales, called *Flow Curvature Method* has been developed [10]. This method gives an implicit non intrinsic equation, because it depends on the euclidean metric. A 'kinetic energy metric' has been introduced in [19] for chemical kinetic systems and an extremum principle for computing *slow invariant manifolds* has been formulated [20, 21] which can be viewed as minimum curvature geodesics. In [14] a curvature-based differential geometry formulation for the *slow manifold* problem has been used for the purpose of a coordinate-independent formulation of the invariance equation. In his famous book entitled *Theory of Oscillations*, the Russian mathematician Nicolas Minorsky [29] wrote:

> "*each time the system absorbs energy the curvature of its trajectory decreases* and *vice versa* when the energy is supplied by the system (e. g. braking) the curvature increases."

Thus, according to Minorsky, *energy* and *curvature of the trajectory* are linked by a relationship that he unfortunately didn't give. So, the aim of this work is to prove this statement in the ε-vicinity of the *slow invariant manifold* of generalized Liénard systems and to establish this relationship for such systems. The paper is organized as follows. In section 2, we present Liénard's assumptions for which the generalized Liénard systems has a *unique stable limit cycle* and so, a *slow invariant manifold*. In section 3, we prove Minorsky's statement for the generalized Liénard systems. In section 4, we exemplify these results with the classical Van der Pol and generalized Liénard *singularly perturbed systems*. Perspectives to this work are presented in the discussion.

2. Generalized Liénard systems and its slow invariant manifold

2.1. Generalized Liénard systems.

Starting from the generalized Liénard equation (2) which is a paradigm for *self-sustained oscillations* and by posing: $t \to \mu t$ and $\mu = 1/\sqrt{\varepsilon}$, we have:

$$\begin{aligned} \varepsilon \dot{x} &= y - F(x), \\ \dot{y} &= -g(x). \end{aligned} \tag{3}$$

where

$$G(x) = \int_0^x g(s)\,\mathrm{d}s > 0 \quad \Longrightarrow \quad G'(x) = g(x)$$

and,

$$F(x) = \int_0^x f(s)\,\mathrm{d}s > 0 \quad \Longrightarrow \quad F'(x) = f(x).$$

Moreover, the generalized Liénard systems (3) is a well-known planar *singularly perturbed dynamical system* [26].

According to Lefschetz [22], under the following assumptions:

I. $f(x)$ is even, $g(x)$ is odd, $xg(x) > 0$ for all $x \neq 0$; $f(0) < 0$;
II. $f(x)$ and $g(x)$ are continuous for all x; $g(x)$ satisfies Lipschitz condition for all x;
III. $F(x) \to \pm\infty$ with x;
IV. $F(x)$ has a single positive zero $x = a$ and is monotone increasing for $x \geqslant a$,

the generalized Liénard systems (3) has a *unique stable limit cycle* and so, possesses a *slow invariant manifold*.

According to the previous assumptions I - II, $g(x)$ is odd and continuous, so we have $g(0) = 0$. Thus, at the crossings with the y-axis the tangents to the *trajectory curves* are horizontal ($\dot{y}/\dot{x} = 0$ since $g(0) = 0$), and at the crossings with the curve $y = F(x)$ they are vertical ($\dot{y}/\dot{x} = \infty$ since $y - F(x) = 0$). Moreover, since this gradient ($\dot{y}/\dot{x}$) is negative below the *critical manifold* ($y = F(x)$) and in the right half part of the xy-plane, i.e., in the ε-vicinity of the *slow invariant manifold*, the *trajectory curve* cannot leave its neighborhood and any tendency for it to move away from it would be counteracted by a rapid growth in magnitude of this negative gradient. Then, according to Lefschetz [22]:

"We see from (3) that with increasing time:

- $x(t)$ increases above the *critical manifold*,
- $x(t)$ decreases below the *critical manifold*,
- $y(t)$ increases to the left of the y axis,
- $y(t)$ decreases to the right of the y axis."

Reciprocally, if $x(t)$ decreases and $y(t)$ decreases below the *critical manifold* for $x \geq a$ and in the right half part of the xy-plane, it follows that we have:

$$\begin{aligned} &\dot{x}(t) < 0 \\ &\dot{y}(t) < 0. \end{aligned} \tag{4}$$

2.2. **Slow invariant manifold of the generalized Liénard systems.** By applying the *Geometric Singular Perturbation Method* to the generalized Liénard systems (3), we can find all functions $Y_n(x)$ involved in the perturbation expansion allowing to build an approximation of the *slow invariant manifold* of such systems up to suitable order in ε. We will show in the next Section that the suitable order must be equal at least to three. So, we deduce for the generalized Liénard systems (3) that at:

order ε^0:

$$Y_0(x) = F(x) \tag{5}$$

where $y = Y_0(x) = F(x)$ is called the *critical manifold*.

order ε^1:

$$Y_1(x) = -\frac{g(x)}{f(x)} \tag{6}$$

where, we recall that, $f(x) = F'(x)$.

order ε^2:

$$Y_2(x) = \frac{g(x)\Delta(x)}{f^4(x)} \tag{7}$$

where

$$\Delta(x) = f'(x)g(x) - f(x)g'(x) \tag{8}$$

order ε^3:

$$Y_3(x) = -\frac{g(x)P(x)}{f^7(x)} \tag{9}$$

where

$$P(x) = 5\Delta^2(x) + 3f(x)g'(x)\Delta(x) - f(x)g(x)\Delta'(x). \tag{10}$$

Thus, the approximation of the *slow invariant manifold* up to order three in ε of the generalized Liénard systems (3) reads

$$y = F(x) - \varepsilon\frac{g(x)}{f(x)} + \varepsilon^2\frac{g(x)\Delta(x)}{f^4(x)} - \varepsilon^3\frac{g(x)P(x)}{f^7(x)} + O\left(\varepsilon^4\right) \tag{11}$$

Remark 1. *According to assumptions I - IV, it follows from (11) that* $y - F(x) < 0$ *in the right half part of the* xy*-plane and below the critical manifold for* $x \geq a$*. This implies that such approximation of the slow invariant manifold (11) is below the critical manifold (5).*

3. Minorsky's statement

In order to establish Minorsky's statement for the generalized Liénard systems (3), we introduce the following propositions.

Proposition 2. *In the right half part of the xy-plane and below the critical manifold for $x \geq a$, the slow invariant manifold, i.e., the curvature of the flow of the generalized Liénard systems* (3) *is positive provided that $g'(x) \geqslant 0$, $\Delta > 0$ and under the assumptions $(I - IV)$.*

Proof. According to the *Flow Curvature Method* developed by Ginoux *et al.* [8, 9, 10, 11, 13], the *slow invariant manifold*, i.e., the *curvature of the flow* of the generalized Liénard systems (3) reads $\phi(x, y, \varepsilon) = \ddot{x}\dot{y} - \ddot{y}\dot{x}$. So, to state Proposition 2, we have to prove that $\ddot{x}\dot{y} > 0$ and $\ddot{y}\dot{x} < 0$.

By replacing (11) in the right hand side of the first equation of (3) and dividing by ε gives:

$$\dot{x} = -\frac{g(x)}{f(x)} + \varepsilon \frac{g(x)\Delta(x)}{f^4(x)} - \varepsilon^2 \frac{g(x)P(x)}{f^7(x)} + O\left(\varepsilon^3\right) \tag{12}$$

From (12), we deduce that:

$$f(x)\dot{x} + g(x) = \varepsilon \frac{g(x)\Delta(x)}{f^3(x)} - \varepsilon^2 \frac{g(x)P(x)}{f^6(x)} + O\left(\varepsilon^3\right) \tag{13}$$

The time derivative of the first equation of (3) reads:

$$\varepsilon\ddot{x} = -\left(g(x) + f(x)\dot{x}\right). \tag{14}$$

By replacing the right hand side of (14) by (13) provides:

$$\ddot{x} = -\frac{g(x)\Delta(x)}{f^3(x)} + \varepsilon \frac{g(x)P(x)}{f^6(x)} + O\left(\varepsilon^2\right) \tag{15}$$

It follows that when $\varepsilon \to 0$, i.e., when the *slow invariant manifold* (11) tends to the *critical manifold* (5), we have:

$$\ddot{x} = -\frac{g(x)\Delta(x)}{f^3(x)} < 0 \tag{16}$$

The time derivative of the second equation of (3) reads:

$$\ddot{y} = -g'(x)\dot{x}. \tag{17}$$

By replacing the right hand side of (17) by (12) provides:

$$\ddot{y} = g'(x)\frac{g(x)}{f(x)} - \varepsilon g'(x)\frac{g(x)\Delta(x)}{f^4(x)} + \varepsilon^2 g'(x)\frac{g(x)P(x)}{f^7(x)} + O\left(\varepsilon^3\right) \tag{18}$$

It follows that when $\varepsilon \to 0$, i.e., when the *slow invariant manifold* (11) tends to the *critical manifold* (5), we have:

$$\ddot{y} = g'(x) \frac{g(x)}{f(x)} \geq 0. \tag{19}$$

Thus we have:

$$\begin{aligned} \ddot{x}(t) &< 0 \\ \ddot{y}(t) &\geq 0. \end{aligned} \tag{20}$$

Taking into account Eqs. (4), the proof of Proposition 2 is stated.

□

Remark 3. *Let's notice that $\Delta = 0$ corresponds to the case $f(x) = kg(x)$ what is inconsistent with assumption I for which $f(x)$ is even and $g(x)$ is odd.*

Now, to prove that the *curvature of the flow* increases, let's state the following proposition:

Proposition 4. *In the right half part of the xy-plane and below the critical manifold for $x \geq a$, the time derivative of the slow invariant manifold, i.e., of the curvature of the flow of the generalized Liénard systems* (3) *is positive provided that $g'(x) \geqslant 0$, $g''(x) \geqslant 0$, $\Delta > 0$ and under the assumptions $(I - IV)$.*

Proof. The time derivative of the *slow invariant manifold* $\phi(x, y, \varepsilon) = \ddot{x}\dot{y} - \ddot{y}\dot{x}$ reads:

$$\frac{d\phi}{dt} = \dddot{x}\dot{y} - \dddot{y}\dot{x}. \tag{21}$$

So, to state Proposition 4, we have to prove that $\dddot{x}\dot{y} > 0$ and $\dddot{y}\dot{x} < 0$.

The second time derivative of the first equation of Eq. (3) reads:

$$\varepsilon^2 \dddot{x} = -\varepsilon \dot{x} \left(g'(x) + f'(x) \dot{x} \right) + f(x) \left(g(x) + f(x) \dot{x} \right). \tag{22}$$

By replacing the right hand side of (22) by Eqs. (12) and (13) and by dividing by ε^2 provides:

$$\dddot{x} = -\frac{g(x)}{f^5(x)} Q(x) + O(\varepsilon). \tag{23}$$

where $Q(x) = P(x) - \Delta^2(x) - g(x) f'(x) \Delta(x)$. By taking into account the expression (10) of $P(x)$, we obtain:

$$Q(x) = 3\Delta^2(x) + 2f(x) g'(x) \Delta(x) - f(x) g(x) \Delta'(x). \tag{24}$$

Let's prove that $Q(x)$ is positive. To this aim, we use Assumption I, according to which $f(x)$ is even and $g(x)$ is odd. It follows that $f'(x)$ is odd and $g'(x)$ is even. From this obvious result, it is easy to prove on the one hand that $\Delta(x)$ is even and so, that $\Delta'(x)$ is odd and, on the other hand that $Q(x)$ is even. Then, by using a famous theorem according to which *any continuous odd function necessary passes through the origin*, we deduce that $g(0) = 0$ since $g(x)$ is odd. So, we have:

$$\Delta(0) = f'(0)g(0) - f(0)g'(0) = -f(0)g'(0).$$

Thus, let's show that $Q(0)$ is positive.

$$Q(0) = 3\Delta^2(0) + 2f(0)\,g'(0)\,\Delta(0) - f(0)\,g(0) = 3\Delta^2(0) + 2f(0)\,g'(0)\,\Delta(0)\,.$$

So, we have:

$$Q(0) = f^2(0)\,g'^2(0) > 0.$$

Thus, since $Q(x)$ is even and only consists of continuous positive functions, it follows that $Q(x)$ is positive. So, when $\varepsilon \to 0$, i.e., when the *slow invariant manifold* (11) tends to the *critical manifold* (5), $\dddot{x} < 0$.

The second time derivative of the first equation of Eq. (3) reads:

$$\varepsilon\dddot{y} = \varepsilon g''(x)\,\dot{x}^2 + g'(x)\,(g(x) + f(x)\,\dot{x})\,. \tag{25}$$

By replacing the right hand side of (25) by Eqs. (12) and (13) and by dividing by ε provides:

$$\dddot{y} = \frac{g(x)}{f^3(x)}\left[f(x)\,g(x)\,g''(x) + g'(x)\,\Delta(x)\right] + O(\varepsilon)\,. \tag{26}$$

Since we have made the assumptions that $g'(x) \geqslant 0$, $g''(x) \geqslant 0$, $\Delta(x) > 0$, it follows that when $\varepsilon \to 0$, i.e., when the *slow invariant manifold* (11) tends to the *critical manifold* (5), $\dddot{y} \geq 0$. Thus we have:

$$\begin{aligned} &\dddot{x}(t) < 0 \\ &\dddot{y}(t) \geq 0. \end{aligned} \tag{27}$$

Taking into account Eqs. (21), the proof of Proposition 4 is stated.

□

Now, let's prove Minorsky's statement for the generalized Liénard systems (3).

Proposition 5. *"each time the system absorbs energy the curvature of its trajectory decreases and vice versa when the energy is supplied by the system (e. g. braking) the curvature increases."*

Proof. According to Bergé *et al.* [2] a classical way to express the variation of energy according to time in generalized Liénard equation (2) (in which we have posed $t \to \mu t$ and $\mu = 1/\sqrt{\varepsilon}$) is to multiply this equation by $\dot{x}(t)$. By doing that, we obtain:

$$\varepsilon\dot{x}\ddot{x} + f(x)\,\dot{x}^2 + g(x)\,\dot{x} = 0. \tag{28}$$

By taking $G'(x) = g(x)$, we find that:

$$\frac{d}{dt}\left(\varepsilon\frac{\dot{x}^2}{2}+G(x)\right)=-f(x)\dot{x}^2 \tag{29}$$

But, from assumption IV, it follows that $F(x)$ is monotone increasing for $x \geqslant a$. So, $F'(x)=f(x)>0$. Let's consider that the *energy* reads:

$$E=\frac{\varepsilon\dot{x}^2}{2}+G(x), \tag{30}$$

where, according to Lefschetz [22], "in the "spring" interpretation $\varepsilon\dot{x}^2/2$ is the kinetic energy and $G(x)$ is the potential energy[3]". So, we have:

$$\frac{dE}{dt}=-f(x)\dot{x}^2<0. \tag{31}$$

Thus, in the right half part of the xy-plane and below the *critical manifold* for $x \geq a$, the energy E of the generalized Liénard systems (3) decreases while from Proposition 7, it follows that the *slow invariant manifold*, i.e. the *curvature of the flow* $\phi(x,y,\varepsilon)$ of the generalized Liénard systems (3) increases. So Minorsky's statement is proved.

□

4. Applications

In this last section we apply the results established in this work to the classical Van der Pol [32] and to the generalized Liénard [26] *planar singularly perturbed systems.*

4.1. **Van der Pol *singularly perturbed system*.**

In his original publication of 1926, Balthasar Van der Pol [32] provided the following prototypic ordinary differential equation for modeling the relaxation oscillations:

$$\frac{d^2x}{dt^2}+\mu\left(x^2-1\right)\frac{dx}{dt}+x=0. \tag{32}$$

By posing: $t\to\mu t$ and $\mu=1/\sqrt{\varepsilon}$, such equation (**??**) can be written as:

$$\begin{aligned}\varepsilon\dot{x}&=y-\left(\frac{x^3}{3}-x\right),\\ \dot{y}&=-x.\end{aligned} \tag{33}$$

Thus, we have: $F(x)=\dfrac{x^3}{3}-x$, $f(x)=F'(x)=x^2-1$, $g(x)=x$ and $g'(x)=1$. This gives: $a=\sqrt{3}$ and $\alpha=1$. From (31) it follows that:

$$\frac{dE}{dt}=-\left(x^2-1\right)\dot{x}^2. \tag{34}$$

The function $x^2-1 \geqslant 0$ for $x \in]-\infty,-1]\bigcup[+1,+\infty[$, and so $dE/dt<0$ within this interval which contains the *curvature of the flow curvature* $\phi(x,y,\varepsilon)$,

[3] In fact Lefschetz [22] provided for the energy E a different expression from the previous one (30). However, it will established in the Appendix that they are exactly the same.

i.e., the first order approximation in ε of the *slow invariant manifold* of Van der Pol *singularly perturbed system* (33). But, since $g'(x) = 1$, this *curvature of the flow* reads:

$$\phi(x, y, \varepsilon) = \ddot{x}\dot{y} + \dot{x}^2. \tag{35}$$

For the Van der Pol *singularly perturbed system* (33) we have:

$$\begin{aligned} &g'(x) = 1 > 0, \quad g''(x) = 0 \geq 0, \quad \Delta(x) = x^2 + 1 > 0, \quad \Delta'(x) = 2x, \\ &P(x) = 2(3x^4 + 6x^2 + 1) \geq 0, \quad Q(x) = 3x^4 + 8x^2 + 1 \geq 0, \end{aligned} \tag{36}$$

Thus, it follows from Proposition 2, 4 & 5, that in the right half part of the xy-plane and below the *critical manifold* for $x \geq a$, the *curvature of the flow* of system (33) increases while its *energy* decreases. On Fig. 1, we have represented, the *trajectory curve*, integral of Van der Pol *singularly perturbed system* (33), i.e. the *limit cycle* (in red), the *critical manifold* $y - F(x) = 0$ with $F(x) = x^3/3 - x$, i.e. the zero order approximation in ε of the *slow invariant manifold* equation of this system (in green), the *curvature of the flow* of this system (in blue) and the roots of the equation $f(x) = x^2 - 1 = 0$ (in dot dashed black). We observe on Fig. 1, that the *curvature of the flow* which represents the first order approximation in ε of the *slow invariant manifold* of Van der Pol *singularly perturbed system* (33) is below the *critical manifold* for $x \geq 1$ and so for $x \geq \sqrt{3}$. Numerical investigations have enabled to show that in the right half part of the xy-plane and for $x \geq 1$, both *critical manifold* and *slow invariant manifold*, i.e., *curvature of the flow* belong a time interval of duration less than 1 unit of time. On Fig. 2 and 3, we have plotted both *energy* (34) and *curvature of the flow* (35) variations for $t \in [97.3, 98.15]$. We observe on these figures (2 & 3), that *energy* is decreasing while *curvature of the flow* is increasing.

4.2. **Generalized Liénard *singularly perturbed system*.** According to Llibre *et al.* [26], an example of generalized Liénard system can be written as follows:

$$\begin{aligned} \varepsilon\dot{x} &= y - \left(\frac{x^5}{5} + \frac{x^3}{3} - x\right), \\ \dot{y} &= -\left(\frac{x^3}{3} + x\right). \end{aligned} \tag{37}$$

Thus we have: $F(x) = \dfrac{x^5}{5} + \dfrac{x^3}{3} - x$, $f(x) = F'(x) = x^4 + x^2 - 1$, $g(x) = \dfrac{x^3}{3} + x$, $g'(x) = x^2 + 1$. From $F(x) = 0$ and $f(x) = 0$ we find that: $a = \sqrt{\dfrac{\sqrt{205} - 5}{6}}$ and $\alpha = \sqrt{\dfrac{\sqrt{5} - 1}{2}}$. From (30) it follows that:

$$\frac{dE}{dt} = -\left(x^4 + x^2 - 1\right)\dot{x}^2. \tag{38}$$

The function $x^4 + x^2 - 1 \geqslant 0$ for $x \in]-\infty, -\alpha] \bigcup [+\alpha, +\infty[$ and so, $dE/dt < 0$ within this interval which contains the *curvature of the flow* $\phi(x, y, \varepsilon)$, i.e., the first order approximation in ε of the *slow invariant manifold* of generalized Liénard

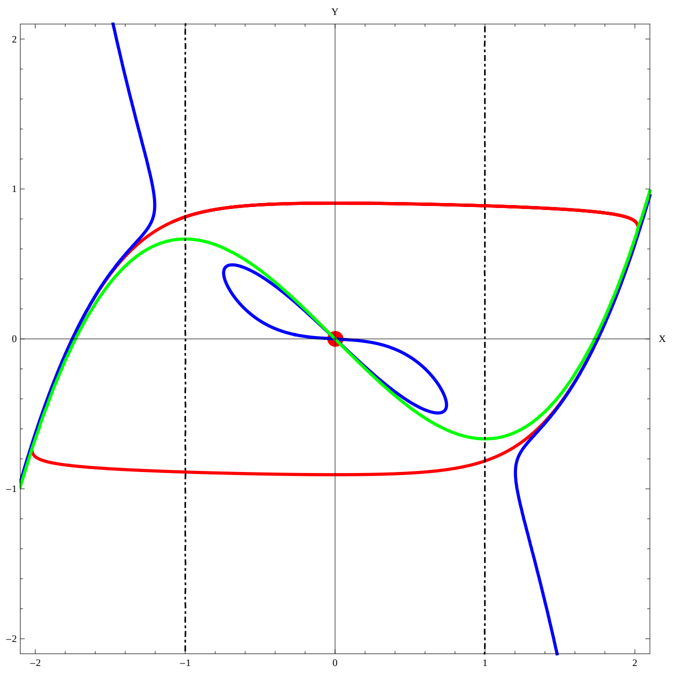

FIGURE 1. Limit cycle and critical manifold of Van der Pol system (33).

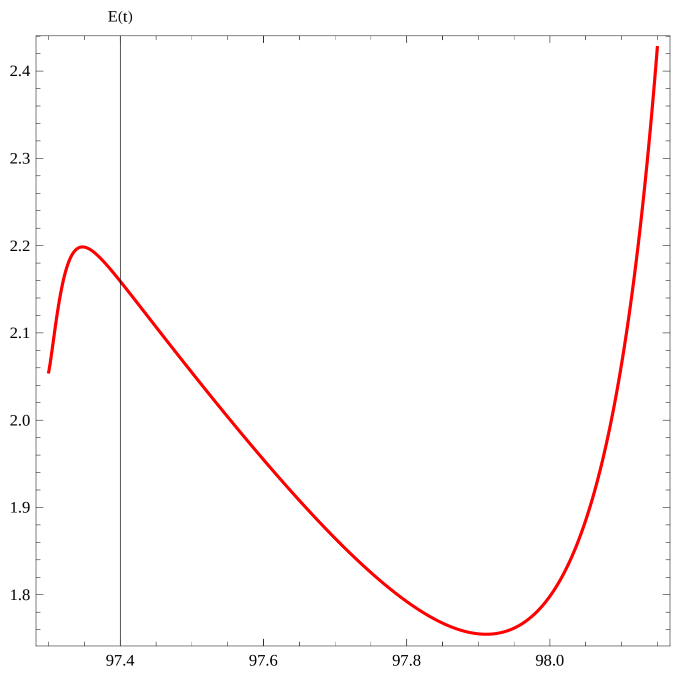

FIGURE 2. Energy of Van der Pol system (33) for $t \in [97.3, 98.15]$.

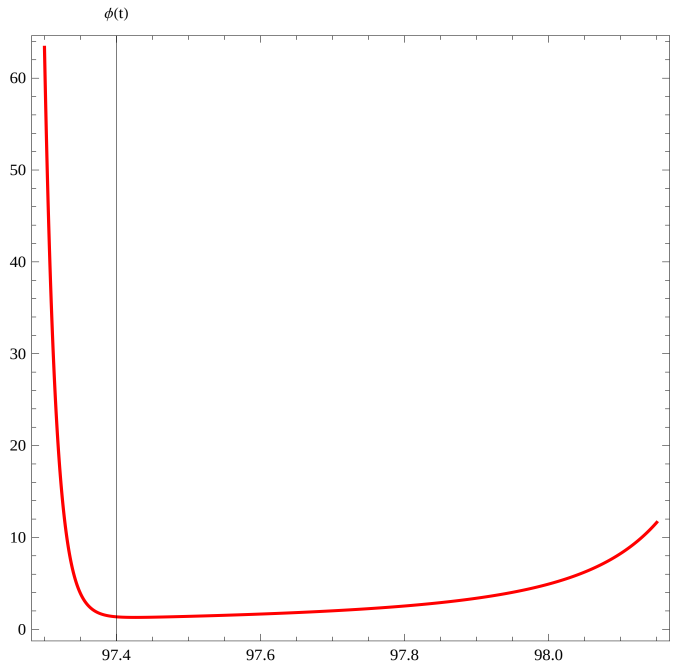

FIGURE 3. Curvature of the flow of Van der Pol system (33) for $t \in [97.3, 98.15]$.

singularly perturbed systems (37). But, since $g'(x) = x^2 + 1$, this *curvature of the flow* reads:

$$\phi(x, y, \varepsilon) = \ddot{x}\dot{y} + \left(x^2 + 1\right)\dot{x}^2. \tag{39}$$

For such *singularly perturbed system* (37) we have:

$$\begin{aligned}
&g'(x) = x^2 + 1 > 0, \quad g''(x) = 2x \geq 0, \\
&\Delta(x) = (x^6 + 8x^4 + 6x^2 + 3)/3 > 0, \\
&\Delta'(x) = (6x^5 + 32x^3 + 12x)/3, \\
&P(x) = (9 + 81x^2 + 237x^4 + 271x^6 + 213x^8 + 57x^{10} + 4x^{12})/9 \geq 0, \\
&Q(x) = (9 + 108x^2 + 312x^4 + 296x^6 + 208x^8 + 52x^{10} + 3x^{12})/9 \geq 0,
\end{aligned} \tag{40}$$

Using the same considerations as previously, it follows from Proposition 2, 4 & 5, that in the right half part of the xy-plane and below the *critical manifold* for $x \geq a$, the *curvature of the flow* of system (37) increases while its *energy* decreases.

5. CONCLUSION

In this work, by using the *Flow Curvature Method*, we have stated that, in the right half part of the xy-plane and below the *critical manifold*, the *slow invariant manifold* of generalized Liénard systems, the *curvature of the flow* increases while the *energy* of such systems decreases. Hence we proved Minorsky's statement for the generalized Liénard systems dating from half a century. Some perspectives to be given to this work should be on the one hand analyze how *curvature* and *energy* could be related to the number of *limit cycles* of such planar *singularly dynamical systems*. On the other hand, it should be interesting to investigate if these results could be extended to higher dimensional *singularly dynamical systems*. It might also be interesting to study the relation between energy considerations and entropy concepts that have been used in the context of *slow manifold* computation, see e.g. [17, 18], where it has been demonstrated that minimum entropy production and minimum curvature are connected. This seems to be a conceptual analogy of classical thermodynamics with entropy characterizing the degree of energy dissipation.

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References

[1] A.A. Andronov & S.E. Chaikin, *Theory of Oscillators*, I. Moscow, 1937; English transl., Princeton Univ. Press, Princeton, N.J. (1949).
[2] P. Bergé, Y. Pomeau & Ch. Vidal *Order within Chaos. Towards a Deterministic Approach to Turbulence*, John Wiley & Sons Ltd., New York, (1987).
[3] J.D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, MA, (1968).
[4] N. Fenichel, "Persistence and Smoothness of Invariant Manifolds for Flows," *Ind. Univ. Math. J.* 21, 193-225 (1971).
[5] N. Fenichel, "Asymptotic stability with rate conditions," *Ind. Univ. Math. J.* 23, 1109-1137 (1974).
[6] N. Fenichel, "Asymptotic stability with rate conditions II," *Ind. Univ. Math. J.* 26, 81-93 (1977).
[7] N. Fenichel, "Geometric singular perturbation theory for ordinary differential equations" *J. Diff. Eq.* 31, 53-98 (1979).
[8] J.M. Ginoux & B. Rossetto, "Differential geometry and mechanics applications to chaotic dynamical systems," *International Journal of Bifurcation & Chaos* 4(16), 887-910 (2006).
[9] J.M. Ginoux, B. Rossetto & L.O. Chua, "Slow invariant manifolds as curvature of the flow of dynamical systems," *International Journal of Bifurcation & Chaos* 11(18), 3409-3430 (2008).
[10] J.M. Ginoux, *Differential geometry applied to dynamical systems*, World Scientific Series on Nonlinear Science, Series A **66** (World Scientific, Singapore), (2009).
[11] J.M. Ginoux "The Slow Invariant Manifold of the Lorenz-Krishnamurthy Model," *Qualitative Theory of Dynamical Systems*, 13(1), 19-37 (2014).
[12] J.M. Ginoux *History of nonlinear oscillations theory*, Archimede, New Studies in the History and Philosophy of Science and Technology, vol. 49, Springer, New York (2017).
[13] J.M. Ginoux, "Slow Invariant Manifolds of Slow-Fast Dynamical Systems," *International Journal of Bifurcation & Chaos*, 31(07), 2150112 [17 pages] (2021).
[14] P. Heiter & D. Lebiedz "Towards differential geometric characterization of slow invariant manifolds in extended phase space: Sectional curvature and flow invariance", *SIAM J. Appl. Dyn. Syst.* 17(1), 732-753 (2018).
[15] M. W. Hirsch, C.C. Pugh & M. Shub, *Invariant Manifolds*, Springer-Verlag, New York (1977).
[16] C.K.R.T. Jones, *Geometric Singular Pertubation Theory, in Dynamical Systems*, Montecatini Terme, L. Arnold, Lecture Notes in Mathematics, vol. 1609, Springer-Verlag, 44-118 (1994).
[17] D. Lebiedz, "Computing minimal entropy production trajectories: An approach to model reduction in chemical kinetics", *J. Chem. Phys.* 120(5), 6890-6897 (2004).
[18] D. Lebiedz, "Entropy-related extremum principles for model reduction of dissipative dynamical systems", *Entropy* 12(4), 706-719 (2010).

[19] D. Lebiedz, V. Reinhardt & J. Siehr "Minimal curvature trajectories: Riemannian geometry concepts for slow manifold computation in chemical kinetics", *J. Comput. Phys.* 229(18), 6512-6533 (2010).
[20] D. Lebiedz & J. Siehr "A continuation method for efficient solution of parametric optimization problems in kinetic model reduction," *SIAM J. Sci. Comput.* 33(2), 703-720 (2011).
[21] D. Lebiedz, J. Siehr & J. Unger "A variational principle for computing slow invariant manifolds in dissipative dynamical systems", *SIAM J. Sci. Comput.* 35(3), A1584-A1603 (2013).
[22] S. Lefschetz *Differential equations: Geometric theory*, Interscience Publishers, John Wiley and Sons, New York, New York (1957).
[23] N. Levinson & O. K. Smith "A general equation for relaxation oscillations," *Duke Mathematical Journal*, 9, 382-403 (1942).
[24] N. Levinson, "A Second order differential equation with singular solutions," *Ann. Math.* 50, 127-153 (1949).
[25] A. Liénard "Étude des oscillations entretenues," *Revue Générale de l'Électricité*, 23, 901-912 and 946-954 (1928).
[26] J. Llibre, A.C. Mereu & M.A. Teixeira "Limit cycles of the generalized polynomial Liénard differential equations," *Math. Proc. Cambridge Philos. Soc.* 148, 363-383 (2010).
[27] R.E. O'Malley, *Introduction to Singular Perturbations*, Academic Press, New York (1974).
[28] R.E. O'Malley, *Singular Perturbations Methods for Ordinary Differential Equations*, Springer-Verlag, New York (1991).
[29] N. Minorsky, *Théorie des oscillations*, Mémorial des sciences mathématiques, fascicule 163 (1967).
[30] L. Perko *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics, Volume 7, Springer-Verlag New York (2001).
[31] A.N. Tikhonov, "On the dependence of solutions of differential equations on a small parameter," *Mat. Sbornik N.S.*, 31, 575-586 (1948).
[32] B. Van der Pol, On "relaxation-oscillations", *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, (VII), **2** 978-992 (1926).
[33] W.R. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Wiley-Interscience, New York (1965).

6. Appendix

Starting from (3) we have the following equation:

$$\varepsilon\ddot{x} + f(x)\,\dot{x} + g(x) = 0.$$

By multiplying this equation by $\dot{x}(t)$ and by considering that $f(x) = F'(x)$, $g(x) = G'(x)$ we obtain:

$$\varepsilon\dot{x}\ddot{x} + \left(F'(x)\dot{x}\right)\dot{x} + \left(G'(x)\dot{x}\right) = 0.$$

Since $\varepsilon\ddot{x} = \dot{y} - F'(x)\dot{x}$ we have:

$$\dot{x}\dot{y} + G'(x)\dot{x} = 0.$$

But $\varepsilon\dot{x} = y - F(x)$, and we find:

$$y\dot{y} + \varepsilon G'(x)\dot{x} = F(x)\dot{y}.$$

Finally we obtain:

$$\frac{d}{dt}\left(\frac{y^2}{2} + \varepsilon G(x)\right) = F(x)\dot{y}.$$

Then starting from $y = \varepsilon\dot{x} + F(x)$ we find that:

$$\varepsilon \frac{d}{dt}\left(\varepsilon \frac{\dot{x}^2}{2} + G(x)\right) + \frac{d}{dt}\left(\varepsilon F(x)\dot{x} + \frac{F^2(x)}{2}\right) = F(x)\dot{y}.$$

After simplifications we obtain the following equation:

$$\frac{d}{dt}\left(\varepsilon \frac{\dot{x}^2}{2} + G(x)\right) = -f(x)\dot{x}^2,$$

which is identical to Eq. (31).

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