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LIMIT CYCLES OF THE DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS ON THE CYLINDER*

Jie Li[1] and Jaume Llibre[2,†]

Dedicated to Professor Jibin Li on the occasion of his 80th birthday.

Abstract In order to understand the dynamics of the differential systems the limit cycles play a main role, but in general their study is not easy. These last years an increasing interest appeared for studying the limit cycles of some classes of discontinuous piecewise differential systems, due to the rich applications of this kind of differential systems.
Very few papers studied the limit cycles of the discontinuous piecewise differential systems in spaces different from the plane $\mathbb{R}^2$. Here we study the limit cycles of a class of discontinuous piecewise differential systems on the cylinder.

Keywords Limit cycles, discontinuous piecewise smooth system, differential systems on the cylinder.

MSC(2010) 34C05, 37G15.

1. Introduction and statement of the main

Consider the following differential equation on the cylinder $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1$

$$\frac{dr}{d\theta} = a_0(\theta) + a_1(\theta)r + a_2(\theta)r^2 + ... + a_n(\theta)r^n. \tag{1.1}$$

All the functions $a_i(\theta)$ are continuous and 2π-periodic in the variable θ. Equation (1.1) with $n = 1$ is a linear differential equation having at most one limit cycle, see for instance [4]. While for $n = 2$ it is a *Riccati* equation with at most two limit cycles, see [6]. For $n = 3$ it is an *Abel* equation. If $a_3(\theta) > 0$ Pliss [11] proved that the Abel equation has at most three limit cycles (see also [3, 8]). For $n \geq 4$ a constant sign in the leading coefficient a_n is not sufficient to bound uniformly the number of limit cycles (see [6, 8]). Lins Neto in [8] gave a example with at least $n + 3$ limit cycles for suitable functions a and f, for the Abel equation

$$\frac{dx}{d\theta} = \varepsilon f(\theta)x^3 + a(\theta)x^2 + \delta x,$$

[†]The corresponding author. Email: jllibre@mat.uab.cat(J. Li)
[1]Department of Mathematics, Sichuan University, 610064 Chengdu, Sichuan, China
[2]Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
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where $|\delta|$ is small, $a(\theta)$ is a trigonometric polynomial of degree 1, and $f(\theta)$ is a trigonometric polynomial of degree $2n$. Calanchi and Ruf [2] proved that if in equation (1.1) n is odd, the leading term is fixed and the remaining terms are small enough, then the number of limit cycles is at most n.

In [1] Bakhshalizadeh and Llibre considered the discontinuous piecewise differential systems of the form

$$\begin{aligned} \dot{x} &= a_0(\theta) + a_1(\theta)x + \cdots + a_n(\theta)x^n, \text{ if } 0 \leq \theta \leq \pi, \\ \dot{x} &= b_0(\theta) + b_1(\theta)x + \cdots + b_m(\theta)x^m, \text{ if } \pi \leq \theta \leq 2\pi, \end{aligned} \tag{1.2}$$

where $a_0(\theta), a_1(\theta), \cdots, a_n(\theta)$ and $b_0(\theta), b_1(\theta), \cdots, b_m(\theta)$ are 2π-periodic, and gave exact bounds for the maximum number of limit cycles. On the lines of discontinuity $\theta = 0$ and $\theta = \pi$ of systems (1.2), the flow is defined following the rules of Filippov [5]. In the rest of the paper always the flow on the lines of discontinuity is defined according with Filippov. The objective of this paper is to extend the results on the maximum number of limit cycles obtained in [1] for the discontinuous piecewise differential systems on the cylinder with two straight lines of separation, to the discontinuous piecewise differential systems on the cylinder with an arbitrary number of lines of separation.

Let C be the cylinder $\{(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}\}$. Consider the discontinuous piecewise differential systems on the cylinder

$$\begin{aligned} \dot{x} &= \sum_{l=0}^{m_1} a_{1l}(\theta)x^l, \quad \text{if } 0 \leq \theta \leq 2\pi/n, \\ \dot{x} &= \sum_{l=0}^{m_2} a_{2l}(\theta)x^l, \quad \text{if } 2\pi/n \leq \theta \leq 2 \cdot 2\pi/n, \\ &\vdots \\ \dot{x} &= \sum_{l=0}^{m_k} a_{kl}(\theta)x^l, \quad \text{if } 2\pi(k-1)/n \leq \theta \leq 2k\pi/n, \\ &\vdots \\ \dot{x} &= \sum_{l=0}^{m_n} a_{nl}(\theta)x^l, \quad \text{if } 2\pi(n-1)/n \leq \theta \leq 2\pi, \end{aligned} \tag{1.3}$$

where $a_{kl}(\theta)$, for $k = 1, \cdots n$ and $l = 0, 1, \cdots, m_k$, are 2π-periodic functions in the variable θ. Then $H(m_1, \cdots, m_n)$ denotes the maximum number of limit cycles that the discontinuous piecewise differential systems (1.3) can exhibit.

Corollary 1.1. *The discontinuous piecewise differential systems on the cylinder C of the form*

$$\begin{aligned} \dot{x} &= a_0(\theta) + a_1(\theta)x, \textit{ if } 0 \leq \theta \leq 2\pi/3, \\ \dot{x} &= b_0(\theta) + b_1(\theta)x, \textit{ if } 2\pi/3 \leq \theta \leq 4\pi/3, \\ \dot{x} &= c_0(\theta) + c_1(\theta)x, \textit{ if } 4\pi/3 \leq \theta \leq 2\pi, \end{aligned} \tag{1.4}$$

where $a_i(\theta), b_i(\theta)$ and $c_i(\theta)$ for $i = 0, 1$ are 2π-periodic functions in the variable θ, have at most one limit cycle, i.e, $H(1, 1, 1) = 1$.

Theorem 1.1. *The discontinuous piecewise differential systems on the cylinder C of the form*

$$\begin{aligned}
\dot{x} &= a_{10}(\theta) + a_{11}(\theta)x, \quad \textit{if } 0 \le \theta \le 2\pi/n,\\
\dot{x} &= a_{20}(\theta) + a_{21}(\theta)x, \quad \textit{if } 2\pi/n \le \theta \le 2\cdot 2\pi/n,\\
&\vdots\\
\dot{x} &= a_{k0}(\theta) + a_{k1}(\theta)x, \quad \textit{if } 2\pi(k-1)/n \le \theta \le 2k\pi/n,\\
&\vdots\\
\dot{x} &= a_{n0}(\theta) + a_{n1}(\theta)x, \quad \textit{if } 2\pi(n-1)/n \le \theta \le 2\pi,
\end{aligned}$$

where $a_{k0}(\theta)$ and $a_{k1}(\theta)$, for $k = 1, \cdots n$, are 2π-periodic functions in the variable θ, have at most one limit cycle, i.e., $H(1, \cdots, 1) = 1$.

Corollary 1.2. *The discontinuous piecewise differential systems on the cylinder C of the form*

$$\begin{aligned}
\dot{x} &= a_0(\theta) + a_1(\theta)x + a_2(\theta)x^2, \textit{ if } 0 \le \theta \le \pi,\\
\dot{x} &= b_0(\theta) + b_1(\theta)x + b_2(\theta)x^2, \textit{ if } \pi \le \theta \le 2\pi,
\end{aligned} \tag{1.5}$$

where $a_i(\theta)$ and $b_i(\theta)$, for $i = 0, 1, 2$, are 2π-periodic functions in the variable θ, have at most two limit cycles, i.e., $H(2, 2) = 2$.

Theorem 1.2. *The discontinuous piecewise differential systems on the cylinder C of the form*

$$\begin{aligned}
\dot{x} &= a_{10}(\theta) + a_{11}(\theta)x + a_{12}x^2, \quad \textit{if } 0 \le \theta \le 2\pi/n,\\
\dot{x} &= a_{20}(\theta) + a_{21}(\theta)x + a_{22}x^2, \quad \textit{if } 2\pi/n \le \theta \le 2\cdot 2\pi/n,\\
&\vdots\\
\dot{x} &= a_{k0}(\theta) + a_{k1}(\theta)x + a_{k2}x^2, \quad \textit{if } 2\pi(k-1)/n \le \theta \le 2k\pi/n,\\
&\vdots\\
\dot{x} &= a_{n0}(\theta) + a_{n1}(\theta)x + a_{n2}x^2, \quad \textit{if } 2\pi(n-1)/n \le \theta \le 2\pi,
\end{aligned}$$

where $a_{k0}(\theta), a_{k1}(\theta)$ and $a_{k2}(\theta)$, for $k = 1, \cdots n$, are 2π-periodic functions in the variable θ, have at most two limit cycles, i.e., $H(2, \cdots, 2) = 2$.

Theorem 1.3. *The discontinuous piecewise differential systems on the cylinder C of the form*

$$\begin{aligned}
\dot{x} &= a_0(\theta) + a_1(\theta)x, && \textit{if } 0 \le \theta \le 2\pi/3,\\
\dot{x} &= b_0(\theta) + b_1(\theta)x + b_2(\theta)x^2, && \textit{if } 2\pi/3 \le \theta \le 4\pi/3,\\
\dot{x} &= c_0(\theta) + c_1(\theta)x, && \textit{if } 4\pi/3 \le \theta \le 2\pi,
\end{aligned} \tag{1.6}$$

where $a_i(\theta), b_i(\theta)$ and $c_i(\theta)$, for $i = 0, 1$ or 2, are 2π-periodic functions in the variable θ, have at most two limit cycles, i.e., $H(1, 2, 1) = 2$.

According to the above Corollaries 1.1, 1.2 and Theorems 1.1, 1.2, 1.3, we can conclude the following corollary.

Corollary 1.3. *The discontinuous piecewise differential systems on the cylinder C of the form (1.3) with $\max\{m_1, \cdots, m_n\} \leq 2$ have at most one limit cycle if $m_1 = \cdots = m_n = 1$, i.e., $H(1, \cdots, 1) = 1$, otherwise $H(m_1, \cdots, m_n) = 2$.*

Theorem 1.4. *For every positive integer k there are discontinuous piecewise differential systems on the cylinder C of the form*

$$\begin{aligned} &\dot{x} = a(\theta)x, && \text{if } 0 \leq \theta \leq 2\pi/3,\\ &\dot{x} = b_2(\theta)x^2 + \varepsilon b_3(\theta)x^3, && \text{if } 2\pi/3 \leq \theta \leq 4\pi/3,\\ &\dot{x} = c(\theta)x, && \text{if } 4\pi/3 \leq \theta \leq 2\pi, \end{aligned} \tag{1.7}$$

where $a(\theta), c(\theta)$ and $b_i(\theta)$ for $i = 2, 3$, are 2π-periodic functions in the variable θ, having at least k limit cycles on the cylinder, i.e., $H(1, 3, 1) = +\infty$.

According to the above Theorem 1.4, we can conclude the following corollary.

Corollary 1.4. *The discontinuous piecewise differential systems on the cylinder C of the form (1.3) with $\max\{m_1, \cdots, m_n\} \geq 3$ have at least k limit cycles for any positive integer k, i.e., $H(m_1, \cdots, m_n) = +\infty$.*

All the results of this section are proved in section 2.

Here we study the limit cycles of piecewise differential systems on the cylinder separated by straight lines. People interested in the limit cycles of piecewise differential systems in the plane $\mathbb{R}^2$ and separated by straight lines can see the papers [9, 10] and the references cited there.

2. Proof of the main results

In this section we will prove the main results as stated in Corollaries 1.1, 1.2, 1.4 and Theorems 1.1, 1.2, 1.3, 1.4.

Proof of Corollary 1.1. Consider the discontinuous piecewise differential systems (1.4). The solution of the first equation of (1.4) satisfying $x(0) = \rho$ is

$$\begin{aligned} &x_1(\theta, \rho) = (I_1(\theta) + \rho)\, e^{K_1(\theta)},\\ &I_1(\theta) = \int_0^\theta a_0(s) e^{-K_1(s)} ds,\\ &K_1(s) = \int_0^s a_1(w) dw. \end{aligned}$$

The solution of the second equation of (1.4) satisfying $x(2\pi/3) = x_1(2\pi/3, \rho)$ is

$$\begin{aligned} x_2(\theta, x_1(\pi/3,\rho)) &= (I_2(\theta) + x_1(\pi/3,\rho))\, e^{K_2(\theta)},\\ I_2(\theta) &= \int_{\frac{2\pi}{3}}^{\theta} b_0(s) e^{-K_2(s)} ds,\\ K_2(s) &= \int_{\frac{2\pi}{3}}^{s} b_1(w) dw, \end{aligned}$$

and the solution of the third equation of (1.4) satisfying $x(4\pi/3)=x_2(4\pi/3, x_1(2\pi/3,\rho))$ is

$$\begin{aligned} x_3(\theta, x_2(4\pi/3, x_1(2\pi/3,\rho))) &= (I_3(\theta) + x_2(4\pi/3, x_1(2\pi/3,\rho)))\, e^{K_3(\theta)},\\ I_3(\theta) &= \int_{\frac{4\pi}{3}}^{\theta} c_0(s) e^{-K_3(s)} ds,\\ K_3(s) &= \int_{\frac{4\pi}{3}}^{s} c_1(w) dw. \end{aligned}$$

Define the function

$$\begin{aligned} \Pi_1(\rho) &= x_3(2\pi, x_2(4\pi/3, x_1(2\pi/3,\rho))) - \rho\\ &= e^{K_3(2\pi)}\big(e^{K_1(2\pi/3)+K_2(4\pi/3)} I_1(2\pi/3) + e^{K_2(4\pi/3)} I_2(4\pi/3) + I_3(2\pi)\big)\\ &\quad + (e^{K_1(2\pi/3)+K_2(4\pi/3)+K_3(2\pi)} - 1)\rho. \end{aligned}$$

Thus the periodic orbits of the discontinuous piecewise differential systems (1.4) are associated with the zeros of the linear equation $\Pi_1(\rho) = 0$. Clearly there is at most one zero. Thus the discontinuous piecewise differential systems (1.4) have at most one limit cycle. □

The proof of Theorem 1.1 is similar to the proof of Corollary 1.1.

Proof of Corollary 1.2. Consider the discontinuous piecewise differential systems (1.5). On the two bands of the cylinder with $\theta \in [0,\pi]$ and $\theta \in [\pi, 2\pi]$ we have a Riccati differential equation.

Suppose that we have a periodic solution $x(\theta) = x_p(\theta)|_{\theta\in[0,\pi]} \cup x_q(\theta)|_{\theta\in[\pi,2\pi]}$. Then doing the change of variable $x \to X_1$ where

$$X_1(\theta) = \frac{1}{x(\theta) - x_p(\theta)},$$

we write the first differential equation in (1.5) with $\theta \in [0,\pi]$ as

$$\frac{dX_1}{d\theta} = -a_2(\theta) - \big(2a_2(\theta)x_p(\theta) + a_1(\theta)\big)X_1. \tag{2.1}$$

Then the solution of the linear equation (2.1) with $\theta \in [0,\pi]$ is written as

$$\begin{aligned} X_1(\theta) &= (N_1(\theta) + X_1(0))\, e^{M_1(\theta)},\\ N_1(\theta) &= \int_0^{\theta} -a_2(s) e^{-M_1(s)} ds,\\ M_1(s) &= \int_0^{s} -\big(2a_2(w)x_p(w) + a_1(w)\big) dw. \end{aligned}$$

Undoing the change of variables we obtain that the solution of the first equation of (1.5) satisfying $x(0) = \rho$ is

$$x_1(\theta, \rho) = \frac{A_1(\theta) + B_1(\theta)\rho}{C_1(\theta) + D_1(\theta)\rho},$$

where

$$\begin{aligned}
A_1(\theta) &= x_p(\theta)e^{M_1(\theta)}\left(1 - x_p(0)N_1(\theta)\right) - x_p(0),\\
B_1(\theta) &= x_p(\theta)N_1(\theta)e^{M_1(\theta)} + 1,\\
C_1(\theta) &= e^{M_1(\theta)}\left(1 - x_p(0)N_1(\theta)\right),\\
D_1(\theta) &= e^{M_1(\theta)}N_1(\theta).
\end{aligned}$$

Similarly we write the second differential equation of (6) in $\theta \in [\pi, 2\pi]$ as

$$\frac{dX_2}{d\theta} = -b_2(\theta) - \big(2b_2(\theta)x_q(\theta) + b_1(\theta)\big)X_2 \tag{2.2}$$

doing the change

$$X_2(\theta) = \frac{1}{x(\theta) - x_q(\theta)}.$$

Both changes of variables the change of variables $x \to X_1$ when $x \in [0, \pi]$ and the change of variables $x \to X_2$ when $x \in [\pi, 2\pi]$ coincide on the periodic orbits intersection with the straight lines $x = 0$ and π, so the structure of the discontinuous piecewise differential systems (1.5) is preserved. Indeed,

$$\begin{aligned}
X_1(0) = X_2(2\pi) &= \frac{1}{x(0) - x_p(0)} = \frac{1}{x(2\pi) - x_q(2\pi)},\\
X_1(\pi) = X_2(\pi) &= \frac{1}{x(\pi) - x_p(\pi)} = \frac{1}{x(\pi) - x_q(\pi)},
\end{aligned}$$

because $x(0) = x(2\pi), x_p(0) = x_q(2\pi)$ and $x_p(\pi) = x_q(\pi)$ on the periodic orbits.

Then the solution of the differential equation (2.2) is

$$\begin{aligned}
X_2(\theta) &= \left(N_2(\theta) + X_2(\pi)\right)e^{M_2(\theta)},\\
N_2(\theta) &= \int_\pi^\theta -b_2(s)e^{-M_2(s)}ds,\\
M_2(s) &= \int_\pi^s -\big(2b_2(w)x_q(w) + b_1(w)\big)dw.
\end{aligned}$$

Undoing the change of variables we get the solution of the second equation of (1.5) satisfying $x(\pi) = x_1(\pi, \rho)$ is

$$\begin{aligned}
x_2(\theta, \rho) &= \frac{A_2(\theta) + B_2(\theta)x_1(\pi, \rho)}{C_2(\theta) + D_2(\theta)x_1(\pi, \rho)}\\
&= \frac{C_1(\pi)A_2(\theta) + A_1(\pi)B_2(\theta) + (D_1(\pi)A_2(\theta) + B_1(\pi)B_2(\theta))\rho}{\big(C_1(\pi)C_2(\theta) + A_1(\pi)D_2(\theta) + (B_1(\pi)D_2(\theta) + D_1(\pi)D_2(\theta))\rho},
\end{aligned}$$

where

$$A_2(\theta) = x_q(\theta)e^{M_2(\theta)}\left(1 - x_q(\pi)N_2(\theta)\right) - x_q(\pi),$$

$$B_2(\theta) = x_q(\theta)N_2(\theta)e^{M_2(\theta)} + 1,$$

$$C_2(\theta) = e^{M_2(\theta)}\left(1 - x_q(\pi)N_2(\theta)\right),$$

$$D_2(\theta) = e^{M_2(\theta)}N_2(\theta).$$

Define the function $\Pi_2(\rho) = x_2(2\pi, x_1(\pi,\rho)) - \rho$, which is

$$\frac{A_2C_1 + A_1B_2 + (A_2D_1 + B_1B_2 - C_1C_2 - A_1D_2)\rho - (B_1D_2 + D_1D_2)\rho^2}{C_1C_2 + A_1D_2 + (B_1D_2 + D_1D_2)\rho},$$

with $A_1 = A_1(\pi), B_1 = B_1(\pi), C_1 = C_1(\pi), D_1 = D_1(\pi)$ and $A_2 = A_2(2\pi), B_2 = B_2(2\pi), C_2 = C_2(2\pi), D_2 = D_2(2\pi)$. Thus the periodic orbits of the discontinuous piecewise differential systems (1.5) are associated with the zeros of the equation $\Pi_2(\rho) = 0$. It follows that the discontinuous piecewise differential systems (1.5) have at most two limit cycles. $\square$

The proof of Theorem 1.2 is similar to the proof of Corollary 1.2.

Proof of Theorem 1.3. Consider the discontinuous piecewise differential systems (1.6). On the second band of the cylinder, i.e., $\theta \in [2\pi/3, 4\pi/3]$, we have a Riccati differential equation.

Suppose that there is a periodic solution

$$x(\theta) = x_r(\theta)|_{\theta\in[0,2\pi/3]} \cup x_s(\theta)|_{\theta\in[2\pi/3,4\pi/3]} \cup x_t(\theta)|_{\theta\in[4\pi/3,2\pi]}.$$

In what follows for studying the limit cycles on the cylinder of systems (7) we will do three changes of variables, in each strip of cylinder defined by the straight lines $\theta = 0$, $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Later on we will show that these changes of variables coincide on the three straight lines $\theta = 0$, $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Then doing the change of variable $x \to X_r$, where

$$X_r(\theta) = \frac{1}{x(\theta) - x_r(\theta)},$$

we write the first differential equation of (1.6) with $\theta \in [0, 2\pi/3]$ as

$$\frac{dX_r}{d\theta} = -a_1(\theta)X_r. \tag{2.3}$$

Then the solution of the linear equation (2.3) with $\theta \in [0, 2\pi/3]$ with an initial value $X_r(0)$ is written as

$$X_r(\theta, X_r(0)) = X_r(0)e^{K_1(\theta)},$$
$$K_1(\theta) = \int_0^\theta (-a_1(s))ds.$$

Correspondingly we obtain the solution of the first differential equation of (1.6) with $\theta \in [0, 2\pi/3]$ satisfying $x(0) = \rho$ is

$$x_1(\theta, \rho) = \frac{x_r(\theta)e^{K_1(\theta)} - x_r(0)}{e^{K_1(\theta)}} + \frac{\rho}{e^{K_1(\theta)}}.$$

Note that on the cylinder with $\theta \in [2\pi/3, 4\pi/3]$ we have a Riccati differential equation. Then doing the change of variable $x \to X_s$, where

$$X_s(\theta) = \frac{1}{x(\theta) - x_s(\theta)},$$

we write the second differential equation of (1.6) with $\theta \in [2\pi/3, 4\pi/3]$ as

$$\frac{dX_s}{d\theta} = -b_2(\theta) - \big(2b_2(\theta)x_s(\theta) + b_1(\theta)\big)X_s. \tag{2.4}$$

The solution of the linear equation (2.4) with $\theta \in [2\pi/3, 4\pi/3]$ is written as

$$\begin{aligned}
X_s(\theta) &= (I_2(\theta) + X_s(2\pi/3))\, e^{K_2(\theta)},\\
I_2(\theta) &= \int_{\frac{2\pi}{3}}^{\theta} -b_2(s)e^{-K_2(s)}ds,\\
K_2(s) &= \int_{\frac{2\pi}{3}}^{s} -\big(2b_2(w)x_s(w) + b_1(w)\big)dw.
\end{aligned}$$

Undoing the change of variables the solution of the second equation of (1.6) satisfying $x(2\pi/3) = x_1(2\pi/3, \rho)$ is

$$x_2(\theta, \rho) = \frac{A(\theta) + B(\theta)\rho}{C(\theta) + D(\theta)\rho}, \tag{2.5}$$

where

$$\begin{aligned}
A(\theta) &= x_s(\theta)e^{K_2(\theta)}\left(e^{K_1(2\pi/3)} - I_2(\theta)x_r(0)\right) - x_r(0)\\
B(\theta) &= I_2(\theta)x_s(\theta)e^{K_2(\theta)} + 1,\\
C(\theta) &= e^{K_2(\theta)}\left(e^{K_1(2\pi/3)} - I_2(\theta)x_r(0)\right),\\
D(\theta) &= I_2(\theta)e^{K_2(\theta)}.
\end{aligned}$$

Similarly, doing the change of variable $x \to X_t$, where

$$X_t(\theta) = \frac{1}{x(\theta) - x_t(\theta)},$$

we write the third differential equation of (1.6) with $\theta \in [4\pi/3, 2\pi]$ as

$$\frac{dX_t}{d\theta} = -c_1(\theta)X_t. \tag{2.6}$$

Then the solution of the linear equation (2.6) with an initial value $X_t(4\pi/3)$ is written as

$$\begin{aligned}
X_t(\theta, X_t(4\pi/3)) &= X_t(4\pi/3)e^{K_3(\theta)},\\
K_3(\theta) &= \int_0^{\theta} (-c_1(s))ds.
\end{aligned}$$

Correspondingly the solution of the third differential equation of (1.6) with $\theta \in [4\pi/3, 2\pi]$ satisfying $x(4\pi/3) = x_2(4\pi/3, \rho)$ is

$$x_3(\theta, \rho) = \frac{x_t(\theta)e^{K_3(\theta)} + x_2(4\pi/3, \rho) - x_t(4\pi/3)}{e^{K_3(\theta)}}. \tag{2.7}$$

Note that we need to check that the change of variables $x \to X_r$ when $x \in [0, 2\pi/3]$, the change of variables $x \to X_s$ when $x \in [2\pi/3, 4\pi/3]$, the change of variables $x \to X_t$ when $x \in [4\pi/3, 2\pi]$ coincide on the periodic orbits intersection with the straight lines $\theta = 0, 2\pi/3$ and $4\pi/3$, and consequently the structure of the discontinuous piecewise differential systems (1.6) is preserved. Indeed,

$$\begin{aligned}
X_r(0) = X_t(2\pi) &= \frac{1}{x(0) - x_r(0)} = \frac{1}{x(2\pi) - x_t(2\pi)},\\
X_r(2\pi/3) = X_s(2\pi/3) &= \frac{1}{x(2\pi/3) - x_r(2\pi/3)} = \frac{1}{x(2\pi/3) - x_s(2\pi/3)},\\
X_s(4\pi/3) = X_t(4\pi/3) &= \frac{1}{x(4\pi/3) - x_s(4\pi/3)} = \frac{1}{x(4\pi/3) - x_t(4\pi/3)},
\end{aligned}$$

because $x(0) = x(2\pi), x_r(0) = x_t(2\pi)$ and $x_r(2\pi/3) = x_s(2\pi/3)$ and $x_s(4\pi/3) = x_t(4\pi/3)$ on the periodic orbits.

Define the function $\Pi_3(\rho) = x_3(2\pi, \rho) - \rho$. Then by (2.5) and (2.7) we obtain

$$\Pi_3(\rho) = \frac{E(2\pi) + F(2\pi)\rho - e^{K_3(2\pi)}D(4\pi/3)\rho^2}{e^{K_3(2\pi)}(C(4\pi/3) + D(4\pi/3)\rho)},$$

where

$$E(\theta) = (x_t(\theta)e^{K_3(\theta)} - x_t(4\pi/3))C(4\pi/3) + A(4\pi/3),$$

$$F(\theta) = (x_t(\theta)e^{K_3(\theta)} - x_t(4\pi/3))D(4\pi/3) + B(4\pi/3) - e^{K_3(\theta)}C(4\pi/3).$$

Thus the periodic orbits of the discontinuous piecewise differential systems (1.6) are associated with the zeros of the equation $\Pi_3(\rho) = 0$. Clearly there is at most two zeros, and therefore the discontinuous piecewise differential systems (1.6) have at most two limit cycles. $\square$

Proof of Theorem 1.4. We consider the discontinuous piecewise differential systems

$$\begin{aligned}
&\dot{x} = a(\theta)x, \quad \text{if } 0 \le \theta \le 2\pi/3,\\
&\dot{x} = b_2(\theta)x^2, \text{ if } 2\pi/3 \le \theta \le 4\pi/3,\\
&\dot{x} = c(\theta)x, \quad \text{if } 4\pi/3 \le \theta \le 2\pi.
\end{aligned} \tag{2.8}$$

The solution of the first differential equation of (2.8) with $\theta \in [0, 2\pi/3]$ satisfying $x(0) = \rho$ is

$$x_1(\theta, \rho) = \rho e^{J_1(\theta)}, \qquad J_1(\theta) = \int_0^\theta a(s)ds.$$

On the other hand the solution of the second differential equation of (2.8) with $\theta \in [2\pi/3, 4\pi/4]$ satisfying $x(2\pi/3) = x_1(2\pi/3, \rho)$ is

$$x_2(\theta, \rho) = \frac{\rho e^{J_1(2\pi/3)}}{1 - \rho e^{J_1(2\pi/3)} J_2(\theta)}, \qquad J_2(\theta) = \int_{\frac{2\pi}{3}}^{\theta} b_2(s) ds.$$

Eventually the solution of the third differential equation of (2.8) with $\theta \in [4\pi/3, 2\pi]$ satisfying $x(4\pi/3) = x_2(4\pi/3, \rho)$ is

$$x_3(\theta, \rho) = \frac{\rho e^{J_1(2\pi/3) + J_3(\theta)}}{1 - \rho e^{J_1(2\pi/3)} J_2(4\pi/3)}, \qquad J_3(\theta) = \int_{\frac{4\pi}{3}}^{\theta} c(s) ds.$$

Define the function

$$\Pi_4(\rho) = x_3(2\pi, \rho) - \rho = \frac{\rho e^{J_1(2\pi/3) + J_3(2\pi)}}{1 - \rho e^{J_1(2\pi/3)} J_2(4\pi/3)} - \rho.$$

Then $\Pi_4(\rho) \equiv 0$ if we assume

$$J_1(2\pi/3) + J_3(2\pi) = J_2(4\pi/3) = 0, \tag{2.9}$$

where we choose the functions $a(\theta)$, $b_2(\theta)$ and $c(\theta)$ in order that the equalities (2.9) hold. We obtain that the discontinuous piecewise differential systems (2.8) has a continuum of periodic solutions in the neighborhood of $\rho = 0$.

In what follows we consider the discontinuous piecewise differential systems (1.7) with a small parameter ε. The solution of the first differential equation of (1.7) with $\theta \in [0, 2\pi/3]$ satisfying $x(0) = \rho$ is $x_1(\theta, \rho)$ given in (12). Let $x_2(\theta, \rho, \varepsilon)$ denote the solution of the second differential equation with initial value $x_2(2\pi/3, \rho, \varepsilon) = x_1(2\pi/3, \rho) = \rho e^{J_1(2\pi/3)} = \bar{x}_1$. Then the solution $x_2(\theta, \bar{x}_1, \varepsilon)$ can be expanded with respect to ε as follows

$$x_2(\theta, \bar{x}_1, \varepsilon) = x_{20}(\theta, \bar{x}_1) + x_{21}(\theta, \bar{x}_1)\varepsilon + O(\varepsilon^2),$$

where

$$x_{20}(\theta, \bar{x}_1) = x_2(\theta, \bar{x}_1, \varepsilon)|_{\varepsilon=0}$$

and

$$x_{21}(\theta, \bar{x}_1) = \partial x_2(\theta, \bar{x}_1, \varepsilon)/\partial \varepsilon|_{\varepsilon=0}.$$

Similarly the solution $x_3(\theta, \rho, \varepsilon)$ of the third differential equation of systems (1.7) with $\theta \in [4\pi/3, 2\pi]$ satisfying $x_3(4\pi/3, \rho, \varepsilon) = x_2(4\pi/3, \bar{x}_1, \varepsilon) = \bar{x}_2$ is

$$x_3(\theta, \bar{x}_2, \varepsilon) = x_2(4\pi/3, \bar{x}_1, \varepsilon) e^{J_3(\theta)}.$$

Then we similarly obtain a function

$$\begin{aligned}
\Pi_4^{\varepsilon}(\rho, \varepsilon) &= x_3(2\pi, \bar{x}_2, \varepsilon) - \rho \\
&= x_2(4\pi/3, \bar{x}_1, \varepsilon) e^{J_3(2\pi)} - \rho \\
&= \Big(x_{20}(4\pi/3, \bar{x}_1) e^{J_3(2\pi)} - \rho \Big) + x_{21}(4\pi/3, \bar{x}_1) e^{J_3(2\pi)} \varepsilon + O(\varepsilon^2).
\end{aligned}$$

In order to find the number of zeros of the equation $\Pi_4^\varepsilon = 0$, associated with the number of limit cycles of the discontinuous piecewise differential systems (1.7), first we have

$$\begin{aligned}\dot{x_2}(\theta, \bar{x}_1, \varepsilon) &= \dot{x}_{20}(\theta, \bar{x}_1) + \dot{x}_{21}(\theta, \bar{x}_1)\varepsilon + O(\varepsilon^2)\\ &= b_2(\theta)\big(x_{20}(\theta, \bar{x}_1) + x_{21}(\theta, \bar{x}_1)\varepsilon + O(\varepsilon^2)\big)^2 + \varepsilon b_3(\theta)\\ &\quad\times\big(x_{20}(\theta, \bar{x}_1) + x_{21}(\theta, \bar{x}_1)\varepsilon + O(\varepsilon^2)\big)^3 + O(\varepsilon^2).\end{aligned}$$

From this equality we get

$$\dot{x}_{20}(\theta, \bar{x}_1) = b_2(\theta)x_{20}^2(\theta, \bar{x}_1), \tag{2.10}$$

with the initial value $x_{20}(2\pi/3, \bar{x}_1) = \rho e^{J_1(2\pi/3)}$ and

$$\dot{x}_{21}(\theta, \bar{x}_1) = 2b_2(\theta)x_{20}(\theta, \bar{x}_1)x_{21}(\theta, \bar{x}_1) + b_3(\theta)x_{20}^3(\theta, \bar{x}_1), \tag{2.11}$$

with an initial value $x_{21}(2\pi/3, \bar{x}_1) = 0$, where $\dot{x}_{2i}(\theta, \bar{x}_1) = \partial x_{2i}(\theta, \bar{x}_1)/\partial\theta$ for $i = 0, 1$. Integrating the differential equation (2.10) in the interval $[2\pi/3, 4\pi/3]$, by (2.9) we get

$$x_{20}(4\pi/3, \rho) = \frac{\rho e^{J_1(2\pi/3)}}{1 - \rho e^{J_1(2\pi/3)}J_2(4\pi/3)} = \rho e^{J_1(2\pi/3)}.$$

On the other hand we get

$$b_2(\theta)x_{20}(\theta, \bar{x}_1) = \frac{\dot{x}_{20}(\theta, \bar{x}_1)}{x_{20}(\theta, \bar{x}_1)},$$

from the differential equation (2.10). Substituting the previous equality in (2.11), we have

$$\frac{\partial}{\partial\theta}\left(\frac{x_{21}(\theta, \bar{x}_1)}{x_{20}^2(\theta, \bar{x}_1)}\right) = b_3(\theta)x_{20}(\theta, \bar{x}_1).$$

Integrating this differential equation in the interval $[2\pi/3, 4\pi/4]$ and combining the assumption (2.9), we obtain

$$\begin{aligned}x_{21}(4\pi/3, \bar{x}_1) &= x_{20}^2(4\pi/3, \bar{x}_1)\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} b_3(\theta)x_{20}(\theta, \bar{x}_1)d\theta\\ &= \rho^3 e^{3J_1(2\pi/3)}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{b_3(\theta)}{1 - \rho e^{J_1(2\pi/3)}J_2(\theta)}d\theta.\end{aligned}$$

We further obtain

$$\begin{aligned}\Pi_4^\varepsilon(\rho, \varepsilon) &= \Big(x_{20}(4\pi/3, x_1)e^{J_3(2\pi)} - \rho\Big) + x_{21}(4\pi/3, x_1)e^{J_3(2\pi)}\varepsilon + O(\varepsilon^2)\\ &= \varepsilon\rho^3 e^{2J_1(2\pi/3)}M(\rho) + O(\varepsilon^2),\end{aligned}$$

where

$$M(\rho) = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{b_3(\theta)}{1 - \rho e^{J_1(2\pi/3)}J_2(\theta)}d\theta,$$

is a *Melnikov function*. It follows from the Implicit Function Theorem that the simple zeros of $M(\rho)$ which are non-zero can be associated with the simple zeros of the function $\Pi_4^\varepsilon(\rho,\varepsilon)$ distinct from zero. More concretely, if $\rho=\rho_0\neq 0$ satisfying $M(\rho_0)=0$ and $M'(\rho_0)\neq 0$, then there exists a differential function ϕ such that $\phi(0)=\rho_0$ and $\Pi_4^\varepsilon(\phi(\varepsilon),\varepsilon)\equiv 0$ for a small enough ε.

For any $n\in\mathbb{N}$ we choose

$$b_2(\theta)=\cos(e^{J_1(2\pi/3)}\theta),\qquad b_3(\theta)=P(\sin(e^{J_1(2\pi/3)}\theta)),$$

where P is a polynomial of degree n in the variable $\sin(e^{J_1(2\pi/3)}\theta)$. We introduce the family of analytic functions

$$I_k(\rho)=\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\frac{\sin^k t}{1-\rho\sin t}dt,$$

where $t=e^{J_1(2\pi/3)}\theta$ and k is a positive integer. By Theorem A of [7] the maximum number of zeros of $M(\rho)$ is the same as the degree n of the polynomial $P(\sin t)$. $\square$

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