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Transcritical Bifurcation at Infinity in Planar Piecewise Polynomial Differential Systems with Two Zones

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Abstract

We present a general mechanism of generation of limit cycles in planar piecewise polynomial differential systems with two zones by means of a transcritical bifurcation at infinity and from a global center.

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1 Introduction and statement of the main results

In the classical Qualitative Theory of Differential Equations it is usual to study the global behavior of the phase portrait of a given planar polynomial differential system by means of the Poincaré compactification [7]. When we apply this construction to a polynomial vector field G on $\mathbb{R}^2$ we obtain a new vector field on $\mathbb{S}^2 \backslash \mathbb{S}^1$ through the central projections and its extension $\mathscr{P}(G)$ to the Poincaré sphere $\mathbb{S}^2$ is everywhere analytic and analytically equivalent to G in each hemisphere.

The vector field $\mathscr{P}(G)$, called *Poincaré compactification* of G, has the equator $\mathbb{S}^1$ as an invariant set which can be either a periodic orbit, a connected union of singular points and arcs of $\mathbb{S}^1$, or even foliated by singular points. In addition, if $\mathbb{S}^1$ is a periodic orbit then it cannot be a semistable one since the central projections provide two identical copies of the dynamics of the vector field G each of them on one hemisphere of $\mathbb{S}^2$.

By means of another projection, for instance, the gnomonic projection such as in [7], we can study the vector field $\mathscr{D}(G)$ obtained by the projection of $\mathscr{P}(G)$ onto $\mathbb{D}$, where $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is the *Poincaré disc*. It follows that there exists a one-to-one correspondence between points placed at infinity of G and points on $\partial\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ of $\mathscr{D}(G)$. In this sense, we say $p \in \partial\mathbb{D}$ is a singular point at infinity of the vector field G, if $\mathscr{D}(G)(p) = 0$. When G has no singular points at infinity we say G has a periodic orbit at infinity which is identified with $\partial\mathbb{D}$.

The Poincaré compactification can be extended to piecewise polynomial differential systems in the plane with two (or more) zones whose separation boundary is a piecewise algebraic curve. Here we consider only the case in which the separation boundary is formed by a specific combination of two half-straight lines having an intersection point as follows.

Let $\Sigma_{\varphi,\beta} = H^{-1}_{\varphi,\beta}(\{0\})$ be the separation boundary defined from the two-parameter family $H_{\varphi,\beta}$ given by

$$H_{\varphi,\beta}(X) = L_\beta(X) M_{\varphi,\beta}(X), \quad X = (x, y) \in \mathbb{R}^2,$$

where

$$L_\beta(x, y) = \Lambda(x, y - \beta) + \frac{\pi}{2}, \quad M_{\varphi,\beta}(x, y) = \frac{\pi}{2} - \varphi - \Lambda(x, y - \beta), \tag{1}$$

$\varphi \in (-\pi, \pi)\backslash\{-\pi/2, 0, \pi/2\}$ and $\beta \in [0, \infty)$. The function Λ in (1) is based on the principal argument function [1] and it is defined by

$$\Lambda(u, v) = \begin{cases} \arctan\left(\dfrac{v}{u}\right), & u > 0, \\ \dfrac{\pi}{2}, & u = 0,\ v > 0, \\ \arctan\left(\dfrac{v}{u}\right) + \pi, & u < 0, \\ -\dfrac{\pi}{2}, & u = 0,\ v < 0, \\ 0, & u = 0,\ v = 0. \end{cases}$$

The set $\Sigma_{\varphi,\beta}$ is the union of two half-straight lines intersecting at $(0, \beta)$ and divides the plane into two unbounded connected components $\Sigma^-_{\varphi,\beta} = \{X \in \mathbb{R}^2 : H_{\varphi,\beta}(X) < 0\}$ and $\Sigma^+_{\varphi,\beta} = \{X \in \mathbb{R}^2 : H_{\varphi,\beta}(X) > 0\}$. One of these half-straight lines is $L_\beta^{-1}(\{0\})$, which corresponds to $x = 0$, $y \leq \beta$, and the other one is $M^{-1}_{\varphi,\beta}(\{0\})$, which is precisely $x - \alpha(y - \beta) = 0$ with (x, y) in a correspondent quadrant defined by the angle φ and $\alpha = \tan(\varphi) \in \mathbb{R}\backslash\{0\}$.

Due to compactification it is easier to work with the parameter α instead of φ. Thus, by an abuse of notation, we write $H_{\alpha,\beta}$ in place of $H_{\varphi,\beta}$ and henceforth we make the following identifications $\Sigma_{\alpha,\beta} \cong \Sigma_{\varphi,\beta}$, $\Sigma^-_{\alpha,\beta} \cong \Sigma^-_{\varphi,\beta}$ and $\Sigma^+_{\alpha,\beta} \cong \Sigma^+_{\varphi,\beta}$. The same occurs with the notation $M_{\alpha,\beta}$.

Consider the planar piecewise polynomial differential system with two zones $(G^-, G^+, \Sigma_{\alpha,\beta})$, that is

$$X' = G(X, \alpha, \beta) = \begin{cases} G^-(X),\ X \in \Sigma_{\alpha,\beta} \cup \Sigma^-_{\alpha,\beta}, \\ G^+(X),\ X \in \Sigma_{\alpha,\beta} \cup \Sigma^+_{\alpha,\beta}, \end{cases} \tag{2}$$

where X' is the derivative of $X = X(t)$ with respect to the variable t, called here time, and the polynomial vector fields

$$\begin{aligned} G^\pm : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ X &\longmapsto G^\pm(X) = (P^\pm(X), Q^\pm(X)) \end{aligned}$$

have degrees $n^\pm = \max\{\text{degree}(P^\pm), \text{degree}(Q^\pm)\}$ and satisfy $P^\pm(0,0) = Q^\pm(0,0) = 0$.

The following two assumptions are essential for the development of this article:

(A) There exists some $\alpha = \alpha_c$ so that the phase portrait of system (2) has a global center around the origin for $\beta = 0$, that is the set $\mathbb{R}^2\backslash\{(0,0)\}$ is filled with periodic orbits. In addition, there are no singular points placed at infinity;

(B) $K^-(\alpha_c, 1) \neq K^+(\alpha_c, 1)$, where

$$K^\pm(x,y) = \frac{P^\pm_{n^\pm}(x,y)x + Q^\pm_{n^\pm}(x,y)y}{P^\pm_{n^\pm}(x,y)y - Q^\pm_{n^\pm}(x,y)x}, \tag{3}$$

with $P^\pm_{n^\pm}$ and $Q^\pm_{n^\pm}$ the highest degree homogeneous polynomials of $P^\pm$ and $Q^\pm$, respectively.

As usual the dynamics of a planar piecewise differential system is ruled by the Filippov's convention. However, we will not deal with some common technical terms in this field of study like escape and sliding points or even sliding vector fields because they are not essential here. The interested reader may find a rigorous explanation of these terms (in addition to others) in [5].

It is enough to say that each point $p = (x_0, y_0) \in \Sigma_{\alpha,\beta}$, $p \neq (0, \beta)$, is of crossing (or sewing) type when $|\alpha - \alpha_c|$ and $\beta \geq 0$ are sufficiently small, meaning that

$$\big\langle N(p), G^+(p) \big\rangle \big\langle N(p), G^-(p) \big\rangle > 0, \tag{4}$$

where $N(p)$ is either $(1, 0)$, if $p \in L^{-1}_\beta(\{0\})$, or $(1, -\alpha)$, if $p \in M^{-1}_{\varphi,\beta}(\{0\})$, and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$. As a result by each point $p \in \Sigma_{\alpha,\beta}$, $p \neq (0, \beta)$, $\gamma_p = \gamma_p^- \cup \gamma_p^+$ is a crossing orbit (see Figure 1), where γ_p^- and γ_p^+ are orbits of the systems $X' = G^-(X)$ and $X' = G^+(X)$ through the crossing point p, restrict to the regions $\Sigma^-_{\alpha,\beta}$ and $\Sigma^+_{\alpha,\beta}$, respectively. So any periodic orbit of the global center treated in the hypothesis **(A)** is a crossing periodic orbit, that is a closed orbit γ_p satisfying $\gamma_p \cap \Sigma_{\alpha_c,0} = \{p, q\}$, where p and q are crossing points with $p \neq q$.

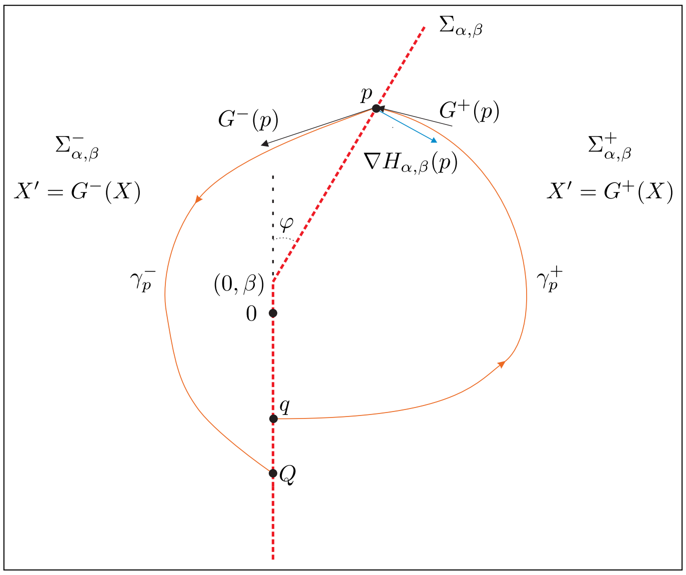

Figure 1: The set $\Sigma_{\alpha,\beta}$ is represented by the union of two thick dashed half-straight lines intersecting at the point $(0,\beta)$ with $\beta \geq 0$. Any point $p \in \Sigma_{\alpha,\beta}$, $p \neq (0,\beta)$, is of crossing type when $|\alpha - \alpha_c|$ and $\beta \geq 0$ are sufficiently small and hence the same holds for the points q and Q. The crossing orbit $\gamma_p = \gamma_p^- \cup \gamma_p^+$, illustrated by solid line, is generated by the flow of system $(G^-, G^+, \Sigma_{\alpha,\beta})$ through the point $p \in \Sigma_{\alpha,\beta}$. When $q = Q$, γ_p is a crossing periodic orbit.

If we apply the Poincaré compactification to the set $\Sigma_{\alpha,\beta} = H^{-1}_{\alpha,\beta}(\{0\})$, a new set $\mathscr{D}(\Sigma_{\alpha,\beta}) = \mathscr{D}(H_{\alpha,\beta})^{-1}(\{0\})$ is obtained on $\mathbb{D}$, where

$$\mathscr{D}(H_{\alpha,\beta})(X) = (1 - x^2 - y^2)H_{\alpha,\beta}(Z), \quad Z = \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}\right).$$

Also we have the vector field $\mathscr{D}(G)$ on $\mathbb{D}$ given by

$$\mathscr{D}(G)(X, \alpha, \beta) = \begin{cases} \mathscr{D}(G^-)(X), \ X \in \mathscr{D}(\Sigma_{\alpha,\beta}) \cup \mathbb{D}^-, \\ \mathscr{D}(G^+)(X), \ X \in \mathscr{D}(\Sigma_{\alpha,\beta}) \cup \mathbb{D}^+, \end{cases} \tag{5}$$

where

$$\begin{aligned} \mathbb{D}^- &= \{X \in \mathbb{R}^2 : \mathscr{D}(H_{\alpha,\beta})(X) < 0\} \cap \mathbb{D}, \\ \mathbb{D}^+ &= \{X \in \mathbb{R}^2 : \mathscr{D}(H_{\alpha,\beta})(X) > 0\} \cap \mathbb{D}, \end{aligned}$$

and

$$\mathscr{D}(G^\pm)(X) = \frac{1}{2}(1 - x^2 - y^2)^{n^\pm} \begin{pmatrix} (1 - x^2 + y^2)P^\pm(Z) - 2xyQ^\pm(Z) \\ -2xyP^\pm(Z) + (1 + x^2 - y^2)Q^\pm(Z) \end{pmatrix}.$$

The Poincaré compactified system on $\mathbb{D}$ corresponding to $(G^-, G^+, \Sigma_{\alpha,\beta})$ will be denoted by $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$. We emphasize that system $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ can be studied in the whole $\mathbb{R}^2$ considering its natural extension but we are mainly interested in its dynamical behavior on the compact set $\mathbb{D}_\eta = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq \eta\}$, where $\eta \geq 1$ is defined by

$$\eta = \begin{cases} 1, & \text{if } \beta = 0, \\ \dfrac{1}{\beta} + \dfrac{\sqrt{1+\beta^2}}{\beta}, & \text{if } 0 < \beta < 1. \end{cases}$$

From the hypothesis **(A)**, we have the first result.

Theorem 1. *Suppose that system* (2) *satisfies the hypothesis* **(A)**. *For* $|\alpha - \alpha_c|$ *and* $\beta \geq 0$ *sufficiently small,* $\partial\mathbb{D}$ *is a periodic orbit of system* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$. *Moreover,* $C = \mathscr{D}(M_{\alpha,\beta})^{-1}(\{0\})$ *is a transversal section for the flow of system* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ *on* $\mathbb{D}_\eta$, *where*

$$\mathscr{D}(M_{\alpha,\beta})(x, y) = 2x - \alpha(2y - \beta(1 - x^2 - y^2)). \tag{6}$$

Thus the previous theorem allows to define a Poincaré map or first return map $\Pi_{\alpha,\beta} : L \to L$ and study the dynamics of system $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ for $|\alpha - \alpha_c|$ and $\beta > 0$ sufficiently small, where $L \subset C$ is a small arc of $C \cap \partial\mathbb{D}$. In this line, our second result is the following one.

Theorem 2. *Suppose that system* (2) *satisfies the hypothesis* **(A)**. *There exists* $\alpha \neq \alpha_c$ *such that the Poincaré map* $\Pi_{\alpha,\beta} : L \to L$ *defined by system* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ *has a fixed point which corresponds to a limit cycle in the phase portrait of* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$, *restricted to* $\mathbb{D}\backslash\partial\mathbb{D}$, *when* $|\alpha - \alpha_c| > 0$ *and* $\beta > 0$ *are sufficiently small.*

The main question to be answered here is how the limit cycle of Theorem 2, appears or disappears in the phase portrait of the vector field (5). In other words, what is the mechanism of generation of limit cycles in the phase portrait of the vector field (5) when we change the parameters α and β in $\Sigma_{\alpha,\beta}$? According to Theorem 3 such mechanism is a transcritical bifurcation of the Poincaré map associated with system $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ close to the periodic orbit $\partial\mathbb{D}$.

Theorem 3. *Suppose that system* (2) *satisfies the hypotheses* **(A)** *and* **(B)**. *Then the Poincaré map* $\Pi_{\alpha,\beta} : L \to L$ *defined by system* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ *undergoes a transcritical bifurcation for* $\alpha = \alpha_c$ *and* $\beta > 0$ *sufficiently small. In other words, there exist two hyperbolic limit cycles in the phase portrait of system* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ *when* $|\alpha - \alpha_c| > 0$ *is sufficiently small, one on* $\mathbb{D}_\eta\backslash\mathbb{D}$ *and the other one being* $\partial\mathbb{D}$. *These limit cycles collide for* $\alpha = \alpha_c$ *and they exist and change their stabilities after this.*

This article is organized as follows. We provide the proofs of Theorems 1, 2 and 3 in Section 2. In Section 3 we analyze examples in Subsections 3.1 and 3.2 involving piecewise homogeneous linear differential systems and third-degree piecewise polynomial differential systems having the origin as a singular point of focus-focus and center-center type, respectively. We also perform some numerical simulations. A degenerate case dealing with an example of second-degree piecewise polynomial differential systems is studied in Section 4 following the same procedure as in Section 3 or, more precisely, that of Subsection 3.1. The article ends with some concluding remarks in Section 5.

2 Proofs of Theorems 1, 2 and 3

The main properties of the planar piecewise polynomial differential system corresponding to the vector field (5) will be obtained performing the change to polar coordinates

$$(r,s) \in (0,1] \times [0,2\pi] \longrightarrow (r\cos(s), r\sin(s)) = (x,y) \in \mathbb{D}_\eta. \tag{7}$$

By means of this change of coordinates, the differential equations $X' = \mathscr{D}(G^-)(X)$ and $X' = \mathscr{D}(G^+)(X)$ are transformed, after a time reparametrization, in $r' = F^-(s,r)$ and $r' = F^+(s,r)$, respectively, where $r' = dr/ds$ and $F^\pm(s,r) = R^\pm(r,s)/S^\pm(r,s)$. The functions $R^\pm$ and $S^\pm$ are given by

$$\begin{aligned} R^\pm(s,r) &= \frac{1}{2}(1-r^2)^{n^\pm+1}\left(P^\pm(Z)\cos(s) + Q^\pm(Z)\sin(s)\right), \\ S^\pm(s,r) &= \frac{1+r^2}{2r}(1-r^2)^{n^\pm}\left(Q^\pm(Z)\cos(s) - P^\pm(Z)\sin(s)\right), \end{aligned} \tag{8}$$

where

$$Z = \left(\frac{2r\cos(s)}{1-r^2}, \frac{2r\sin(s)}{1-r^2}\right).$$

Since $P^\pm$ and $Q^\pm$ are polynomial functions in two variables of degrees $n^\pm$, we can write

$$\begin{aligned} P^\pm(x,y) &= P^\pm_{m^\pm}(x,y) + P^\pm_{m^\pm+1}(x,y) + \cdots + P^\pm_{n^\pm}(x,y), \\ Q^\pm(x,y) &= Q^\pm_{m^\pm}(x,y) + Q^\pm_{m^\pm+1}(x,y) + \cdots + Q^\pm_{n^\pm}(x,y), \end{aligned}$$

where $P^\pm_i$ and $Q^\pm_i$ are homogeneous polynomials in two variables of degrees $i = m^\pm, \ldots, n^\pm$, with $m^{\pm} \leq n^{\pm}$. So the expressions in (8) are rewritten as

$$
\begin{aligned}
R^{\pm}(s,r) &= \frac{1}{2}\sum_{i=m^{\pm}}^{n^{\pm}} 2^{i}(1-r^2)^{n^{\pm}-i+1} M_i^{\pm}(s,r),\\
S^{\pm}(s,r) &= \frac{1+r^2}{2r}\sum_{i=m^{\pm}}^{n^{\pm}} 2^{i}(1-r^2)^{n^{\pm}-i} N_i^{\pm}(s,r),
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
M_i^{\pm}(s,r) &= P_i^{\pm}(r\cos(s), r\sin(s))\cos(s) + Q_i^{\pm}(r\cos(s), r\sin(s))\sin(s),\\
N_i^{\pm}(s,r) &= Q_i^{\pm}(r\cos(s), r\sin(s))\cos(s) - P_i^{\pm}(r\cos(s), r\sin(s))\sin(s).
\end{aligned}
$$

Instead of studying the properties of the Poincaré map $\Pi_{\alpha,\beta} : L \to L$, it is often easier to work in polar coordinates with an equivalent function called displacement function. Note that the equation $\mathscr{D}(M_{\alpha,\beta})(a\cos(s), a\sin(s)) = 0$ defines implicitly a unique function

$$
s = \phi(a,\alpha,\beta), \quad \mod 2\pi, \quad \forall (a,\alpha,\beta) \in D,
\tag{10}
$$

where $a = \sqrt{x_0^2 + y_0^2}$ and (x_0, y_0) is any point in L, for $|\alpha - \alpha_c|$ and $\beta \geq 0$ sufficiently small. The domain is $D = I_a \times I_\alpha \times I_\beta$, where I_a, I_α and I_β are small appropriated intervals containing $r = 1$, $\alpha = \alpha_c$ and $\beta = 0$, respectively.

From the function ϕ, we can define the *displacement function* Δ associated with the vector field (5) by

$$
\Delta(a,\alpha,\beta) = r^{-}\left(\frac{3\pi}{2}, \phi(a,\alpha,\beta), a\right) - r^{+}\left(\frac{3\pi}{2}, 2\pi + \phi(a,\alpha,\beta), a\right), \ (a,\alpha,\beta) \in D,
\tag{11}
$$

where $r^{\pm} = r^{\pm}(s, s_0, r_0)$ are the maximal solutions of the following two Cauchy problems, namely

$$
\begin{cases}
\dfrac{\partial}{\partial s} r^{\pm}(s,s_0,r_0) = F^{\pm}(s, r^{\pm}(s,s_0,r_0)) = \dfrac{R^{\pm}(s, r^{\pm}(s,s_0,r_0))}{S^{\pm}(s, r^{\pm}(s,s_0,r_0))},\\
r^{\pm}(s_0,s_0,r_0) = r_0.
\end{cases}
\tag{12}
$$

See Figure 2 for details about the function ϕ and the displacement function Δ.

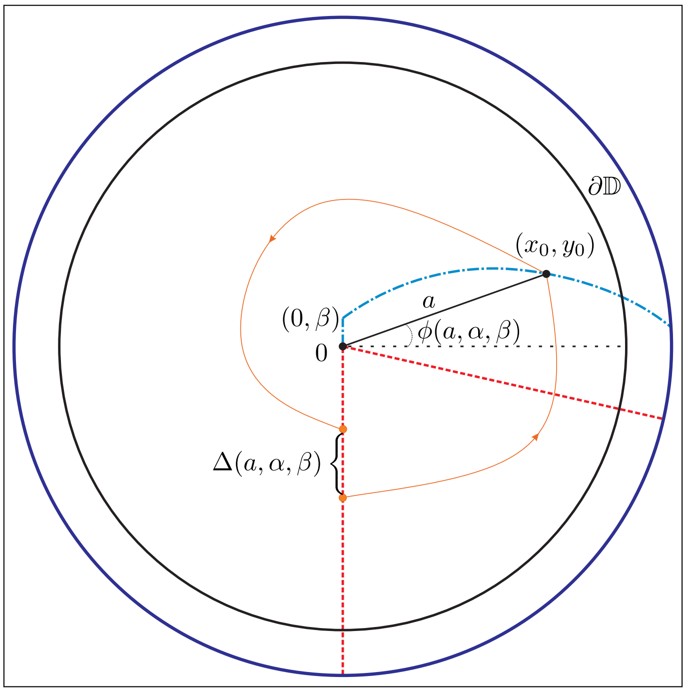

Figure 2: Construction of the function ϕ and of the displacement function Δ as in (11). The set $\Sigma_{\alpha,\beta}$, for $\alpha = \alpha_c$ and $\beta = 0$, is represented by the union of the two thick dashed line segments. The union of the vertical thick dashed line segment and the thick dash-dot curve illustrates the set $\Sigma_{\alpha,\beta}$ when $\beta > 0$ is sufficiently small. The boundary $\partial\mathbb{D}$ of the Poincaré disc $\mathbb{D}$ is illustrated by the inner circle and the outer circle represents the boundary of the compact set $\mathbb{D}_\eta$.

2.1 Proofs of Theorems 1 and 2

The first part of Theorem 1 follows from (8). The second one is based on the hypothesis **(A)** and the continuity arguments on (4) when $|\alpha - \alpha_c|$ and $\beta \geq 0$ are small enough.

From (6) any intersection point between the sets $\mathscr{D}(M_{\alpha_c,0})$ and $\mathscr{D}(M_{\alpha,\beta})$ is a solution of the following nonlinear system

$$\begin{cases} 2x - \alpha\left(2y - \beta(1 - x^2 - y^2)\right) = 0, \\ x - \alpha_c y = 0. \end{cases}$$

Fix a value of $\beta > 0$ small enough. Since the sets $\mathscr{D}(M_{\alpha_c,0})$ and $\mathscr{D}(M_{\alpha_c,\beta})$ have transversal intersection on $\partial\mathbb{D}$ when $\alpha = \alpha_c$, the same holds for $|\alpha - \alpha_c| > 0$ sufficiently small. Moreover, there exists α such that the intersection point belongs to $\mathbb{D}\backslash\partial\mathbb{D}$.

Denote by $p_{\alpha,\beta} = (x_{\alpha,\beta}, y_{\alpha,\beta})$ the corresponding intersection point for $|\alpha - \alpha_c| > 0$ sufficiently small. We can rewritten it in polar coordinates as

$$p_{\alpha,\beta} = (a_{\alpha,\beta}\cos(s_{\alpha,\beta}), a_{\alpha,\beta}\sin(s_{\alpha,\beta}))$$

with $a_{\alpha,\beta} = \sqrt{x_{\alpha,\beta}^2 + y_{\alpha,\beta}^2}$ and $s_{\alpha,\beta} = \phi(a_{\alpha,\beta}, \alpha, \beta)$, where ϕ is defined in (10). Thus from hypothesis **(A)** and (11), it follows that $\Delta(a_{\alpha_c,0}, \alpha_c, 0) = 0$. But since $p_{\alpha,\beta} \in \mathscr{D}(\Sigma_{\alpha_c,0}) \cap \mathscr{D}(\Sigma_{\alpha,\beta})$, we conclude that $a_{\alpha,\beta} = a_{\alpha_c,0}$, $s_{\alpha,\beta} = s_{\alpha_c,0}$ and $\Delta(a_{\alpha,\beta}, \alpha, \beta) = 0$. Therefore, $p_{\alpha,\beta} \in L$ and $\Pi_{\alpha,\beta}(p_{\alpha,\beta}) = p_{\alpha,\beta}$ and this completes the proof of Theorem 2.

2.2 Proof of Theorem 3

Under the hypothesis **(A)** and using (11), we generalize an analogous result of [3] to our planar piecewise polynomial differential system (2).

Theorem 4. *Suppose that system* (2) *satisfies the hypothesis* **(A)**. *For* $|\alpha - \alpha_c|$ *and* $\beta \geq 0$ *sufficiently small, the first order partial derivatives of the displacement function* (11) *with respect to* a *and* α, *evaluated at* $(1, \alpha, \beta)$, *are given by*

$$\begin{aligned} \Delta_a(1, \alpha, \beta) &= \kappa(\alpha, \beta)\left(\exp(\rho(1, \alpha, \beta)) - 1\right), \\ \Delta_\alpha(1, \alpha, \beta) &= 0, \end{aligned} \tag{13}$$

where κ *is some positive function,* $\phi(1, \alpha, \beta) = \pi/2 - \arctan\alpha$ *and*

$$\rho(1, \alpha, \beta) = \int_{\phi(1,\alpha,\beta)}^{\frac{3\pi}{2}} K^-\left(\cos(u), \sin(u)\right) \mathrm{d}u + \int_{\frac{3\pi}{2}}^{2\pi+\phi(1,\alpha,\beta)} K^+\left(\cos(u), \sin(u)\right) \mathrm{d}u \tag{14}$$

with the functions $K^\pm$ *defined in* (3).

Proof. Once the solutions $r^\pm = r^\pm(s, s_0, r_0)$ of the Cauchy problems (12) are smooth functions on $[0, 2\pi] \times [0, 2\pi] \times I_a$, we can differentiate partially the function (11) with respect to the variables a and α obtaining

$$\begin{aligned} \Delta_a(a, \alpha, \beta) &= \Delta^{s_0}(a, \alpha, \beta)\phi_a(a, \alpha, \beta) + \Delta^{r_0}(a, \alpha, \beta), \\ \Delta_\alpha(a, \alpha, \beta) &= \Delta^{s_0}(a, \alpha, \beta)\phi_\alpha(a, \alpha, \beta), \end{aligned} \tag{15}$$

where

$$\Delta^k(a, \alpha, \beta) = r_k^-\left(\frac{3\pi}{2}, \phi(a, \alpha, \beta), a\right) - r_k^+\left(\frac{3\pi}{2}, 2\pi + \phi(a, \alpha, \beta), a\right), \tag{16}$$

and $r_k^\pm = \partial r^\pm/\partial k$, $k \in \{s_0, r_0\}$. In addition, $r_{s_0}^\pm = r_{s_0}^\pm(s, s_0, r_0)$ and $r_{r_0}^\pm = r_{r_0}^\pm(s, s_0, r_0)$ are solutions of

$$\frac{\partial}{\partial s} r_k^\pm(s, s_0, r_0) = \frac{\partial}{\partial r} F^\pm(s, r^\pm(s, s_0, r_0)) r_k^\pm(s, s_0, r_0), \quad k \in \{r_0, s_0\}, \tag{17}$$

with the initial conditions

$$r_{s_0}^\pm(s_0, s_0, r_0) = -\left.\frac{\partial}{\partial s} r^\pm(s, s_0, r_0)\right|_{s=s_0} = -F^\pm(s_0, r_0), \tag{18}$$

$$r_{r_0}^\pm(s_0, s_0, r_0) = 1, \tag{19}$$

due to the smooth dependence of the solutions of the Cauchy problems (12) on parameters and initial conditions.

Note that $r^\pm(s, s_0, 1) = 1$ for all $(s, s_0) \in [0, 2\pi] \times [0, 2\pi]$, since $\partial\mathbb{D}$ is invariant under the flow of system $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$. Therefore, from (9), $F^\pm(s_0, 1) = 0$ for any $s_0 \in [0, 2\pi]$ and $r_{s_0}^\pm(s, s_0, 1) = 0$ for all $(s, s_0) \in [0, 2\pi] \times [0, 2\pi]$. It follows from (16) that the function Δ^{s_0} is identically zero when $a = 1$ and the equalities in (15) ensure that $\Delta_a(1, \alpha, \beta) = \Delta^{r_0}(1, \alpha, \beta)$ and $\Delta_\alpha(1, \alpha, \beta) = 0$. It only remains to find an expression for $\Delta^{r_0}(1, \alpha, \beta)$.

From (9) $R^\pm(s, 1) = 0$ for all $s \in [0, 2\pi]$ and an easy calculation shows that

$$\begin{aligned} \left.\frac{\partial}{\partial r} F^\pm(s, r)\right|_{r=1} &= \left.\frac{\partial}{\partial r}\left(\frac{R^\pm(s, r)}{S^\pm(s, r)}\right)\right|_{r=1} \\ &= \left.\frac{R_r^\pm(s, r) S^\pm(s, r) - R^\pm(s, r) S_r^\pm(s, r)}{S^\pm(s, r)^2}\right|_{r=1} \\ &= \frac{R_r^\pm(s, 1)}{S^\pm(s, 1)} = \frac{M_{n^\pm}^\pm(s, 1)}{N_{n^\pm}^\pm(s, 1)}. \end{aligned} \tag{20}$$

So if we define the functions $A^\pm$ as

$$A^\pm(s) = \left.\frac{\partial}{\partial r} F^\pm(s, r)\right|_{r=1} = \frac{M_{n^\pm}^\pm(s, 1)}{N_{n^\pm}^\pm(s, 1)} = K^\pm(\cos(s), \sin(s)), \quad s \in [0, 2\pi], \tag{21}$$

then, solving (17) with (19) when $r_0 = 1$, it follows that

$$r_{r_0}^\pm(s, s_0, 1) = \exp\left(\int_{s_0}^{s} A^\pm(u)\,\mathrm{d}u\right), \quad \forall (s, s_0) \in [0, 2\pi] \times [0, 2\pi],$$

and the theorem is proved after some straightforward calculations. ■

The stability of the infinity $\partial\mathbb{D}$ is a consequence of the previous theorem.

Corollary 1. *Suppose that* $|\alpha - \alpha_c| > 0$ *and* $\beta > 0$ *are sufficiently small and system* (2) *satisfies the hypothesis* **(A)**. *Then the phase portrait of system* $\mathscr{D}(G^-, G^+, \Sigma_{\alpha,\beta})$ *has* $\partial\mathbb{D}$ *as:*

(a) *An atractor periodic orbit when* $\rho(1, \alpha, \beta) < 0$*;*

(b) *A repellor periodic orbit when* $\rho(1, \alpha, \beta) > 0$.

Proof. The result follows from the sign of (13). ■

The case $\Delta_a(1, \alpha, \beta) = 0$ is not covered by Corollary 1 but it is related to $\alpha = \alpha_c$ and may imply the existence of a semistable periodic orbit. In fact, from hypothesis **(A)** it follows that $\Delta_a(1, \alpha_c, 0) = 0$ but since the expression for ρ given in (14) does not depend on β we conclude that $\Delta_a(1, \alpha_c, \beta) = 0$ for $\beta > 0$ sufficiently small. The proof that $\partial\mathbb{D}$ is a semistable periodic orbit when $\alpha = \alpha_c$ and $\beta > 0$ is sufficiently small is based on the next theorem and the Transcritical Bifurcation Theorem for maps (see [8] for more details).

As stated in the introduction the phase portrait on $\mathbb{S}^2$ of a compactified vector field cannot have the equator $\mathbb{S}^1$ as a semistable periodic orbit. Thus the existence of a semistable periodic orbit in the phase portrait of the compactification of system (2), when $\alpha = \alpha_c$ and $\beta > 0$ is sufficiently small, is a consequence of the study done not in $\mathbb{S}^2$ but in $\mathbb{D}_\eta$ when $\eta > 1$.

Theorem 5. *Suppose that system* (2) *satisfies the hypotheses* **(A)** *and* **(B)**. *For* $\alpha = \alpha_c$ *and* $\beta \geq 0$ *sufficiently small, the following second order partial derivatives of the displacement function* (11) *with respect to* a *and* α*, evaluated at* $(1, \alpha_c, \beta)$*, satisfy*

$$\Delta_{aa}(1, \alpha_c, \beta) \neq 0,$$
$$\Delta_{a\alpha}(1, \alpha_c, \beta) \neq 0.$$

Proof. We follow the same line of the proof of Theorem 4. First we differentiate partially the equalities (15) with respect to the variable a. By doing this we obtain

$$\begin{aligned}\Delta_{aa}(a, \alpha, \beta) = {} & \Delta^{s_0}(a, \alpha, \beta)\phi_{aa}(a, \alpha, \beta) + \Delta^{s_0 s_0}(a, \alpha, \beta)\phi_a(a, \alpha, \beta)^2 + \\ & 2\Delta^{s_0 r_0}(a, \alpha, \beta)\phi_a(a, \alpha, \beta) + \Delta^{r_0 r_0}(a, \alpha, \beta),\end{aligned}$$

$$\begin{aligned}\Delta_{a\alpha}(a, \alpha, \beta) = {} & \Delta^{s_0}(a, \alpha, \beta)\phi_{a\alpha}(a, \alpha, \beta) + \Delta^{s_0 s_0}(a, \alpha, \beta)\phi_a(a, \alpha, \beta) + \\ & \Delta^{s_0 r_0}(a, \alpha, \beta)\phi_\alpha(a, \alpha, \beta),\end{aligned}$$

where Δ^{s_0} and Δ^{r_0} are given in (16) and

$$\Delta^{jk}(a, \alpha, \beta) = r^-_{jk}\left(\frac{3\pi}{2}, \phi(a, \alpha, \beta), a\right) - r^+_{jk}\left(\frac{3\pi}{2}, 2\pi + \phi(a, \alpha, \beta), a\right),$$

with $r^{\pm}_{jk} = \partial^2 r^{\pm}/\partial j \partial k$ for $j,k \in \{s_0, r_0\}$. The functions $r^{\pm}_{s_0} = r^{\pm}_{s_0}(s, s_0, r_0)$ are solutions of the equations in (17) with the initial conditions (18) and, in addition, the functions $r^{\pm}_{r_0 r_0} = r^{\pm}_{r_0 r_0}(s, s_0, r_0)$, $r^{\pm}_{s_0 s_0} = r^{\pm}_{s_0 s_0}(s, s_0, r_0)$ and $r^{\pm}_{s_0 r_0} = r^{\pm}_{s_0 r_0}(s, s_0, r_0)$ are solutions of

$$
\begin{aligned}
\frac{\partial}{\partial s} r^{\pm}_{jk}(s, s_0, r_0) = {} & \frac{\partial}{\partial r} F^{\pm}(s, r^{\pm}(s, s_0, r_0)) r^{\pm}_{jk}(s, s_0, r_0) + \\
& \frac{\partial^2}{\partial r^2} F^{\pm}(s, r^{\pm}(s, s_0, r_0)) r^{\pm}_{j}(s, s_0, r_0) r^{\pm}_{k}(s, s_0, r_0), \quad j,k \in \{r_0, s_0\},
\end{aligned} \tag{22}
$$

with the initial conditions

$$
\begin{aligned}
r^{\pm}_{r_0 r_0}(s_0, s_0, r_0) &= 0, \\
r^{\pm}_{s_0 s_0}(s_0, s_0, r_0) &= - \left. \frac{\partial^2}{\partial s^2} r^{\pm}(s, s_0, r_0) \right|_{s=s_0} - 2 \left. \frac{\partial}{\partial s} r^{\pm}_{s_0}(s, s_0, r_0) \right|_{s=s_0} \\
&= -\frac{\partial}{\partial s} F^{\pm}(s_0, r_0) - \frac{\partial}{\partial r} F^{\pm}(s_0, r_0) F^{\pm}(s_0, r_0) + 2 \frac{\partial}{\partial r} F^{\pm}(s_0, r_0) F^{\pm}(s_0, r_0) \\
&= -\frac{\partial}{\partial s} F^{\pm}(s_0, r_0) + \frac{\partial}{\partial r} F^{\pm}(s_0, r_0) F^{\pm}(s_0, r_0), \\
r^{\pm}_{s_0 r_0}(s_0, s_0, r_0) &= - \left. \frac{\partial}{\partial s} r^{\pm}_{r_0}(s, s_0, r_0) \right|_{s=s_0} \\
&= -\frac{\partial}{\partial r} F^{\pm}(s_0, r_0),
\end{aligned}
$$

respectively, all of them obtained from (12), (17), (18) and (19).

Note that differentiating partially $F^{\pm}$ with respect to the variable s we obtain

$$
\left. \frac{\partial}{\partial s} F^{\pm}(s, r) \right|_{r=1} = \frac{R^{\pm}_s(s,1)}{S^{\pm}(s,1)} = 0.
$$

Thus from the previous calculation and from (20) the initial conditions are

$$
r^{\pm}_{r_0 r_0}(s_0, s_0, 1) = 0, \quad r^{\pm}_{s_0 s_0}(s_0, s_0, 1) = 0, \quad r^{\pm}_{s_0 r_0}(s_0, s_0, 1) = A^{\pm}(s_0), \tag{23}
$$

when $r_0 = 1$ because $F^{\pm}(s_0, 1) = 0$.

Now we are able to solve the equations in (22) with the initial conditions (23) taking into account that

$$\begin{aligned}
\left.\frac{\partial^2}{\partial r^2}F^{\pm}(s,r)\right|_{r=1} &= \left.\frac{\partial^2}{\partial r^2}\left(\frac{R^{\pm}(s,r)}{S^{\pm}(s,r)}\right)\right|_{r=1} \\
&= \frac{R^{\pm}_{rr}(s,1)}{S^{\pm}(s,1)} - \frac{2R^{\pm}_{r}(s,1)S^{\pm}_{r}(s,1)}{S^{\pm}(s,1)^2} \\
&= \frac{2M^{\pm}_{n^{\pm}-1}(s,1) - 2(M^{\pm}_{n^{\pm}})_r(s,1) - M^{\pm}_{n^{\pm}}(s,1)}{N^{\pm}_{n^{\pm}}(s,1)} - \\
&\quad \frac{2M^{\pm}_{n^{\pm}}(s,1)(N^{\pm}_{n^{\pm}-1}(s,1) - (N^{\pm}_{n^{\pm}})_r(s,1))}{N^{\pm}_{n^{\pm}}(s,1)^2},
\end{aligned}$$

where $(M^{\pm}_{n^{\pm}})_r = \partial M^{\pm}_{n^{\pm}}/\partial r$ and $(N^{\pm}_{n^{\pm}})_r = \partial N^{\pm}_{n^{\pm}}/\partial r$.

If we define the functions $B^{\pm}$ as

$$\begin{aligned}
B^{\pm}(s) = \left.\frac{\partial^2}{\partial r^2}F^{\pm}(s,r)\right|_{r=1} &= \frac{2M^{\pm}_{n^{\pm}-1}(s,1) - 2(M^{\pm}_{n^{\pm}})_r(s,1) - M^{\pm}_{n^{\pm}}(s,1)}{N^{\pm}_{n^{\pm}}(s,1)} - \\
&\quad \frac{2M^{\pm}_{n^{\pm}}(s,1)(N^{\pm}_{n^{\pm}-1}(s,1) - (N^{\pm}_{n^{\pm}})_r(s,1))}{N^{\pm}_{n^{\pm}}(s,1)^2}, \qquad s \in [0,2\pi],
\end{aligned}$$

then

$$\begin{aligned}
r^{\pm}_{r_0 r_0}(s,s_0,1) &= \int_{s_0}^{s} B^{\pm}(u)\exp\left(-\int_{s}^{u} A^{\pm}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}u, \\
r^{\pm}_{s_0 s_0}(s,s_0,1) &= 0, \\
r^{\pm}_{s_0 r_0}(s,s_0,1) &= \exp\left(\int_{s_0}^{s} A^{\pm}(u)\,\mathrm{d}u\right)A^{\pm}(s_0), \qquad s \in [0,2\pi],
\end{aligned}$$

where $A^{\pm}$ are defined in (21).

Differentiating implicitly the equation

$$\mathscr{D}(M_{\alpha,\beta})\left(a\cos(\phi(a,\alpha,\beta)), a\sin(\phi(a,\alpha,\beta))\right) = 0,$$

with respect to the variables a and α and evaluating at $a = 1$ we obtain

$$\phi_a(1,\alpha,\beta) = -\frac{\beta\alpha}{\sqrt{1+\alpha^2}}, \quad \phi_\alpha(1,\alpha,\beta) = -\frac{1}{1+\alpha^2},$$

where $\mathscr{D}(M_{\alpha,\beta})$ is given in (6). So for $(a,\alpha,\beta) = (1,\alpha_c,\beta)$, the function $\Delta^{s_0 s_0}$ is identically zero and

$$\Delta^{s_0 r_0}(1,\alpha_c,\beta) = \exp\left(\int_{2\pi+\phi(1,\alpha_c,\beta)}^{\frac{3\pi}{2}} A^{+}(u)\,\mathrm{d}u\right)\big(K^{-}(\alpha_c,1) - K^{+}(\alpha_c,1)\big) \neq 0.$$

Moreover, we have $\phi_a(1,\alpha_c,0)=0$ and, therefore, $\Delta_{aa}(1,\alpha_c,0)=\Delta^{r_0r_0}(1,\alpha_c,0)=0$, because in this case we have a global center. But the expression for $\Delta^{r_0r_0}$ does not depend on β which implies that $\Delta^{r_0r_0}(1,\alpha_c,\beta)=0$ for $\beta>0$ sufficiently small and we conclude the proof of the theorem. ■

Theorems 4 and 5 lead to the existence of a transcritical bifurcation of the Poincaré map $\Pi_{\alpha,\beta}: L\to L$ defined by system $\mathscr{D}(G^-,G^+,\Sigma_{\alpha,\beta})$ when $\alpha-\alpha_c$ change its sign and hence $\partial\mathbb{D}$ is a semistable periodic orbit for $\alpha=\alpha_c$ and $\beta>0$ sufficiently small. In fact, the outcomes in Theorems 4 and 5 allow to show that, for $a=1$, $\alpha=\alpha_c$, and $\beta>0$ sufficiently small, the parametrized Poincaré map defined by system $\mathscr{D}(G^-,G^+,\Sigma_{\alpha,\beta})$ and denoted also by $\Pi_{\alpha,\beta}$ satisfies

$$\Pi_{\alpha,\beta}(1)=1,\quad \frac{\partial}{\partial a}\Pi_{\alpha,\beta}(1)=1,\quad \frac{\partial}{\partial \alpha}\Pi_{\alpha,\beta}(1)=0,\quad \frac{\partial^2}{\partial a^2}\Pi_{\alpha,\beta}(1)\neq 0,\quad \frac{\partial^2}{\partial a\partial\alpha}\Pi_{\alpha,\beta}(1)\neq 0.$$

Therefore, Theorem 3 is proved.

3 Nondegenerate cases

In this section we study two examples of systems that exhibit the dynamical behavior shown in Theorem 3. Also we perform some numerical calculations and when it is necessary the results are displayed with only six decimals.

Firstly, to prove that both systems satisfy the latter part of the hypothesis **(A)**, we resort to the Poincaré compactification and to the polar coordinate change (7) to show that the infinity is indeed a periodic orbit. After that in Subsection 3.1 we obtain the conditions for the existence of a global center by analyzing the solutions of the same compactified system in polar coordinates except by a time reparametrization and a suitable time rescaling in the angular variable. This is equivalent to the study of the original system (noncompactified one) in polar coordinates. Keeping this in mind in Subsection 3.1 we use the expression in (11) for the displacement function but with a different domain $D_R=\{(a,\alpha,\beta): a\in[0,\infty),\alpha\in\mathbb{R}\backslash\{0\},\beta\in[0,\infty)\}$.

In Subsection 3.2 we take a different approach showing the existence of a global center from the analysis of the displacement function constructed through the properties of the Hamiltonian systems.

3.1 Piecewise homogeneous linear differential systems

Consider the planar piecewise homogeneous linear systems with two zones of the form

$$X' = G(X, \alpha, \beta) = \begin{cases} A^- X, \ X \in \Sigma_{\alpha,\beta} \cup \Sigma^-_{\alpha,\beta}, \\ A^+ X, \ X \in \Sigma_{\alpha,\beta} \cup \Sigma^+_{\alpha,\beta}, \end{cases} \tag{24}$$

where $\Sigma_{\alpha,\beta}$, $\Sigma^-_{\alpha,\beta}$ and $\Sigma^+_{\alpha,\beta}$ are the sets defined in system (2). We assume the following assumption:

(M) The real matrices

$$A^\pm = \begin{pmatrix} a^\pm_{11} & a^\pm_{12} \\ a^\pm_{21} & a^\pm_{22} \end{pmatrix}$$

have complex eigenvalues $\lambda^\pm = \mu^\pm \pm i\omega^\pm$ and $\bar{\lambda}^\pm$, with $\mu^\pm, \omega^\pm \in \mathbb{R}$, $\mu^-\mu^+ < 0$ and $\omega^\pm > 0$, where the bar stands for the complex conjugate. Moreover, $a^-_{12}a^+_{12} > 0$.

Proceeding as in Section 3 and by means of the polar coordinate change (7), we can rewrite the compactified systems, denoted by $X' = \mathscr{D}(A^-X)$ and $X' = \mathscr{D}(A^+X)$, as

$$r' = R^\pm(s, r) = \frac{dr}{dt} = r(1 - r^2)g^\pm(s), \quad s' = S^\pm(s, r) = \frac{ds}{dt} = 1 + r^2, \tag{25}$$

or through a time reparametrization as

$$r' = \frac{dr}{ds} = F^\pm(s, r) = \frac{r(1 - r^2)}{1 + r^2} g^\pm(s), \tag{26}$$

with the functions $g^\pm$ given by

$$g^\pm(s) = \frac{a^\pm_{11} + a^\pm_{22} + (a^\pm_{11} - a^\pm_{22})\cos(2s) + (a^\pm_{12} + a^\pm_{21})\sin(2s)}{a^\pm_{21} - a^\pm_{12} + (a^\pm_{12} + a^\pm_{21})\cos(2s) + (a^\pm_{22} - a^\pm_{11})\sin(2s)}, \quad s \in [0, 2\pi]. \tag{27}$$

In addition, performing the time rescaling $ds = ((1 - r^2)/(1 + r^2))d\bar{s}$ on $\mathbb{D}\backslash\partial\mathbb{D}$, omitting the bar, and using the same notation $F^\pm$, we obtain $r^\pm = r^\pm(s, s_0, r_0)$ as solutions of

$$r' = \frac{dr}{ds} = F^\pm(s, r) = rg^\pm(s), \tag{28}$$

where

$$r^\pm(s, s_0, r_0) = \exp(h^\pm(s, s_0))r_0, \tag{29}$$

and

$$h^{\pm}(s,s_0)=\int_{s_0}^{s} g^{\pm}(u)\,\mathrm{d}u,\quad (s,s_0)\in[0,2\pi]\times[0,2\pi].$$

Note that (28) is just the rewriting of the linear systems $X'=A^{-}X$ and $X'=A^{+}X$ in polar coordinates.

Thus from (29) and (11) with the domain D_R we obtain

$$\Delta(a,\alpha,\beta)=r^{+}\left(\frac{3\pi}{2},2\pi+\theta,a\right)\left(\exp(h(\theta))-1\right), \tag{30}$$

where $\theta=\phi(a,\alpha,\beta)$ and the function h is defined by

$$\begin{aligned} h(\theta) &= h^{-}\left(\frac{3\pi}{2},\theta\right)-h^{+}\left(\frac{3\pi}{2},2\pi+\theta\right)\\ &= \int_{\theta}^{\frac{3\pi}{2}} g^{-}(u)\,\mathrm{d}u+\int_{\frac{3\pi}{2}}^{2\pi+\theta} g^{+}(u)\,\mathrm{d}u,\quad \theta\in I=\left(-\frac{\pi}{2},\frac{3\pi}{2}\right). \end{aligned} \tag{31}$$

Based on the hypothesis **(M)**, the origin of system (24) can be either an attractor or a repellor focus or even a center depending on the values of α when $\beta=0$. We are mainly interested in the latter case according to the next lemma.

Lemma 1. *If system* (24) *satisfies the hypothesis* **(M)** *and $\beta=0$, then there exists $\alpha=\alpha_c$ such that the phase portrait of* (24) *has a global center at the origin.*

Proof. The displacement function Δ in (30) will be identically zero for $\beta=0$ and for all $a>0$ if and only if $\theta=\theta_c$ is a zero of the function h defined in (31). Since h is a continuous function on I and $h^{\pm}(s,s_0)=-h^{\pm}(s_0,s)$ for all $(s,s_0)\in[0,2\pi]\times[0,2\pi]$, we will show that

$$h_*\left(-\frac{\pi}{2}\right)h_*\left(\frac{3\pi}{2}\right)=h_*^{-}\left(\frac{3\pi}{2},-\frac{\pi}{2}\right)h_*^{+}\left(\frac{7\pi}{2},\frac{3\pi}{2}\right)<0,$$

where the asterisk stands for the limit in the variable θ of the respective functions.

In fact, we have that $G^{\pm}$ defined by

$$\begin{aligned} G^{\pm}(u) = & -\frac{t^{\pm}}{\sqrt{4d^{\pm}-(t^{\pm})^2}}\arctan\left(\frac{a_{11}^{\pm}-a_{22}^{\pm}+2a_{12}^{\pm}\tan(u)}{\sqrt{4d^{\pm}-(t^{\pm})^2}}\right)-\\ & \frac{1}{2}\ln\left|a_{12}^{\pm}-a_{21}^{\pm}-(a_{12}^{\pm}+a_{21}^{\pm})\cos(2u)+(a_{11}^{\pm}-a_{22}^{\pm})\sin(2u)\right|,\quad u\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \end{aligned}$$

are primitive functions of $g^{\pm}$, where $t^{\pm} = \mathrm{Tr}(A^{\pm}) = 2\mu^{\pm}$ and $d^{\pm} = \det(A^{\pm}) = (\mu^{\pm})^2 + (\omega^{\pm})^2$. Finally, as $g^{\pm}$ are periodic functions of period π, we obtain

$$h_*\left(-\frac{\pi}{2}\right) = h_*^-\left(\frac{3\pi}{2}, -\frac{\pi}{2}\right) = 2h_*^-\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = -\frac{2\pi t^- \mathrm{sign}(a_{12}^-)}{\sqrt{4d^- - (t^-)^2}},$$
$$h_*\left(\frac{3\pi}{2}\right) = h_*^+\left(\frac{7\pi}{2}, \frac{3\pi}{2}\right) = 2h_*^+\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = -\frac{2\pi t^+ \mathrm{sign}(a_{12}^+)}{\sqrt{4d^+ - (t^+)^2}},$$

ending the proof of the lemma. ■

The analysis of (25) shows the absence of singular points placed at infinity for both systems $X' = A^-X$ and $X' = A^+X$. Thus from Lemma 1 we conclude that system (24) satisfies the hypothesis **(A)**. Furthermore, from (28), (20), (21) and (27) it follows that $K^{\pm}(\cos(s), \sin(s)) = -g^{\pm}(s)$ and $K^-(\alpha_c, 1) \neq K^+(\alpha_c, 1)$ when $s = \phi(1, \alpha_c, \beta)$ and the hypothesis **(B)** is also satisfied.

It is obvious that the equations in (26) are integrable and hence we have an expression for the displacement function Δ associated with the compactification of (24) in which we can show directly the existence of a transcritical bifurcation at infinity. However, we will not follow this way but we will get the same conclusion by making use of Theorem 3.

We conclude this subsection showing some numerical results when

$$A^- = \begin{pmatrix} \frac{4}{3} & -\frac{20}{3} \\ \frac{377}{750} & -\frac{26}{15} \end{pmatrix}, \quad A^+ = \begin{pmatrix} \frac{19}{50} & -1 \\ 1 & \frac{19}{50} \end{pmatrix} \tag{32}$$

in (24). These are the same matrices found in [2] and [6].

Figure 3 shows the graph of the function h as in (31) and its three zeros $\theta_1 = 0.999982$, $\theta_2 = 3.135174$ and $\theta_3 = 3.471306$ obtained numerically.

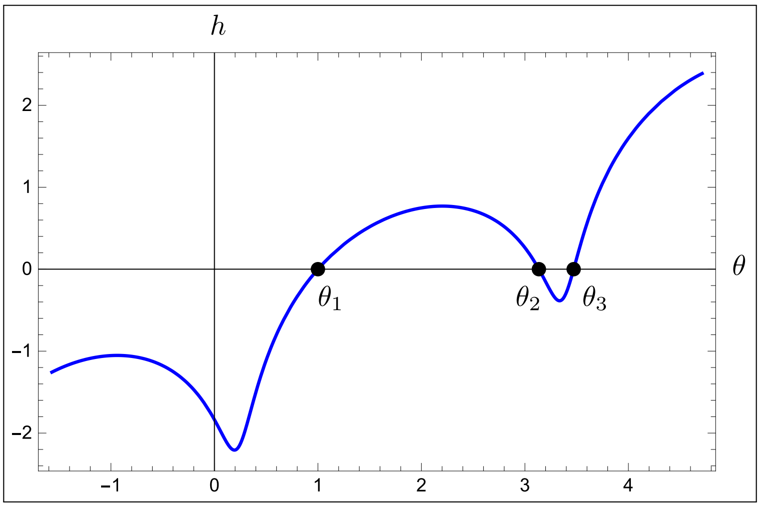

Figure 3: Graph of the function h defined in (31), when $A^{\pm}$ are the ones given in (32). The numerical values of the three zeros are $\theta_1 = 0.999982$, $\theta_2 = 3.135174$ and $\theta_3 = 3.471306$.

For example, choosing $\theta_1 = 0.999982$ it follows that $\alpha_c = 1/\tan(\theta_1) = 0.642117$ and from this we have Figures 4, 5 and 6.

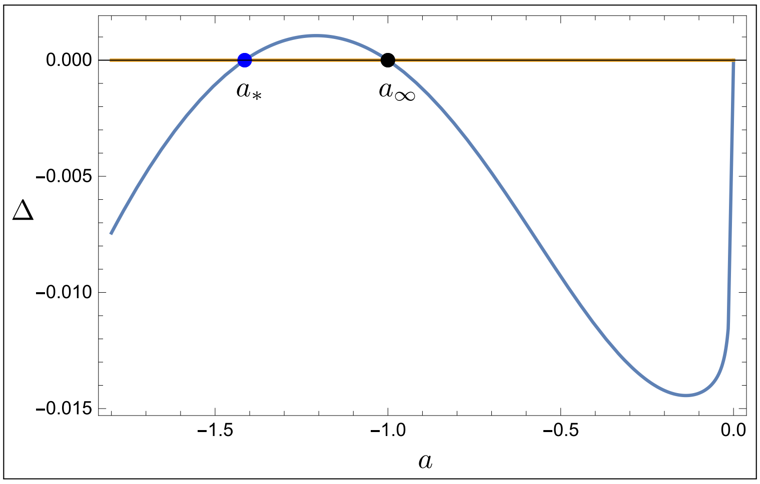

Figure 4: Graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (24). Such graph was obtained numerically using the matrices $A^{\pm}$ of (32), $\beta = 0.1$ and $\alpha = \alpha_c - 0.01$, with $\alpha_c = 1/\tan(\theta_1) = 0.642117$. In this case, the zeros $a_* = -1.414411$ and $a_\infty = -1$ of Δ correspond to an unstable and a stable periodic orbits in the phase portrait of the compactification of (24) on $\mathbb{D}_\eta$, respectively.

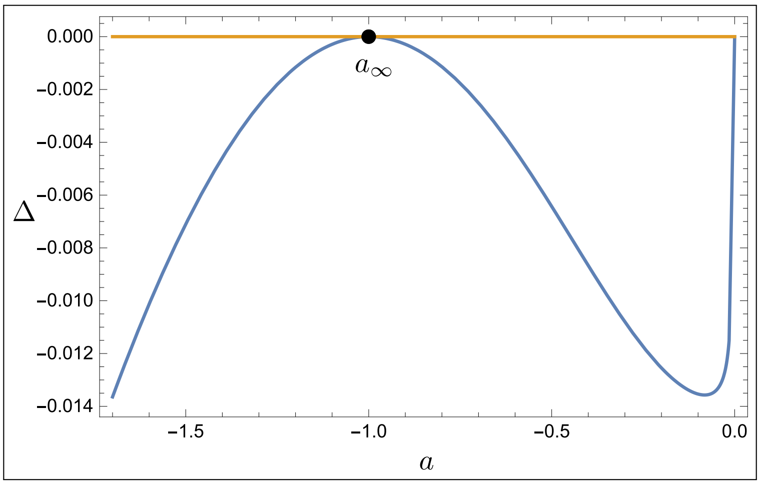

Figure 5: Graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (24). Such graph was obtained numerically using the matrices $A^{\pm}$ of (32), $\beta = 0.1$ and $\alpha = \alpha_c$, with $\alpha_c = 1/\tan(\theta_1) = 0.642117$. In this case, the zero $a_\infty = -1$ of Δ corresponds to a semistable periodic orbit in the phase portrait of the compactification of (24) on $\mathbb{D}_\eta$.

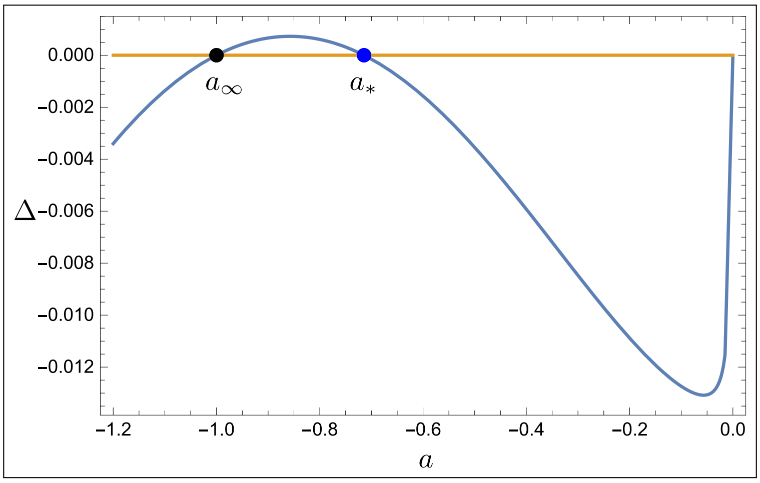

Figure 6: Graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (24). Such graph was obtained numerically using the matrices $A^{\pm}$ of (32), $\beta = 0.1$ and $\alpha = \alpha_c + 0.01$, with $\alpha_c = 1/\tan(\theta_1) = 0.642117$. In this case, the zeros $a_\infty = -1$ and $a_* = -0.714295$ of Δ correspond to an unstable and a stable periodic orbits in the phase portrait of the compactification of (24) on $\mathbb{D}_\eta$, respectively.

3.2 Third-degree polynomial piecewise differential systems

Consider the function S defined by $S(x,y) = x^4 + y^4$ when $(x,y) \in \mathbb{R}^2$, and suppose that the polynomial vector fields $G^{\pm}$ in (2) are Hamiltonians of the form

$$G^-(x,y) = \left(-\frac{\partial S(x,y)}{\partial y}, \frac{\partial S(x,y)}{\partial x}\right), \quad G^+(x,y) = \left(-\frac{\partial S(u,v)}{\partial y}, \frac{\partial S(u,v)}{\partial x}\right), \tag{33}$$

where $(u,v) = (x\sin(\theta) + y\cos(\theta), x\cos(\theta) - y\sin(\theta))$ and $\theta \in (0, \pi/2)$.

From (33) and the polar coordinates change (7) we can rewrite the compactified systems $X' = \mathscr{D}(G^-)(X)$ and $X' = \mathscr{D}(G^+)(X)$ as

$$\begin{aligned} r' &= R^-(s,r) = 4r^3(1-r^2)\sin(4s), \\ s' &= S^-(r,s) = 4r^2(1+r^2)(3+\cos(4s)), \\ r' &= R^+(s,r) = 4r^3(1-r^2)\sin(4(s+\theta)), \\ s' &= S^+(r,s) = 4r^2(1+r^2)(3+\cos(4(s+\theta))). \end{aligned} \tag{34}$$

Thus from (34) the infinity is a periodic orbit. The next lemma gives the global behavior of the phase portrait of (2) when the vector fields $G^{\pm}$ are the ones given in (33).

Lemma 2. *There exists $\alpha_c = \alpha_c(\theta)$ and $\theta \in (0, \pi/2)$ such that the origin of system (2) with the vector fields $G^{\pm}$ given by (33) is a global center for $\alpha = \alpha_c$ and $\beta = 0$. Moreover, if $|\alpha - \alpha_c| > 0$ is sufficiently small and $\beta = 0$, then the origin is a global focus, an attractor when $\alpha < \alpha_c$ and a repellor when $\alpha > \alpha_c$.*

Proof. First note that the nonempty level sets of the function S, excepting the set $\{(0,0)\}$, are closed curves around the origin. Thereby the first return map defined by system (2) is well defined.

For $\beta = 0$ we can study the stability of the origin through the displacement function obtained here by means of the properties of Hamiltonian systems. Thus supposing that $\alpha = \tan(\vartheta)$ where $\vartheta \in (0, \pi/2)$ then for $(0,a) \in \Sigma_{\alpha,\beta}$ with $a < 0$ the displacement function Δ follows from

$$G^+(0,a) - G^+(\alpha b, b) = 0, \tag{35}$$

$$G^-(\alpha b, b) - G^-(0,c) = 0, \tag{36}$$

where $b, c \in \mathbb{R}$, $b > 0$ and $c < 0$. Solving (35) for a^4 and (36) for c^4, we obtain

$$\begin{aligned} c^4 - a^4 &= (c-a)(c+a)(c^2+a^2) \\ &= \frac{4b^4\alpha \sin(2\theta)\left(2\left(\alpha^2-1\right)\cos(2\theta) - 3\alpha\sin(2\theta)\right)}{3+\cos(4\theta)}. \end{aligned} \tag{37}$$

Since $\Delta(a, \alpha, 0) = c(\alpha) - a(\alpha)$ it follows that the sign of $-\Delta(a, \alpha, 0)$ is given by the right-hand side sign of (37) and we conclude the proof of the lemma. ■

As we have seen before there are no singular points placed at infinity and the hypothesis **(A)** is satisfied using Lemma 2. Moreover, from (21) and (34) it follows that

$$-\frac{\cos\left(4\left(\arctan(\alpha_c)\right)\right)}{3+\sin\left(4\left(\arctan(\alpha_c)\right)\right)} = K^-(\alpha_c, 1) \neq K^+(\alpha_c, 1) = -\frac{\sin\left(4\left(\theta - \arctan(\alpha_c)\right)\right)}{3+\cos\left(4\left(\theta - \arctan(\alpha_c)\right)\right)},$$

which is precisely the hypothesis **(B)**.

Now we present a numerical example. For $\theta = \pi/8$ the phase portrait of system (2) with (33) has the origin as a global center if $\beta = 0$ and $\alpha_c = 2.0$ which is a nonzero zero of the right-hand side of (37). Figures 7, 8 and 9 show the local features of the graphs of the displacement function defined by system (2) that occur around its zeros. In this case, the polynomial vector fields are as in (33) with $\theta = \pi/8$, $\alpha_c = 2.0$ and $\beta = 0.1$.

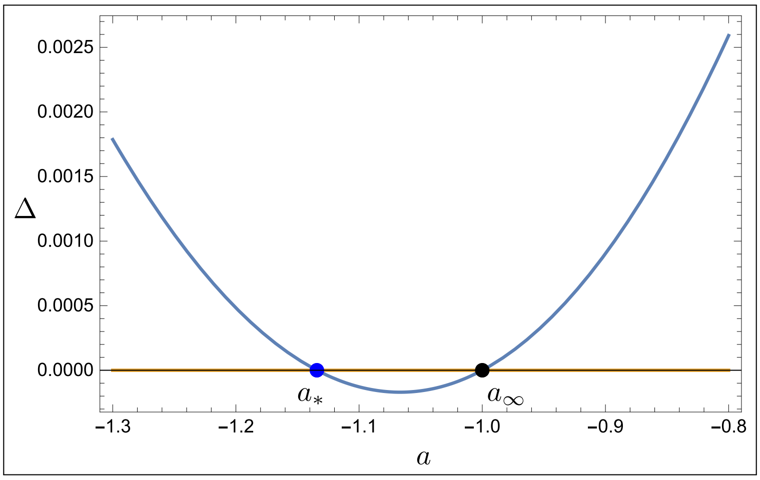

Figure 7: Local graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (2). Such graph was obtained numerically using the vector fields $G^{\pm}$ of (33) with $\theta = \pi/8$, $\beta = 0.1$, $\alpha = \alpha_c - 0.05$ and $\alpha_c = 2.0$. In this case, the zeros $a_* = -1.134261$ and $a_\infty = -1$ of Δ correspond to a stable and an unstable periodic orbits in the phase portrait of the compactification of (2) on $\mathbb{D}_\eta$, respectively.

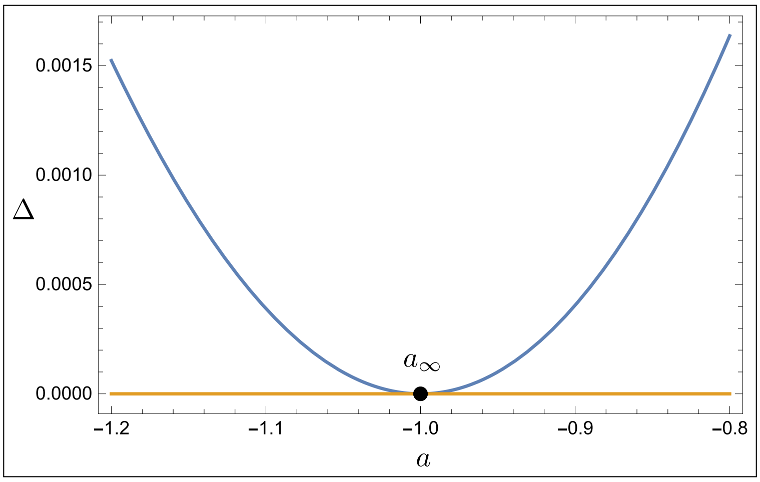

Figure 8: Local graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (2). Such graph was obtained numerically using the vector fields $G^{\pm}$ of (33) with $\theta = \pi/8$, $\beta = 0.1$, $\alpha = \alpha_c$ and $\alpha_c = 2.0$. In this case, the zero $a_\infty = -1$ of Δ corresponds to a semistable periodic orbit in the phase portrait of the compactification of (24) on $\mathbb{D}_\eta$.

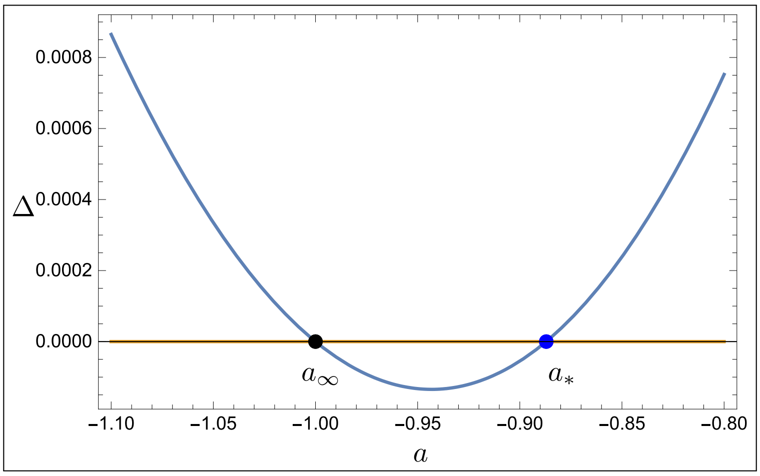

Figure 9: Local graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of (2). Such graph was obtained numerically using the vector fields $G^{\pm}$ of (33) with $\theta = \pi/8$, $\beta = 0.1$, $\alpha = \alpha_c + 0.05$ and $\alpha_c = 2.0$. In this case, the zeros $a_\infty = -1$ and $a_* = -0.887044$ of Δ correspond to a stable and an unstable periodic orbits in the phase portrait of the compactification of (24) on $\mathbb{D}_\eta$, respectively.

4 A degenerate case

Here we analyze a degenerate case, namely a particular type of second-degree polynomial piecewise differential systems, following the same procedure used in the Subsection 3.1. Such systems satisfy only the hypothesis **(A)**.

Suppose that the polynomial vector fields $G^{\pm}$ in (2) are of the form

$$G^{\pm}(X) = (\eta^{\pm}x - y(1+x^2+y^2), \eta^{\pm}y + x(1+x^2+y^2)), \quad X = (x,y), \tag{38}$$

where $\eta^{\pm} \in \mathbb{R}$ and $\eta^{-}\eta^{+} < 0$. The unique singular point of both systems $X' = G^{-}(X)$ and $X' = G^{+}(X)$ is the origin which is a global focus. In fact the eigenvalues of the Jacobian matrices associated with the vector fields (38) and evaluated at $(0,0)$ are complex of the form $\lambda^{\pm} = \eta^{\pm} \pm i$ and $\bar{\lambda}^{\pm}$. Furthermore, the derivative of the function L given by $L(x,y) = x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$ along the solutions of both systems $X' = G^{-}(X)$ and $X' = G^{+}(X)$ shows that the origin is a global attractor or repellor focus depending on the sign of $\eta^{\pm}$.

Through the polar coordinate change (7) the compactified systems $X' = \mathscr{D}(G^{-})(X)$ and $X' = \mathscr{D}(G^{+})(X)$ are rewritten as

$$r' = R^{\pm}(s,r) = \frac{dr}{dt} = \eta^{\pm} r(1-r^2)^3, \quad s' = S^{\pm}(s,r) = \frac{ds}{dt} = (1+r^2)^3, \tag{39}$$

which show that there are no singular points placed at infinity of system (2) when $G^{\pm}$ are the vector fields given in (38).

After a time reparametrization and the time rescaling $ds = ((1-r^2)^3/(1+r^2)^3)d\bar{s}$, we can study systems $X' = \mathscr{D}(G^{-})(X)$ and $X' = \mathscr{D}(G^{+})(X)$ on $\mathbb{D}\backslash\partial\mathbb{D}$ by means of

$$r' = \frac{dr}{ds} = F^{\pm}(s,r) = \frac{\eta^{\pm} r}{1+r^2}. \tag{40}$$

In this case the solutions $r^{\pm} = r^{\pm}(s, s_0, r_0)$ of (40) are given by

$$r^{\pm}(s,s_0,r_0) = \sqrt{W_0\left(\exp\left(2\eta^{\pm}(s-s_0)\right) r_0^2 \exp\left(r_0^2\right)\right)}, \; (s,s_0,r_0) \in [0,2\pi] \times [0,2\pi] \times (0,\infty),$$

where $W_0 = W_0(u)$ for $u \geq 0$ is the principal branch of the Lambert W function [4]. So we obtain

$$\Delta(a,\alpha,\beta) = r^{+}\left(\frac{3\pi}{2}, 2\pi+\theta, a\right)\left(\delta(\theta,a) - 1\right), \tag{41}$$

where

$$\delta(\theta, a) = \sqrt{\frac{W_0\left(\exp\left(\eta^-(3\pi - 2\theta)\right) a^2 \exp\left(a^2\right)\right)}{W_0\left(\exp\left(-\eta^+(\pi + \theta)\right) a^2 \exp\left(a^2\right)\right)}}.$$

By the injectivity of W_0 on $[0, \infty)$ we conclude that the displacement function Δ in (41) will be identically zero for $\beta = 0$ and for all $a > 0$ if and only if

$$\theta = \theta_c = \frac{\pi(3\eta^- + \eta^+)}{2(\eta^- - \eta^+)}.$$

Thereby we may have a global center and of course from (39) the infinity is a periodic orbit as stated earlier. However the hypothesis **(B)** is not satisfied because from (21) and (34) we have $K^-(\alpha_c, 1) = K^+(\alpha_c, 1) = 0$.

Although in these degenerate cases higher order partial derivatives of the displacement function Δ, evaluated at $(1, \alpha_c, \beta)$, are necessary to prove an analogous result to Theorem 3, it is not difficult to see that the Poincaré map defined by this particular example undergoes a Transcritical Bifurcation. Indeed, through a time reparametrization and the time rescaling $ds = ((1-r^2)^2/(1+r^2)^2)d\bar{s}$, the compactified systems $X' = \mathscr{D}(G^-)(X)$ and $X' = \mathscr{D}(G^+)(X)$ are rewritten as

$$r' = \frac{dr}{ds} = F^{\pm}(s, r) = \frac{\eta^{\pm} r(1 - r^2)}{1 + r^2}, \tag{42}$$

which are topologically equivalent to systems (26) on $\mathbb{D}_\eta \backslash \partial\mathbb{D}$, with $\eta > 1$, when the matrices $A^{\pm}$ satisfying the hypothesis **(M)** are in the Jordan Canonical Form and $\eta^-\eta^+ < 0$. Besides that, systems in (42) are well defined on $\mathbb{D}_\eta$.

For example, choosing $\eta^- = -3$ and $\eta^+ = 5$ it follows that

$$\alpha_c = \tan\left(\frac{1}{\theta_c}\right) = \tan\left(\frac{\pi(\eta^+ + \eta^-)}{\eta^+ - \eta^-}\right) = 1.0.$$

The next three figures show the behavior of the displacement function when $\alpha < \alpha_c$, $\alpha = \alpha_c$ and $\alpha > \alpha_c$, respectively.

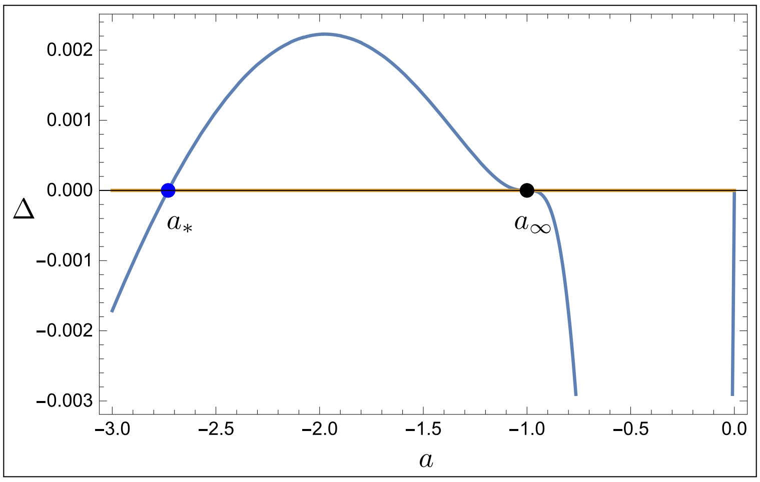

Figure 10: Graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (2). Such graph was obtained numerically using the vector fields $G^{\pm}$ of (38), $\beta = 0.1$, $\alpha = \alpha_c - 0.03$ and $\alpha_c = 1/\tan(\theta_c) = 1.0$. In this case, the zeros $a_* = -2.730492$ and $a_\infty = -1$ of Δ correspond to an unstable and a stable periodic orbits in the phase portrait of the compactification of (2) on $\mathbb{D}_\eta$, respectively. The function Δ is continuous for $a \leq 0$.

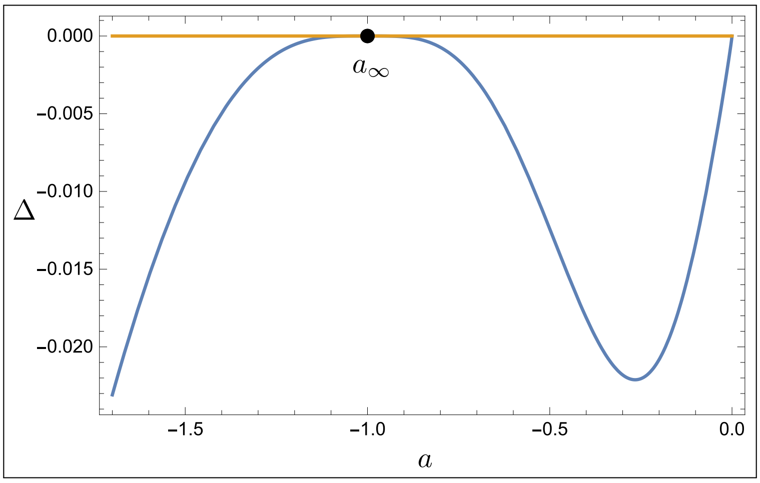

Figure 11: Graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (2). Such graph was obtained numerically using the vector fields $G^{\pm}$ of (38), $\beta = 0.1$, $\alpha = \alpha_c$ and $\alpha_c = 1/\tan(\theta_c) = 1.0$. In this case, the zero $a_\infty = -1$ of Δ corresponds to a semistable periodic orbit in the phase portrait of the compactification of (2) on $\mathbb{D}_\eta$.

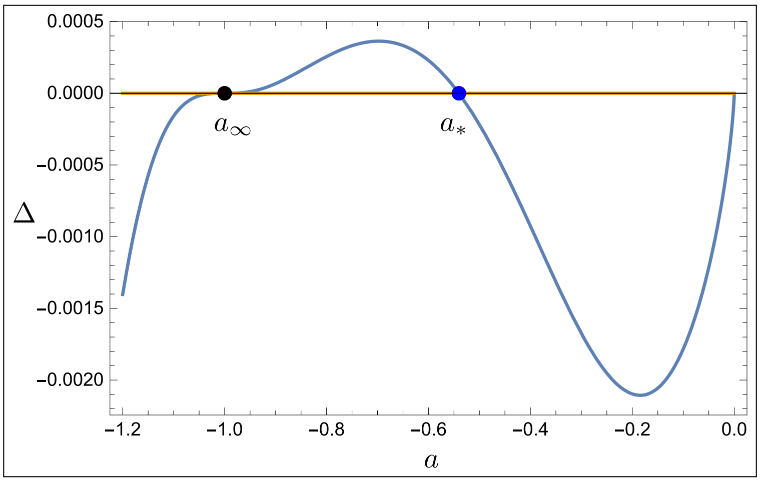

Figure 12: Graph of the displacement function Δ given by $\Delta(a, \alpha, \beta) = \Pi_{\alpha,\beta}(a) - a$ and defined by the compactification of system (2). Such graph was obtained numerically using the vector fields $G^{\pm}$ of (38), $\beta = 0.1$, $\alpha = \alpha_c + 0.03$ and $\alpha_c = 1/\tan(\theta_c) = 1.0$. In this case, the zeros $a_\infty = -1$ and $a_* = -0.540230$ of Δ correspond to an unstable and a stable periodic orbits in the phase portrait of the compactification of (2) on $\mathbb{D}_\eta$, respectively.

5 Concluding remarks

There are several planar piecewise polynomial differential systems with two zones of the form (2) satisfying the main hypotheses and consequently covered by Theorem 3. For instance, starting from a smooth Hamiltonian system whose phase portrait exhibits a global center around the origin, we can think about a similar construction to that one made in Subsection 3.2.

This work does not deal with the simplest case of obtaining a global center which is the one in which the separation boundary is a straight line and also with degenerate cases, providing only one example in Section 4. Nevertheless, the theory presented here can be easily adapted to such cases. Obviously a more general theory requires a piecewise algebraic curve as a separation boundary. Possibly with more hypotheses about the behavior of such a curve at infinity and a little effort such cases may also be analyzed.

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