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NON-EXISTENCE AND UNIQUENESS OF LIMIT CYCLES IN A CLASS OF GENERALIZED LIÉNARD EQUATIONS

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Abstract. We provide a sharp upper bound for the number of limit cycles of the generalized Liénard differential systems

$$\dot{x} = y + ax^n + bx^k, \quad \dot{y} = cx^m$$

where n, k, m are positive integers, $1 < n < k$ and $a, b, c \in \mathbb{R}$ with $bc \neq 0$. We also provide examples realizing the upper bounds

1. Introduction and statement of the main results

The so called Liénard differential systems introduced by Liénard in [6] and its generalizations have been studied by many different authors, nowadays in MathSciNet appear 1285 papers related with these systems.

This paper deals with the problem of finding upper bounds for the number of limit cycles of some generalized Liénard systems. The number of limit cycles for polynomial differential systems is part of the 16th problem of the 23 problems proposed by Hilbert in 1900, see [5]. Over the last century the uniqueness of limit cycles have been studied very thoroughly. Also in his list of mathematical problems for the next century, Smale in [7] mentioned that before solving the difficult 16th Hilbert problem it will be convenient to solve first it restricted to the class of polynomial Liénard differential systems.

A *limit cycle* of a differential system is a periodic orbit of the system isolated in the set of all periodic solutions of the system.

In this paper we study the maximum number of limit cycles of the Liénard systems

$$\dot{x} = y + ax^n + bx^k, \quad \dot{y} = cx^m \tag{1}$$

where n, k, m are positive integers, $1 < n < k$ and $a, b, c \in \mathbb{R}$ with $bc \neq 0$, and as usual the dot means derivative in the variable t.

We state the main results of the paper.

Proposition 1. *With a change of variables and a rescaling of the time, system* (1) *is equivalent to the following two systems*

(i) $\dot{x} = y + ax^n + x^k$, $\dot{y} = x^m$ *with* $a \in \mathbb{R}$.
(ii) $\dot{x} = y + ax^n + x^k$, $\dot{y} = -x^m$ *with* $a \in \mathbb{R}$.

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The proof of Proposition 1 is given in section 2. In view of Proposition 1 we will work with systems (i) and (ii).

Proposition 2. *For m even, neither systems (i) nor systems (ii) have limit cycles.*

The proof of Proposition 2 is given in section 3. In view of Proposition 2 in order that system (1) has limit cycles we must have m odd.

Proposition 3. *For m odd systems (i) do not have limit cycles.*

The proof of Theorem 3 is also given in section 3. In view of Propositions 2 and 3 in order that system (1) has limit cycles we must study system (ii) with m odd.

We recall tha a limit cycle $(x(t), y(t))$ of period T of systems (ii) is *hyperbolic* if $\int_0^T \operatorname{div}(x(t), y(t))\, dt \neq 0$ where div denotes the divergence of systems (ii).

Theorem 4. *For m odd systems (ii) have limit cycles if and only if k and n are odd and $a < 0$. In this case the maximum number of limit cycles that system (ii) can have is one and this upper bound is reached. Moreover, whenever the limit cycle exists, it is hyperbolic.*

The proof of Theorem 4 is given in section 4. The hyperbolicity of the limit cycle is relevant. Without having this property, the limit cycle could bifurcate under small perturbations, but if a limit cycle is hyperbolic, it will persist under small C^1-perturbations and this implies the non-appearance of new periodic solutions near it. Theorem 4 in the particular case of $m = 1$ was proved in [3].

2. Proof of Proposition 1

Introducing the change of variables $X = \alpha x$, $Y = \beta y$ and reparameterization of time $dT = \gamma\, dt$ with

$$\begin{aligned}
\alpha &= \operatorname{sign}(b)\operatorname{sign}(c)|b|^{2/(1-2k+m)}|c|^{-(1/(1-2k+m))},\\
\beta &= \operatorname{sign}(b)\operatorname{sign}(c)|b|^{(1+m)/(1-2k+m)}|c|^{-(k/(1-2k+m))},\\
\gamma &= \operatorname{sign}(b)|b|^{(1-m)/(1-2k+m)}\operatorname{sign}(c)|c|^{(-1+k)/(1-2k+m)},
\end{aligned}$$

we get that system (1) becomes the following four systems with only one parameter $a \in \mathbb{R}$,

$$(2)\qquad \begin{array}{llll}
x' = y + ax^n + x^k, & y' = x^m & \text{if } b, c > 0, & (a)\\
x' = y + ax^n + x^k, & y' = -x^m & \text{if } b > 0, c < 0, & (b)\\
x' = y + ax^n - x^k, & y' = x^m & \text{if } b < 0,\ c > 0, & (c)\\
x' = y + ax^n - x^k, & y' = -x^m & \text{if } b, c < 0, & (d)
\end{array}$$

where the prime means derivative in the new independent variable T.

Note that if we apply to system (a) in (2) the change

$$(x, y, a, t) \mapsto (x, -y, -a, -t)$$

we obtain system (c) in (2). Moreover, if we apply to system (b) in (2) the same change we get system (d) in (2). This completes the proof of the proposition.

3. Proofs of Propositions 2 and 3

In order to prove Propositions 2 and 3 we recall that it is known that in the region R limited by a limit cycle the sum of the topological indices of the equilibria contained in R must be 1, see [1, Section 6.4]. So we need the topological index of the origin be 1.

Proof of Proposition 2. By [1, Theorem 3.5] if m is even, the origin of systems (i) or (ii) is a cusp or a saddle-node. Since a cusp and a saddle-node has topological index 0 (see [1, Section 6.4]), we get that neither systems (i) nor (ii) have limit cycles whenever m is even. This completes the proof of the proposition. □

Proof of Proposition 3. By [1, Theorem 3.5] if m is odd the origin of system (i) is a saddle. Since a saddle has topological index -1 (see [1, Section 6.4]), we get that system (i) with m odd do not have limit cycles. This completes the proof of the proposition. □

4. Proof of Theorem 4

We will separate the proof of Theorem 4 in several propositions.

Proposition 5. *Systems (ii) with $a = 0$ do not have limit cycles.*

In order to prove the proposition we will first recall two results that will be used during the proof. The proof for the first one can be obtained for example in [1, Theorem 7.10].

Theorem 6 (Bendixson's Theorem)**.** *Assume that the divergence function $\partial P/\partial x + \partial Q/\partial y$ of system $x' = P(x, y)$, $y' = Q(x, y)$, with P, Q functions of class C^1, satisfies that it is either greater than or equal zero, or less than or equal zero in a simply connected region R, and is not identically zero on any open subset of R. Then system $x' = P(x, y)$, $y' = Q(x, y)$, does not have a periodic orbit which lies entirely in R.*

Proposition 7. *Assume that system*

$$\dot{x} = y - F(x), \quad \dot{y} = g(x)$$

has a unique equilibrium point which is a center. Then it cannot have limit cycles.

We recall that the period annulus of a center p is the maximal neighborhood P of p such that all the orbits contained in P are periodic, except of course, the point p.

Proof of Proposition 7. Let p be the unique equilibrium point of the system which is a center.

If the period annulus of p is contained in a compact set then it must be a periodic orbit. Indeed, the period annulus can only be a periodic orbit or a graph (a closed orbit formed by periodic points and pieces of orbits) but since the unique singular point is p, it must be a periodic orbit. In this case we consider the Poincaré map π defined in a transversal section Π through γ. Since the vector field is analytic, it follows that π is also analytic. However as π is the identity map in the piece of Π contained in the bounded region limited by γ, it must also be the identity map in the piece of Π contained in the unbounded region delimited by γ near the period annulus but then the orbits contained in the unbounded region delimited by γ near the period annulus are also periodic in contradiction with the fact that γ is the boundary of the period annulus.

If the period annulus is not contained in a compact set, then the orbits scape to infinity in some direction and so it cannot be a limit cycle surrounding the origin. This completes the proof of the proposition. $\square$

Proof of Proposition 5. Consider systems (ii) with $a = 0$.

If k is even systems (ii) with $a = 0$ are time-reversible because they are invariant under the change

$$(x, y, t) \mapsto (-x, y, -t) \tag{3}$$

and consequently (since it is monodromic) it has a center at the origin, see [4]. By Proposition 7, systems (ii) with $a = 0$ in this case have no limit cycles.

If k is odd then the divergence of systems (ii) with $a = 0$ is $kx^{k-1} \geq 0$. By the Bendixson's Theorem there are no limit cycles in this case. This concludes the proof of the proposition. $\square$

From now on we assume that $a \neq 0$.

Proposition 8. *Systems (ii) with $a \neq 0$ and either n or k even do not have limit cycles.*

In order to prove the proposition we introduce a result whose proof can be found in [3]. To introduce it, we need first one definition adapted from [3] to our systems (ii). Given a positive real number z, we will say that $(-w_1, w_1)$ with $-w_1 < 0 < w_1$ is a *solution of* (F, G) if $F(w_1) = F(w_2)$ and $G(w_1) = G(w_2)$, where

$$F(x) = -ax^n - x^k, \quad G(x) = \frac{x^{m+1}}{m+1}.$$

Theorem 9. *Assume that there does not exist a solution of $(F, G) = 0$. Then system (ii) has no periodic orbits.*

Proof of Proposition 8. If n and k are even, then systems (ii) are reversible because they are invariant under the change

$$(x, y, t) \mapsto (-x, y, -t)$$

and so it is time-reversible and monodromic and consequently has a center at the origin, see [4]. By Proposition 7, systems (ii) in this case have no limit cycles.

Note that the solutions of $(F,G)=0$ are given by $x=\pm\big((m+1)z\big)^{1/(m+1)}$ with z some positive real number such that

$$
\begin{aligned}
&F\big(\big((m+1)z\big)^{1/(m+1)}\big)-F\big(-\big((m+1)z\big)^{1/(m+1)}\big)\\
&=-z^{n/(m+1)}(m+1)^{n/(m+1)}\big(-a(1-(-1)^n)\\
&\quad -z^{(k-n)/(m+1)}(m+1)^{(k-n)/(m+1)}(1-(-1)^k)\big)=0.
\end{aligned}
\tag{4}
$$

If n is even and k is odd then equation (4) becomes $-2(m+1)^{k/(m+1)}z^{k/(m+1)}=0$, which is not possible since $z>0$. So, in view of Theorem 9 there are no limit cycles.

If n is odd and m is even then equation (4) becomes $-2a(m+1)^{k/(m+1)}z^{k/(m+1)}=0$, which is not possible since $z>0$ and $a\neq 0$. So, in view of Theorem 9 there are no limit cycles. This concludes the proof of the proposition. □

Proposition 10. *Consider systems (ii) with n and k odd. Then the following statements hold:*

(a) if $a>0$ they do not have limit cycles.
(b) if $a<0$ and either $m=1$, or $m>1$ and $m<2n-1$, or $m>1$, $m=2n-1$ with $(an)^2-4n<0$, they have at least one limit cycle which is hyperbolic.

Before proving the proposition we state and prove an auxiliary result.

Proposition 11. *System*

$$\dot{x}=y-F(x),\quad \dot{y}=g(x) \tag{5}$$

with $g(x)$ being an odd function and F being an analytic function, has a center at the origin if and only if F is even.

In view of Proposition 11 the Lyapunov constants are the odd coefficients of F.

Proof. We first prove sufficiency. System (5) with F even is invariant by the symmetry $(x,y,t)\to(-x,y,-t)$. So, it is time-reversible and monodromic and consequently has a center at the origin.

Now we prove necessity. We expand $F(x)$ in Taylor series in the form $F(x)=\sum_{i\geq 0}a_ix^i$. Consider the wedge product of system (5) that we denote by X and of system

$$\dot{x}=y-\sum_{i\geq 0}a_{2i}x^{2i},\quad \dot{y}=g(x)$$

which we already know that it has a center and we denote it as X_c.

Doing so we obtain

$$X\wedge X_c=g(x)\sum_{i\geq 0}a_{2i+1}x^{2i+1}. \tag{6}$$

If $a_{2i+1}\neq 0$ for some i, from equation (6) we deduce that, in a neighborhood of the origin, the level curves of the solutions of X_c do not have contact with the flow of X

giving the impossibility of having a center for X. This shows that $a_{2i+1} = 0$ for $i \geq 0$ is necessary for having a center. Note that this argument is independent of i and so it concludes the proof of the theorem. $\square$

Proof of Proposition 10. Assume first $a > 0$. Note that the divergence of systems (ii) is

$$anx^{n-1} + kx^{k-1} = x^{n-1}(an + kx^{k-n}) > 0$$

if $a > 0$ because $n-1$ and $k-n$ are even. By Bendixson's theorem there are no limit cycles in this case. This concludes the proof of statement (a).

To prove statement (b) note that if $m = 1$ the origin is either a center or a focus and if $m > 1$ in view of [1, Theorem 3.5] then if either $m < 2n-1$ or $m = 2n-1$ and $(an)^2 - 4n < 0$ then the origin is either a center or a focus.

In these cases, in view of Proposition 11 the first Lyapunov constant of systems (ii) is a and the second Lyapunov constant is 1. Therefore, if $a \geq 0$ the origin is unstable and if $a < 0$ the origin is stable. Therefore, for $a \lesssim 0$ it is possible to generate one and only one small amplitude limit cycle around the origin with an Andronov-Hopf bifurcation. The limit cycle exists only when $a < 0$ and is unstable and hyperbolic. This concludes the proof of the proposition. $\square$

Proof of Theorem 4. In order to prove Theorem 4 we need two auxiliary results. The first one is proved using the Bendixson-Dulac criterion for ℓ-connected sets and was given in [3, Proposition 2.3].

An open subset U of $\mathbb{R}^2$ with smooth boundary is said to be *ℓ-connected* if its fundamental group $\pi_1(U)$ is $\mathbb{Z} \times |Z$ ℓ-times, that is, U has ℓ gaps. Given an open subset W with smooth boundary and a smooth function $f\colon W \to \mathbb{R}$ we denote by $\ell(W, f)$ the sum of $\ell(U)$ where U ranges over all connected components of $W \setminus \{f = 0\}$. Finally, we denote by $c(W, f)$ the number of closed ovals of $\{f = 0\}$ contined in W.

Theorem 12. *Consider the C^1 differential system*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{7}$$

and set $X = (P, Q)$. Assume there exists a real number s and an analytic function f in $\mathbb{R}^2$ such that

$$M_s = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial x}Q + sf\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = \langle \nabla f, X\rangle + sf\mathrm{div}\,(X)$$

does not change sign in an open region $W \subset \mathbb{R}^2$ with regular boundary and vanishes only in a zero measure Lebesgue set. Then system (7) has two types of limit cycles in W,

(i) Limit cycles totally contained in $\{f = 0\}$, and
(ii) Limit cycles which do not cut $\{f = 0\}$.

Moreover, the following statements hold.

(iii) The number of limit cycles described in (i) is at most $c(W, f)$.

(iv) The number of limit cycles described in (ii) is at most $\ell(W)$ if $s > 0$ and zero if $s = 0$. When $s < 0$, this number is bounded from above by $\ell(w, f)$. Moreover, for any value of s, all the limit cycles are hyperbolic.

The second one was proved in [8].

Theorem 13. *Assume that there exist $d' \in \mathbb{R}^-$, $d \in \mathbb{R}^+$, $x_0 < 0$ and $\alpha \le x_0$ such that system*

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{8}$$

satisfies

(i) $xg(x) > 0$ for $x \in (d', d)$ and $x \neq 0$;
(ii) $(x - x_0)f(x) < 0$ for $x \in (d', d)$, $x \neq x_0$ with $f(x) = F'(x)$;
(iii) $\sqrt{G(x)}f(x)/g(x)$ is nonincreasing for $x \in (0, d)$, where $G(x) = \int_0^x g(\tau)\, d\tau$;
(iv) The system of equations

$$F(u) = F(v), \quad G(u) = G(v)$$

has at most one solution (u, v) with $d' < u' < b$, $0 < v < d$ where $\alpha \le x_0$ is such that $F(\alpha) = 0$.

Then system (8) has at most one limit cycle and when it exists is hyperbolic and stable.

First note that in view of Propositions 5–10 in order that systems (ii) can have limit cycles we must have k, n, m odd and $a < 0$. Moreover in this case whenever $m = 1$ or $m > 1$ and either $m < 2n - 1$, or $m = 2n - 1$ and $(an)^2 - 4n < 0$ there exists at least one hyperbolic limit cycle.

We consider different cases.

Case 1*:* $(1 + m - 2k)(1 + m - 2n) \le 0$. Consider

$$f(x, y) = y^2 - (ax^n + x^k)y + \frac{2x^{m+1}}{m+1}, \quad s = -1,$$

with $P = y + ax^n + x^k$ and $Q = -x^m$ in Theorem 12. Then

$$\begin{aligned} M_{-1} &= \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial x}Q + sf\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \\ &= \frac{x^m}{m+1}((1 + m - 2k)x^k + a(1 + m - 2n)x^n). \end{aligned}$$

Assume first that $m - 2k + 1 = 0$ then

$$M_{-1} = \frac{2a(k - n)}{m + 1}x^{m+n}$$

which taking into account that $m + n$ is even and $k - n > 0$ we conclude that $M_{-1} \le 0$ and it is zero just at $x = 0$.

If $m - 2n + 1 = 0$ then

$$M_{-1} = \frac{2(n - k)}{m + 1}x^{m+k}$$

which taking into account that $m+k$ is even and $k-n<0$ we conclude that $M_{-1}\le 0$ and it is zero just at $x=0$.

Finally, if $(1+m-2k)(1+m-2n)<0$ then it is clear that $m-2k+1<0$ and $m-2n+1>0$ (note that since $k>n$ then $m-2k+1<m-2n+1$ and so if $m-2n+1<0$ we will have $m-2k+1<0$ which yields $m-2n+1>0$ and $m-2k+1<0$).

$$\begin{aligned}M_{-1}&=\frac{x^{m+n}}{m+1}((1+m-2k)x^{k-n}+a(1+m-2n))\\&=\frac{x^{m+n}}{(m+1)(1+m-2k)}\Big(x^{k-n}+\frac{a(1+m-2n)}{1+m-2k}\Big).\end{aligned}$$

Since $a<0$, $k-n$ is even, $m+n$ is even and $(1+m-2n)/(1+m-2k)<0$ with $m+1-2k<0$ we conclude that $M_{-1}\le 0$ and it is zero just at $x=0$.

In the three cases we have that $M_{-1}\le 0$ and it is zero just at $x=0$. We analyse now the set $\{f=0\}$. In view of the fact that $M_{-1}\le 0$ and it is zero just at $x=0$, the curves contained in this set are simple (they do not have singular points) and transversal (except at $x=0$) to the flow defined by $(P,Q)=(y+ax^n+x^k,-x^m)$ since they are crossed by the flow just in one direction. Then, any closed component of $\{f=0\}$ does not contain limit cycles and must surround the unique critical point of systems (ii) at the origin.

The fact that systems (ii) have at least one limit cycle for some values of m,n,k forces that $\{f=0\}$ hast at least one closed component. Taking into account that $f(x,y)$ is a second degree polynomial in the variable y, we get that $\{f=0\}$ contains exactly one closed component. Then, from Theorem 12 we conclude that systems (ii) have exactly one limit cycle which is hyperbolic and unstable. This limit cycle is contained in the 1-connected component of $\mathbb{R}^2\setminus\{f=0\}$. This concludes the proof of Theorem 4 in this case.

Case 2: $(1+m-2k)(1+m-2n)>0$ with $1+m-2k>0$.

Consider

$$f(x,y)=y^2-(ax^n+x^k)y+\frac{2x^{m+1}}{m+1}+c_0,\quad s=-1$$

with $P=y+ax^n+x^k$, $Q=-x^m$ in Theorem 12 and

$$c_0=\frac{1+m-2k}{k(m+1)}\left(\frac{-a(1+m-2n)}{1+m-2k}\right)^{(m+1)/(k-n)}.$$

Note that c_0 is well defined since $-a(1+m-2n)/(1+m-2k)>0$, $k(m+1)\neq 0$ and $k\neq n$. Moreover

$$
\begin{aligned}
M_{-1} &= \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial x}Q + sf\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \\
&= \frac{x^m}{m+1}((1+m-2k)x^k + a(1+m-2n)x^n) - c_0kx^{k-1} - ac_0nx^{n-1} \\
&= x^{n-1}\left(\frac{1+m-2k}{m+1}x^{k-n+m+1} + \frac{1+m-2k}{m+1}x^{k-n} + \frac{1+m-2k}{m+1}d_2x^{m+1}\right. \\
&\quad \left. + \frac{1+m-2k}{m+1}d_1d_2 + d_3\right)
\end{aligned} \tag{9}
$$

where

$$
d_1 = -\left(\frac{-a(1+m-2n)}{1+m-2k}\right)^{(m+1)/(k-n)} < 0, \quad d_2 = \frac{a(1+m-2n)}{1+m-2k} < 0 \tag{10}
$$

and

$$
d_3 = \frac{ad_1}{k}(n-k) > 0.
$$

The function M_{-1} given in (9) can be written as

$$
M_{-1} = x^{n-1}\left(\frac{1+m-2k}{m+1}((x^{m+1}+d_1)(x^{k-n}+d_2)) + d_3\right).
$$

From the explicit expression of d_1 and d_2 given in (10) we see that the real roots of the polynomials $x^{m+1}+d_1$ and $x^{k-n}+d_2$ are both

$$
x = (-d_2)^{1/(k-n)} = \left(\frac{-a(1+m-2n)}{1+m-2k}\right)^{1/(k-n)},
$$

that is they are the same. So taking into account that $d_3 > 0$ we have

$$
M_{-1} = x^{n-1}\left(\frac{1+m-2k}{m+1}\left(x - \left(\frac{-a(1+m-2n)}{1+m-2k}\right)^{1/(k-n)}\right)^2 + d_3\right) \geq 0
$$

for all $x \in \mathbb{R}$ and is zero just at $x = 0$.

We analyse now the set $\{f = 0\}$. In view of the fact that $M_{-1} \geq 0$ and it is zero just at $x = 0$, the curves contained in this set are simple (they do not have singular points) and transversal (except at $x = 0$) to the flow defined by $(P, Q) = (y + ax^n + x^k, -x^m)$ since they are crossed by the flow just in one direction. Then, any closed component of $\{f = 0\}$ does not contain limit cycles and must surround the unique critical point of systems (ii) at the origin.

The fact that systems (ii) have at least one limit cycle for some values of m, n, k forces that $\{f = 0\}$ hast at least one closed component. Taking into account that $f(x, y)$ is a second degree polynomial in the variable y, we get that $\{f = 0\}$ contains exactly one closed component. Then, from Theorem 12 we conclude that systems (ii) have exactly one limit cycle which is hyperbolic and unstable. This limit cycle is contained in the

1-connected component of $\mathbb{R}^2 \setminus \{f = 0\}$. This concludes the proof of Theorem 4 in this case.

Case 3: $(1 + m - 2k)(1 + m - 2n) > 0$ with $1 + m - 2k < 0$. In this case we have that both $1 + m - 2k < 0$ and $1 + m - 2n < 0$. We introduce the change of variables $(x, y, a, t) \mapsto (x, -y, -a, -t)$ and we get that systems (ii) become

$$\dot{x} = y - x^k + ax^n, \quad \dot{y} = -x^{m+1}$$

with $a > 0$. Therefore, in the notation of Theorem 13 we have

$$F(x) = x^k - ax^n, \quad f(x) = kx^{k-1} - anx^{n-1}, \quad g(x) = x^m, \quad G(x) = \frac{x^{m+1}}{m+1}.$$

Now we will show that we are under the assumptions of Theorem 13.

Take

$$d' = x_0 = -\Big(\frac{an}{k}\Big)^{1/(k-n)}, \quad d = \Big(\frac{an(2n+1-m)}{k(2k+1-m)}\Big)^{1/(k-n)} \quad \text{and} \quad \alpha = -a^{1/(k-n)}.$$

Since m is odd, we have

$$xg(x) = x^{m+1} > 0 \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

So assumption (i) holds.

Furthermore

$$(x - x_0)f(x) = \Big(x + \Big(\frac{an}{k}\Big)^{1/(k-n)}\Big)(kx^{k-1} - anx^{n-1}).$$

Taking into account that $x - x_0 > 0$ in the interval $\big(-\big(\frac{an}{k}\big)^{1/(k-n)}, \big(\frac{an(2n+1-m)}{k(2k+1-m)}\big)^{1/(k-n)}\big)$ and that k, n are odd, we get that

$$kx^{k-1} - anx^{n-1} = x^{n-1}(kx^{k-n} - an) = |x|^{n-1}(k|x|^{k-n} - an) < 0$$

and so $(x - x_0)f(x) < 0$ for $x \in (d', d)$, which implies that assumption (ii) is satisfied.

Moreover, we note that fr $x \in (0, d)$,

$$\begin{aligned}(\sqrt{G(x)}f(x)/g(x))' &= (kx^{(2k+1-m)/2} - anx^{(2n+1-m)/2})' \\ &= \frac{1}{2}(k(2k+1-m)x^{(2k-1-m)/2} - an(2n+1-m)x^{(2n-1-m)/2}) \\ &= \frac{1}{2}x^{(2n-1-m)/2}(k(2k+1-m)x^{k-n} - an(2n+1-m)) < 0,\end{aligned}$$

where in the last inequality we have used that $a > 0$, $2k+1-m, 2k-1-m > 0$, $k > n$ and $x \in (0, d) = \big(0, \big(\frac{an(2n+1-m)}{k(2k+1-m)}\big)^{1/(k-n)}\big)$. This shows that assumption (iii) holds.

Finally, $G(u) = G(v)$ yields $u^{m+1} = v^{m+1}$ and since m is odd and $v \neq u$ we obtain $v = -u$. Then $F(u) = F(v) = F(-u)$ yields $u^k - au^n = -u^k + au^n$ and so $2(u^k - au^n) = 2u^n(u^{k-n} - a) = 0$ which yields $u = \alpha$ implying that statement (iv) is satisfied. Since we are under the assumptions of Theorem 13 we conclude that systems (ii) have exactly one limit cycle which is hyperbolic and unstable. This concludes the proof of the theorem. □

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