This is a preprint of: "The improved Euler-Jacobi formula and the planar cubic polynomial vector fields in $R^2$", Jaume Llibre, Clàudia Valls, *Dyn. Syst.*, vol. 37(3), 403–443, 2022.
DOI: [10.1080/14689367.2022.2066508]

# THE IMPROVED EULER-JACOBI FORMULA AND THE PLANAR CUBIC POLYNOMIAL VECTOR FIELDS IN $\mathbb{R}^2$

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

ABSTRACT. The new Euler-Jacobi formula for points with multiplicity two provides an algebraic relation between the singular points of a polynomial vector field and their topological indices. Using this formula we obtain the configuration of the singular points together with their topological indices for the planar cubic polynomial differential systems when these systems have eight finite singular points, being one of them with multiplicity two. The case with nine finite singular points has already been solved using the classical Euler-Jacobi formula.

## 1. Introduction and statement of the main results

Consider the planar polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

in $\mathbb{R}^2$ where $P(x, y)$ and $Q(x, y)$ are real polynomials of degree $n$ and $m$ respectively. The vector field associated to the planar polynomial differential system (1) will be denoted by $X$.

When $n = m = 2$ and system (1) has four singular points the possible configurations of the (topological) indices of these points where characterized by Berlinskii in [2] in 1960. Later on in 1999 Gasull and Torregrosa [7] also for systems (1) with $n = m = 2$ but with three singular points one with multiplicity two extended the Berlinskii theorem to these systems. More recently, the authors in [9] with $n = m = 3$ (i.e. for cubic polynomial differential systems) classify all the possible configurations of the indices of these systems having nine singular points. The objective of this paper is to classify all the configurations of indices of systems (1) with $n = m = 3$ having eight singular points one of them with multiplicity two.

When system (1) has $nm$ finite singular points the classical Euler-Jacobi formula (see [1] for its proof) gives an algebraic relation between the indices of these finite singular points. When system (1) has points with multiplicity two, the classical Euler-Jacobi formula is no longer valid. However in [7] the

2010 *Mathematics Subject Classification.* Primary 34A05. Secondary 34C05, 37C10.
*Key words and phrases.* Euler-Jacobi formula, singular points, topological index, polynomial differential systems.

authors (Gasull and Torregrosa) provided a generalization of the classical Euler-Jacobi formula when the system has points with multiplicity two. Using this new formula and the index theory we obtain the distribution of the singular points of cubic polynomial differential systems when this system has eight finite singularities, being one of them with multiplicity two.

We denote by $A_X = A$ the set of points $p \in \mathbb{R}^2$ such that $X(p) = 0$. Given a finite subset $B$ of $\mathbb{R}^2$, we denote by $\hat{B}$, $\partial\hat{B}$ and $\#B$ its convex hull, the boundary of the convex hull, and its cardinal, respectively.

Set $A_0 = A$ and $A_i = A \cap \partial\big(\widehat{A \setminus A_0 \cup \cdots \cup A_{i-1}}\big)$ for $i \geq 1$. Note that there exists $q \in \mathbb{N}$ such that $A_q \neq \emptyset$ and $A_{q+1} = \emptyset$.

We say that $A$ has the *configuration* $(K_0; K_1; K_2; \ldots; K_q)$ if $K_i = \#(A_i \cap \partial\hat{A}_i)$. We say that $A$ has the configuration $(K_0; K_1; K_2; \ldots; K_r; *)$ if we do not specify for the values of $K_i$ for $i$ between $r+1$ and $q$. We also say that the singular points of $X$ belonging to $A_i \cap \partial\hat{A}_i$ are on the *i-th level.*

We want to be more precise and study also the indices of the singular points of $X$. Then we substitute each $K_i$ by the sign of the indices of the points of $A_i \cap \partial\hat{A}_i$, i.e. instead of $K_i$ in the configuration we write the string $(s_1^i, s_2^i, \ldots, s_{K_i}^i)$ where $s_i^j \in \{+, -, 0\}$. When $A_i \cap \partial\hat{A}_i$ is a polygon, the starting $s_1^i$ will be the point with multiplicity two (denoted by $p_0$) if such point is in the $i$-th level and the signs $s_2^i, \ldots, s_{K_i}^i$ are the signs of the list of positive or negative indices that follow point with multiplicity two in counterclockwise or clockwise sense according with the largest list of points between the two lists closest to the point with multiplicity two. In case that both closest lists have the same length, we choose the one with the second largest closest list, and so one. In fact when there are $\ell$ equal consecutive signs, for instance if they are $+$, then instead of $+ \cdots +$ $\ell$-times we shall write $\ell+$. In the case that the $i$-th level does not contain the point with multiplicity two, then $s_1^i$ is the length of the largest list of positive or negative indices of the singular point in the $i$-th level. The numbers $s_2^i, \ldots, s_{K_1}^i$ are chosen following the previous criteria changing the point with multiplicity two by $s_1^i$.

When $A_i \cap \partial\hat{A}_i$ is a segment, which does not contain the point with multiplicity two, the signs of the strings are ordered starting at one of the endpoints. Then we start for the endpoint having the larger list of signs independently if this list is formed by plus or minus signs. In case that the length of the list of signs of both endpoints are equal, then we choose to start with the endpoint whose second list is larger, and so on.

If $A_i \cap \partial\hat{A}_i$ contains the point with multiplicity two we identify all the list of signs of this segment cyclically, i.e. after one endpoint it follows the other endpoint. Then the starting sign in the list is the sign 0 of the point of multiplicity two, and after it we choose the largest list closest to $p_0$. In case that the two lists of signs separated by $p_0$ have the same length, then we choose to start with the list near $p_0$ whose second list is larger, and so on.

With this notation we can state the main result of the paper. We denote by $i_X(a)$ the index of a singular point $a \in A$ of a planar cubic polynomial vector field $X$.

It was proved in [6] and [8] that in the case of cubic polynomial differential systems the absolute value of the sum of the indices of the points is either 1 or 3. In the next theorem, which is our first main theorem we characterize all the possible configurations for cubic polynomial differential systems when the absolute value of the sum of the indices is three.

**Theorem 1.** *For planar cubic polynomial differential systems having* 8 *singular points with one of multiplicity two, and* $|\sum_{a\in A} i_X(a)| = 3$*, the only possible configurations for the eight topological indices of their singularities are*

(5;3) $(5+;0,2-)$, $(0,4+;2-,+)$, $(5-;0,2+)$, $(0,4-;2+,-)$.
(4;4) $(4+;0,2-,+)$, $(4+;0,2-,+)$, $(0,3+;+,-,+,-)$, $(0,3+;2+,2-)$, $(4-;0,2+,-)$, $(4-;0,+,-,+)$, $(0,3-;+,-,+,-)$, $(0,3-;2+,2-)$.
(4;3;1) $(4+;+,2-;0)$, $(4+;0,2-;+)$, $(0,3+;2-,+;+)$, $(0,3+;2+,-;-)$, $(4-;2+,-;0)$, $(4-;0,2+;-)$, $(0,3-;2+,-;+)$, $(0,3-;2-,+;+)$.
(3;5) $(3+;0,+,-,+,-)$, $(0,2+;2+,-,+,-)$, $(3-;0,-,+,-,+)$, $(0,2-;2-,+,-,+)$.
(3;4;1) $(3+;2+,2-;0)$, $(3+;+,-,+,-;0)$, $(0,2+;+,-,+,-;+)$, $(3-;2+,2-;0)$, $(3-;-,+,-,+;0)$, $(0,2-;-,+,-,+;-)$.
(3;3;2) $(3+;2+,-;0,-)$, $(3+;0,-,+;+,-)$, $(0,2+;2+,-;+,-)$, $(3-;2-,+;0,+)$, $(3-;0,+,-;+,-)$, $(0,2-;2-,+;+,-)$.

*and there exist examples of such cubic polynomial differential systems with these configurations.*

The proof of Theorem 1 is given in section 3. To complete the cubic case it is necessary to characterize all possible configurations when $|\sum_{a\in A} i_X(a)| = 1$ which is done in the following theorem.

**Theorem 2.** *For planar cubic polynomial differential systems having* 8 *singular points with one of multiplicity two, and* $|\sum_{a\in A} i_X(a)| = 1$*, the only possible configurations for the eight topological indices of their singularities are*

(8) $(0,+,-,+,-,+,-,+)$, $(0,2+,-,+,-,+,-)$, $(0,-,+,-,+,-,+,-)$ *and* $(0,2-,+,-,+,-,+)$.
(7;1) $(0,+,-,+,-,+,-;+)$, $(0,2+,-,+,-,+;-)$, $(0,3+,-,+,-;-)$, $(2+,-,+,-,+,-;0)$, $(0,-,+,-,+,-,+;-)$, $(0,2+,-,+,-,+;-)$, $(0,3-,+,-,+;+)$ *and* $(2-,+,-,+,-,+;0)$.

(6;2) $(0,+,-,+,-,+;+,-)$, $(0,-,2+,-,+;+,-)$, $(0,2+,-,+,-;+,-)$, $(0,2+,-,2+;2-)$, $(2+,-,2+,-;0,-)$, $(3+,-,+,-;0,-)$, $(+,-,+,-,+,-;0,+)$, $(0,-,+,-,+,-;+,-)$, $(0,+,2-,+,-;+,-)$, $(0,2-,+,-,+;+,-)$, $(0,2-,+,2-;2+)$, $(2-,+,2-,+;0,+)$, $(3-,+,-,+;0,+)$ *and* $(+,-,+,-,+,-;0,-)$.

(5;3) $(0,-,2+,-;2+,-)$, $(0,+,-,+,-;2+,-)$, $(0,2+,-,+;2-,+)$, $(0,3+,-;2-,+)$, $(2+,-,+,-;0,+,-)$, $(4+,-;0,2-)$, $(0,+,2-,+;2-,+)$, $(0,-,+,-,+;2-,+)$, $(0,2-,+,-;2+,-)$, $(0,3-,+;2+,-)$, $(2-,+,-,+;0,-,+)$ *and* $(4-,+;0,2+)$.

(4;4) $(+,-,+,-;0,+,-,+,-)$, $(+,-,+,-;0,2+,-)$, $(0,-,+,-;3+,-)$, $(0,+,-,+;+,-,+,-)$, $(0,+,-,+;2+,2-)$, $(0,2+,-;+,-,+,-)$, $(0,2+,-;2+,2-)$, $(0,3+;3-,+)$, $(3+,-;0,-,+,-)$, $(3+,-;0,2-,+)$, $(4+;0,3-)$, $(+,-,+,-;0,-,+,-,+)$, $(+,-,+,-;0,2-,+)$, $(0,+,-,+;3-,+)$, $(0,-,+,-;+,-,+,-)$, $(0,-,+,-;2+,2-)$, $(0,2-,+;-,+,-,+)$, $(0,2-,+;2+,2-)$, $(0,3-;3+,-)$, $(3-,+;0,+,-,+)$, $(3-,+;0,2+,-)$ *and* $(4-;0,3+)$.

(4;3;1) $(0,+,-,+;2-,+;+)$, $(+,-,+,-;0,+,-;+)$, $(+,-,+,-;0,2+;-)$, $(+,-,+,-;2+,-;0)$, $(0,-,+,-;2+,-;+)$, $(0,-,+,-;3+;-)$, $(0,2+,-;2+,-;-)$, $(0,3+;3-;+)$, $(3+,-;2-,+;0)$, $(3+,-;0,+,-;-)$, $(3+,-;0,2-;+)$, $(4+;3-;0)$, $(4+;0,2-;-)$, $(0,2+,-;2-,+;+)$, $(0,-,+,-;2+,-;+)$, $(0,-,+,-;2+,-;-)$, $(-,+,-,+;0,-,+;-)$, $(-,+,-,+;0,2-;+)$, $(-,+,-,+;2-,+;0)$, $(0,-,+,-;2-,+;-)$, $(0,+,-,+;3-;+)$, $(0,2-,+;2-,+;+)$, $(0,3-;3+;-)$, $(3-,+;2+,-;0)$, $(3-,+;0,-,+;+)$, $(3-,+;0,2+;-)$, $(4-;3+;0)$, $(4-;0,2+;+)$, $(0,2-,+;2+,-;-)$ *and* $(0,+,-,+;2-,+;-)$.

(3;5) $(0,+,-;2+,-,+,-)$, $(0,2+;2-,+,-,+)$, $(2+,-;0,-,2+,-)$, $(2+,-;0,+,-,+,-)$, $(3+;0,2-,+,-)$, $(0,+,-;2-,+,-,+)$, $(0,2-;2+,-,+,-)$, $(2-,+;0,+,2-,+)$, $(2-,+;0,-,+,-,+)$ *and* $(3-;0,2+,-,+)$.

(3;4;1) $(0,+,-;+,-,+,-;+)$, $(0,+,-;3+,-;-)$, $(0,2+;3-,+;-)$, $(2+,-;0,2-,+;+)$, $(2+,-;0,2+,-;-)$, $(2+,-;2+,2-;0)$, $(3+;0,2-,+;-)$, $(3+;0,3-;+)$, $(3+;3-,+;0)$, $(0,-,+;-,+,-,+;-)$, $(0,-,+;3-,+;+)$, $(0,2-;3+,-;+)$, $(2-,+;0,2+,-;-)$, $(2-,+;0,2-,+;+)$, $(2-,+;2-,2+;0)$, $(3-;0,2+,-;+)$, $(3-;0,3+;-)$, *and* $(3-;3+,-;0)$,

(3;3;2) $(0,+,-;2+,-;+,-)$, $(0,2+;2-,+;+,-)$, $(0,2+;3-;2+)$, $(2+,-;2-,+;0,+)$, $(2+,-;0,+,-;+,-)$, $(2+,-;2+,-;0,-)$, $(3+;0,2+;+,-)$, $(2+,-;0,2-;2+)$, $(3+;3-;0,+)$ $(0,-,+;2-,+;+,-)$, $(0,2-;2+,-;-,+)$, $(0,2-;3+;2-)$, $(2-,+;2+,-;0,-)$, $(2-,+;0,-,+;-,+)$, $(2-,+;2-,+;0,+)$, $(3-;0,2-;+,-)$, $(2-,+;0,2+;2-)$ *and* $(3-;3+;0,-)$.

*and there exist examples of such cubic polynomial differential systems with these configurations.*

The proof of Theorem 2 is given in section 4. As mentioned above with Theorems 1 and 2 we provide the classification of the configurations of the singular points and their topological indices for all planar polynomial differential systems with eight finite singular points being one (that we will translate it at the origin) double.

## 2. Preliminaries

First of all we observe that if a configuration exists for a cubic polynomial vector field $X$ with $\#A_X = 8$, being one of them double then it is possible to construct the same configuration but interchanging points with index $+1$ with points with index $-1$. For doing that it is enough to take $Y = (-P, Q)$ instead of $X = (P, Q)$. So we can restrict ourselves to the cases in which $\sum_{a\in A} i_X(a) \geq 0$. Moreover it was proved in [6] and [8] that in this case the sum of the absolute value of the indices of the points is either 3 or 1. Since there is only one point with multiplicity two we must have five points with positive index and two points with negative index when the sum of the indices of the finite singular points is three, and four points with positive index and three points with negative index when the sum of the indices of the finite singular points is one.

It follows from index theory (see for instance [3]) that if two points collide then the indices add. Since the indices of the semi-hyperbolic points can only be $-1, 1, 0$ and the indices of the hyperbolic points when there are nine finite singular points can only be $-1, 1$ it follows that only points with indices $-1$ and $+1$ can collide. In short, the only possible configurations in Theorem 1 can be from colliding two hyperbolic points with index $+1$ and $-1$ in a configuration of a planar cubic polynomial differential system such that $\#A = 9$ and $|\sum_{a\in A} i_X(a)| = 3$. In [9] the authors obtained such configurations in the following theorem.

**Theorem 3.** *For planar cubic polynomial differential systems having* 9 *singular points and* $|\sum_{a\in A} i_X(a)| = 3$ *the only possible configurations for the nine topological indices of their singularities are*

(5;3;1) *with* $(5+; 3-; +)$, $(5-; 3+; -)$;
(4;5) *with* $(4+; +, 2-, +, -)$, $(4-; 2+, -, +, -)$;
(4;4;1) *with* $(4+; +, 3-; +)$, $(4-; 3+, -; -)$;
(4;3;2) *with* $(4+; 3-; 2+)$, $(4+; +, 2-; +, -)$, $(4-; 3+; 2-)$, $(4-; 2+, -; +, -)$;
(3;6) *with* $(3+; +, -, +, -, +, -)$, $(3-; +, -, +, -, +, -)$;
(3;5;1) *with* $(3+; +, 2-, +, -; +)$, $(3-; 2+, -, +, -; -)$;
(3;4;2) *with* $(3+; 2+, 2-; +, -)$, $(3-; 2+, 2-; +, -)$;
(3;3;3) *with* $(3+; 2+, -; +, 2-)$, $(3-; +, 2-; 2+, -)$;

*and there exist examples of such cubic polynomial differential systems with these configurations.*

**Lemma 4.** *For planar cubic polynomial differential systems with* 8 *singular points being one with multiplicity two, and* $|\sum_{a\in A} i_X(a)| = 3$*, the only possible configurations for the eight topological indices of their singularities are of the form* $(K+;*)$ *where* $K \leq 5$ *or* $(0, L+;*)$ *where* $L \leq 4$.

*Proof.* It follows directly from Theorem 3 and the fact that the point with multiplicity two can only formed by colliding two hyperbolic points in a configuration of planar cubic polynomial differential systems such that $\#A = 9$ and $|\sum_{a\in A} i_X(a)| = 3$. □

It follows that a singular point $p$ of system (1) is *simple* if and only if the determinant of the Jacobian matrix of $P$ and $Q$ at $p$ is different from zero, i.e.

$$J(P,Q)(p) := J(p) = \left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}\right)\Big|_p \neq 0.$$

A singular point $p$ of system (1) is *double* if and only if $J(p) = 0$ and $I(P,Q)(p) := I(p) \neq 0$ where

$$\begin{aligned} I(P,Q)(p) := I(p) = {} & \left(\frac{\partial P}{\partial y}\right)^2 \left(\frac{\partial P}{\partial x}\frac{\partial^2 Q}{\partial x \partial x^2} - \frac{\partial Q}{\partial x}\frac{\partial^2 P}{\partial x^2}\right) \\ & - 2\frac{\partial P}{\partial x}\frac{\partial P}{\partial y}\left(\frac{\partial P}{\partial x}\frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial Q}{\partial x}\frac{\partial^2 P}{\partial x \partial y}\right) \\ & + \left(\frac{\partial P}{\partial x}\right)^2 \left(\frac{\partial P}{\partial x}\frac{\partial^2 Q}{\partial y^2} - \frac{\partial Q}{\partial x}\frac{\partial^2 P}{\partial y^2}\right)\Big|_p . \end{aligned}$$

For a proof see Lemma 2.2 of [7]. Moreover it is well-known that for planar polynomial differential systems (1), a simple point $p$ has index 1 (if $J(p) > 0$) or $-1$ (if $J(p) < 0$) (see for instance [10]) and that points with multiplicity two of our system (1) has index zero.

In this paper we consider the case in which one of the 8 singular points, that we will denote by $p_0$, has multiplicity two. We will use the new Euler-Jacobi formula for points with multiplicity two proved by Gasull and Torregrosa in [7] which can be stated as follows. To state it we need the following notation. We write

$$\begin{aligned} P(x,y) &= P_{10}x + P_{01}y + P_{20}x^2 + P_{11}xy + P_{02}y^2 + \dots, \\ Q(x,y) &= Q_{10}x + Q_{01}y + Q_{20}x^2 + Q_{11}xy + Q_{02}y^2 + \dots \end{aligned}$$

and given a polynomial $R$ we also write it as

$$R(x,y) = R_{00} + R_{10}x + R_{01}y + R_{20}x^2 + R_{11}xy + R_{02}y^2 + \dots$$

The next result is proved in Theorem 3.2 of [7] for two real polynomials of degrees $n$ and $m$. We state it here for the case in which $n = m = 3$ and when the system has eight finite singular points one with multiplicity two.

**Theorem 5.** *Consider a cubic system of two real polynomials in the variables $x$ and $y$. If the set of zeroes of that system (that we denote by $A$) contains exactly eight elements (seven being simple and one with multiplicity two $p_0$ that without loss of generality we can assume it is at the origin), then for any polynomial $R$ of degree less than or equal to 3 we have*

$$\sum_{a\in A_S} \frac{R(a)}{J(a)} + S(0) = 0 \tag{2}$$

*where $A_S$ denotes the set of simple zeroes of the system and $S(0)$ is equal to*

$$S(0) = \frac{4P_{10}R_{00}N}{I(0)^2} + \frac{2P_{10}(P_{10}R_{01} - P_{01}R_{10})}{I(0)},$$

*where*

$$\begin{aligned} N = {} & P_{10}^3(Q_{10}P_{03} - P_{10}Q_{03}) - P_{10}^2P_{01}(Q_{10}P_{12} - P_{10}Q_{12}) \\ & + P_{10}P_{01}^2(Q_{10}P_{21} - P_{10}Q_{21}) - P_{01}^3(Q_{10}P_{30} - P_{10}Q_{30}) \\ & + P_{10}^3(Q_{11}P_{02} - P_{11}Q_{02}) - 2P_{10}^2P_{01}(Q_{20}P_{02} - P_{20}Q_{02}) \\ & + P_{10}P_{01}^2(Q_{20}P_{11} - P_{20}Q_{11}). \end{aligned}$$

In the proof of Theorems 1 and 2 we will denote by $L_{ij}$ the straight line through the points $p_i$ and $p_j$ where $i, j \in \{1, \ldots, 8\}$ and also $L_{ij}^2 = L_{ij}L_{ij}$. Moreover during all the paper we will consider straight lines with only two singular points because if there are straight-lines with three singular points the arguments become even easier. Furthermore, we note that when proving that some configurations are not possible, we will take $p_0 = (0, 0)$ and a conic (generally formed by two straight lines) passing through $p_0$. Doing so, we have that $R_{00} = R_{10} = R_{01} = 0$, and consequently $S(0) = 0$ in formula (2).

**Lemma 6.** *For planar cubic polynomial differential systems such that $\#A_X = 8$, and $|\sum_{a\in A} i_X(a)| = 1$, there are no configurations of the form $(K*, 2-, *; *)$, that is, there are no configurations with two consecutive points with negative index in the 0-level.*

*Proof.* Assume that there are two consecutive points in the 0-level with negative index and denote them by $p_1, p_2$. Since the absolute value of the sum of the indices is one and we can assume that the sum is positive, we have an additional point with negative index, that we denote by $p_3$. Hence, applying the Euler-Jacobi formula (2) with $R = L_{0,p_3}^2 L_{p_1,p_2}$ we reach a contradiction. □

## 3. Proof of Theorem 1

It follows from Lemma 4 that the possible configurations are: $(5+; 3)$, $(0, 4+; 3)$, $(4+; 4)$, $(0, 3+; 4)$, $(4+; 3; 1)$, $(0, 3+; 4; 1)$, $(3+; 5)$, $(0, 2+; 5)$, $(3+; 4; 1)$,

$(0, 2+; 4; 1)$, $(3+; 3; 2)$ and $(0, 2+; 3; 2)$. We will study each of them separately.

*Configuration* $(5+; 3)$ Since there are five points with positive index and two points with negative index the unique possible configuration is $(5+; 0, 2-)$. The cubic system (1) with

$$\begin{aligned} P(x, y) = & \quad -6y + 5y^2 + 2x^2y - y^3, \\ Q(x, y) = & \quad \frac{1}{5}(48 + 32x - 40y - 12x^2 - 26xy + 8y^2 - 8x^3 - x^2y + 5xy^2), \end{aligned}$$

has the singular points

$$\begin{array}{ccccc} (-1., 4.), & (0, 3), & (-2, 0), & (2, 0), & (1, 4), \\ (-1, 1), & (-3/2, 0), & (0, 2), & (0, 2) & \end{array}$$

in the configuration $(5+; 0, 2-)$.

*Configuration* $(0, 4+; 3)$ Since there are five points with positive index and two points with negative index the unique possible configuration is $(0, 4+; 2-, +)$. The cubic system (1) with

$$\begin{aligned} P(x, y) = & \quad 6y - 5y^2 - 2x^2y + y^3, \\ Q(x, y) = & \quad -\frac{1}{10}(48x + 3y - y^2 - 46xy - 12x^3 - 2x^2y + 10xy^2), \end{aligned}$$

has the singular points

$$\begin{array}{cccccc} (0, 3), & (0, 3), & (-2, 0), & (2, 0), & (1, 4), & (-1, 1), \\ (0.428571, 1.71429), & (1, 1), & (0, 0) & & & \end{array}$$

in the configuration $(0, 4+; 2-, +)$.

*Configuration* $(4+; 4)$ Since there are five points with positive index and two points with negative index the unique possible configurations of the form $(4+; 4)$ are $(4+; 0, -, +, -)$ and $(4+; 0, 2-, +)$. The cubic system (1) with

$$\begin{aligned} P(x, y) = & \ 1.4523 + 1.23689x + 3.08149y + 3.84729x^2 + 1.05231xy \\ & - 3.52134y^2 + 0.168176x^3 + 0.910772x^2y - y^3, \\ Q(x, y) = & \ 1.2082 + 11.867x + 14.1512y + 2.98455x^2 + 7.00776xy \\ & + 3.81719y^2 + 0.0233874x^3 + 0.827415x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{ccc} (-3.9, -3.6), & (-3.92, -3.5), & (-3.84.., -3.5..), \\ (-3.84.., -3.5..), & (16.24.., -13.16..), & (3.99.., -3.74..), \\ (-2.99.., 2.82..), & (2.00., -1.73..), & (0.86.., -0.80..) \end{array}$$

in the configuration $(4+; 0, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -2.14508 - 2.27088x - 0.61618y - 1.93249x^2 + 0.32733xy \\ &\quad + 2.46863y^2 + 0.26016x^3 + 1.06646x^2y - y^3, \\ Q(x,y) &= -5.23367 - 4.92581x - 0.81582y - 3.21993x^2 - 0.9972xy \\ &\quad + 2.79274y^2 - 0.78125x^3 - 0.24175x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-4,3), \quad (3,-3), \quad (-3.29..,2.66..), \quad (-3,2.82..), \quad (4,5), \\ &(-0.95..,1.25..), \quad (-0.95..,1.25..), \quad (-3,-2.82..), \quad (-2,-1.73..) \end{aligned}$$

in the configuration $(4+;0,2-,+)$.

*Configuration* $(0,3+;4)$ Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,3+;+,-,+,-)$ and $(0,3+;2+,2-)$. The cubic system (1) with

$$\begin{aligned} P(x,y) &= -2.14507 - 2.27088x - 0.61617y - 1.93248x^2 + 0.32733xy \\ &\quad + 2.46863y^2 + 0.26016x^3 + 1.06646x^2y - y^3, \\ Q(x,y) &= -5.23368 - 4.92581x - 0.81582y - 3.21993x^2 - 0.99719xy \\ &\quad + 2.79274y^2 - 0.78125x^3 - 0.24175x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-4,3), \quad (3,-3), \quad (-3.29..,2.66..), \quad (-3,2.82..), \quad (4,5), \\ &(-0.95..,1.25..), \quad (-0.95..,1.25..),(-3,-2.82), \quad (-2,-1.73) \end{aligned}$$

in the configuration $(0,3+;+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.00003 - 0.00002x + 0.19126y - 0.00002x^2 + 0.46652xy \\ &\quad + 0.11736y^2 + 0.98626x^2y - y^3, \\ Q(x,y) &= 0.00001 - 1.09067y + 0.16407xy + 2.18129y^2 \\ &\quad + 0.43317x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-3.00..,2.82..), \quad (-5.05..,0.00..), \quad (4.00..,0.00..), \\ &(4.00..,0.00..), \quad (-0.73..,0.67..), \quad (-1.50..,0), \\ &(0.99..,0), \quad (0,0.49..), \quad (-2.00..,-1.73..) \end{aligned}$$

in the configuration $(0,3+;2+,2-)$.

*Configuration* $(4+;3;1)$ Since there are five points with positive index and two points with negative index the unique possible configurations are $(4+;2-,+;0)$, $(4+;0,2-;+)$ and $(4+;0,+,-;-)$.

We will show that the configuration $(4+;0,+,-;-)$ is not possible. Denote by $p_1, p_2, p_3, p_4$ the four points in the 0-level (with positive index), by $p_0, p_5, p_6$ the points in the 1st-level (being $p_0$ the point with multiplicity two, $p_5$ the point with positive index and $p_6$ the singular point with negative index) and by $p_7$ the singular point in the 2nd level. There exists $k_0, k_1 \in \{1,2,3,4\}$ so that $L_{k_0,6}$ leaves $p_{k_1}$ on one side and the other five points in the other side. We denote by $p_{k_2}$ and $p_{k_3}$ the two remaining points in the 0-level. Note that we can consider, without loss of generality, that $L_{0,k_2}$ leaves $k_3$ on one side and the other five points in the other side. Applying the Euler-Jacobi formula to $S = L_{0,5}L_{0,k_2}L_{k_0,6}$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.01043 + 0.08997x - 0.00631y - 0.0381x^2 + 0.29991xy\\&\quad + 0.16308y^2 - 0.0623x^3 - 0.69456x^2y + y^3,\\Q(x,y) &= 0.03992 - 0.38324x - 0.55026y + 0.0967x^2 - 1.57747xy\\&\quad - 2.27773y^2 + 0.24662x^3 - 0.41836x^2y - xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(-3.,-2.82..),\quad (-1.5,0),\quad (1,0),\quad (-2,1.73..),\quad (4,-3),\\(0,-0.3),\quad (0,-0.3),\quad (0.44..,-0.41..),\quad (-0.1,0.12)\end{gathered}$$

in the configuration $(4+;2-,+;0)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -0.94486 - 1.31372x - 1.34391y - 2.35727x^2 + 0.14990xy\\&\quad + 2.92903y^2 + 0.19308x^3 + 1.08818x^2y - y^3,\\Q(x,y) &= -7.09812 - 5.92471x + 0.59531y - 1.43097x^2 - 0.98248xy\\&\quad + 1.45502y^2 - 0.53935x^3 - 0.39364x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(-4,3),\quad (-3,2.82..),\quad ,(-3,2.82..),\quad (3,-2.82..),\quad (3,5),\\(0.02..,2.03..),\quad (-0.95..,1.25..),\quad (-3,-2.82..),\quad (-2,-1.73..)\end{gathered}$$

in the configuration $(4+;0,2-;+)$.

*Configuration* $(0,3+;3;1)$ Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,3+;2-,+;+)$ and $(0,3+;2+,-;-)$. The cubic system (1) with

$$\begin{aligned}P(x,y) &= 8.01755 + 13.6686x + 13.329y + 8.63431x^2 + 18.4872xy\\&\quad + 6.48002y^2 + 3.14207x^3 + 4.69624x^2y - y^3,\\Q(x,y) &= -1.58427 - 1.41114x + 1.3319y - 0.934538x^2 - 0.607739xy\\&\quad + 2.32775y^2 - 0.692771x^3 - 0.130663x^2y + xy^2,\end{aligned}$$

has the singular points

$$(-4,-4),\quad (-4,5),\quad (-4,5),\quad (-13.50..,11.16..),\quad (-1.5,-0.5),$$
$$(0.57..,-1.16..),\quad (-1,0.2),\quad (2,-1.73..),\quad (4,-3)$$

in the configuration $(0,3+;2-,+;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 13.31010 + 6.56921x + 10.49492y + 0.55371x^2 + 6.40611xy \\ &\quad + 4.24196y^2 + 0.94893x^3 + 1.993x^2y - y^3, \\ Q(x,y) &= -1.51411 + 0.83433x + 7.4396y + 1.98657x^2 + 0.93647xy \\ &\quad + 1.48236y^2 - 0.36592x^3 - 0.0735x^2y + xy^2, \end{aligned}$$

has the singular points

$$(-119.62..,69.71..),\quad (-4,-4),\quad (45.73..,-24.47..),\quad (-2.52..,-2.17..),$$
$$(-4,5),\quad (-4,5),\quad (-1.5,-0.5),\quad (1.85,-2),\quad (2,-1.73..)$$

in the configuration $(0,3+;2+,-;-)$.

*Configuration* $(3+;5)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(3+;0,+,-,+,-)$, $(3+;0,+,2-,+)$ and $(3+;0,2+,2-)$. We show that the configurations $(3+;0,+,2-,+)$ and $(3+;0,2+,2-)$ are not possible.

For the configuration $(3+;0,+,2-,+)$ denote by $p_1,p_2,p_3$ the points in the 0-level and by $p_4,p_7$ and $p_5,p_6$ the points in the 1st level with positive and negative indices, respectively. The curve $C = L_{0,5}L_{0,6}$ has different signs on $p_7$ and on $p_4$. Moreover there exits $k_0 \in \{1,2,3\}$ so that $C(p_{k_0})$ has either the same sign than $C(p_4)$, or the same sign than $C(p_7)$. Applying formula (2) to $L_{k_1,k_2}C$ (note that here $R_{0,0} = R_{1,0} = R_{0,1} = 0$ and so $S(0) = 0$) with $k_1,k_2 \in \{1,2,3\}$ and $k_i \neq k_0$ for $i = 1,2$ we reach to a contradiction. Hence this case is not possible.

For the configuration $(3+;0,2+;2-)$ denote by $p_1,p_2,p_3$ the points in the 0-level and by $p_4,p_5$ the points in the 1st level with negative index. Note that there exists $k_0 \in \{1,2,3\}$ so that the curve $L_{0,4}L_{0,5}$ evaluated on $p_{k_0}$ has the same sign as $C$ evaluated on two positive indices of the 1st level. Applying formula (2) with $C = L_{0,4}L_{0,5}L_{k_1,k_2}$ where $k_1,k_2 \in \{1,2,3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach to a contradiction.

In short the unique possible configuration is $(3+;0,+,-,+,-)$. The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.10892 - 0.06352x - 1.52889y + -0.06353x^2 - 2.2135xy \\ &\quad + 0.09531y^2 + 0.01816x^3 - 1.45684x^2y + y^3, \\ Q(x,y) &= -1.43214 + 0.83542x + 1.82603y + x0.83542x^2 + 0.31892xy \\ &\quad + 1.74687y^2 - 0.23869x^3 - 0.09658x^2y + xy^2, \end{aligned}$$

has the singular points

$$
\begin{aligned}
&(-3.00.., 2.83..), \quad (4.00.., 0.00..), \quad (0.15.., -1.45..),\\
&(-0.73.., 0.67..), \quad (-0.73.., 0.67..), \quad (-1.5.., 0),\\
&(-2.99.., -2.82..), \quad (0.99.., 0), \quad (-2.00.., -1.73..)
\end{aligned}
$$

in the configuration $(3+; 0, +, -, +, -)$.

*Configuration* $(0, 2+; 5)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(0, 2+; 3+, 2-)$ and $(0, 2+; 2+, -, +, -)$. We will show that the configuration $(0, 2+; 3+, 2-)$ is not possible. Indeed, denote by $p_1, p_2$ the points in the 0 level and by $p_3, p_4, p_5$ the points in the 1st level with positive index and by $p_6, p_7$ the points in the 1st level with negative index. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{6,7}$ we reach to a contradiction. In short only the configuration $(0, 2+; 2+, -, +, -)$ is possible. The cubic system (1) with

$$
\begin{aligned}
P(x, y) &= 0.24337 + 0.04054x - 0.28334y - 0.06592x^2 + 0.51321xy\\
&\quad + 0.09318y^2 + 0.01014x^3 + 1.09144x^2y - y^3,\\
Q(x, y) &= -1.69782 - 0.2829x + 2.22066y + 0.45983x^2 - 0.1617xy\\
&\quad + 2.35005y^2 - 0.07074x^3 - 0.300636x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$
\begin{aligned}
&(-3.00.., 2.82..), \quad (4.00.., 0.00..), \quad (4.00.., 0.00..),\\
&(0.99.., -1.00..), \quad (-1.5.., 0), \quad (-0.49.., 0.46..),\\
&(-2.99.., -2.82..), \quad (0, 0.499..), \quad (-2.00.., -1.73..)
\end{aligned}
$$

in the configuration $(0, 2+; 2+, -, +, -)$.

*Configuration* $(3+; 4; 1)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(3+; 2+, 2-; 0)$, $(3+; +, -, +, -; 0)$, $(3+; 0, 2-, +; +)$, $(3+; 0, -, +, -; +)$, $(3+; 0, +, -, +; -)$ and $(3+; 0, 2+, -; -)$. We will show that the configuration $(3+; 0, +, -, +; -)$ is not possible. Take the curve $C = L_{0,4}L_{0,6}$. Note that there exists $k_0 \in \{1, 2, 3\}$ so that it is the unique singularity contained in a connected component of $\mathbb{R}^2 \setminus C$. Applying formula (2) with $C = L_{0,4}L_{0,5}L_{k_1,k_2}$ where $k_1, k_2 \in \{1, 2, 3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

The cubic system (1) with

$$
\begin{aligned}
P(x, y) &= 11.55589 - 4.10670x + 7.72417y + 1.08554x^2 - 1.88945xy\\
&\quad - 1.13607y^2 + 0.0411x^3 + 0.42839x^2y - y^3,\\
Q(x, y) &= 7.98585 + 5.44717x + 11.84708y + 0.7037x^2 + 4.54680xy\\
&\quad + 3.93911y^2 - 0.05877x^3 + 0.40006x^2y + xy^2,
\end{aligned}
$$

has the singular points

$$(45.73.., -24.47..),\quad (-4,5),\quad (1.85,-2),\quad (2,-1.73..),\quad (1,-1.5),$$
$$(-4,-4),\quad (-1.42..,-2.11..),\quad (-2.52..,-2.17..),\quad (-2.52..,-2.17..)$$

in the configuration $(3+;0,-,+,-;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 8.98459 + 6.6871x + 10.86426y + 2.87879x^2 + 6.1108xy \\ &\quad + 2.50325y^2 + 0.91190x^3 + 2.00473x^2y - y^3, \\ Q(x,y) &= 3.99511 + 1.65625x + 8.11454y - 0.114601x^2 + 1.88344xy \\ &\quad + 3.332y^2 - 0.27851x^3 + 0.00545x^2y + xy^2, \end{aligned}$$

has the singular points

$$(45.73.., -24.47..),\quad (14.69..,-8.08..),\quad (-4,5),\quad (2,-1.73..),$$
$$(1,-1.5),\quad (1,-1.5),\quad (-4,-4),\quad (-1.5,-0.5),\quad (-2.52..,-2.17..)$$

in the configuration $(3+;2+,2-;0)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 18.01525 - 0.56436x + 12.6791y + 1.95534x^2 + 0.51759xy \\ &\quad - 0.9431y^2 + 0.19838x^3 + 0.7084x^2y - y^3, \\ Q(x,y) &= 7.63906 + 5.25699x + 11.58106y + 0.657x^2 + 4.41758xy \\ &\quad + 3.92875y^2 - 0.06722x^3 + 0.38503x^2y + xy^2, \end{aligned}$$

has the singular points

$$(45.73.., -24.47..),\quad (-4,5),\quad (4,-3),\quad (2,-1.73..),\quad (1,-1.5),$$
$$(-4,-4),\quad (-2.52..,-2.17..),\quad (-2.52..,-2.17..),\quad (-2.96..,-2.14..)$$

in the configuration $(3+;+,-,+,-;0)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 12.53185 + 5.04636x + 15.68935y + 0.30004x^2 + 2.48145xy \\ &\quad + 2.9495y^2 + 0.03753x^3 + 0.34625x^2y - y^3, \\ Q(x,y) &= 7.74477 + 4.89906x + 11.2205y + 0.77834x^2 + 4.32362xy \\ &\quad + 3.69906y^2 - -0.04556x^3 + 0.4297x^2y + xy^2, \end{aligned}$$

has the singular points

$$(45.73..., -24.47...),\quad (-4,5),\quad (4,-3),\quad (1.85,-2),\quad (2,-1.73..),$$
$$(2,-1.73..),\quad (-1.5,-0.5),\quad (-2.52..,-2.17..),\quad (-3.51..,-2.36..)$$

in the configuration $(3+;0,2-,+;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 8.69530 + 2.74101x + 10.71702y + 0.56511x^2 + 1.26514xy \\ &\quad + 1.61948y^2 + 0.29941x^3 + 0.85775x^2y - y^3, \\ Q(x,y) &= 5.5855 + 4.1302x + 9.47293y + 0.9203x^2 + 4.29167xy \\ &\quad + 3.7649y^2 - 0.01426x^3 + 0.49555x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll} (-4,-4), & (45.73..,-24.47..), & (-4,5), & (-1.5,-0.5), & (0.8,-1.6), \\ (0,8,-1.6), & (1.044..,-1.59..), & (2,-1.73..), & (4.2,-2.9) & \end{array}$$

in the configuration $(3+;0,2+,-;-)$.

*Configuration* $(0,2+;4;1)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,2+;2+,2-;+)$, $(0,2+;+,-,+,-;+)$ and $(0,2+;3+,-;-)$. We will show that the configuration $(0,2+;2+,2-;+)$ is not possible. Indeed, denote by $p_1$ and $p_2$ the points in the 0-level and by $p_3, p_4$ the points in the 1st level with negative index. Applying formula (2) with $C = L_{0,1}L_{0,2}L_{3,4}$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 9.39252 - 15.4142x - 2.77515y + 3.34204x^2 - 4.776xy \\ &\quad - 4.97594y^2 + 0.07817x^3 + 0.60294x^2y - y^3, \\ Q(x,y) &= 4.63046 - 7.21976x - 0.67095y + 3.4358x^2 + 1.63839xy \\ &\quad - 0.27316y^2 + 0.04718x^3 + 0.71861x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll} (-4,-4), & (-4,-4), & (-3.9,-4), & (-4,5), & (1,-1.5), \\ (45.73..,-24.47..), & (1.85,-2), & (2,-1.73..), & (1.03..,-0.71..) & \end{array}$$

in the configuration $(0,2+;+,-,+,-;+)$

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 7.86131 - 22.585x - 10.2257y + 5.15825x^2 - 5.92109xy \\ &\quad - 7.34354y^2 + 0.21660x^3 + 0.9303x^2y - y^3, \\ Q(x,y) &= 2.50283 - 12.253x - 6.27431y + 4.96135x^2 + 0.88691xy \\ &\quad - 2.02498y^2 + 0.16245x^3 + 0.99043x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll} (-4,-4), & (-4,-4), & (-4.1,5.1), & (-4,5), & (45.73..,-24.47..), \\ (1.85,-3), & (1.88..,-2.65..), & (1,-1.85), & (2,-1.73..) & \end{array}$$

in the configuration $(0,2+;3+,-;-)$.

*Configuration* $(3+;3;2)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(3+;0,2-;2+)$, $(3+;0,2+;2-)$, $(3+;0,+,-;+,-)$, $(3+;+,2-;0,+)$ and also $(3+;2+,-;0,-)$. We will show that the configurations $(3+;0,2-;2+)$ is not possible. Indeed, we denote by $p_1,p_2,p_3$ the points in the 0-level and by $p_4,p_5$ the points in the 1st level with negative index. Note that there exists $k_0\in\{1,2,3\}$ so that $C=L_{0,4}L_{0,5}$ evaluated at $p_{k_0}$ has the same sign than $C$ evaluated to the two positive points in the 2nd index. Applying formula (2) with $C=L_{0,4}L_{0,5}L_{k_1,k_2}$ where $k_1,k_2\in\{1,2,3\}$ with $k_1\neq k_0$ and $k_2\neq k_0$ we have a contradiction.

The cubic system (1) with

$$\begin{aligned}P(x,y)&=7.80543+5.53466x+11.72129y+1.75211x^2+3.947459xy\\&\quad+2.83894y^2+0.4850x^3+1.23003x^2y-y^3,\\Q(x,y)&=3.70643+4.38387x+6.9673y+1.88746x^2+5.093xy\\&\quad+3.2218y^2+0.31337x^3+1.135x^2y+xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(-4,5.20..),\quad(4.92..,-3.88..),\quad(4.00..,-2.99..),\\(3.82..,-2.91..),\quad(1-85,-2),\quad(1.85,-2),\\(1,-1.49..),\quad(-1.96..,-0.00..),\quad(-2.52..,-2.17..)\end{gathered}$$

in the configuration $(3+;0,2+;2-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y)&=17.676+12.71329x+23.36367y+1.89862x^2+5.68740xy\\&\quad+3.79344y^2+0.00005x^3+0.45244x^2y-y^3,\\Q(x,y)&=10.45245+9.29009x+14.92425y+1.9876x^2+6.28217xy\\&\quad+3.87415y^2-0.01806x^3+0.60356x^2y+xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(-54.84..,-4.87..),\quad(-4,5.2),\quad(4.92..,-3.88..)\\(4.00..,-2.99..),\quad(4.00..,-2.99..),\quad(1.85,-2),\\(1,-1.49..),\quad(-1.96..,-0.00..),\quad(-2.52..,-2.17..)\end{gathered}$$

in the configuration $(3+;0,+,-;+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y)&=19.5475+16.6124x+18.99y+6.43719x^2+10.846xy\\&\quad+2.22562y^2+1.25946x^3+2.69803x^2y-y^3,\\Q(x,y)&=0.292006-5.41301x+4.45102y-2.412x^2-1.63724xy\\&\quad+2.91582y^2-0.598931x^3-0.619738x^2y+xy^2,\end{aligned}$$

has the singular points

$$\begin{array}{lll}(-4,-4), & (45.73..,-24.47..), & (-2.52..,-2.17..),\\ (-4,5), & (-0.9..,-1.35..), & (-0.9,-1.35),\\ (-0.9,-1.35), & (-0.9,-1.34), & (4,-3)\end{array}$$

in the configuration $(3+;+,2-;0,+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 6.23278 + 4.39091x + 9.86634y + 1.72876x^2 + 3.67023xy\\ &\quad + 2.68686y^2 + 0.56225x^3 + 1.35392x^2y - y^3,\\ Q(x,y) &= 3.14549 + 3.97591x + 6.30566y + 1.87914x^2 + 4.99412xy\\ &\quad + 3.16756y^2 + 0.34093x^3 + 1.1792x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{array}{lll}(-4,5.20), & (4.92..,-3.88), & (4.00..,-2.99..),\\ (1.85,-2), & (1,-1.49..), & (1,-1.49..),\\ (-1.96..,-0.00..), & (-1.36..,-0.31..), & (-2.52..,-2.17..)\end{array}$$

in the configuration $(3+;2+,-;0,-)$.

*Configuration* $(0,2+;3;2)$. Since there are five points with positive index and two points with negative index the unique possible configurations are $(0,2+;2-,+;2+)$, $(0,2+;3+;2-)$ and $(0,2+;2+,-;+,-)$. We show that the configurations $(0,2+;+,2-;2+)$ and $(0,2+;3+;2-)$ are not possible.

For the configuration $(0,2+;2-,+;2+)$ denote by $p_1,p_2$ the points in the 0 level and by $p_3,p_4$ the two points with negative index in the 1st level. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{3,4}$ we reach a contradiction .

For the configuration $(0,2+;3+;2-)$ we denote by $p_1,p_2$ the points in the 0 level, by $p_3,p_4,p_5$ the points in the 1st level and by $p_6,p_7$ the points in the 2nd level. Note that there exists $k_0 \in \{1,2\}$ and $k_1 \in \{3,4,5\}$ so that $C = L_{0,k_1}L_{k_2,k_3}$ with $k_2,k_3 \in \{3,4,5\}$ such that $k_2 \neq k_1$ and $k_3 \neq k_1$ satisfies that $C(p_{k_0})$ has different sign than $C(p_6)$. Applying formula (2) with $R = CL_{0,k_4}$ with $k_4 \in \{1,2\}$ and $k_4 \neq k_0$, we have a contradiction.

In short only the configuration $(0,2+;2+,-;+,-)$ is possible. The cubic system (1) with

$$\begin{aligned}P(x,y) &= 6.80256 - 22.2257x + 10.2134y - 4.99883x^2 + 6.01111xy\\ &\quad + 7.14855y^2 - 0.19868x^3 - 0.89485x^2y + y^3,\\ Q(x,y) &= 1.09836 - 11.7764x - 6.25806y + 4.74987x^2 + 0.76750xy\\ &\quad - 1.76632y^2 + 0.13868x^3 + 0.9434x^2y + xy^2,\end{aligned}$$

has the singular points

$$(-4., -4), \quad (-4, -4), \quad (-4, 5), \quad (-4.1, 5.1), \quad (45.73.., -24.47..),$$
$$(0.92.., -3.93..), \quad (1, -1.85), \quad (1.85, -2), \quad (2, -1.73..)$$

in the configuration $(0, 2+; 2+, -; +, -)$.

## 4. Proof of Theorem 2

The possible configurations are (8), $(7; 1)$, $(6; 2)$, $(5; 3)$, $(4; 4)$, $(4; 3; 1)$, $(3; 5)$, $(3; 4; 1)$ and $(3; 3; 2)$. We will study each of them separately.

*Configuration* (8) Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -, +, -, +, -, +)$ and $(0, 2+, -, +, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 2 - y + 2x^2 - 2y^2 - x^2 y + y^3, \\ Q(x, y) &= 2.64643 - 0.41589x + 0.91936y - 2.46187x^2 + 0.54440xy \\ &\quad - 1.7054y^2 + 0.58412x^3 - 0.22935x^2 y + xy^2, \end{aligned}$$

has the singular points

$$(2, 2), \quad (2, 2), \quad (1, 2), \quad (-0.58.., -0.80..), \quad (-0.70.., -0.70..),$$
$$(-0.86.., -0.5), \quad (-0.97.., -0.22..), \quad (0.58.., -0.80..), \quad (0.86.., -0.5)$$

in the configuration $(0, +, -, +, -, +, -, +)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 2 - 2x^2 - y + x^2 y - 2y^2 + y^3, \\ Q(x, y) &= 2.61768 - 0.68657x + 1.04665y - 2.44229x^2 + 0.30222xy \\ &\quad - 1.56953y^2 + 0.78357x^3 - 0.346x^2 y + xy^2, \end{aligned}$$

has the singular points

$$(2, 2), \quad (1, 2), \quad (1, 2), \quad (-0.58.., -0.80..), \quad (-0.70.., -0.70..),$$
$$(-0.14.., -0.98..), \quad (-0.86.., -0.5), \quad (0.42.., -0.90..), \quad (0.86.., -0.5)$$

in the configuration $(0, 2+, -, +, -, +, -)$.

*Configuration* $(7; 1)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -, +, -, +, -; +)$, $(0, 2+, -, +, -, +; -)$, $(0, 3+, -, +, -; -)$ and $(2+, -, +, -, +, -; 0)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& 128.006 + 265.684x - 73.0216y - 93.2801x^2 + 260.74xy\\ &- 188.824y^2 - 180.419x^3 + 125.935x^2y + y^3,\\ Q(x,y) =& 151.198 + 312.596x - 83.878y - 110.076x^2 + 307.755xy\\ &- 221.865y^2 - 211.959x^3 + 146.972x^2y + xy^2,\end{aligned}$$

has the singular points

$$(2,2),\quad (1,2),\quad (-0.58..,-0.80..),\quad (-0.70..,-0.70..),\quad (-0.86..,-0.5),$$
$$(-0.86..,-0.5),\quad (-0.84..,-0.49..),\quad (0.42..,-0.90..),\quad (0.86..,-0.5)$$

in the configuration $(0,+,-,+,-,+,-;+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& 2 - y - 2x^2 - 2y^2 + x^2y + y^3,\\ Q(x,y) =& -8.08705 - 1.52929x - 3.76842y + 4.96361x^2 - 1.23536xy\\ &+ 3.90598y^2 - 2.48180x^2y + xy^2,\end{aligned}$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (1,2),\quad (-1.11..,2),$$
$$(-0.70..,-0.70..),\quad (-0.86..,-0.5),\quad (0.42..,-0.90..),\quad (0.58..,-0.80..)$$

in the configuration $(0,2+,-,+,-,+;-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& 4.64583 + 5.30182x - 1.50138y - 1.3072x^2 + 0.45597xy\\ &- 6.5235y^2 - 6.98704x^3 + 7.47839x^2y + y^3,\\ Q(x,y) =& 1.21683 + 2.34265x - 1.07425y - 0.99325x^2 + 2.17057xy\\ &- 2.65934y^2 - 2.10737x^3 + 1.08679x^2y + xy^2,\end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.5,2),\quad (1,1.5),\quad (1.24..,0.57..),\quad (-0.86..,-0.5),$$
$$(-0.70..,-0.5),\quad (0.24..,-0.94),\quad 0.42..,-0.90..),\quad (0.58..,-0.80..)$$

in the configuration $(0,3+,-,+,-;-)$ .

The cubic system (1) with

$$\begin{aligned}P(x,y) =& -5.45495 - 15.7188x + 3.26104y + 3.40044x^2 - 15.4262xy\\ &+ 9.05317y^2 + 10.6742x^3 - 6.3916x^2y + y^3,\\ Q(x,y) =& -4.94356 - 16.6295x + 5.36845y + 3.03514x^2 - 15.344xy\\ &+ 9.64123y^2 + 11.61x^3 - 7.84299x^2y + xy^2,\end{aligned}$$

has the singular points

$$(2,2),\quad (1,2),\quad (-0.58..,-0.80..),\quad (-0.71..,-0.71..),\quad (-0.70..,-0.70..),$$
$$(-0.70..,-0.70..),\quad (-0.86..,-0.5),\quad (0.42..,-0.90..),\quad (0.86..,-0.5)$$

in the configuration $(2+,-,+,-,+,-;0)$.

*Configuration* $(6;2)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,3+,-,+;2-)$, $(0,4+,-;2-)$, $(0,-,2+,-,+;+,-)$, $(0,+,-,+,-,+,-;+,-)$, $(0,2+,-,+,-;+,-)$, $(0,2+,-,2+;2-)$, $(2+,-,2+,-;0,-)$, $(3+,-,+,-;0,-)$ and $(+,-,+,-,+,-;0,+)$.

We will show that the configurations $(0,3+,-,+;2-)$ and $(0,4+,-;2-)$ are not possible.

For the configuration $(0,3+,-,+;2-)$ we denote by $p_1,p_2,p_3$ the three consecutive points with positive index being $p_1$ the closest point to $p_0$ and by $p_5$ the remaining point with positive index. Applying the Euler-Jacobi formula (2) to $R=L_{0,1}L_{0,5}L_{2,3}$ we reach a contradiction.

For the configuration $(0,4+,-;2-)$ we denote by $p_1,p_2,p_3,p_4$ the three consecutive points with positive index and by $p_5,p_6,p_7$ the points with negative index being $p_5$ the point in the 0-level. Consider the straight line $S=L_{6,7}$. Consider the straight line $L_{0,2}$. We have three possible cases. First the points $p_6,p_7$ are on the right-hand side of $L_{0,2}$, second the points $p_6,p_7$ are on the left-hand side of $L_{0,2}$, and third the points $p_6,p_7$ are one of the left-hand side, for instance $p_6$, and the other is on the right-hand side of $L_{0,2}$. In the first case applying (2) to $L_{0,1}L_{0,2}L_{3,4}$ we have a contradiction. In the second case applying (2) to $L_{0,1}L_{0,2}L_{4,5}$ we reach a contradiction, and finally in the third case applying (2) to $L_{0,1}L_{0,6}L_{5,7}$ we have a contradiction.

The cubic system (1) with

$$\begin{aligned}P(x,y)&=-0.63728-1.01236x-1.77583y+0.35343x^2-1.55544xy\\&\quad-0.36373y^2-0.29452x^3+0.85409x^2y+y^3,\\Q(x,y)&=0.05170-0.95142x-0.03677y-0.22530x^2-0.45898xy\\&\quad-0.29892y^2+0.15219x^3-0.42001x^2y+xy^2,\end{aligned}$$

has the singular points

$$(1,2),\quad (0.58..,-0.80..),\quad (-0.59..,0.80..),\quad (0.42..,-0.90..),\quad (4,2),$$
$$(2,2),\quad (0.01..,-0.45..),\quad (-0.70..,-0.70..),\quad (-0.70..,-0.70..)$$

in the configuration $(0,-,2+,-,+;+,-)$.

The cubic system (1) with

$$P(x,y) = 2 - y - 2x^2 - 2y^2 + x^2y + y^3,$$
$$Q(x,y) = -1.45224 - 1.29953x - 0.3233y + 1.58257x^2 - 1.35024xy + 0.52471y^2 - 0.79129x^2y + xy^2,$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (1,2),\quad (0.03..,2),\quad (-0.70..,-0.70..),$$
$$(0.42..,-0.90..),\quad (-0.59..,0.80..),\quad (0.86..,-0.5)$$

in the configuration $(0,+,-,+,-,+,-;+,-)$.

The cubic system (1) with

$$P(x,y) = -0.63728 - 1.01236x - 1.77583y + 0.35343x^2 - 1.55544xy - 0.36373y^2 - 0.29452x^3 + 0.8541x^2y + y^3,$$
$$Q(x,y) = 0.0517 - 0.95142x - 0.03677y - 0.2253x^2 - 0.45898xy - 0.29892y^2 + 0.15219x^3 - 0.42001x^2y + xy^2,$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (-0.70..,-0.70..),\quad (-0.70..,-0.70..),$$
$$(0.42..,-0.90..),\quad (0.01..,-0.45..),\quad (0.58..,-0.8..),\quad (-0.59..,0.80..)$$

in the configuration $(0,2+,-,+,-;+,-)$,

The cubic system (1) with

$$P(x,y) = -1.05986 - 1.17457x - 1.90014y + 0.73053x^2 - 1.80467xy - 0.10155y^2 - 0.34171x^3 + 0.83072x^2y + y^3,$$
$$Q(x,y) = 0.39154 - 0.82097x + 0.06321y - 0.52855x^2 - 0.25854xy - 0.50976y^2 + 0.19014x^3 - 0.40121x^2y + xy^2,$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (-0.70..,-0.70..),\quad (0.42..,-0.90..),$$
$$(0.41..,-0.88..),\quad (0.58..,-0.80..),\quad (0.58..,-0.80..),\quad (-0.59..,0.80..)$$

in the configuration $(0,2+,-,2+;2-)$.

The cubic system (1) with

$$P(x,y) = -0.89569 + 1.6814x - 3.69593y + 0.46307x^2 + 1.50863xy - 1.85672y^2 - 1.2361x^3 + 1.89441x^2y + y^3,$$
$$Q(x,y) = -0.15127 + 1.1644x - 1.54493y - 0.13917x^2 + 1.94772xy - 1.4716y^2 - 0.58739x^3 + 0.39711x^2y + xy^2,$$

has the singular points

$$(4,2),\quad (1,2),\quad (-1.18..,-0.93..),\quad (-0.70..,-0.70..),\quad (-0.70..,-0.70..),$$
$$(-0.86..,-0.5),\quad (0.42..,-0.90..),\quad (0.58..,-0.80..),\quad (0.86..,-0.5)$$

in the configuration $(2+,-,2+,-;0,-)$.

The cubic system (1) with

$$P(x,y)=2-y-2x^2-2y^2+x^2y+y^3,$$
$$Q(x,y)=-1.31108-1.47454x-0.43766y+0.99081x^2-1.26273xy$$
$$+0.5466y^2-0.49541x^2y+xy^2,$$

has the singular points

$$(4,2),\quad (2,2),\quad (2,2),\quad (1.14..,2),\quad (1,2),\quad (-0.70..,-0.70..),$$
$$(0.42..,-0.95..),\quad (0.58..,-0.80..),\quad (-0.59..,0.80..)$$

in the configuration $(3+,-,+,-;0,-)$.

The cubic system (1) with

$$P(x,y)=0.22343x-3.70599y+1.91805x^2-0.91189xy$$
$$+2.33258y^2+0.20499x^3-1.65986x^2y+y^3,$$
$$Q(x,y)=0.33464x-4.37889y+2.26518x^2-1.91377xy$$
$$+4.02829y^2+0.29371x^3-2.21375x^2y+xy^2$$

has the singular points

$$(2.68..,1.46..),\quad (1.7,1.3),\quad (1.7,1.3),\quad (1.2,0.64),\quad (0.1,1.1),$$
$$(0.6,0.2),\quad (-0.5,0.1),\quad (-1.1,0.65),\quad (0,0)$$

in the configuration $(+,-,+,-,+,-;0,+)$.

*Configuration* $(5;3)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,-,2+,-;2+,-)$, $(0,+,-,+,-;2+,-)$, $(0,2+,-,+;2-,+)$, $(0,3+,-;2-,+)$, $(2+,-,+,-;0,+,-)$ and $(4+,-;0,2-)$.

The cubic system (1) with

$$P(x,y)=-169.055+488.263x-94.7783y-431.31x^2+174.286xy$$
$$-5.32869y^2+116.227x^3+-63.2897x^2y+y^3,$$
$$Q(x,y)=-24.6823+73.049x-14.8523y-66.5269x^2+29.1387xy$$
$$-1.43955y^2+18.7953x^3-11.8453x^2y+xy^2,$$

has the singular points

$$(2,2),\quad (2,2),\quad (1.93..,3.44..),\quad (0.62..,-1.07..),\quad (1.91..,2.85..),$$
$$(0.8,2),\quad (0.57,-0.9),\quad (0.58..,-0.80..),\quad (0.86..,-0.5)$$

in the configuration $(0,-,2+,-;2+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -0.26624 - 0.127x + 1.42367y + 0.21549x^2 + 4.60997xy \\ &\quad + 3.27449y^2 - 2.03241x^3 - 2.64176x^2y + y^3, \\ Q(x,y) &= 0.26673 + 0.74274x - 1.54738y + 0.42201x^2 - 4.81204xy \\ &\quad - 2.55609y^2 + 1.32549x^3 + 1.2861x^2y + xy^2,\end{aligned}$$

has the singular points

$(1.7, 1.3)$, $(1.2, 0.64)$, $(0.2, -1.2)$, $(-0.15.., -0.41..)$, $(-1.63.., 1.29..)$, $(-1.63.., 1.29..)$, $(0, 0.14)$, $(-0.83.., 0.45..)$, $(-0.9, 0.65)$

in the configuration $(0, +, -, +, -; 2+, -)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =&2 - y - 2x^2 + xy - 2y^2 + x^2y + y^3, \\ Q(x,y) =& - 1.45224 - 1.29953x - 0.3233y + 1.58257x^2 - 1.35024xy \\ &+ 0.52471y^2 - 0.79129x^2y + xy^2,\end{aligned}$$

has the singular points

$(4, 2)$, $(4, 2)$, $(2, 2)$, $(1, 2)$, $(0.18.., 2)$,
$(-0.70.., -0.70..)$, $(0.42.., -0.90..)$, $(-0.59.., 0.80)$, $(0.86.., -0.5)$

in the configuration $(0, 2+, -, +; 2-, +)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& - 0.32744 - 0.74422x - 1.5832y + 0.38015x^2 - 1.37547xy \\ &- 0.62723y^2 - 0.24965x^3 + 0.68371x^2y + y^3, \\ Q(x,y) =&0.29506 - 0.74081x + 0.11453y - 0.2043x^2 - 0.31762xy \\ &- 0.50588y^2 + 0.18743x^3 - 0.55384x^2y + xy^2,\end{aligned}$$

has the singular points

$(4, 2)$, $(2, 2)$, $(1, 2)$, $(-0.70.., -0.70..)$, $(-0.70.., -0.70)$,
$(0.42.., -0.90..)$, $(-0.59.., 0.80..)$, $(0.40.., -0.34..)$, $(0.86.., -0.5)$

in the configuration $(0, 3+, -; 2-, +)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =& - 2.65273 - 1.49182x - 3.75008y + 1.40703x^2 - 4.00078xy \\ &+ 1.89442y^2 - 0.6781x^3 + 1.66983x^2y + y^3, \\ Q(x,y) =& - 1.53136 - 1.32802x - 1.58746y + 0.60227x^2 - 2.37969xy \\ &+ 1.47477y^2 - 0.1491x^3 + 0.22071x^2y + xy^2,\end{aligned}$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (-1.077..,-0.89..),\quad (-0.70..,-0.70..),$$
$$(-0.70..,-0.70..),\quad (-0.86..,-0.5),\quad (-0.59..,0.80..),\quad (0.86..,-0.5)$$

in the configuration $(2+,-,+,-;0,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) =& -0.66741 - 0.85292x - 1.66839y + 0.72781x^2 - 1.57638xy \\ & - 0.42671y^2 - 0.28612x^3 + 0.63752x^2y + y^3, \\ Q(x,y) =& 0.4278 - 0.69837x + 0.14779y - 0.34005x^2 - 0.23917xy \\ & - 0.58417y^2 + 0.20166x^3 - 0.5358x^2y + xy^2, \end{aligned}$$

has the singular points

$$(4,2),\quad (2,2),\quad (1,2),\quad (-0.70..,-0.70..),\quad (0.42..,-0.90..),$$
$$(0.21..,-0.75..),\quad (-0.59..,0.80..),\quad (0.86..,-0.5),\quad (0.86..,-0.5)$$

in the configuration $(4+,-;0,2-)$.

*Configuration* $(4;4)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,-,+,-;3+,-)$, $(0,+,-,+;+,-,+,-)$, $(0,+,-,+;2+,2-)$, $(0,2+,-;+,-,+,-)$, $(0,2+,-;2+,2-)$, $(0,3+;3-,+)$, $(+,-,+,-;0,+,-,+,-)$, $(+,-,+,-;0,2+,-)$, $(3+,-;0,-,+,-)$, $(3+,-;0,2-,+)$ and $(4+;0,3-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.81818 + 0.06798x - 1.1351y + 2.57036x^2 + 0.06798xy \\ & - 1.95328y^2 - 0.71989x^3 - 0.75432x^2y + y^3, \\ Q(x,y) &= 1.40791 - 0.95813x - 0.04956y + 1.71975x^2 + 0.04187xy \\ & - 1.45747y^2 - 1.31898x^3 + 0.09218x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.33..,1.66..),\quad (1.31..,1.67..),\quad (0.-1),\quad (0,-1),$$
$$(0.79..,1.3),\quad (0.15..,-1.01),\quad (0.53..,-1.13..),\quad (-0.53..,1.13..)$$

in the configuration $(0,-,+,-;3+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.22404 + 1.65337x - 1.64431y + 1.96013x^2 - 1.71128xy \\ & - 0.58091y^2 + 1.00465x^3 - 1.94643x^2y + y^3, \\ Q(x,y) &= 0.68907 + 0.70571x - 0.70214y + 2.06091x^2 - 1.41422xy \\ & - 0.32726y^2 - 1.35748x^3 - 0.18828x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.33..,1.66..),\quad (-2.68..,3.93..),\quad (0.79..,1.5),\quad (0.53..,1.13..),$$
$$(-0.53..,-1.13..),\quad (-0.53..,-1.13..),\quad (-0.53..,1.13..),(-0.22..,0.78..)$$

in the configuration $(0,+,-,+;+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 7.55853 - 10.0716x - 1.24914y + 5.0247x^2 + 10.2185xy \\ &\quad - 7.80767y^2 - 6.16078x^3 + 2.69095x^2y + y^3, \\ Q(x,y) &= 2.06465 - 2.88002x - 0.05144y + 2.29759x^2 + 1.7866xy \\ &\quad - 2.11609y^2 - 1.76752x^3 + 0.05748x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.47..,2.02..),\quad (1.5,2),\quad (1.5,2),\quad (1.33..,1.66..),\quad (1.31..,1.67..),$$
$$(0.84..,1.3),\quad (0.53..,1.10..),\quad (0,-1),\quad (-0.53..,1.13..)$$

in the configuration $(0,+,-,+;2+,2-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.05521 + 0.04668x - 2.22167y + 1.23859x^2 - 1.12651xy \\ &\quad - 0.9332y^2 + 0.49019x^3 - 0.36319x^2y + y^3, \\ Q(x,y) &= 0.25156 + 0.58542x - 4.02491y + 1.58971x^2 - 8.43119xy \\ &\quad - 5.19507y^2 + 3.01039x^3 + 2.43304x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.7,1.3),\quad (1.2,0.64),\quad (0.2,-1.2),\quad (0.6,0.2),\quad (-1.63..,1.29..),$$
$$(-1.63..,1.29..),\quad (-0.75..,0.28..),\quad (-1.3,0.65),\quad (-0.35..,0.09..)$$

in the configuration $(0,2+,-;+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -10.3655 - 8.98918x - 27.6779y + 9.36214x^2 + 6.33352xy \\ &\quad + 6.75136y^2 + 0.8147x^3 + 7.51867x^2y + y^3, \\ Q(x,y) &= 8.32828 + 3.6089x + 13.3552y - 4.42819x^2 + 3.32785xy \\ &\quad - 4.90853y^2 - 1.30944x^3 - 7.55403x^2y + xy^2, \end{aligned}$$

has the singular points

$$(-2.72..,-6.63..),\quad (-0.06..,3.14..),\quad (1.47..,0.49..),$$
$$(2.94..,-1.33..),\quad (2.94..,-1.33..),\quad (2.17..,-1.78..),$$
$$(-1.09..,0.42..),\quad (1.09..,-0.52..),\quad (2.35..,-1.46..)$$

in the configuration $(0,2+,-;2+,2-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.59672 + 0.29792x - 1.21191y + 2.50785x^2 + 0.29792xy \\ &\quad - 1.80862y^2 - 0.96331x^3 - 0.81568x^2y + y^3, \\ Q(x,y) &= 0.9278 - 0.5773x - 0.30067y + 2.22057x^2 + 0.42271xy \\ &\quad - 1.22846y^2 - 2.3668x^3 + 0.26153x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.5, 2), \quad (0, -1), \quad (0, -1), \quad (1.33.., 1.66..), \quad (0.79.., 1.5), \\ (0.53.., 1.13..), \quad (0.53.., -1.13..), \quad (-0.77.., 1.26..), \quad (-0.22.., 0.78..) \end{gathered}$$

in the configuration $(0, 3+; 3-, +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.66741 - 0.85292x - 1.66839y + 0.72781x^2 - 1.57638xy \\ &\quad - 0.42671y^2 - 0.28612x^3 + 0.63752x^2y + y^3, \\ Q(x,y) &= 0.4278 - 0.69837x + 0.14779y - 0.34005x^2 - 0.23917xy \\ &\quad - 0.58417y^2 + 0.20167x^3 - 0.5358x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (4, 2), \quad (2, 2), \quad (1, 2), \quad (-0.70.., -0.70..), \quad (0.42.., -0.90..), \\ (0.21.., -0.75..), \quad (-0.59.., 0.80..), \quad (0.86.., -0.5), \quad (0.86.., -0.5) \end{gathered}$$

in the configuration $(+, -, +, -; 0, +, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.23671 - 1.23041x - 0.91629y + 2.60321x^2 + 0.82175xy \\ &\quad - 1.153y^2 + 1.03428x^3 - 2.83549x^2y + y^3, \\ Q(x,y) &= 0.70928 - 2.21598x + 0.07739y + 2.09491x^2 + 0.81736xy \\ &\quad - 0.63189y^2 - 0.00953x^3 - 1.81021x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.5, 2), \quad (1.31.., 1.67..), \quad (0.79.., 1.3), \quad (0.79.., 1.28..), \quad (0, -1), \\ (0.53.., 1.10..), \quad (0.53.., 1.10..), \quad (0.53.., -1.13..), \quad (-0.53.., 1.13..) \end{gathered}$$

in the configuration $(+, -, +, -; 0, 2+, -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.32167 + 1.43788x - 1.57293y + 2.16797x^2 - 1.37002xy \\ &\quad - 0.86391y^2 0.40451x^3 - 1.5578x^2y + y^3, \\ Q(x,y) &= 0.37924 + 1.38958x - 0.92865y + 1.40129x^2 - 2.49725xy \\ &\quad + 0.57086y^2 + 0.54712x^3 - 1.42164x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned}&(1.5, 2), \quad (1.33.., 1.66..), \quad (0.79.., 1.5),\\&(-0.53.., -1.13..), \quad (0.53.., 1.13..), \quad (-0.53.., 1.13..),\\&(-0.22364.., 0.78..), \quad (-0.22393.., 0.78..), \quad (-0.22393.., -0.78..)\end{aligned}$$

in the configuration $(3+, -; 0, -, +, -)$.

The cubic system (1) with

$$\begin{aligned}P(x, y) = & -0.03342 + 0.13443x + 0.1563y + 0.34256x^2 - 1.88158xy\\& + 0.97155y^2 + 0.62347x^3 - 1.09981x^2y + y^3,\\Q(x, y) = & -0.02792 + 0.01194x + 0.24347y + 0.1814x^2 - 0.64693xy\\& - 0.12559y^2 + 0.3054x^3 - 0.94879x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.7, 1.3), \quad (0.3, 0.2), \quad (0.3, 0.2), \quad (1.2, 0.64), \quad (0.2, -1.2),\\&(0.62.., 0.50..), \quad (0.40.., 0.31..), \quad (-0.5, 0.1), \quad (-0.1, 0.1)\end{aligned}$$

in the configuration $(3+, -; 0, 2-, +)$.

The cubic system (1) with

$$\begin{aligned}P(x, y) = & \, 3.1825 + 0.66392x + 1.92747y + 1.30631x^2 - 4.16906xy\\& - 2.89871y^2 - 1.25514x^3 + 2.09021x^2y + y^3,\\Q(x, y) = & -0.0862 + 0.15006x - 0.09326y + 0.0254x^2 + 0.76885xy\\& - 0.60459y^2 + 0.82558x^3 - 1.92383x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(2, 2), \quad (1.55.., 1.90..), \quad (1.5, 1.8), \quad (1.33.., 1.81..), \quad (1.33.., 1.81..),\\&(1, 2), \quad (-0.86.., -0.5), \quad (-0.70.., -0.5), \quad (0.42.., -0.90..)\end{aligned}$$

in the configuration $(4+; 0, 3-)$.

*Configuration* $(4; 3; 1)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -, +; 2-, +; +)$, $(0, +, -, +; 2+, -; -)$, $(0, 2+, -; 2+, -; -)$, $(0, 2+, -; 2-, +; +)$, $(0, 3+; 3-; +)$, $(0, 3+; 2-, +; -)$, $(0, -, +, -; 2+, -; +)$, $(0, -, +, -; 3+; -)$, $(+, -, +, -; 0, +, -; +)$, $(+, -, +, -; 0, 2+; -)$, $(3+, -; 0, +, -; -)$, $(3+, -; 0, 2-; +)$, $(+, -, +, -; 2+, -; 0)$, $(3+, -; 2-, +; 0)$, $(4+; 3-; 0)$, and $(4+; 0, 2-; -)$.

No we show that the configurations $(0, 2+, -; 2+, -; -)$ and $(0, 3+; 2-, +; -)$ are not possible.

For the configuration $(0, 2+, -; 2+, -; -)$, we denote by $p_0, p_1, p_2, p_3$ the points in the 0 level (being $p_3$ the point with negative index), by $p_4, p_5, p_6$ the points in the 1st level (being $p_6$ the negative index) and by $p_7$ the point in the

2nd level. Consider the straight line $L_{6,7}$. The points $p_1, p_2, p_4$ and $p_5$ have positive index. The line $L_{6,7}$ can separate the set of points $\{p_1, p_2, p_4, p_5\}$ in two ways. First one point $p_\ell$ is on one side of $L_{6,7}$ and the other three points are on the other side of $L_{6,7}$. Second two points are on one side of $L_{6,7}$ and the other two points are on the other side of $L_{6,7}$. In the first case applying (2) to $L_{0,3}L_{0,\ell}L_{6,7}$ we have a contradiction. In the second case denote by $p_k$ the closest point of the set $\{p_1, p_2, p_4, p_5\}$ to $L_{6,7}$. Applying (2) to $L_{0,3}L_{0,k}L_{6,7}$ we reach a contradiction.

For the configuration $(0, 3+; 2-, +; -)$ denote by $p_0, p_1, p_2, p_3$ the points in the 0 level, by $p_4, p_5, p_6$ the points in the 1st level (being $p_4$ with positive index) and by $p_7$ the point in the 2nd level. Consider the straight line $L_{0,4}$. If the three singular points with negative index are in the same side of $L_{0,4}$ then applying (2) to $L_{0,1}L_{0,4}L_{2,3}$ we have a contradiction. If the three singular points with negative index are not all in the same side of $L_{0,4}$ then applying (2) to $L_{0,1}L_{0,3}L_7$ being $L_7$ the straight line passing to $p_7$ being parallel to $L_{5,6}$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 2.66278 - 2.03402x - 3.32725y + 0.13241x^2 + 5.23787xy \\ &\quad - 2.25453y^2 - 1.23728x^3 - 4.87209x^2y + y^3, \\ Q(x,y) &= 1.07346 - 1.81894x - 3.55618y + 0.48056x^2 + 4.04944xy \\ &\quad + 1.16146y^2 + 0.42202x^3 + 0.58928x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-2.72.., -6.63..), \quad (2.94.., -1.33..), \quad (2.17.., -1.78..), \\ &(-0.06.., 3.14..), \quad (1.09.., -0.52..), \quad (-1.09.., 0.42..), \\ &(-1.09.., 0.42..), \quad (1.026.., -0.25..), \quad (0.45.., 0.56..) \end{aligned}$$

in the configuration $(0, +, -, +; 2-, +; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 9.65409 - 1.51164x - 5.00183y + 6.8444x^2 + 1.47388xy \\ &\quad - 7.15989y^2 - 11.7997x^3 + 9.61314x^2y + y^3 \\ Q(x,y) &= 1.21914 - 0.151691x + 0.432748y + 2.77906x^2 - 1.75697xy \\ &\quad - 1.74289y^2 - 2.74733x^3 + 1.64549x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} &(-0.79.., 2), \quad (0.53.., -1.13..), \quad (1.45.., 2.14..), \quad (1.5, 2), \quad (1.5, 2), \\ &(1.31.., 1.67..), \quad (0.81.., 1.3), \quad (0.53.., 1.10..), \quad (1.33.., 1.66..) \end{aligned}$$

in the configuration $(0, +, -, +; 2+, -; -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.58089 - 3.87837x - 8.08004y + 0.71079x^2 + 5.41967xy \\ &\quad - 0.64709y^2 - 0.58716x^3 - 2.83003x^2y + y^3 \\ Q(x,y) &= -1.29881 - 0.50969x - 2.43789y + 2.54356x^2 + 4.06429xy \\ &\quad + 1.05356y^2 - 0.17975x^3 + 0.78554x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lll} (-2.72.., -6.63..), & (2.31.., -1.98..), & (2.94.., -1.33..), \\ (2.94.., -1.33..), & (2.17.., -1.78..), & (-0.06.., 3.14..), \\ (2.03.., -1.46..), & (1.09.., -0.52..), & (-1.09.., 0.42..) \end{array}$$

in the configuration $(0, 2+, -; 2-, +; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.83083 + 0.06503x - 1.12582y + 2.53496x^2 + 0.06503xy \\ &\quad - 1.95665y^2 - 0.6756x^3 - 0.77083x^2y + y^3, \\ Q(x,y) &= 1.29796 - 0.93243x - 0.13022y + 2.02728x^2 + 0.06757xy \\ &\quad - 1.42818y^2 - 1.70372x^3 + 0.23551x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{l} (1.5, 2), \quad (0, -1), \quad (0, -1), \quad (1.38.., 1.89..), \quad (1.31.., 1.67..), \\ (0.79.., 1.3), \quad (0.53.., 1.10..), (0.53.., -1.13..), \quad (-0.53.., 1.13..) \end{array}$$

in the configuration $(0, 3+; 3-; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -19.764x - 20y - y^2 + 4.38561x^2 + 3.55687xy - 0.238337x^3 \\ &\quad - 0.136263x^2y + 0.118392xy^2, \\ Q(x,y) &= -395.28x - 400y + 109.356x^2 + 113.222xy - 6.63869x^3 \\ &\quad - 5.42657x^2y + 2.58145xy^2 + y^3, \end{aligned}$$

has the singular points

$$\begin{array}{l} (0,0), \quad (0,0), \quad (0,-20), \quad (10,-9), \quad (9.1,-6), \\ (5,-7.87..), \quad (9,-7.87..), \quad (9,-7), \quad (8.9,-7.86..) \end{array}$$

in the configuration $(0, -, +, -; 2+, -; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.4391 - 0.77711x - 0.99307y + 2.5933x^2 + 0.5588xy \\ &\quad - 1.43217y^2 + 0.4201x^3 - 2.10839x^2y + y^3, \\ Q(x,y) &= 0.96301 - 1.94997x + 0.11714y + 1.74667x^2 + 0.61791xy \\ &\quad - 0.84587y^2 + 0.01895x^3 - 1.49701x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.80..,2.37..),\quad (1.5,2),\quad (1.33..,1.66..),\quad (1.31..,1.67..),\quad (0,-1),$$
$$(0.79..,1.3),\quad (-0.53..,1.13..),\quad (-0.53..,1.13..),\quad (0.53..,-1.13..)$$

in the configuration $(0,-,+,-;3+;-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 2.18148 + 0.82957x - 1.24948y + 2.49468x^2 - 0.38158xy \\ &\quad - 2.43096y^2 - 1.68474x^3 + 0.4477x^2y + y^3, \\ Q(x,y) &= 4.93445 + 6.99657x - 1.41267y + 1.60951x^2 - 4.56412xy \\ &\quad - 6.34712y^2 - 12.1694x^3 + 12.8727x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.31..,1.67..),\quad (1.31..,1.67..),\quad (0.79..,1.3),\quad (0,-1),$$
$$(0.53..,1.10..),\quad (-0.68..,1.53..),\quad (-0.53..,1.13..),\quad (0.53..,-1.13..)$$

in the configuration $(+,-,+,-;0,+,-;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) =& 1.2053 - 1.29832x - 0.90547y + 2.60744x^2 + 0.86151xy \\ & - 1.11077y^2 + 1.12318x^3 - 2.94351x^2y + y^3, \\ Q(x,y) =& 0.84068 - 2.22268x + 0.16297y + 1.75407x^2 + 0.7763xy \\ & - 0.67771y^2 + 0.38682x^3 - 1.93396x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.5,2),\quad (1.31..,1.70..),\quad (1.33..,1.66..),\quad (1.31..,1.67..),\quad (0,-1),$$
$$(0.79..,1.3),\quad (0.79..,1.3),\quad (0.53..,-1.13..),\quad (-0.53..,1.13..)$$

in the configuration $(+,-,+,-;0,2+;-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.21847 - 3.23165x - 1.86475y + 8.41861x^2 + 4.65148xy \\ &\quad + 2.12508y^2 - 5.14896x^3 - 3.22687x^2y + y^3, \\ Q(x,y) &= -1.88545 - 9.51989x - 4.2883y + 29.3482x^2 + 5.9392xy \\ &\quad + 5.94989y^2 - 18.5052x^3 - 1.99464x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.74..,1.89..),\quad (-0.2,0.5),\quad (1.74..,-1.89..),$$
$$(0.85..,-0.10..),\quad (0.85..,-0.10..),\quad (-0.19..,0.53..),$$
$$(0.85..,0.20..),\quad (0.69..,-0.21..),\quad (0.55..,-0.33..)$$

in the configuration $(3+,-;0,+,-;-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -3.98259 + 3.15619x7.96671y - 2.13359x^2 + 8.36509xy- \\ &\quad 0.53375y^2 - 0.59508x^3 - -1.55334x^2y + y^3, \\ Q(x,y) &= 1.96752 - 0.56429x + 2.74268y + 1.01136x^2 - 2.82378xy \\ &\quad - 1.28245y^2 + 0.63638x^3 - 0.87949x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(2,2), \quad (1.5,1.8), \quad (1.33..,1.81..), \quad (1,2), \quad (1.23..,1.84..), \\ (-0.86..,-0.5), \quad (-0.70..,-0.5), \quad (-0.70..,-0.5), \quad (0.42..,-0.90..)\end{gathered}$$

in the configuration $(3+,-;0,2-;+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 1.20565 - 1.29812x - 0.90534y + 2.60678x^2 + 0.8613xy \\ &\quad - 1.11099y^2 + 1.12365x^3 - 2.9434x^2y + y^3, \\ Q(x,y) &= 0.65655 - 2.33095x + 0.09598y + 2.10096x^2 + 0.88451xy \\ &\quad - 0.56057y^2 + 0.14222x^3 - 1.99345x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.5,2), \quad (1.31..,1.67..), \quad (0.79..,1.3), \quad (0.79..,1.3), \quad (0,-1), \\ (0.52..,1.11..), \quad (0.53..,1.10..), \quad (0.53..,-1.13..), \quad (-0.53..,1.13..)\end{gathered}$$

in the configuration $(+,-,+,-;2+,-;0)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -0.70509 - 0.69397x + 4.27287y - 0.13448x^2 + 4.13789xy \\ &\quad - 3.43606y^2 - 1.67007x^3 - 0.16262x^2y + y^3, \\ Q(x,y) &= -2.11104 - 2.03548x + 12.7928y + 1 - 0.81903x^2 + 2.2063xy \\ &\quad - 9.51663y^2 - 5.57219x^3 + 1.30903x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7,1.3), \quad (-0.2,0.2), \quad (-0.2,0.2), \quad (-1.29..,-1.05..), \quad (1.2,0.64), \\ (0.3,0.2), \quad (-0.5,0.1), \quad (-1.63..,1.29..), \quad (-0.8,0.65),\end{gathered}$$

in the configuration $(3+,-;2-,+;0)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 1.1096 + 0.1224x - 1.02991y + 1.24418x^2 - 1.13951y^2 \\ &\quad - 0.42839x^3 - 0.89532x^2y + y^3, \\ Q(x,y) &= 0.51363 - 0.8668x - 0.03189y + 0.65715x^2 - 0.54553y^2 \\ &\quad - 1.46621x^3 + 0.11163x^2y + xy^2,\end{aligned}$$

has the singular points

$$(1.5, 2), \quad (0.62.., 1.20..), \quad (0.53, 1.133),$$
$$(0.534.., 1.1338..), \quad (0.534.., 1.1338..), \quad (-0.534.., -1.1338..),$$
$$(0, -1), \quad (0.534.., -1.1338..), \quad (-0.534.., 1.1338..)$$

in the configuration $(4+; 3-; 0)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 3.25912 + 0.55182x + 1.8978y + 1.18903x^2 - 4.42068xy \\ &\quad - 2.91515y^2 - 1.59916x^3 + 2.65276x^2y + y^3, \\ Q(x, y) &= -0.10693 + 0.18039x - 0.08524y + 0.05712x^2 + 0.83693xy \\ &\quad - 0.60014y^2 + 0.91865x^3 - 2.07602x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1, 2), \quad (0.42.., -0.90..), \quad (1.331.., 1.8113..),$$
$$(1.33, 1.81), \quad (1.336.., 1.812..), \quad (1.336.., 1.812..),$$
$$(2, 2), \quad (-0.86.., -0.5), \quad (-0.70.., -0.5)$$

in the configuration $(4+; 0, 2-; -)$.

*Configuration* $(3; 5)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -; 2+, -, +, -)$, $(0, +, -; 3+, 2-)$, $(0, 2+; 2-, +, -, +)$, $(0, 2+; 3-, 2+)$, $(2+, -; 0, -, 2+, -)$, $(2+, -; 0, +, -, +, -)$, $(2+, -; 0, 2+, 2-)$, $(3+; 0, 2-, +, -)$ and $(3+; 0, 3-, +)$.

We will show that the configurations $(0, +, -; 3+, 2-)$, $(0, 2+; 3-, 2+)$, $(2+, -; 0, 2+, 2-)$ and $(3+; 0, 3-, +)$ are not possible.

For the configuration $(0, +, -; 3+, 2-)$ denote by $p_0, p_1, p_2$ the points in the 0 level and by $p_3, p_4, p_5, p_6, p_7$ the points in the 1st level being $p_6, p_7$ the points with negative index. Applying (2) to $R = L_{0,1}L_{0,2}L_{6,7}$ we reach a contradiction.

For the configuration $(0, 2+; 3-, 2+)$, denote by $p_0, p_1, p_2$ the points in the 0 level and by $p_3, p_4, p_5, p_6, p_7$ the points in the 1st level being $p_6, p_7$ the points with positive index. Applying (2) to $R = L_{0,1}L_{0,2}L_{6,7}$ we have a contradiction.

For the configuration $(2+, -; 0, 2+, 2-)$ denote by $p_1, p_2, p_3$ the points in the 0 level in counterclockwise sense being $p_3$ the one with negative index, and by $p_0, p_4, p_5, p_6, p_7$ the points in the 1st level in counterclockwise sense (being $p_6, p_7$ with negative index). Note that there exists $k_1, k_2 \in \{1, 2, 3\}$ so the conic $C = L_{0,6}L_{0,7}$ satisfies that $C(p_{k_1})C(p_{k_2}) > 0$. Note that at least $p_{k_1}$ or $p_{k_2}$ has positive index. Without loss of generality we assume it is $p_{k_1}$. Applying formula (2) to $R = L_{0,6}L_{0,7}L_{k_2k_3}$ with $k_3 \in \{1,2,3\}$ so that $k_3 \neq k_1$ and $k_3 \neq k_2$ we reach a contradiction.

For the configuration $(3+; 0, 3-, +)$ denote by $p_1, p_2, p_3$ the points in the 0 level and by $p_0, p_4, p_5, p_6, p_7$ the points in the 1st level in counterclockwise sense. Consider $C = L_{0,4}L_{0,7}$. Note that there exists $\ell \in \{1,2,3\}$ so that the signs of $C(p_\ell)$ and $C(p_5)$ are different. Applying (2) to $R = CL_{k_0,k_1}$ being $k_0, k_1 \in \{1,2,3\}$ with $k_0 \neq \ell$ and $k_1 \neq \ell$, we get a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.07010 - 0.01445x - 0.73135y + 0.32004x^2 + 0.70005xy \\ &\quad + 0.60163y^2 + 0.15478x^3 - 1.47483x^2y + y^3, \\ Q(x,y) &= 0.06439 - 0.12082x - 0.54808y + 0.16133x^2 + 1.65512xy \\ &\quad - 0.45545y^2 - 0.11253x^3 - 1.2832x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.2, -1.2), \quad (-0.1, 0.1), \\ (0.400.., 0.31..), \quad (0.3, 0.2), \quad (-0.5, 0.1), \quad (0.09.., 0.12..) \end{gathered}$$

in the configuration $(0, +, -; 2+, -, +, -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -6.8337 - 23.5185x - 3.02691y + 25.4219x^2 + 25.1177xy \\ &\quad + 4.80679y^2 - 15.2956x^3 - 13.3315x^2y + y^3, \\ Q(x,y) &= 2.77818 + 10.0337x + 0.59214y - 6.70244x^2 - 11.4714xy \\ &\quad - 2.18605y^2 + 5.10052x^3 + 3.55814x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.5, 2), \quad (1.33.., 1.66..), \quad (-0.79.., 2), \quad (-0.79.., 2), \quad (0, -1), \\ (-0.78, 1.99), \quad (0.53.., 1.10..), \quad (-0.67.., 1.61..), \quad (-0.53.., 1.13..) \end{gathered}$$

in the configuration $(0, 2+; 2-, +, -, +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.16718x - 0.84559y + 0.67325x^2 - 0.13985xy \\ &\quad + 0.40119y^2 - 0.07882x^3 - 0.88085x^2y + y^3, \\ Q(x,y) &= 0.2161x + 0.17696y + 0.25951x^2 - 0.94753xy \\ &\quad - 0.26355y^2 + 0.01953x^3 - 0.47296x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7, 1.3), \quad (1.2, 0.64), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0, 0), \\ (0.41.., 0.71..), \quad (0.40.., 0.71..), \quad (-0.5, 0.1), \quad (-1, 0.55) \end{gathered}$$

in the configuration $(2+, -; 0, -, 2+, -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -1.95227 + 1.56068x - 9.84555y + 0.02429x^2 + 6.40808xy \\ &\quad - 0.55368y^2 - 0.11501x^3 - 1.07423x^2y + y^3, \\ Q(x,y) &= 1.82196 - 1.19243x - 4.12989y - 0.03222x^2 + 4.35035xy \\ &\quad - 2.42007y^2 + 0.08933x^3 - 1.0964x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll} (4.82..,2.80..), & (3,1), & (-4.10..,0), & (2.60..,0.66..), & (2.51..,0.69..), \\ (2.53..,0.65..), & (2.55,0.63..), & (2.55..,0.63..), & (2.4,-0.8) & \end{array}$$

in the configuration $(2+,-;0,+,-,+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.73639 - 0.38881x - 0.96698y + 2.16264x^2 + 0.17322xy \\ &\quad - 1.70337y^2 + 0.53463x^3 - 1.68794x^2y + y^3, \\ Q(x,y) &= 0.30263 + 3.3525x - 0.00289y + 0.01622x^2 - 4.26596xy \\ &\quad - 0.30552y^2 + 0.52779x^3 + 0.1948x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll} (1.5,2), & (1.33..,1.66..), & (-0.79..,2), & (-0.78,1.99), & (0,-1), \\ (0.53..,1.10..), & (0.29..,1.04..), & (-0.53..,1.13..), & (-0.53..,1.13..) & \end{array}$$

in the configuration $(3+;0,2-,+,-)$.

*Configuration* $(3;4;1)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0,+,-;+,-,+,-;+)$, $(0,+,-;2+,2-;+)$, $(0,+,-;3+,-;-)$, $(0,2+;3-,+;+)$, $(0,2+;2+,2-;-)$, $(0,2+;+,-,+,-;-)$, $(2+,-;0,2-,+;+)$, $(2+,-;0,2+,-;-)$, $(2+,-;0,-,+,-;+)$, $(2+,-;0,+,-,+;-)$, $(3+;0,2-,+;-)$, $(3+;0,3-;+)$, $(3+;0,-,+,-;-)$, $(2+,-;2+,2-;0)$, $(2+,-;+,-,+,-;0)$, and $(3+;3-,+;0)$.

We will show that the configurations $(0,+,-;2+,2-;+)$, $(0,2+;2+,2-;-)$, $(0,2+;+,-,+,-;-)$, $(2+,-;0,-,+,-;+)$, $(2+,-;0,+,-,+;-)$, $(3+;0,-,+,-;-)$ and $(2+,-;+,-,+,-;0)$ are not possible.

For the configuration $(0,+,-;2+,2-;+)$, denote by $p_0,p_1,p_2$ the points in the 0 level and by $p_3,p_4,p_5,p_6$ the points in the first level being $p_3$ and $p_4$ the two consecutive points with positive index and $p_5,p_6$ the two consecutive points with negative index. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{5,6}$ we reach a contradiction.

For the configuration $(0,2+;2+,2-;-)$, denote by $p_0,p_1,p_2$ the points in the 0 level and by $p_3,p_4,p_5,p_6$ the points in the first level being $p_3$ and $p_4$ the two consecutive points with positive index and $p_5,p_6$ the two consecutive points with negative index. Applying formula (2) to $R = L_{0,1}L_{0,2}L_{3,4}$ we reach a contradiction.

For the configuration $(0, 2+; +, -, +, -; -)$ denote by $p_0, p_1, p_2$ the points in the 0 level, by $p_3, p_4, p_5, p_6$ the points in the 1st level in counterclockwise sense and by $p_7$ the point in the 2nd level. Denote by $p_{k_1}$ the closest point to $p_0$ in the set $\{p_3, p_5\}$. Denote by $p_\ell$ the closest point to $L_{0,k_1}$ in the set $\{p_4, p_6, p_7\}$. Denote by $p_{k_2}$ the point in $\{p_3, p_5\}$ different from $p_{k_1}$. There exists $j_1, j_2 \in \{1, 2\}$ so that $L_{j_1,k_2}$ leaves $p_{j_2}$ on one side of $L_{j_1,k_2}$ and the remaining points on the other side of $L_{j_1,k_2}$. Applying formula (2) to $L_{0,k_1}L_{0,\ell}L_{j_1,k_2}$ we reach a contradiction.

For the configuration $(2+, -; 0, -, +, -; +)$, we denote by $p_1, p_2, p_3$ the points in the 0 level in counterclockwise sense and by $p_0, p_4, p_5, p_6$ the points in the first level also in counterclockwise sense. If $L_{0,4}L_{0,6}$ leave a point with positive index $p_\ell$ of the 0-level (that without loss of generality we can assume that it is $p_1$) then applying formula (2) to $L_{0,4}L_{0,6}L_{2,3}$ we reach a contradiction. Otherwise, $p_\ell$ has negative index (that we assume that it is $p_1$). Note that either $L_{1,4}$ leaves the points $p_5, p_7$ on the same side of $L_{1,4}$, or $L_{1,6}$ leaves the points $p_5, p_7$ on the same side of $L_{1,4}$. Without loss of generality we assume that it is $L_{1,4}$. Then there exists $k_1 \in \{2, 3\}$ so that $L_{0,6}L_{0,k_1}L_{1,4}$ leaves the other point in the set $\{p_1, p_2\}$ on the same side of $p_5, p_7$. Applying formula (2) to $L_{0,6}L_{0,k_1}L_{1,4}$ we reach a contradiction.

For the configuration $(2+, -; 0, +, -, +; -)$, denote by $p_1, p_2, p_3$ the points in the 0 level (being $p_1, p_2$ with positive index and $p_3$ with negative index) and denote by $p_0, p_4, p_5, p_6$ the points in the 1st level in counterclockwise sense and by $p_7$ the point in the 2nd level. There exists $k_0 \in \{1, 2\}$ so that the conic $C = L_{0,5}L_{0,7}$ evaluated on $p_{k_0}$ and on $p_4$ have the same sing, or the conic $C$ evaluated on $p_{k_0}$ and on $p_6$ have the same sign (note that $C$ evaluated on $p_4$ and on $p_6$ have different sign). Applying (2) to $R = CL_{k_1,k_2}$ with $k_1, k_2 \in \{1, 2, 3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

For the configuration $(3+; 0, -, +, -; -)$, denote by $p_1, p_2, p_3$ the points in the 0 level, by $p_0, p_4, p_5, p_6$ the points in the 1st level in counterclockwise sense and by $p_7$ the point in the 2nd level. There exits $k_1, k_2 \in \{1, 2, 3\}$ so that $L_{k_1,5}$ leaves $k_2$ on one side of $L_{k_1,5}$ and the rest of the points in the other side of $L_{k_1,5}$. Denote by $k_3$ the remaining point in the 0 level (different from $k_0$ and $k_1$). Denote by $p_\ell$ the closest point to $L_{0,k_3}$ in the set $\{4, 6, 7\}$. Applying formula (2) to $L_{0,k_3}L_{0,\ell}L_{k_1,5}$ we reach a contradiction.

For the configuration and $(2+, -; +, -, +, -; 0)$ denote by $p_1, p_2, p_3$ the points in the 0 level (being $p_1, p_2$ with positive index and $p_3$ with negative index) by $p_4, p_5, p_6, p_7$ the points in the 1st level in counterclockwise sense (being $p_4$ and $p_6$ with positive index) and by $p_0$ the point in the 2nd level. There exists $k_0 \in \{1, 2\}$ so that the conic $C = L_{0,5}L_{0,7}$ evaluated on $p_{k_0}$ and on $p_4$ have the same sign, or the conic $C$ evaluated on $p_{k_0}$ and on $p_6$

have the same sign (note that the conic $C$ evaluated on $p_4$ and on $p_6$ have different signs). Applying (2) to $R = CL_{k_1,k_2}$ with $k_1, k_2 \in \{1,2,3\}$ with $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.0629x - 0.62613y + 0.71196x^2 + 0.66221xy \\ &\quad + 0.73704y^2 - 0.2665x^3 - 1.2599x^2y + y^3, \\ Q(x,y) &= 0.50188x - 1.27453y + 0.92819x^2 + 1.64485xy \\ &\quad - 1.06428y^2 - 1.57961x^3 + 0.31205x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7, 1.3), \quad (1.41.., 0.95..), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.2, -1.2), \\ (0.40.., 0.31..), \quad (-1.63.., 1.29..), \quad (-1.1, 0.65), \quad (0,0) \end{gathered}$$

in the configuration $(0, +, -; +, -, +, -; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 3851.43 - 7426.34x - 7061.69y + 2914.87x^2 + 12386.9xy \\ &\quad - 813.074y^2 + 589.519x^3 - 4778.06x^2y + y^3, \\ Q(x,y) &= -1388.63 + 2677.53x + 2546.87y - 1050.86x^2 - 4466.74xy \\ &\quad + 291.434y^2 - 212.616x^3 + 1722.41x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.74.., 1.89..), \quad (1.74.., 1.89..), \quad (1.74.., -1.89..), \quad (1.10.., 0), \quad (0.1, 0.5), \\ (0.85.., 0.10..), \quad (0.85.., -0.08..), \quad (-0.19.., 0.53..), \quad (-0.39.., 0.54..) \end{gathered}$$

in the configuration $(0, +, -; 3+, -; -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.14321 - 0.04623x - 0.28656y + 0.69539x^2 + 0.70614xy \\ &\quad + 1.11684y^2 - 0.2855x^3 - 1.50591x^2y + y^3, \\ Q(x,y) &= 0.44569 + 0.84699x - 0.88299y - 0.11115x^2 - 3.36763xy \\ &\quad - 1.88701y^2 + 0.63082x^3 + 1.12932x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{gathered} (1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.40.., 0.31..), \quad (-0.5, 0.1), \\ (-1.63.., 1.29..), \quad (-1.1, 0.65), \quad (-1.1, 0.65), \quad (-1.08.., 0.65..) \end{gathered}$$

in the configuration $(0, 2+; 3-, +; +)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.06804x - 0.06224y + 0.27383x^2 - 1.1449xy + 0.89605y^2\\ &\quad + 0.53725x^3 - 1.23682x^2y + y^3,\\ Q(x,y) &= 0.17249x - 0.23800y - 0.2008x^2 + 0.52425xy - 0.31002y^2\\ &\quad - 0.91859x^3 + 0.42637x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7,1.3),\quad (0.2,-1.2),\quad (-0.30..,-0.39..),\quad (0.3,0.2),\quad (0.3,0.2),\\ (0.40..,0.31..),\quad (-1.63..,1.29..),\quad (-1.1,0.65),\quad (0,0)\end{gathered}$$

in the configuration $(2+,-;0,2-,+;+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.22436y - 0.02526x^2 - 1.54369xy + 1.08593y^2\\ &\quad + 0.75301x^3 - 1.20812x^2y + y^3,\\ Q(x,y) &= 0.48854y - 0.959x^2 - 0.4867xy + 0.17134y^2\\ &\quad - 0.37163x^3 + 0.49914x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7,1.3),\quad (1.18..,0.99..),\quad (0.2,-1.2),\quad (0,0),\quad (0,0),\\ (0.40..,0.31..),\quad (0.3,0.2),\quad (-1.63..,1.29..),\quad (-1.1,0.65)\end{gathered}$$

in the configuration $(2+,-;0,2+,-;-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= -0.39665y + 0.53911x^2 - 0.11032xy + 0.79396y^2 + 0.07436x^3\\ &\quad - 1.25172x^2y + y^3,\\ Q(x,y) &= -3.10565y + 2.30737x^2 + 7.80916xy - 1.5185y^2 - 4.29942x^3\\ &\quad + 0.24677x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{gathered}(1.7,1.3),\quad (1.2,0.64),\quad (0.2,-1.2),\quad (0,0),\quad (0,0)\\ (0.40..,0.31..),\quad (0.32..,0.13..),\quad (-1.63..,1.29..),\quad (-1.1,0.65)\end{gathered}$$

in the configuration $(3+;0,2-,+;-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 0.09811x - 0.282y + 0.40366x^2 - 0.85096xy + 0.7576y^2\\ &\quad + 0.36605x^3 - 1.16101x^2y + y^3,\\ Q(x,y) &= 0.22473x + 0.2107y + 0.25304x^2 - 1.27757xy - 0.29072y^2\\ &\quad + 0.13727x^3 - 0.43116x^2y + xy^2,\end{aligned}$$

has the singular points

$$(1.7, 1.3), \quad (1.2, 0.64), \quad (0, 0), \quad (-0.5, 0.1), \quad (-0.5, 0.1),$$
$$(0.2, -1.2), \quad (0.40.., 0.43..), \quad (-1.1, 0.65), \quad (1.07.., 0.58..)$$

in the configuration $(3+; 0, 3-; +)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= 0.1436x - 0.01373y + 0.21951x^2 - 1.67597xy \\ &\quad + 0.83957y^2 + 0.74789x^3 - 1.18366x^2y + y^3, \\ Q(x, y) &= -0.13225x + 0.19145y + 0.47895x^2 - 0.57423xy \\ &\quad - 0.15865y^2 + 0.06518x^3 - 0.81562x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.3, 0.2), \quad (0.3, 0.2),$$
$$(0.56.., 0.59..), \quad (0.40.., 0.43..), \quad (-1.1, 0.65), \quad (0, 0)$$

in the configuration $(2+, -; 2+, 2-; 0)$.

The cubic system (1) with

$$\begin{aligned} P(x, y) &= -7.77285 - 26.0531x - 3.25264y + 27.9708x^2 + 27.8513xy \\ &\quad + 5.52021y^2 - 17.0304x^3 - 14.6074x^2y + y^3, \\ Q(x, y) &= 3.1688 + 11.088x + 0.68603y - 7.76257x^2 - 12.6084xy \\ &\quad - 2.48277y^2 + 5.82205x^3 + 4.08885x^2y + xy^2, \end{aligned}$$

has the singular points

$$(1.5, 2), \quad (1.33.., 1.66..), \quad (-0.94.., 2.55..), \quad (-0.79.., 2), \quad (0, -1),$$
$$(-0.78, 1.99), \quad (-0.78, 1.99), \quad (0.53.., 1.10..), \quad (-0.53.., 1.13..)$$

in the configuration $(3+; 3-, +; 0)$.

*Configuration* $(3; 3; 2)$ Since there are four points with positive index and three points with negative index, taking into account Lemma 6 we can have only the configurations $(0, +, -; 2+, -; +, -)$, $(0, +, -; 2-, +; 2+)$ $(0, +, -; 3+; 2-)$, $(0, 2+; 2-, +; +, -)$, $(0, 2+; 3-; 2+)$, $(0, 2+; 2+, -; 2-)$, $(2+, -; 0, +, -; +, -)$, $(2+, -; 0, 2-; 2+)$, $(2+, -; 0, 2+; 2-)$, $(3+; 0, +, -; 2-)$, $(3+; 0, 2-; +, -)$, $(2+, -; 2-, +; 0, +)$, $(3+; 3-; 0, +)$, $(2+, -; 2+, -; 0, -)$, and $(3+; 2-, +; 0, -)$.

We will see that the configurations $(0, +, -; 2-, +; 2+)$, $(0, +, -; 3+; 2-)$, $(0, 2+; 2+, -; 2-)$, $(2+, -; 0, 2+; 2-)$, $(3+; 0, +, -; 2-)$ and $(3+; 2-, +; 0, -)$ are not possible.

For the configuration $(0, +, -; 2-, +; 2+)$ denote by $p_1, p_2$ the points in the 0-level and by $p_3, p_4$ the points in the 1st level with negative index. Applying (2) with $L_{0,1}L_{0,2}L_{3,4}$ we have a contradiction.

For the configuration $(0, 2*; 3*; 2-)$ (which in particular contains the configurations $(0, +, -; 3+; 2-)$ and $(0, 2+; 2+, -; 2-)$) we note that there exists at least one point $p_\ell$ in the 0 level that has positive index. Without loss of generality we call it $p_1$. We denote by $p_0, p_1, p_2$ the points in the 0 level, by $p_3, p_4, p_5$ the points in the first level and by $p_6, p_7$ the points in the 2nd level. There exists $k_1, k_2 \in \{3, 4, 5\}$ so that $C = L_{0,k_1} L_{0,k_2}$ evaluated on $p_6, p_7, p_1$ has the same sign. Denote by $k_3$ the remaining index in $\{3, 4, 5\}$ different from $k_1, k_2$. Applying (2) to $L_{0,k_1} L_{0,k_2} L_{2,k_3}$ we reach a contradiction.

For the configuration $(2+, -; 0, 2+; 2-)$ denote by $p_1, p_2, p_3$ the points in the 0-level being $p_3$ the negative, by $p_0, p_4, p_5$ the points in the 1st-level and by $p_6, p_7$ the points in the 2nd level. The two straight lines $C = L_{0,4} L_{0,5}$ intersect the sides of the triangle with vertices $p_1, p_2, p_3$ in four points. So one side of this triangle has two of these four points. If $p_3$ is in the same component of $\mathbb{R}^2 \setminus C$ then applying (2) with $CL_{1,2}$ we get a contradiction. If $p_3$ is not in the same component than $\mathbb{R}^2 \setminus C$ there exists $\ell \in \{1, 2\}$ such that $p_\ell$ is in the same component of $\mathbb{R}^2 \setminus C$ than $p_6$ and $p_7$. Then applying (2) to $CL_{3,\ell}$ we have a contradiction.

For the configuration $(3+; 0, +, -; 2-)$ we denote by $p_1, p_2, p_3$ the points in the 0 level, by $p_4, p_5$ the points in the 1st level (being $p_4$ the point with positive index) and by $p_6, p_7$ the points in the 2nd level. We can rename $p_1, p_2, p_3$ so that $R = L_{0,1} L_{0,2} L_{3,4}$ evaluated on $p_5, p_6$ and $p_7$ have the same sign. Applying formula (2) to $R$ we reach a contradiction.

Finally, for the configuration $(3+; 2-, +; 0, -)$ denote by $p_1, p_2, p_3$ the points in the 0-level, by $p_4, p_5, p_6$ the points in the first level (being $p_6$ the point with positive index) and by $p_0, p_7$ the points in the 2nd level. There exists $\ell_0, \ell_1, \ell_2 \in \{4, 5, 7\}$ so that $C = L_{0,\ell_0} L_{0,\ell_1}$ evaluated on $p_6$ and on $p_{\ell_2}$ have the same sign. Moreover, there exists $k_0 \in \{1, 2, 3\}$ so that $C$ evaluated on $p_6$ and on $p_{k_0}$ has different sign. Applying the Euler-Jacobi formula (2) to $R = CL_{k_1,k_2}$ with $k_1 \neq k_2$, $k_1, k_2 \in \{1, 2, 3\}$ being $k_1 \neq k_0$ and $k_2 \neq k_0$ we reach a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x, y) = {} & -0.63917 - 0.03915x + 1.54354y + 0.0948x^2 + -0.09761xy \\ & - 0.9535y^2 + 0.01618x^3 - 0.20338x^2y + y^3, \\ Q(x, y) = {} & -1.41632 - 0.11903x + 10.7244y + 0.19431x^2 - 5.71385xy \\ & - 2.63511y^2 + 0.03392x^3 + 0.57344x^2y + xy^2, \end{aligned}$$

has the singular points

$$\begin{aligned} & (269.39.., -130.87..), \quad (-4.10.., 0), \quad (-4.10.., 0), \quad (2.49.., -0.27..), \\ & (2.48.., 0.55..), \quad (2.55.., 0.63..), \quad (2.60.., 0.66..), \quad (3, 1), \quad (2.34.., 0.65..) \end{aligned}$$

in the configuration $(0, +, -; 2+, -; +, -)$.

The cubic system (1) with

$$P(x,y) = 1.74803 - 0.42482x - 1.08034y + 2.55396x^2 + 0.46753xy - 1.82837y^2 - 0.29212x^3 - 1.28883x^2y + y^3,$$
$$Q(x,y) = 0.13374 + 0.26732x + 0.08368y + 2.04637x^2 - 2.03967xy - 0.05005y^2 - 0.10762x^3 - 1.02755x^2y + xy^2,$$

has the singular points

$$(1.63.., 2.14..),\quad (1.5, 2),\quad (1.33.., 1.66..),\quad (1.31.., 1.67..),\quad (0, -1),$$
$$(0, 79.., 1.3),\quad (-0.79.., 2),\quad (-0.79.., 2),\quad (0.53.., 1.10..)$$

in the configuration $(0, 2+; 2-, +; +, -)$.

The cubic system (1) with

$$P(x,y) = -50.1135x - y + 11.1057x^2 - 25.6451xy + 0.0683157x^3 + 2.20461x^2y - 3.23766xy^2 + 0.101397y^3,$$
$$Q(x,y) = 17.5763x^2 + 0.082767xy + y^2 - 0.029669x^3 + 3.41881x^2y + 0.0242532xy^2 + 0.318429y^3,$$

has the singular points

$$(0,0),\quad (0,0),\quad (-5,-5),\quad (7,-5),\quad (-2,-4.5),$$
$$(2,-4.5),\quad (0,-3.14..),\quad (-1/2,-3.5),\quad (1/2,-3.5)$$

in the configuration $(0, 2+; 3-; 2+)$.

The cubic system (1) with

$$P(x,y) = 4.51755 + 4.06796x - 1.97571y + 2.2832x^2 - 1.57532xy - 5.49326y^2 - 7.53714x^3 + 7.1848x^2y + y^3,$$
$$Q(x,y) = 1.15404 + 1.92249x - 0.24618y + 1.94662x^2 - 2.79227xy - 1.40022y^2 - 2.77672x^3 + 2.09418x^2y + xy^2,$$

has the singular points

$$(1.58.., 2.03..),\quad (1.5, 2),\quad (1.5, 2),\quad (1.33.., 1.66..),\quad (0, -1)$$
$$(1.31.., 1.67..),\quad (0.79.., 1.3),\quad (-0.79.., 2),\quad (0.53.., 1.10..)$$

in the configuration $(2+, -; 0, +, -; +, -)$.

The cubic system (1) with

$$P(x,y) = 1.67312x - y - 0.826899x^2 + 1.90157xy - 0.987919x^3 - 1.24626x^2y + 1.42219xy^2 + y^3,$$
$$Q(x,y) = -1.6012x^2 - 0.222771xy + y^2 + 0.820301x^3 + 0.865023x^2y - 0.850119xy^2 - y^3,$$

has the singular points

$$\begin{array}{lllll}(0,0), & (0,0), & (-5,-5), & (7,-5), & (0,1),\\ (-2,-2,1), & (2,-2.1), & (1.89..,-2), & (-1.89..,-2) & \end{array}$$

in the configuration $(2+,-;0,2-;2+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -4.34579+10.04x+39.1074y-22.3386x^2-61.6263xy\\ &\quad+25.3103y^2+25.8492x^3-9.25676x^2y+y^3,\\ Q(x,y) &= -5.04257+11.6444x+45.5646y-25.7126x^2-72.0976xy\\ &\quad+27.8699y^2+29.631x^3-9.55345x^2y+xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll}(1.7,1.3), & (1.2,0.64), & (0.2,-1.2), & (0.1,0.1), & (0.1,0.1),\\ (0.79..,0.23..), & (0.6,0.2), & (0.7,0.1), & (-1.63..,1.29..) & \end{array}$$

in the configuration $(3+;0,2+;+,-)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.04785-3.26417x-2.41163y+3.37543x^2+6.19441xy\\ &\quad+0.14689y^2-1.158x^3-3.93093x^2y+y^3,\\ Q(x,y) &= -0.47186+1.4699x+1.8721y-1.52x^2-3.50447xy\\ &\quad-1.81296y^2+0.52146x^3+1.39268x^2y+xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll}(1.74..,1.89..), & (1.74..,-1.89..), & (1.10..,0), & (0.90..,0), & (0.90..,0),\\ (0.1,0.5), & (0.850..,-0.1008..), & (0.856..,-0.109..), & (-0.19..,0.53..) & \end{array}$$

in the configuration $(2+,-;2-,+;0,+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 0.06826x-0.04133y+0.28897x^2-1.25411xy\\ &\quad+0.89334y^2+0.59877x^3-1.2708x^2y+y^3,\\ Q(x,y) &= 0.17828x+0.30521y+0.19261x^2-2.31316xy\\ &\quad-0.38029y^2+0.67984x^3-0.45642x^2y+xy^2, \end{aligned}$$

has the singular points

$$\begin{array}{lllll}(1.7,1.3), & (1.2,0.64), & (0.2,-1.2), & (0.3,0.2), & (0.3,0.2),\\ (-0.5,0.1), & (-1.1,0.65), & (-0.13..,0.03..), & (0,0) & \end{array}$$

in the configuration $(3+;3-;0,+)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) =&0.05945x - 0.05036y + 0.28675x^2 - 1.16699xy\\ &+ 0.90087y^2 + 0.58306x^3 - 1.29606x^2y + y^3,\\ Q(x,y) =&0.05071x + 0.07078y + 0.13478x^2 - 0.04988xy\\ &- 0.18464y^2 + 0.27164x^3 - 1.11276x^2y + xy^2,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.3, 0.2), \quad (0.3, 0.2),\\ &(0.62.., 0.50..), \quad (0.40.., 0.31..), \quad (-0.5, 0.1), \quad (0, 0)\end{aligned}$$

in the configuration $(2+, -; 2+, -; 0, -)$.

## Acknowledgements

The first author is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

## References

[1] V. Arnold, A. Varchenko and S. Goussein-Zude, *Singularités des applications différentialbes*, Mir, Moscow, 1982.
[2] A. N. Berlinskii, *On the behavior of the integral curves of a differential equation*, Izv. Vyssh. Uchebn. Zaved. Mat. **2** (1960), 3–18.
[3] R.F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
[4] C. Chicone and T. Jinghuang, *On general properties of quadratic systems*, Amer. Math. Monthly **81** (1982), 167–178.
[5] A. Cima, A. Gasull and F. Mañosas, *Some applications of the Euler-Jacobi formula to differential equations*, Proc. Amer. Math. **118** (1993), 151–163.
[6] A. Cima and J. Llibre, *Configurations of fans and nests of limit cycles for polynomial vector fields in the plane*, J. Differential Equations **82** (1989), 71–97.
[7] A. Gasull and J. Torregrosa, *Euler-Jacobi formula for double points and applications to quadratic and cubic systems*, Bull. Belg. Math. Soc. **6** (1999), 337–346.
[8] A. G. Khovanskii, *Index of a polynomial vector field*, Funktsional Anal. i Prilozhen **13** (1979), 49–58.
[9] J. Llibre and C. Valls, *On the singularities of the planar cubic polynomial differential systems and the Euler Jacobi formula*, to appear in Qualitative Theory of Dynamical Systems, (2020).
[10] N.G. Lloyd, *Degree Theory*, Cambridge Tracks in Mathematics, **73**, Cambridge University Pres, New York, 1978.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

*Email address*: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

*Email address*: cvalls@math.ist.utl.pt