ON THE CROSSING LIMIT CYCLES FOR PIECEWISE LINEAR DIFFERENTIAL SYSTEMS SEPARATED BY A STRAIGHT LINE AND HAVING SYMMETRIC EQUILIBRIUM POINTS

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Abstract. In this paper we study the maximum number of crossing limit cycles that can have the planar piecewise linear differential systems separated by a straight line Σ and formed by two linear differential systems X^-, X^+ which singularities are symmetrical with respect to the straight line of discontinuity Σ. More precisely, the singularities points of the linear differential systems X^-, X^+ considered can be a center (C), a focus (F), a diagonalizable node (N), an improper node (iN) or a saddle (S), which can be real or virtual. Then we have fourteen cases depending of the type and the position of the singularities of X^- and X^+. Here we provide lower or upper bounds for the maximum number of crossing limit cycles for each case.

1. Introduction and statement of the main results

The qualitative theory of discontinuous piecewise differential systems arose in a natural way in the study of nonlinear oscillations by Andronov, Vitt and Khaikin in [1]. Moreover in these last years this qualitative theory is a matter of great interest for many researchers because these systems are used to investigate nonlinear dynamics, to model several real phenomena like cell activity and processes appearing in electronics, mechanics, economy, etc., see for instance [3, 5, 21, 24] and references quoted therein.

We recall that a *crossing limit cycle* is a periodic orbit isolated in the set of all periodic orbits of the piecewise linear differential system, which only have isolated points of intersection with the discontinuity curve.

The class of piecewise linear differential systems in $\mathbb{R}^2$ with two zones separated by a straight line Σ is the simplest class of piecewise differential systems. We can consider without loss of generality that the discontinuity straight line is $\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$. It separates the plane into two regions, namely

$$\Sigma^- = \{(x,y) \in \mathbb{R}^2 : x < 0\} \text{ and } \Sigma^+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}.$$

Therefore we obtain the piecewise linear differential system

$$\dot{X} = \begin{cases} X^- = A^- X + B^-, & \text{if } (x,y) \in \Sigma^-, \\ X^+ = A^+ X + B^+, & \text{if } (x,y) \in \Sigma^+, \end{cases} \tag{1}$$

where

$$A^\pm = \begin{pmatrix} a_{11}^\pm & a_{12}^\pm \\ a_{21}^\pm & a_{22}^\pm \end{pmatrix}, \quad B^\pm = \begin{pmatrix} b_1^\pm \\ b_2^\pm \end{pmatrix} \text{ and } X = (x,y)^T \in \mathbb{R}^2$$

In [20] Lum and Chua conjectured that a continuous piecewise linear differential system (1) has at most one crossing limit cycle. In [9] Freire et al. proved this conjecture. There are several papers tried to investigate the problem of Lum and Chua for the class of discontinuous piecewise linear differential systems in the plane. For instance in [10] Han and

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Zhang conjectured that discontinuous piecewise linear differential systems (1) have at most two crossing limit cycles. Via a numerical example with three crossing limit cycles in a discontinuous piecewise linear differential system, Huan and Yang gave a negative answer to this conjecture, see [11]. Later on in [18, 8] were given analytical proofs for the existence of these three crossing limit cycles. Nevertheless until today it is an open problem to know if three is the upper bound for the maximum number of crossing limit cycles of discontinuous piecewise linear differential systems (1).

Due to the difficulty of this problem several researchers study the upper bounds of crossing limit cycles of system (1) under some special conditions, see [2,6–8,11–13,15–17,19,22,23]. In [16] the authors proved that when one of linear differential systems of (1) has the equilibrium point on Σ, systems (1) have at most two crossing limit cycles and this upper bound is reached. In [7] the authors studied systems (1) such that have a maximal crossing set, and with a focus-focus dynamics, they proved that if $a_{12}^{-}a_{12}^{+} > 0$, then systems (1) have at most one crossing limit cycle. In [22] it was proved that systems (1) with focus-saddle type with $b_1^{+} = 0$ have at most one crossing limit cycle. Recently in [15] it was proved that systems (1) having a unique non-degenerated equilibrium can have at least three crossing limit cycles depending on the configurations of the equilibrium points for each linear differential system in (1).

The objective of this paper is to study the maximum number of crossing limit cycles that can have the planar piecewise linear differential systems(PWLS) (1) when the equilibrium points of the differential linear systems X^- and X^+ are symmetric with respect to the line of discontinuity Σ and these singularities can be real or virtual.

We recall that the singularity $P_- = (x_0, y_0)$ is a *real singularity* (P_-^r) for the linear differential system X^- if $x_0 < 0$ and it is a *virtual singularity* (P_-^v) for the linear differential system X^- if $x_0 > 0$. Considering the linear differential system X^+, we have that $P_+ = (x_1, y_1)$ is a real singularity (P_+^r) if $x_1 > 0$ and it is a virtual singularity (P_+^v) if $x_1 < 0$.

We analyze the possible configurations that can arise when the equilibrium points of the linear differential systems X^- and X^+ are symmetric with respect to the straight line Σ. We denote those configurations like (P_-, P_+) depending of type and the position of the equilibrium points, $P_-, P_+ \in \{C^r, C^v, F^r, F^v, N^r, N^v, iN^r, iN^v, S^r, S^v\}$.

We observed that the equilibrium points P_- and P_+ can not be a saddle S^v, a diagonalizable node N^r or an improper node iN^r because the first return map for the linear differential systems X^- or X^+ is not defined on the discontinuity straight line Σ.

We assume that the equilibrium points P_- and P_+ of linear differential systems X^- and X^+, respectively are symmetric with respect to the line of discontinuity Σ. Then we obtain two options, first the case when the singularities of X^- and X^+ are symmetric with respect Σ and they are on the straight line $y = \epsilon$, $\epsilon \in \mathbb{R}$, this is, the singularities are $(-k, \epsilon)$ or (k, ϵ), with $k \in \mathbb{R}^+$. Second we have the case when the singularities of linear differential systems X^- and X^+ are symmetric with respect Σ and they are on the straight line $y = sx$, with $s \in \mathbb{R}$, this is, the equilibrium points are $(-k, -sk)$ and (k, sk).

In Theorem 1 we assume that the singularities P_- and P_+ are on the straight line $y = sx$, with $s \in \mathbb{R}$ and we observe that this condition is sufficient to analyze the above two cases because when $\epsilon = 0$ the equilibrium points are $(-k, 0)$ and $(k, 0)$ which are on the straight line $y = sx$, with $s = 0$ and it is possible to verify that the number of crossing limit cycles when the equilibrium point are on the straight line $y = \epsilon$ independent of the epsilon.

If the linear differential system X^- has a center (C) we have the following options of configurations: $(C^r, C^r), (C^r, F^r), (C^r, S^r), (C^v, C^v), (C^v, F^v), (C^v, N^v)$ and (C^v, iN^v). In the paper [16] it was proved that if the planar PWLS (1) has the configuration (C^r, C^r) or

(C^v, C^v), then there are no crossing limit cycles. Therefore in statement (i) of Theorem 1 we study the remaining five cases.

When the singularity P_- of the linear differential system X^- is a focus (F) we have the following options: $(F^r, C^r), (F^r, F^r), (F^r, S^r), (F^v, C^v), (F^v, F^v), (F^v, N^v)$ and (F^v, iN^v), here we observed that due to that having symmetric equilibrium points with respect the discontinuity straight line Σ, the configurations (F^r, C^r) and (C^r, F^r); (F^v, C^v) and (C^v, F^v) are equivalent. Then we study the remaining five cases in statement (ii) of Theorem 1.

If P_- is a saddle (S) we have the configurations $(S^r, C^r), (S^r, F^r)$ and (S^r, S^r), but the configurations (S^r, C^r) and (C^r, S^r) are equivalent, and the configurations (S^r, F^r) and (F^r, S^r) are equivalent, then in this case we only have one possible new configuration (S^r, S^r) which is analyzed in statement (iii) of Theorem 1.

When P_- is a diagonalizable node (N), we have the following configurations: (N^v, C^v), (N^v, F^v), (N^v, N^v) and (N^v, iN^v), since the previous two cases have been already studied, we only need to study the cases (N^v, N^v) and (N^v, iN^v) in statement (iv) of Theorem 1. The configuration (N^v, F^v) is in the statement (ii) of Theorem because it is equivalent to the configuration (F^v, N^v) due to that having symmetric equilibrium points with respect the discontinuity straight line Σ.

When the singularity P_- is an improper node (iN), we only study the configuration (iN^v, iN^v) in statement (v) of Theorem 1, because having symmetric equilibrium points with respect to discontinuity straight line Σ, the configurations $(iN^v, C^v), (iN^v, F^v), (iN^v, N^v)$ are considered in the above cases.

We denote the maximum number of crossing limit cycles of planar PWLS (1) by $\mathcal{N}(P_-, P_+)$.

Theorem 1. *Consider that the linear differential systems X^- and X^+ in* (1) *have symmetric equilibrium points with respect the discontinuity straight line Σ and they are on the straight line $y = sx$, $s \in \mathbb{R}$. Then the following statements hold.*

(i) $\mathcal{N}(C^r, F^r) = \mathcal{N}(C^r, S^r) = \mathcal{N}(C^v, F^v) = \mathcal{N}(C^v, N^v) = \mathcal{N}(C^v, iN^v) = 1$. *Moreover these upper bounds are reached and the crossing limit cycles are stables.*
(ii) $\mathcal{N}(F^r, F^r) \geq 2$, $\mathcal{N}(F^r, S^r) \geq 2$, $\mathcal{N}(F^v, F^v) \geq 2$, $\mathcal{N}(F^v, N^v) \geq 1$ *and* $\mathcal{N}(F^v, iN^v) \geq 2$. *See Figures* 1*,* 2*,* 3*,* 4 *and* 6*, respectively.*
(iii) $\mathcal{N}(S^r, S^r) \geq 1$. *See Figure* 7*.*
(iv) $\mathcal{N}(N^v, N^v) \geq 2$ *and* $\mathcal{N}(N^v, iN^v) \geq 2$. *See Figures* 8, 10*.*
(v) $\mathcal{N}(iN^v, iN^v) \geq 1$. *See Figure* 11*.*

Theorem 1 is proved in Section 3.

Proposition 1. *The upper bound for the maximum number of crossing limit cycles provided in statement* (i) *of Theorem 1 is reached and the crossing limit cycle in each configuration of statement* (i) *it is hyperbolic. See Figures* $12 - 16$*.*

2. Canonical forms and basic results

We observe that piecewise linear differential system (1) depend on twelve parameters. In order to reduce the number of parameters on which the PWLS (1) depends we use the canonical forms in the Propositions 2 and 3.

Proposition 2. *There exists a topological equivalence between the phase portrait of the discontinuous PWLS* (1) *and the phase portrait of the discontinuous PWLS* (2) *for all the* *orbits not having points in common with the sliding set.*

$$\dot{X}(x,y) = \begin{cases} X^-(x,y) = \begin{pmatrix} 2l & -1 \\ l^2-\alpha^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}, & \text{if } (x,y)\in\Sigma^-, \\ X^+(x,y) = \begin{pmatrix} 2r & -1 \\ r^2-\beta^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}, & \text{if } (x,y)\in\Sigma^+, \end{cases} \tag{2}$$

where $\alpha,\beta\in\{i,0,1\}$. If $\alpha=i$, we have that the equilibrium point of a linear differential system X^- has eigenvalues $\lambda^-_{1,2}=l\pm i$, so it is a focus if $l\neq 0$ or a center if $l=0$. When $\alpha=0$, then the equilibrium point of a linear differential system X^- has one eigenvalue of multiplicity 2, namely $\lambda^-=l\neq 0$, so it is a non-diagonalizable node. If $\alpha=1$ the equilibrium point of a linear differential system X^- has eigenvalues $\lambda^-_1=l-1$ and $\lambda^-_2=l+1$, then we have that the equilibrium point of X^- is a saddle if $|l|<1$ or it is a diagonalizable node if $|l|>1$. Analogously for the linear differential system X^+.

For a proof of Proposition 2 see [8].

Other normal form which is independent of the change of coordinates it is provide in the following proposition.

Proposition 3. *Consider the linear differential system*

$$\dot{X}(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{3}$$

it has a singularity

(a) *of type focus(F) (resp. a center(C)) if*

$$\dot{X}(x,y) = \begin{pmatrix} A & B \\ \dfrac{-(A-\tilde{C})^2-d^2}{B} & 2\tilde{C}-A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{4}$$

with $B<0$ and $\tilde{C}\neq 0$ (resp. $\tilde{C}=0$ and $B<0$);

(b) *of type diagonalizable node(N) (resp. an improper node(iN)) if*

$$\dot{X}(x,y) = \begin{pmatrix} A & B \\ \dfrac{-(A-\tilde{C})^2+d^2}{B} & 2\tilde{C}-A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{5}$$

with $\tilde{C}^2>d^2>0$ and $B<0$ (resp. $d=0$ and $B<0$);

(c) *of type saddle(S) if*

$$\dot{X}(x,y) = \begin{pmatrix} A & B \\ \dfrac{-(A-\tilde{C})^2+d^2}{B} & 2\tilde{C}-A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{6}$$

with $0<\tilde{C}^2<d^2$ and $B<0$.

Where the parameters $\tilde{C}, A$ and B in (4), (5) and (6) are such that $2\tilde{C}=a_{11}+a_{22}$, $A=a_{11}$ and $B=a_{12}$.

Proof. We know that the eigenvalues of linear differential system (3) are

$$\lambda_{1,2} = \frac{a_{11}+a_{22}\pm\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}}{2}. \tag{7}$$

(a) If we consider $a_{11}+a_{22}=2\tilde{C}$, this is $a_{22}=2\tilde{C}-a_{11}$, with $\tilde{C}, a_{11}\in\mathbb{R}$ and $(a_{11}-a_{22})^2+4a_{12}a_{21}=-4d^2$, this is $a_{21}=(-(a_{11}-\tilde{C})^2-d^2)/a_{12}$, with $d, a_{12}\in\mathbb{R}$. Then the eigenvalues (7) are $\lambda_{1,2}=\tilde{C}\pm id$, therefore the singularity of linear differential system (3) is a focus (F) if $\tilde{C}\neq 0$, and a center (C) if $\tilde{C}\neq 0$. Considering $a_{11}=A$ and $a_{12}=B$, we obtain system (4).

(b) We consider $a_{11}+a_{22}=2\tilde{C}$, then analogously to the above case $a_{22}=2\tilde{C}-A$, and we assume that $(a_{11}-a_{22})^2+4a_{12}a_{21}=4d^2$, then $a_{21}=(-(A-\tilde{C})^2+d^2)/B$. Then the eigenvalues (7) are $\lambda_{1,2}=\tilde{C}\pm d$, therefore the singularity of linear differential system (3) is a diagonalizable node (N), if $\tilde{C}^2>d^2>0$ and $B<0$, because the two eigenvalues would have the same sign, and it is a improper node (iN), if $d=0$, because the two eigenvalues would be equals. Therefore we obtain system (5).

(c) Analogously to the previous case we consider $a_{22}=2\tilde{C}-A$ and $a_{21}=(-(A-\tilde{C})^2+d^2)/B$. Then the eigenvalues (7) are $\lambda_{1,2}=\tilde{C}\pm d$, therefore the singularity of linear differential system (3) is a saddle(S), if $0<\tilde{C}^2<d^2$ and $B<0$, because with this condition we have that $\lambda_1\lambda_2<0$. Therefore we obtain system (6). □

We shall use the following tools for proving our results.

The functions $f_0, f_1, ..., f_n$, defined on an open set $U\subset\mathbb{R}$ are *linearly independent* functions if

$$\text{for every } t\in U,\ \sum_{i=0}^{n}\alpha_i f_i(t)=0 \text{ implies that } \alpha_0=\alpha_1=...=\alpha_n=0.$$

Proposition 4. *Let $f_0, f_1, ..., f_n$ be analytic functions defined on an open interval $U\subset\mathbb{R}$. If the functions $f_0, f_1, ..., f_n$ are linearly independent then there exists $\tilde{t}_1, ..., \tilde{t}_n\in U$ and $\tilde{\alpha}_0, \tilde{\alpha}_1, ..., \tilde{\alpha}_n\in\mathbb{R}$ such that $\sum_{i=0}^{n}\tilde{\alpha}_i f_i(\tilde{t}_j)=0$, for every $j\in\{1, ..., n\}$.*

For a proof of Proposition 4 see [14] or [18].

Now we recall the concept of Chebyshev systems. For more details see [14].

Definition 1. *Let $\mathcal{F}=\{f_0, f_1, ..., f_n\}$ be an ordered set of smooth real functions defined on an interval $I\subset\mathbb{R}$. The set $\mathcal{F}$ is an Extended Chebyshev system (ET-system) on I if and only if the maximum number of zeros counting multiplicities by any non-trivial linear combination of functions in $\mathcal{F}$ is at most n, and this number is reached. The family $\mathcal{F}$ is an Extended Complete Chebyshev system (ECT-system) on I if and only if for any $k\in\{0, 1, ..., n\}$ the set $\mathcal{F}_k=\{f_0, f_1, ..., f_k\}$ is an Extended Chebyshev system.*

In the proof of Lemma 1 we will use the following proposition, for a proof see [14].

Proposition 5. *The ordered set of functions $\mathcal{F}$ is an ECT-system on I if and only if the Wronskians $W_k(f_0, f_1, ..., f_k)(t)\neq 0$, on I for each $k\in\{0, 1, ..., n\}$.*

For a proof see [14].

The following lemma will be used later on in the proof of statement (i) of Theorem 1 to establish a sharp upper bound for the maximum number of crossing limit cycles that system (1) can have.

Lemma 1. *We consider the functions*

$$f_0(t_2)=\sin(t_2),\quad f_1(t_2)=\sinh(rt_2),\quad f_2(t_2)=\sinh(t_2),\quad f_3(t_2)=t_2.$$

The following statements hold.

(a) *The set of functions $\mathcal{F}^1=\{f_0, f_1\}$ is an ECT-system on the intervals $(0, 2\pi)\setminus\{\pi\}$ for every $r\neq 0$;*

(b) *The set of functions* $\mathcal{F}^2 = \{f_2, f_1\}$ *is an ECT-system for every* $t_2 \neq 0$ *and* $r \neq 1$;
(c) *The set of functions* $\mathcal{F}^3 = \{f_3, f_1\}$ *is an ECT-system for every* $t_2 \neq 0$ *and* $r \neq 0$.

Proof.

(a) Considering the functions f_0 and f_1 the Wronskian is
$$W(t_2) = r\cosh(rt_2)\sin(t_2) - \cos(t_2)\sinh(rt_2).$$
Since $W(0) = 0$ and $W'(t_2) = (1+r^2)\sin(t_2)\sinh(rt_2)$ does not vanish for any $t_2 \in (0, 2\pi)\setminus\{\pi\}$ and $r \neq 0$. Then $W(t_2) \neq 0$ for $t_2 \in (0, 2\pi)\setminus\{\pi\}$ and $r \neq 0$, therefore by Proposition 5, statement (a) is proved.

(b) The Wronskian of the functions f_1 and f_2 is
$$W(t_2) = r\cosh(rt_2)\sinh(t_2) - \cosh(t_2)\sinh(rt_2),$$
and we observed that $W(0) = 0$ and $W'(t_2) = (-1+r^2)\sinh(t_2)\sinh(rt_2)$, then $W'(t_2)$ does not vanish for every $t_2 \neq 0$ and $r \neq 1$. Therefore $W(t_2) \neq 0$ for $t_2 \neq 0$ and $r \neq 1$, then by Proposition 5, statement (b) is proved.

(c) The Wronskian of the functions f_1 and f_3 is
$$W(t_2) = rt_2\cosh(rt_2) - \sinh(rt_2),$$
and we observed that $W(0) = 0$ and $W'(t_2) = r^2 t_2 \sinh(rt_2)$, we have that $W'(t_2)$ does not vanish if $t_2 \neq 0$ and $r \neq 0$, then $W(t_2) \neq 0$ for $t_2 \neq 0$ and $r \neq 0$. Therefore by Proposition 5, statement (c) is proved. □

In order to analyze the existence of periodic orbits which intersect both zones $\Sigma^{\pm}$ and Σ at the two points $\mathbf{p} = (0, y_0)$ and $\mathbf{q} = (0, y_1)$ we use the *closing equations* provide in the following Proposition.

Proposition 6. *Assume that the PWLS* (1) *has a crossing periodic orbit that transversely intersecting the straight line* Σ *in the points* $\mathbf{p} = (0, y_0)$ *and* $\mathbf{q} = (0, y_1)$ *where* $y_1 = y^-(t_1)$ *and* $y_0 > y_1$*, with flight times* $t_1 > 0$ *and* $t_2 > 0$ *in the zones* Σ^- *and* Σ^+*, respectively. Then* (t_1, t_2, y_0) *are real solutions of the closing equations:*

$$\begin{aligned} e_1: &\quad x^-(t_1) = 0,\\ e_2: &\quad x^+(-t_2) = 0,\\ e_3: &\quad y^+(-t_2) - y^-(t_1) = 0. \end{aligned} \tag{8}$$

3. Proof of Theorem 1

Proof of statement (i) ***of Theorem 1.*** We have that the equilibrium point of linear differential system X^- is a center, then using Proposition 3, we consider that the linear differential system X^- is in the canonical form (4) with $\tilde{C} = 0$. Then the equilibrium point of linear differential system X^- is

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2}{d^2}, -\frac{A^2 b_1 + ABb_2 + b_1 d^2}{Bd^2}\right). \tag{9}$$

We separate the proof of statement (i) of Theorem 1 in two cases.

Case 1: P_- is a real singularity of X^-. We assume that $P_- = (-k, -sk)$, for this we must consider $b_1 = Ak + Bsk$ and $b_2 = -k(A^2 + d^2 + ABs)/B$. Therefore, linear differential system X^- is

$$X^-(x, y) = \begin{pmatrix} A(x+k) + B(y+sk) \\ -\dfrac{(A^2+d^2)(k+x) + AB(y+sk)}{B} \end{pmatrix}. \tag{10}$$

When we have an equilibrium point of type C^r for the linear differential system X^-, by hypothesis, we have two possible configurations for the equilibrium points of the PWLS (1), namely, we can have the configurations (C^r, F^r) and (C^r, S^r).

We consider that linear differential system X^+ is in the canonical form (2) which has the equilibrium point

$$P_+ = (x_1, y_1) = \left(-\frac{c}{r^2-\beta^2}, \frac{-2cr + b(r^2-\beta^2)}{r^2-\beta^2}\right). \tag{11}$$

Therefore the equilibrium point P_+ is a real singularity of X^+ if

$$b = -k(2r-s),\ c = -k(r^2-\beta^2); \tag{12}$$

and P_+ is a virtual singularity of X^+ if

$$b = k(2r-s),\ c = k(r^2-\beta^2). \tag{13}$$

Configuration (C^r, F^r)**:** For the linear differential system X^+, we consider the condition (12) with $\beta = i$ and $r \neq 0$.

The linear differential system X^+ in this case is

$$X^+(x,y) = \begin{pmatrix} -y + 2r(x-k) + sk \\ (1+r^2)(x-k) \end{pmatrix}. \tag{14}$$

With those conditions the solution of system (10) starting at the point $(x, y) = (0, y_0) \in \Sigma$ is

$$\begin{aligned} x^-(t) &= k(-1+\cos(dt)) + \frac{(Ak + B(y_0+sk))\sin dt}{d}, \\ y^-(t) &= -sk + (y_0+sk)\cos(dt) - \frac{((A^2+d^2+ABs)k + ABy_0)\sin(dt)}{Bd}, \end{aligned}$$

and the solution of system (14) starting at the point $(x, y) = (0, y_0) \in \Sigma$ is

$$\begin{aligned} x^+(t) &= k - e^{rt}(k\cos(t) + ((r-s)k + y_0)\sin(t)), \\ y^+(t) &= sk + e^{rt}((y_0-sk)\cos(t) - (k + r((r-s)k+y_0))\sin(t)). \end{aligned}$$

Considering that there exists $t_1, t_2 > 0$ the finite times defined in Proposition 6. We have that system (8) is equivalent to system

$$\begin{aligned} e_1 :\ & kd(-1+\cos(dt_1)) + (Ak + B(y_0+sk))\sin(dt_1) = 0, \\ e_2 :\ & k + e^{-rt_2}(-k\cos(t_2) + ((r-s)k + y_0)\sin(t_2)) = 0, \\ e_3 :\ & 2sk - (y_0+sk)\cos(dt_1) + \frac{((A^2+d^2+ABs)k + ABy_0)\sin(dt_1)}{Bd} \\ & + e^{-rt_2}((y_0-sk)\cos(t_2) + (k + r((r-s)k+y_0))\sin(t_2)) = 0. \end{aligned} \tag{15}$$

From the first equation we obtain

$$\begin{aligned} \cos(dt_1) &= \frac{(-A^2+d^2)k^2 - 2ABk(y_0+ks) - B^2(y_0+sk)^2}{(A^2+d^2)k^2 + 2ABk(y_0+ks) + B^2(y_0+ks)^2}, \\ \sin(dt_1) &= \frac{2kd(Ak + B(y_0+ks))}{(A^2+d^2)k^2 + 2ABk(y_0+ks) + B^2(y_0+ks)^2}, \end{aligned} \tag{16}$$

from equation e_2 we get $y_0 = -k(r - s - \cot(t_2) + e^{rt_2}\csc(t_2))$. Substituting y_0 in equation e_3 we have $e_3 = 2k(A/B - r + 2s - \csc(t_2)\sinh(rt_2))$, and to determine the solutions for this equation is equivalent to determine the solutions for the following equation

$$\frac{2k}{B\sin(t_2)}((A - rB + 2Bs)f_0(t_2) - Bf_1(t_2)) = 0, \quad \text{with} \quad t_2 \in (0, 2\pi) \setminus \{\pi\}, \tag{17}$$

and we can conclude that equation (17) has at most one real solution for $t_2 \in (0, 2\pi) \setminus \{\pi\}$, because by statement (a) of Lemma 1 the set of functions $\mathcal{F}^1 = \{f_0, f_1\}$ is an extended complete Chebyshev system for $t_2 \in (0, 2\pi) \setminus \{\pi\}$ for every $r \neq 0$ and even more the coefficients $A-rB+2Bs$ and B can be chosen arbitrarily. Therefore we have proved that a PWLS (1) with the configuration (C^r, F^r) formed by the linear differential systems (10) and (14) has at most one crossing limit cycle. □

Configuration (C^r, S^r)**:** The equilibrium point P_+ of system X^+ satisfies the condition (12) with $\beta = 1$ and $|r| < 1$. Therefore

$$X^+(x,y) = \begin{pmatrix} -y + 2r(x-k) + sk \\ (-1+r^2)(x-k) \end{pmatrix}. \tag{18}$$

The solution of system (18) starting at the point $(x,y) = (0,y_0) \in \Sigma$ is

$$\begin{aligned} x^+(t) &= \frac{e^{-t}}{2}\left(2ke^t + e^{rt}((-1+r-s)k + y_0) - e^{(2+r)t}(k(1+r-s)+y_0)\right), \\ y^+(t) &= \frac{e^{-t}}{2}\left(2e^t sk + e^{rt}(1+r)(y_0 + (r-1-s)k - e^{(2+r)t}(r-1)(y_0 + (1+r-s)k)\right). \end{aligned}$$

Let t_1 and t_2 be the finite times defined in Proposition 6. In this case we have that system (8) is equivalent to system

(19)
$$\begin{aligned} e_1 :&\quad kd(-1+\cos(dt_1)) + (Ak + B(y_0 - sk))\sin(dt_1) = 0, \\ e_2 :&\quad 2ke^{-t_2} + e^{-rt_2}((r-1-s)k + y_0) - e^{-(2+r)t_2}(k(1+r-s)+y_0)) = 0, \\ e_3 :&\quad sk + (y_0+sk)\cos(dt_1) + \frac{((A^2+d^2+ABs)k + ABy_0)\sin(dt_1)}{Bd} + \frac{e^{t_2}}{2}\left(2e^{-t_2}sk\right. \\ &\quad \left. + e^{-rt_2}(1+r)(y_0 + (r-1-s)k) - e^{-(2+r)t_2}(r-1)(y_0+(1+r-s)k)\right) = 0. \end{aligned}$$

Then the real solutions of system (19) generate crossing limit cycles of PWLS (1) formed by the linear differential systems (10) and (18). Similar to Case (C^r, Fr), from equation e_1 we obtain equations (16), from e_2 we get

$$y_0 = -k\frac{-1 + 2e^{t_2+rt_2} - r + e^{2t_2}(-1+r-s) + s}{-1+e^{2t_2}},$$

then $e_3 = 2k(A/B - r + 2s - \operatorname{csch}(t_2)\sinh(rt_2))$. To determine the solutions for equation e_3 is equivalent to determine the solutions for the following equation

$$\frac{2k}{B\sinh(t_2)}\left((A - rB + 2sB)f_2(t_2) - Bf_1(t_2)\right) = 0, \quad \text{with} \quad t_2 \neq 0. \tag{20}$$

By statement (*b*) of Lemma 1 the set of functions $\mathcal{F}^2 = \{f_2, f_1\}$ is an extended complete Chebyshev system for $t_2 \neq 0$ and $r \neq 1$ and moreover the coefficients $A - rB + 2sB$ and B can be chosen arbitrarily. Then we can conclude that equation (20) has at most one real solution for $t_2 \neq 0$ and $|r| < 1$. Therefore PWLS (1) with the configuration (C^r, S^r) formed by the linear differential systems (10) and (18) has at most one crossing limit cycle. □

Case 2: P_- **is a virtual singularity of** X^-**.** We consider that the equilibrium point P_- in (9) is a center C^v, this is $P_- = (k, sk)$, for this we must consider $b_1 = -Ak - Bks$ and $b_2 = k(A^2 + d^2 + ABs)/B$. Therefore linear differential system X^- is

$$X^-(x,y) = \begin{pmatrix} A(x-k) + B(y-ks) \\ \dfrac{(A^2+d^2)(-x+k) + AB(-y+sk)}{B} \end{pmatrix}. \tag{21}$$

When the equilibrium point P_- is a C^v for the linear differential system X^-, then we have three possible configurations for the equilibrium points (P_-^-, P_+) of the PWLS (1), namely, we have the configurations (C^v, F^v), (C^v, N^v) and (C^v, iN^v).

Configuration (C^v, F^v)**:** We consider that the configuration of the equilibrium points of the linear differential systems X^- and X^+ in (1) is (C^v, F^v), then the equilibrium point P_+

satisfies (13) with $\beta = i$ and $r \neq 0$. Therefore

$$X^+(x,y) = \begin{pmatrix} -y + 2r(k+x) - ks \\ (1+r^2)(k+x) \end{pmatrix}. \tag{22}$$

The solutions of systems (21) and (22) starting at the point $(x,y) = (0, y_0) \in \Sigma$ are

$$\begin{aligned}
x^-(t) &= k(1-\cos(dt)) + \frac{(By_0 - (A+Bs)k)\sin dt}{d},\\
y^-(t) &= \frac{Bdks + Bd(y_0 - ks)\cos(dt) + ((A^2+d^2+ABs)k - ABy_0)\sin(dt)}{Bd},\\
x^+(t) &= -k - e^{rt}(k\cos(t) - (k(s-r)+y_0)\sin(t)),\\
y^+(t) &= -ks + e^{rt}((y_0+ks)\cos(t) + (k + r^2k - r(y_0+ks))\sin(t)).
\end{aligned}$$

Let t_1 and t_2 be the finite times defined in Proposition 6. Here we have that system (8) is equivalent to system

$$\begin{aligned}
e_1:\ & kd(1-\cos(dt_1)) + (-(A+bs)k + By_0)\sin(dt_1) = 0,\\
e_2:\ & -k + e^{-rt_2}(k\cos(t_2) + ((-r+s)k + y_0)\sin(t_2)) = 0,\\
e_3:\ & -2ks + (-y_0+ks)\cos(dt_1) - \frac{((A^2+d^2+ABs)k - ABy_0)\sin(dt_1)}{Bd}\\
& + e^{-rt_2}((y_0+ks)\cos(t_2) + (-k + r(y_0 + (-r+s)k))\sin(t_2)) = 0.
\end{aligned} \tag{23}$$

Similar to case (C^r, F^r), we obtain that e_3 is equivalent to equation (17) then we can conclude that PWLS (1) with the configuration (C^v, F^v) formed by the linear differential systems (21) and (22) has at most one crossing limit cycle. □

We observe that in the previous cases the constant k does not influence the number of solutions of system (8) and in the following cases the same thing happens, therefore without loss of generality we can assume that $k = 1$, this is the singularities of systems X^- and X^+ are in $(-1, -s)$ or $(1, s)$, with $s \in \mathbb{R}$.

Configuration (C^v, N^v)**:** We consider that the configuration of the equilibrium points of the linear differential systems X^- and X^+ in (1) is (C^v, N^v), then the equilibrium point P_+ satisfies (13) with $\beta = 1$ and $|r| > 1$. Therefore the linear differential system X^+ is

$$X^+(x,y) = \begin{pmatrix} -y + 2r(1+x) - s \\ (-1+r^2)(1+x) \end{pmatrix}. \tag{24}$$

The solution of system (24) starting at the point $(x,y) = (0, y_0) \in \Sigma$ is

$$\begin{aligned}
x^+(t) &= \frac{e^{-t}}{2}\left(-2e^t + e^{(2+r)t}(1+r-s-y_0) + e^{rt}(1-r+s+y_0)\right),\\
y^+(t) &= \frac{e^{-t}}{2}\left(-2e^ts + e^{(2+r)t}(-1+r)(1+r-s-y_0) + e^{rt}(1+r)(1-r+s+y_0)\right).
\end{aligned}$$

Considering t_1 and t_2 the finite times defined in Proposition 6, we obtain that system (8) is equivalent to system

$$\begin{aligned}
e_1:\ & d(1-\cos(dt_1)) - (A + Bs - By_0)\sin(dt_1) = 0,\\
e_2:\ & -2e^{-t_2} + e^{-(2+r)t_2}(1+r-s-y_0) + e^{-rt_2}(1-r+s+y_0) = 0,\\
e_3:\ & -s + (s-y_0)\cos(dt_1) - (A^2+d^2+AB(s-y_0))\sin(dt_1) - \frac{e^{t_2}}{2}\left(-2e^{-t_2}s\right.\\
& \left. + e^{-(2+r)t_2}(-1+r)(1+r-s-y_0) + e^{-rt_2}(1+r)(1-r+s+y_0)\right) = 0.
\end{aligned} \tag{25}$$

From equation e_1 we obtain that

$$\cos(dt_1) = \frac{-A^2 + d^2 + 2AB(-s+y_0) - B^2(s-y_0)^2}{A^2 + d^2 + 2AB(s-y_0) + B^2(s-y_0)^2},$$

$$\sin(dt_1) \quad = \frac{2d(A+B(s-y_0))}{A^2+d^2+2AB(s-y_0)+B^2(s-y_0)^2},$$

and from e_2 we get

$$y_0 = \frac{-1+2e^{t_2+rt_2}-r+e^{2t_2}(-1+r-s)+s}{-1+e^{2t_2}},$$

then substituting in e_3 we obtain that e_3 is equivalent to equation (20), therefore we can conclude that PWLS (1) with the configuration (C^v, N^v) formed by the linear differential systems (21) and (24) has at most one crossing limit cycle. □

Configuration (C^v, iN^v)**:** We consider that the configuration of the equilibrium points of the linear differential systems X^- and X^+ in (1) is (C^v, iN^v). We consider that equilibrium point P_+ satisfies (13) with $\beta = 0$ and $r \neq 0$. Then

$$X^+(x,y) = \begin{pmatrix} -y+2r(1+x)-s \\ r^2(1+x) \end{pmatrix}. \tag{26}$$

The solution of system (26) starting at the point $(x,y) = (0,y_0) \in \Sigma$ is

$$x^+(t) = -1+e^{rt}(1-t(y_0-r+s)), \qquad y^+(t) = -s+e^{rt}(y_0-rty_0+s+r(r-s)t), \tag{27}$$

Considering t_1 and t_2 the finite times defined in Proposition 6, we obtain that system (8) is equivalent to system

$$\begin{aligned} e_1 :&\ d(1-\cos(dt_1))+(By_0-(A+Bs))\sin(dt_1) = 0, \\ e_2 :&\ -1+e^{-rt_2}(1+t_2(y_0-r+s)) = 0, \\ e_3 :&\ -2s+e^{-rt_2}(y_0+rt_2y_0+(s+r(-r+s)t_2)) \\ &+\frac{Bd(-y_0+s)\cos(dt_1)-(-ABy_0+(A^2+d^2+ABs))\sin(dt_1)}{Bd} = 0. \end{aligned} \tag{28}$$

From equation e_1 we obtain the expression (3) and from e_2 we get

$$y_0 = \frac{-1+e^{rt_2}+(r-s)t_2}{t_2},$$

then

$$e_3 = 2\left(-\frac{A}{B}+r-2s+\frac{\sinh(rt_2)}{t_2}\right) = 0,$$

and to determine the solutions for equation e_3 is equivalent to determine the solutions for the equation

$$\frac{-2}{Bt_2}\left((A-rB-2sA)f_3(t_2)-Bf_1(t_2)\right) = 0, \quad \text{with} \quad t_2 \neq 0. \tag{29}$$

By statement (c) of Lemma 1 the set of functions $\mathcal{F}^3 = \{f_3, f_1\}$ is an extended complete Chebyshev system for $t_2 \neq 0$ and $r \neq 0$ and moreover the coefficients $A-rB-2sA$ and B can be chosen arbitrarily. Then we can conclude that equation (29) has at most one real solution for $t_2 \neq 0$ and $r \neq 0$. Therefore the PWLS (1) with configuration (C^v, iN^v) formed by the linear differential systems (21) and (26) has at most one crossing limit cycle. □

Moreover the upper bound provided in the above cases is reached, see the examples in the proof of Proposition 1.

Proof of statement (ii) ***of Theorem*** **1.** Here we analyze the number of crossing limit cycles of PWLS (1) when the equilibrium point of linear differential system X^- is a real or virtual focus (F^r) or (F^v). We consider that system X^- is in the canonical form (4) with $\tilde{C} \neq 0$. Then the equilibrium point of system X^- is

$$P_- = (x_0, y_0) = \left(\frac{Ab_1+Bb_2-2b_1\tilde{C}}{\tilde{C}^2+d^2}, -\frac{A^2b_1+ABb_2-2Ab_1\tilde{C}+b_1\tilde{C}^2+b_1d^2}{B\tilde{C}^2+Bd^2}\right).$$

We separate the proof of statement (ii) of Theorem 1 in two cases, first we study the case when P_- is a real focus and second we assume that P_- is a virtual focus. We consider that linear differential system X^+ is in canonical form (2) then the equilibrium point is (11).

Case 1: P_- is a real focus of X^-. We assume that $P_- = (-1,-s)$, for this we must consider that

$$(30) \qquad b_1 = A + Bs, \qquad b_2 = -\frac{A^2 - 2A\tilde{C} + \tilde{C}^2 + d^2 + ABs - 2B\tilde{C}s}{B}.$$

Then linear differential system X^- is

$$(31) \quad X^-(x,y) = \begin{pmatrix} A(x+1) + B(y+s) \\ -\dfrac{(A^2+c^2+d^2)(x+1) - 2Bc(y+s) + A(-2c(x+1) + B(y+s))}{B} \end{pmatrix}.$$

The solution of linear differential system (31) starting at the point $(x,y) = (0,y_0) \in \Sigma$ is

$$\begin{aligned} x^-(t) = &\quad -1 + \frac{e^{\tilde{C}t}(d\cos(dt) + (By_0 + A - \tilde{C} + Bs)\sin(dt))}{d}, \\ y^-(t) = &\quad -s + \frac{e^{\tilde{C}t}}{Bd}\Big((-(B(A-\tilde{C})y_0 + (d^2 + (A-\tilde{C})(A-\tilde{C}+Bs))\sin(dt)) \\ &\quad +(Bd(y_0+s)\cos(dt))\Big). \end{aligned}$$

When P_- is a real focus then we have two possible configurations for the equilibrium points of the PWLS (1), namely we obtain the configurations (F^r, F^r) and (F^r, S^r).

Configuration (F^r, F^r)**:** We assume that the equilibrium point P_- satisfies the conditions (30) and the equilibrium point P_+ satisfies the conditions (12) with $\beta = i$, $r \neq 0$, then we have the configuration (F^r, F^r).

In the following example we provide a PWLS having two crossing limit cycles. We consider

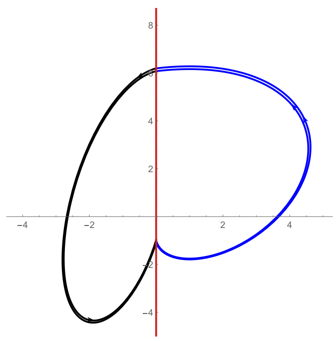

FIGURE 1. The two crossing limit cycles Γ_1 and Γ_2 of the discontinuous PWLS (32) with configuration (F^r, F^r).

that $A = 1/2$, $B = -1/2$, $\tilde{C} = -67/500$, $d = 123/100$, $r = 2/5$ and $s = 0$, then we obtain the PWLS formed by

$$(32)$$
$$X^-(x,y) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{239357}{62500} & -\frac{96}{125} \end{pmatrix} X + \begin{pmatrix} \frac{1}{2} \\ \frac{239357}{62500} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \frac{4}{5} & -1 \\ \frac{29}{25} & 0 \end{pmatrix} X + \begin{pmatrix} -\frac{4}{5} \\ -\frac{29}{25} \end{pmatrix}.$$

For this PWLS we have that system (8) is equivalent to system

$$
\begin{aligned}
-1+\frac{1}{615}e^{-67t_1/500}(615\cos(123t_1/100)+(317-250y_0)\sin(123t_1/100))=0,\\
1+e^{-2t_2/5}\left(-\cos(t_2)+\left(\frac{2}{5}+y_0\right)\sin(t_2)\right)=0,\\
e^{-67t_1/500}(-76875y_0\cos(123t_1/100)+(-239357+39625y_0)\sin(123t_1/100))\\
+3075e^{-2t_2/5}(25y_0\cos(t_2)+(29+10y_0)\sin(t_2))=0.
\end{aligned}
$$

Which has two real solutions with $t_1, t_2 \in (0, 2\pi)$, namely $(t_1^1, t_2^1, y_0^1) = (3.586636.., 4.260216.., 6.196201..)$ and $(t_1^2, t_2^2, y_0^2) = (3.614645.., 4.344295.., 6.078132..)$. Therefore the PWLS (32) has two crossing limit cycles Γ_1 and Γ_2 which intersect Σ in $(0, y_0^1) = (0, 6.196201..)$ and $(0, y_1^1) = (0, y_{11}^-(t_1^1)) = (0, -1.088003..)$ with flight times $t_1^1 = 3.586636..$ and $t_2^1 = 4.260216..$ in the regions Σ^- and Σ^+, respectively; and $(0, y_0^2) = (0, 6.078132..)$ and $(0, y_1^2) = (0, y_{11}^-(t_1^2)) = (0, -0.974222..)$ with flight times $t_1^2 = 3.614645..$ and $t_2^2 = 4.344295..$ in the regions Σ^- and Σ^+, respectively. See Figure 1.

Configuration (F^r, S^r)**:** If the equilibrium point P_- is a focus F^r and the equilibrium point P_+ satisfies the conditions (12) with $\beta = 1$, $|r| < 1$, then we have the configuration (F^r, S^r). In what follows we provide a PWLS having two crossing limit cycles. Considering

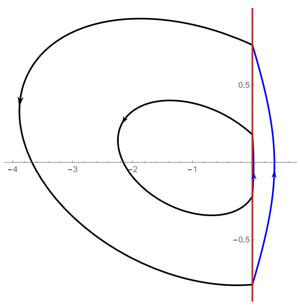

FIGURE 2. The two crossing limit cycles Γ_1 and Γ_2 of the discontinuous PWLS (33) with configuration (F^r, S^r).

$A = -2/5$, $B = -7/2$, $\tilde{C} = 1/20$, $d = -1$, $r = 1/100$ and $s = 0$, we obtain the PWLS formed by

(33)
$$
X^-(x,y)=\begin{pmatrix}-\frac{2}{5} & -\frac{7}{2}\\ \frac{449}{1400} & \frac{3}{10}\end{pmatrix}X+\begin{pmatrix}-\frac{2}{5}\\ \frac{449}{1400}\end{pmatrix},\quad X^+(x,y)=\begin{pmatrix}\frac{1}{50} & -1\\ -\frac{9999}{10000} & 0\end{pmatrix}X+\begin{pmatrix}-\frac{1}{50}\\ \frac{9999}{10000}\end{pmatrix}.
$$

For this PWLS we have that system (8) has two real solutions with $t_1, t_2 \in (0, 2\pi)$, namely $(t_1^1, t_2^1, y_0^1) = (3.854989.., 2.065073.., 0.759545..)$ and $(t_1^2, t_2^2, y_0^2) = (5.114523.., 0.403781.., 0.1794388..)$. Therefore the PWLS (33) has two crossing limit cycle Γ_1 and Γ_2 which intersect Σ in $(0, y_0^1) = (0, 0.759545..)$ and $(0, y_1^1) = (0, -0.790192..)$; and$(0, y_0^2) = (0, 0.1794388..)$ and $(0, y_1^2) = (0, -0.218905..)$, respectively. See Figure 2. □

Case 2: P_- **is virtual focus of** X^-**.** We consider that P_- is a focus (F^v), this is, $P_- = (1, s)$, therefore

$$
b_1=-A-Bs,\qquad b_2=-\frac{-A^2+2A\tilde{C}-\tilde{C}^2-d^2-ABs+2B\tilde{C}s}{B}.
$$

Then linear differential system X^- is

(34)
$$X^-(x,y) = \begin{pmatrix} A(x-1)+B(y-s) \\ \dfrac{-A^2x+2A\tilde{C}x-\tilde{C}^2x-d^2x-ABy+2B\tilde{C}y+((A-\tilde{C})^2+d^2+B(A-2\tilde{C})s)}{B} \end{pmatrix}.$$

The solution of linear differential system (34) starting at the point $(x,y)=(0,y_0)\in\Sigma$ is

(35)
$$\begin{aligned} x^-(t) =&\ 1+\frac{e^{\tilde{C}t}(-d\cos(dt)+(By_0-(A-\tilde{C}+Bs)\sin(dt))}{d}, \\ y^-(t) =&\ s+\frac{e^{\tilde{C}t}}{Bd}\Big(((B(-A+\tilde{C})y_0+(d^2+(A-\tilde{C})(A-\tilde{C}+Bs))\sin(dt)) \\ &+(Bd(y_0-s)\cos(dt)))\Big). \end{aligned}$$

When P_- is a focus (F^v) we have three possible configurations for the equilibrium point of PWLS (1), namely we have the configurations (F^v,F^v), (F^v,N^v) and (F^v,iN^v).

Configuration (F^v,F^v)**:** The equilibrium point P_+ is a focus F^v and the equilibrium point P_+ satisfies the condition (13) with $\beta=i$ and $r\neq 0$, then we have the configuration (F^v,F^v). We provide a PWLS with two crossing limit cycles. Considering $A=-7/10$, $B=-1/2$, $\tilde{C}=-2$, $d=-1$, $r=6/10$ and $s=0$, we obtain the PWLS formed by

(36)
$$X^-(x,y)=\begin{pmatrix} -\dfrac{7}{10} & -\dfrac{1}{2} \\ \dfrac{269}{50} & -\dfrac{33}{10}\end{pmatrix}X+\begin{pmatrix}\dfrac{7}{10}\\ -\dfrac{269}{50}\end{pmatrix}, \quad X^+(x,y)=\begin{pmatrix}\dfrac{6}{5} & -1 \\ \dfrac{34}{25} & 0\end{pmatrix}X+\begin{pmatrix}\dfrac{6}{5}\\ \dfrac{34}{25}\end{pmatrix}.$$

For this PWLS we have that system (8) has two real solution with $t_1,t_2\in(0,2\pi)$, namely $(t_1^1,t_2^1,y_0^1)=(0.903052..,\ 2.593104..,\ 11.325957..)$ and $(t_1^2,t_2^2,y_0^2)=(0.276244..,\ 1.538684..,\ 3.086535..)$. Therefore the PWLS (36) has two crossing limit cycle Γ_1 and Γ_2 which intersect Σ in $(0,y_0^1)=(0,11.325957..)$ and $(0,y_1^1)=(0,-1.441285..)$; and $(0,y_0^2)=(0,3.086535..)$ and $(0,y_1^2)=(0,0.234677..)$, respectively. See Figure 3. Therefore we have that a PWLS

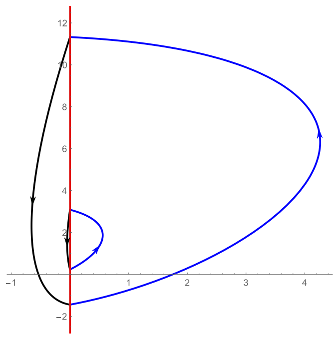

FIGURE 3. The two crossing limit cycles Γ_1 and Γ_2 of the discontinuous PWLS (36) with configuration (F^v,F^v).

(1) with the configuration (F^v,F^v) it has at least two crossing limit cycles. □

Configuration (F^v,N^v)**:** The equilibrium point P_- is a focus F^v and the equilibrium point P_+ satisfies the conditions (13) with $\beta=1$ and $|r|>1$, then we have the configuration (F^v,N^v). We provide a PWLS with this configuration and with one crossing limit cycle. If $A=-3$, $B=-1/2$, $\tilde{C}=-3/10$, $d=1$, $r=2$ and $s=0$, we obtain the PWLS formed by

(37)
$$X^-(x,y)=\begin{pmatrix} -3 & -\dfrac{1}{2} \\ \dfrac{829}{50} & \dfrac{12}{5}\end{pmatrix}X+\begin{pmatrix}3\\ -\dfrac{829}{50}\end{pmatrix}, \quad X^+(x,y)=\begin{pmatrix}4 & -1 \\ 3 & 0\end{pmatrix}X+\begin{pmatrix}4\\ 3\end{pmatrix}.$$

For this PWLS we have that system (8) has one real solution with $t_1, t_2 \in (0, 2\pi)$, namely $(t_1, t_2, y_0) = (2.073656.., \ 1.547693.., \ 10.752069..)$. Then the PWLS (37) has one crossing limit cycle which intersects Σ in $(0, y_0) = (0, 10.752069..)$ and $(0, y_1) = (0, 3.074636..)$. See Figure 4. Therefore we can conclude that a PWLS (1) with the configuration (F^v, N^v) it

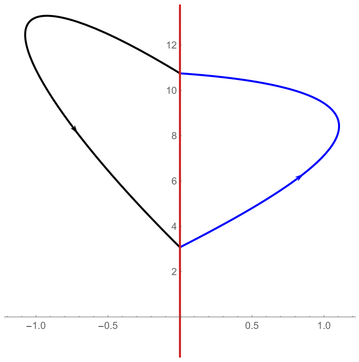

FIGURE 4. The crossing limit cycle of the discontinuous PWLS (37) with configuration (F^v, N^v).

has at least one crossing limit cycles. □

Configuration (F^v, iN^v)**:** The equilibrium point P_- is a focus F^v and the equilibrium point P_+ satisfies the conditions (13) with $\beta = 0$ and $r \neq 0$, then we have the configuration (F^v, iN^v). Then considering t_1 and t_2 as in Proposition 6 and from equations (27) and (35), system (8) is equivalent to system

$$(38)\qquad \begin{array}{ll} e_1: & d + e^{\tilde{C}t_1}(-d\cos(dt_1) + (By_0 - A + \tilde{C} - Bs)\sin(dt_1)) = 0, \\ e_2: & -1 + e^{-rt_2}(1 + t_2(y_0 - r + s)) = 0, \\ e_3: & Bde^{-rt_2}(y_0 + rt_2y_0 + (s + r(-r + s)t_2)) - e^{ct_1}(Bd(y_0 - s)\cos(dt_1) \\ & +(B(-A + \tilde{C})y_0 + (d^2 + (A - \tilde{C})(A - \tilde{C} + Bs)))\sin(dt_1)) - 2Bds = 0. \end{array}$$

From equation e_1 we get

$$y_0 = \frac{A - \tilde{C} + Bs + d\cot(dt_1) - de^{-\tilde{C}t_1}\csc(dt_1)}{B},$$

and from e_2 we get

$$(39)\qquad t_2 = -\frac{1}{y_0 - r + s} - \frac{\mathcal{W}\left(-\dfrac{e^{-\frac{r}{y_0 - r + s}}r}{y_0 - r + s}\right)}{r},$$

then substituting y_0 and t_2 in e_3 we obtain that

$$(40)\qquad e_3 = \frac{1}{B}\Big((-A + \tilde{C} + Br - 2Bs)\tilde{f}_0(t_1) + d\tilde{f}_1(t_1) - rB\tilde{f}_2(t_1)\Big) = 0.$$

Here $\tilde{f}_0(t_1) = 1, \ \tilde{f}_1(t_1) = \cot(dt_1) - e^{\tilde{C}t_1}\csc(dt_1)$, and

$$\tilde{f}_2(t_1) = \frac{1}{\mathcal{W}\left(\dfrac{Bre^{\tilde{C}t_1 + \frac{Bre^{\tilde{C}t_1}}{e^{\tilde{C}t_1}(-A + \tilde{C} + Br - 2Bs - d\cot(dt_1)) + d\csc(dt_1)}}}{e^{\tilde{C}t_1}(-A + \tilde{C} + Br - 2Bs - d\cot(dt_1)) + d\csc(dt_1)}\right)},$$

where $\mathcal{W}$ is the *Lambert Function*, for more details see [4]. When

$$(41)\quad t_1 \in (0, \pi/d) \text{ and } \eta(t_1) = e^{\tilde{C}t_1}(-A + \tilde{C} + Br - 2Bs - d\cot(dt_1)) + d\csc(dt_1) \neq 0,$$

we can conclude that equation (40) has at least two real solutions by Proposition 4. Thus system (38) has at least two real solutions, that is, a PWLS with the configuration (F^v, iN^v) has at least two crossing limit cycles.

In what follows we provide a PWLS with configuration (F^v, iN^v) having two crossing limit cycles. Considering $A = -25/2$, $B = -13/10$, $\tilde{C} = -6/5$, $d = 13/10$, $r = 5$ and $s = 0$, we have that condition (41) is not empty.

$$\eta(t_1) = \frac{1}{10}\left(e^{-6t_1/5}\left(48 - 13\cot\left(\frac{13t_1}{10}\right)\right) + 13\csc\left(\frac{13t_1}{10}\right)\right), \quad t_1 \in \left(0, \frac{10\pi}{13}\right).$$

It is possible verify that in the interval $\left(0, \frac{10\pi}{13}\right)$ the unique critical value is $t_1^* = 1.501574..$, and it is a minimum value of the function $\eta(t_1)$ for $t_1 \in \left(0, \frac{10\pi}{13}\right)$, moreover $\eta(t_1^*) = 2.278475.. > 0$, therefore $\eta(t_1) > 0$ for $t_1 \in \left(0, \frac{10\pi}{13}\right)$. See Figure 5. With these parameters

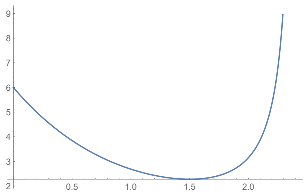

FIGURE 5. The graphic of the function $\eta(t_1)$ in the interval $(0, 10\pi/13)$.

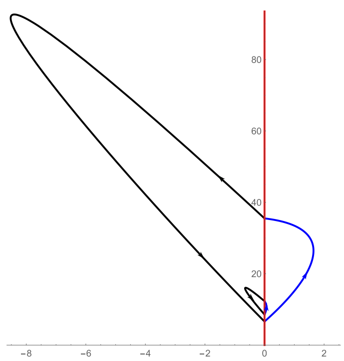

FIGURE 6. Two crossing limit cycles of the discontinuous PWLS (42) with configuration (F^v, iN^v).

we obtain the PWLS formed by

$$(42)\quad X^-(x,y) = \begin{pmatrix} -\frac{25}{2} & -\frac{13}{10} \\ \frac{6469}{65} & \frac{101}{10} \end{pmatrix} X + \begin{pmatrix} \frac{25}{2} \\ -\frac{6469}{65} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} 10 & -1 \\ 25 & 0 \end{pmatrix} X + \begin{pmatrix} 10 \\ 25 \end{pmatrix}.$$

For this PWLS we have that system (38) has two real solutions, namely $(t_1^1, t_2^1, y_0^1) = (1.096629.., 0.143589.., 12.314051..)$; and $(t_1^2, t_2^2, y_0^2) = (2.043521.., 0.588292.., 35.501071..)$. Then the PWLS (42) has two crossing limit cycles Γ_1 and Γ_2 which intersect Σ in $(0, y_0^1) =$

$(0, 12.314051..)$ and $(0, y_1^1) = (0, 2.476508..)$, $(0, y_0^2) = (0, 35.501071..)$ and $(0, y_1^2) = (0, 6.610102..)$, respectively. See Figure 6. □

***Proof of statement** (iii) **of Theorem** 1.* In this case we analyze the maximum number of crossing limit cycles of PWLS (1) when the equilibrium point of linear differential system X^- is a real saddle (S^r). We consider that system X^- is in the canonical form (6), then

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2 - 2b_1\tilde{C}}{\tilde{C}^2 - d^2}, -\frac{A^2b_1 + ABb_2 - 2Ab_1\tilde{C} + b_1\tilde{C}^2 - b_1d^2}{B\tilde{C}^2 - Bd^2} \right), \tag{43}$$

with $0 < \tilde{C}^2 < d^2$ and $B < 0$. This equilibrium point is a $S^r = (-1, -s)$, if $b_1 = A + Bs$, $b_2 = -(A^2 - 2A\tilde{C} + \tilde{C}^2 - d^2 + ABs - 2B\tilde{C}s)/B$. When system X^- is a S^r we have that linear differential system X^+ must be a saddle S^r, then we consider that system X^+ is in the canonical form (2) and the equilibrium point P_+ satisfies (12) with $\beta = 1$, $|r| < 1$. Therefore we obtain the configuration (S^r, S^r). In the following example we provide a PWLS (1) such that the equilibrium points of the linear differential systems X^- and X^+ have the configuration (S^r, S^r) and it has one crossing limit cycle.

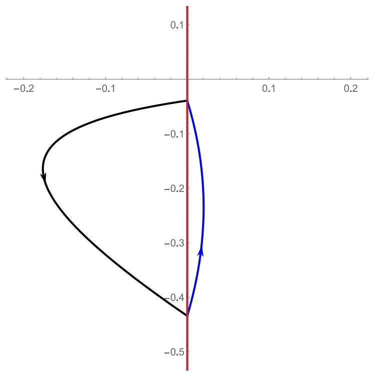

FIGURE 7. The crossing limit cycle of the discontinuous PWLS (44) with configuration (S^r, S^r).

Considering the parameters $A = -1$, $B = -5$, $\tilde{C} = 4/5$, $d = -19/10$, $r = 6/50$ and $s = 0$, we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -1 & -5 \\ -\dfrac{37}{500} & \dfrac{12}{5} \end{pmatrix} X + \begin{pmatrix} -1 \\ -\dfrac{37}{500} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{6}{25} & -1 \\ -\dfrac{616}{625} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{6}{25} \\ \dfrac{616}{625} \end{pmatrix}. \tag{44}$$

For this PWLS we have that system (8) has one real solution, namely $(t_1, t_2, y_0) = (0.754087.., 0.406189.., -0.039307..)$. Then the PWLS (44) has one crossing limit cycle which intersects Σ in $(0, y_0) = (0, -0.039307..)$ and $(0, y_1) = (0, -0.434309..)$. See Figure 7. □

***Proof of statement** (iv) **of Theorem** 1.* In this case we analyze the maximum number of crossing limit cycles of PWLS (1) when the equilibrium point P_- is a virtual diagonalizable node (N^v). We consider that system X^- is in the canonical form (5), then P_- is equal to (43) with $0 > \tilde{C}^2 > d^2$ and $B < 0$. This equilibrium point is a N^v if $b_1 = -A - Bs$, $b_2 = -(-A^2 + 2A\tilde{C} - \tilde{C}^2 + d^2 - ABs + 2B\tilde{C}s)/B$. We consider that system X^+ is in the canonical form (2) and the equilibrium point P_+ can be a diagonalizable node N^v or an improper node iN^v. Then we have two possible configurations (N^v, N^v) and (N^v, iN^v).

Configuration (N^v, N^v)**:** We assume that P_- is a diagonalizable node N^v and that P_+ satisfies (13) with $\beta = 1$ and $|r| > 1$. Then we obtain the configuration (N^v, N^v).

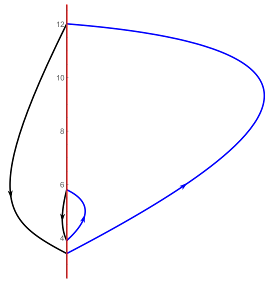

FIGURE 8. Two crossing limit cycles of the discontinuous PWLS (45) with configuration (N^v, N^v).

Considering the parameters $A = -23/10$, $B = -1/2$, $\tilde{C} = -41/10$, $d = 7/2$, $r = 57/25$ and $s = 0$, we obtain the PWLS formed by

$$(45)\quad X^-(x,y) = \begin{pmatrix} -\dfrac{23}{10} & -\dfrac{1}{2} \\ -\dfrac{901}{50} & -\dfrac{59}{10} \end{pmatrix} X + \begin{pmatrix} \dfrac{23}{10} \\ \dfrac{59}{10} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{114}{25} & -1 \\ \dfrac{2624}{625} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{114}{25} \\ \dfrac{2624}{625} \end{pmatrix}.$$

For this PWLS we have that system (8) has two real solutions, namely $(t_1^1, t_2^1, y_0^1) = (0.796618.., 1.259611.., 12.011789..)$; and $(t_1^2, t_2^2, y_0^2) = (0.205065.., 0.425140.., 5.805536..)$. Then the PWLS (45) has two crossing limit cycles which intersect Σ in $(0,\ y_0^1) = (0, 12.011789..)$ and $(0, y_1^1) = (0, 3.420218..)$; and $(0, y_0^2) = (0, 5.805536..)$ and $(0, y_1^2) = (0, 3.906249..)$, respectively. See Figure 8. Therefore we have that PWLS with the configuration (N^v, N^v) have at least two crossing limit cycles.

Configuration (N^v, iN^v)**:** The equilibrium point P_- is a diagonalizable node N^v and P_+ satisfies (13) with $\beta = 0$ and $r \neq 0$. Then we obtain the configuration (N^v, iN^v). The solution of system X^- starting in $(0, y_0) \in \Sigma$ is

(46)
$$\begin{aligned} x^-(t) = {} & \frac{d + e^{\tilde{C}t}(-d\cosh(dt) + (By_0 - (A - \tilde{C}) + Bs))\sinh(dt)}{d}, \\ y^-(t) = {} & s + \frac{e^{\tilde{C}t}}{Bd}(Bd(y_0 - s)\cosh(dt) + (B(-A + \tilde{C})y_0 + (-d^2 + (A - \tilde{C}) \\ & (A - \tilde{C} + Bs)))\sinh(dt)). \end{aligned}$$

By (27) and (46) we obtain that system (8) is equivalent to system

(47)
$$\begin{aligned} e_1 :\ & d + e^{\tilde{C}t_1}(-d\cosh(dt_1) + (By_0 - (A - \tilde{C} + Bs))\sinh(dt_1)) = 0, \\ e_2 :\ & -1 + e^{-rt_2}(1 + t_2(y_0 - r + s)) = 0, \\ e_3 :\ & -2Bds + Bde^{-rt_2}(y_0 + rt_2y_0 + (s + r(-r + s)t_2)) - e^{\tilde{C}t_1}(Bd(y_0 - s)\cosh(dt_1) \\ & +(B(-A + \tilde{C})y_0 + (-d^2 + (A - \tilde{C})(A - \tilde{C} + Bs)))\sinh(dt_1)) = 0. \end{aligned}$$

From equation e_1 we get $y_0 = (A - \tilde{C} + Bs + d\coth(dt_1) - de^{-\tilde{C}t_1}\mathrm{csch}(dt_1))/B$, and from e_2 we get the expression (39) for t_2, then substituting y_0 and t_2 in e_3 we obtain that

$$e_3 = \frac{1}{B}\Big((-A + \tilde{C} + Br - 2Bs)\tilde{f}_0(t_1) + d\tilde{f}_3(t_1) - rB\tilde{f}_4(t_1)\Big) = 0,$$

where $\tilde{f}_0(t_1) = 1$, $\tilde{f}_3(t_1) = \coth(dt_1) - e^{\tilde{C}t_1}\mathrm{csch}(dt_1)$ and

$$\tilde{f}_4(t_1)=\frac{1}{\mathcal{W}\left(\dfrac{Bre^{\tilde{C}t_1+\dfrac{Bre^{\tilde{C}t_1}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\coth(dt_1))+d\operatorname{csch}(dt_1)}}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\coth(dt_1))+d\operatorname{csch}(dt_1)}\right)}.$$

If

$$t_1\in(0,\infty)\quad\text{and}\quad \tilde{\eta}(t_1)=e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\coth(dt_1))+d\operatorname{csch}(dt_1)\neq 0, \tag{48}$$

by Proposition 4 we can conclude that a system (47) has at least two real solutions therefore a PWLS with the configuration (N^v, iN^v) has at least two crossing limit cycles. In what follows we provide a PWLS with configuration (N^v, iN^v) and having two crossing limit cycles.

Considering $A=-23/10$, $B=-8/5$, $\tilde{C}=-24/5$, $d=37/10$, $r=3/5$ and $s=0$, we have that

$$\tilde{\eta}(t_1)=e^{-24t_1/5}\left(-\frac{173}{50}-\frac{37}{10}\coth\left(\frac{37t_1}{10}\right)\right)+\frac{37}{10}\operatorname{csch}\left(\frac{37t_1}{10}\right).$$

Substituting

$$\coth(x)=\frac{e^x+e^{-x}}{e^x-e^{-x}},\quad\text{and}\quad \operatorname{csch}(x)=\frac{2}{e^x-e^{-x}},$$

in the equation $\tilde{\eta}(t_1)$ we obtain that

$$\tilde{\eta}(t_1)=\frac{e^{37t_1/10}(-370+12e^{-17t_1/2}+358e^{-11t_1/10})}{50(1-e^{37t_1/5})}>0,\quad\text{for}\quad t_1>0.$$

Therefore the condition (48) is satisfied. See the graphic of this function in Figure 9. Moreover we obtain the PWLS formed by

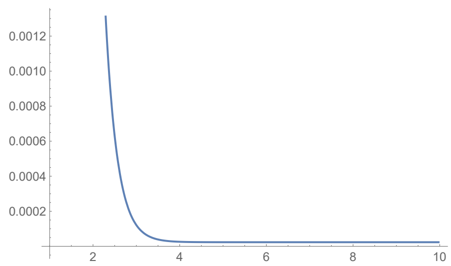

FIGURE 9. The graphic of the function (48) for $t_1>0$.

$$X^-(x,y)=\begin{pmatrix}-\dfrac{23}{10} & -\dfrac{8}{5}\\ -\dfrac{93}{20} & -\dfrac{73}{10}\end{pmatrix}X+\begin{pmatrix}\dfrac{23}{10}\\ \dfrac{93}{20}\end{pmatrix},\quad X^+(x,y)=\begin{pmatrix}\dfrac{6}{5} & -1\\ \dfrac{9}{25} & 0\end{pmatrix}X+\begin{pmatrix}\dfrac{6}{5}\\ \dfrac{9}{25}\end{pmatrix}. \tag{49}$$

For this PWLS we have that system (47) has two real solutions, namely $(t_1^1,t_2^1,y_0^1)=(0.564675..,\ 5.217342..,\ 4.794330..)$; and $(t_1^2,t_2^2,y_0^2)=(0.763740..,\ 6.119198..,\ 6.860880..)$. Then the PWLS (49) has two crossing limit cycles which intersect Σ in $(0,y_0^1)=(0,4.794330..)$ and $(0,y_1^1)=(0,0.783292..)$ and $(0,y_0^2)=(0,6.860880..)$ and $(0,y_1^2)=(0,0.759263..)$, respectively. See Figure 10. □

Proof of statement (v) ***of Theorem*** 1. In this case we analyze the maximum number of crossing limit cycles of PWLS (1) when the equilibrium point of linear differential system

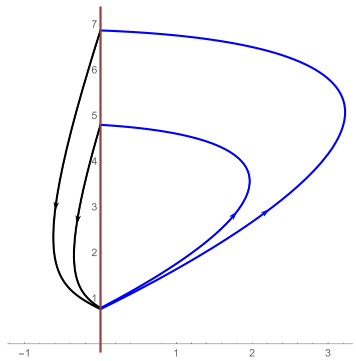

FIGURE 10. Two crossing limit cycles of the discontinuous PWLS (49) with configuration (N^v, iN^v).

X^- is a virtual improper node (iN^v). We consider that system X^- is in the canonical form (5) with $d = 0$ and $B < 0$, then equilibrium point P_- is

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2 - 2b_1\tilde{C}}{\tilde{C}^2}, -\frac{A^2b_1 + ABb_2 - 2Ab_1\tilde{C} + b_1\tilde{C}^2}{B\tilde{C}^2} \right).$$

This equilibrium point is a virtual improper node iN^v if $P_- = (1, s)$, then $b_1 = -A - Bs$, $b_2 = -(-A^2 + 2A\tilde{C} - \tilde{C}^2 - ABs + 2B\tilde{C}s)/B$. With these condition the solution of system X^- starting in $(0, y_0) \in \Sigma$ is

$$\text{(50)} \qquad \begin{aligned} x^-(t) &= 1 + e^{\tilde{C}t}(Bty_0 - (1 + (A - \tilde{C} + Bs)t)), \\ y^-(t) &= \frac{Bs + (A - \tilde{C})^2 e^{\tilde{C}t} t + Be^{\tilde{C}t}(1 - At + \tilde{C}t)(y_0 - s)}{B}. \end{aligned}$$

We consider that linear differential system X^+ is an improper node iN^v, then we consider that system X^+ is in the canonical form (2) and the equilibrium point P_+ satisfies (13) with $\beta = 0$ and $r \neq 0$. Therefore we obtain the configuration (iN^v, iN^v).

Now considering t_1 and t_2 as in Proposition 6 and by equations (50) and (27), system (8) is equivalent to

$$\text{(51)} \qquad \begin{aligned} e_1 &: \ 1 + e^{\tilde{C}t_1}(Bt_1y_0 - (1 + (A - \tilde{C} + Bs)t_1)) = 0, \\ e_2 &: \ -1 + e^{-rt_2}(1 + t_2(y_0 - r + s)) = 0, \\ e_3 &: \ 2Bs + (A - \tilde{C})^2 e^{\tilde{C}t_1} t_1 + Be^{\tilde{C}t_1}(1 - At_1 + \tilde{C}t_1)(y_0 - s) \\ &\quad\ -Be^{-rt_2}(y_0 + rt_2y_0 + (s + r(-r + s)t_2)) = 0. \end{aligned}$$

By the equation e_1, we get $y_0 = (1 - e^{-\tilde{C}t_1} + (A - \tilde{C} + Bs)t_1)/Bt_1$, and from e_2 we obtain the expression (39) for t_2. Substituting these expressions in e_3, we get

$$e_3 = \frac{1}{B}\Big((-A + \tilde{C} + Br - 2Bs)\tilde{f}_0(t_1) + \tilde{f}_5(t_1) - Br\tilde{f}_6(t_1)\Big) = 0,$$

where $\tilde{f}_0(t_1) = 1$, $\tilde{f}_5(t_1) = \dfrac{1 - e^{\tilde{C}t_1}}{t_1}$ and

$$\tilde{f}_6(t_1) = \frac{1}{\mathcal{W}\left(\dfrac{Brt_1 e^{t_1\left(\tilde{C}+\dfrac{Bre^{\tilde{C}t_1}}{1+e^{\tilde{C}t_1}(-1+(-A+\tilde{C}+Br-2Bs)t_1)}\right)}}{1+e^{\tilde{C}t_1}(-1+(-A+\tilde{C}+Br-2Bs)t_1)}\right)}.$$

Therefore by Proposition 4 we can conclude that system (51) has at least two real solutions for

$$t_1 \in (0,\infty) \quad \text{and} \quad \bar{\eta}(t_1) = 1 + e^{\tilde{C}t_1}(-1 + (-A + \tilde{C} + Br - 2Bs)t_1) \neq 0. \tag{52}$$

Due to symmetry we have that if (t_1, t_2, y_0) is a real solution of system (51) then $(-t_1, -t_2, y_1)$ also it is a real solution of system (51), where $y_1 = y^-(t_1) = y^+(-t_2)$, we observed that the real solutions (t_1, t_2, y_0) and $(-t_1, -t_2, y_1)$ of system (51) provide the same crossing limit cycle of PWLS with the configuration (iN^v, iN^v). Therefore a PWLS with the configuration (iN^v, iN^v) has at least one crossing limit cycle.

In what follows we provide a example of a PWLS with the configuration (iN^v, iN^v) having one crossing limit cycle.

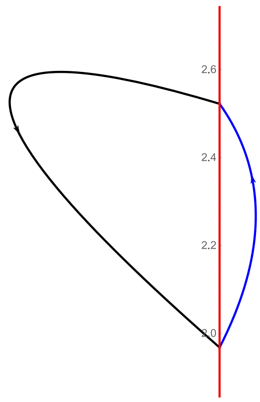

FIGURE 11. One crossing limit cycle of the discontinuous PWLS (53) with configuration (iN^v, iN^v).

Considering $A = -6$, $B = -14/5$, $\tilde{C} = -6/5$, $r = 11/10$ and $s = 0$, we have that

$$\bar{\eta}(t_1) = 1 + e^{-6t_1/5}\left(-1 + \frac{43}{25}t_1\right) > 0, \quad \text{for} \quad t_1 > 0.$$

Therefore the condition (52) is satisfied.

Moreover we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -6 & -\dfrac{14}{5} \\ \dfrac{288}{35} & \dfrac{126}{35} \end{pmatrix} X + \begin{pmatrix} 6 \\ -\dfrac{288}{35} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{11}{5} & -1 \\ \dfrac{121}{100} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{11}{5} \\ \dfrac{121}{100} \end{pmatrix}. \tag{53}$$

For this PWLS we have that system (47) has two real solutions, namely $(t_1, t_2, y_0) = (0.964798.., 0.448780.., 2.522296..)$ and $(-t_1, -t_2, y_1) = (-0.964798.., -0.448780.., 1.968154..)$, which provide one crossing limit cycle such that intersects Σ in $(0, y_0) = (0, 2.522296..)$ and $(0, y_1) = (0, 1.968154..)$. See Figure 11.

The proof of Proposition 1 is provide by the following examples, where we prove that the upper bound provided in statement (i) of Theorem 1 is reached in each case.

Example 1. *We consider PWLS* (1) *with the configuration* (C^r, F^r) *formed by the linear differential systems* (10) *and* (14)*, with* $A = -2$, $B = -8/10$, $d = 7/10$, $r = -2/10$, $k = 1$ *and* $s = 0$ *then we obtain that*

$$(54)\quad X^-(x,y) = \begin{pmatrix} -2 & -\dfrac{4}{5} \\ \dfrac{449}{80} & 2 \end{pmatrix} X + \begin{pmatrix} -2 \\ \dfrac{449}{80} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} -\dfrac{2}{5} & -1 \\ \dfrac{26}{25} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{2}{5} \\ -\dfrac{26}{25} \end{pmatrix}.$$

For this PWLS we have that closing equations (15) *are*

$$(55)\quad \begin{aligned} &-1 + \cos\left(\frac{7t_1}{10}\right) - \frac{4}{7}(5 + 2y_0)\sin\left(\frac{7t_1}{10}\right) = 0,\\ &1 + e^{t_2/5}\left(-\cos(t_2) + \left(-\frac{1}{5} + y_0\right)\sin(t_2)\right) = 0,\\ &-y_0\cos\left(\frac{7t_1}{10}\right) - \frac{1}{56}(449 + 160y_0)\sin\left(\frac{7t_1}{10}\right)\\ &+\frac{1}{25}e^{t_2/5}(25y_0\cos(t_2) + (26 - 5y_0)\sin(t_2)) = 0. \end{aligned}$$

Taking into account that $t_1t_2 > 0$ *and that* $t_1, t_2 \in (0, 2\pi)$ *it is possible verify computationally that the system* (55) *has two real solutions, namely* $(t_1^1, t_2^1, y_0^1) = (4.796799..,$ $3.418539.., 5.564042..)$ *and* $(t_1^2, t_2^2, y_0^2) = (5.859455.., 5.731792.., -0.819335..)$. *Nevertheless the orbit of linear differential system* X^+ *starting at the point* $(x, y) = (0, y_0^2) = (0, -0.819335..)$ *and with flight time* $t_2^2 = 5.731792..$ *it is such that intersects the region* Σ^- *which cannot happen to obtain a crossing limit cycle of PWLS* (54)*, therefore we have the unique real solution that generates one crossing limit cycle* Γ_1 *of the PWLS* (54) *is* $(t_1^1, t_2^1, y_0^1) = (4.796799.., 3.418539.., 5.564042..)$*, and that crossing limit cycle starts at the point* $(0, y_0^1) = (0, 5.564042..)$*, enters in the half-plane* Σ^- *and after a time* $t_1^1 = 4.796799..$ *reaches the discontinuity line* Σ *at the point* $(0, y_1^1) = (0, -10.564042..)$*, enters in the half-plane* Σ^+ *and after a time* $t_2^1 = 3.418539..$ *reaches the point* $(0, y_0^1)$. *See Figure* 12.

Now we analyze the stability of the crossing limit cycle Γ_1. *We consider the PWLS* (54) *and we analyze the flow of PWLS around of the crossing limit cycle* Γ_1 *which intersects the discontinuity straight line* Σ *at the points* $y_0 = 5.564042..$ *and* $y_1 = -10.564042...$

We consider a point $W_0 \in \Sigma$ *and within the region limited by the crossing limit cycle* Γ_1*, this is,* $W_0 = (0, w_0)$ *with* $-10.564042.. < w_0 < 5.564042...$ *For example we consider that* $w_0 = 5$*, then the solution of linear differential system* X^- *in* (54) *starting at the point* $W_0 = (0, 5) \in \Sigma$ *is*

$$x^-(t) = -1 + \cos\left(\frac{7t}{10}\right) - \frac{60}{7}\sin\left(\frac{7t}{10}\right), \quad y^-(t) = 5\cos\left(\frac{7t}{10}\right) + \frac{1249}{56}\sin\left(\frac{7t}{10}\right),$$

and the flight time in the region Σ^- *is*

$$t^- = \frac{10}{7}\left(-\pi + \arctan\left(\frac{840}{3551}\right) + 2\pi\right),$$

then the intersection point with Σ *is* $W_1 = (0, w_1) = (0, y^-(t^-))$*, where* $y^-(t^-) = -10$. *Now the solution of linear differential system* X^+ *in* (54) *starting at the point* $W_1 = (0, -10)$ *is*

$$x^+(t) = 1 + \frac{e^{-t/5}}{5}\left(-5\cos(t) + 51\sin(t)\right), \quad y^+(t) = -\frac{2}{25}e^{-t/5}(125\cos(t) + 38\sin(t)),$$

the flight time in the region Σ^+ *is* $t^+ = 3.434483..$ *and the intersection point of this orbit with the discontinuity straight line is the point* $W_2 = (0, w_2) = (0, y^+(t^+)) = (0, 5.258689..)$*, then* $5 = w_0 < w_2 = 5.258689$. *Therefore we obtain that the flow of PWLS* (54) *spirals in*

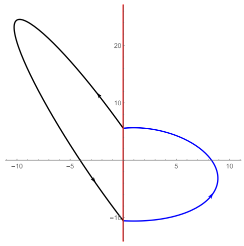

FIGURE 12. The crossing limit cycle of the discontinuous PWLS (54) with configuration (C^r, F^r).

the counterclockwise outward for points $W_0 = (0, w_0)$ *with* $-10.564042.. < w_0 < 5.564042..$. *Now we consider a point on* Σ *and outside the region limited by* Γ_1*, namely* $Z_0 = (0, z_0)$ *with* $z_0 > y_0$*. We consider* $Z_0 = (0, 6)$ *and similarly to above case we determine the solution* $(x^-(t), y^-(t))$ *of linear differential system* X^- *in* (54) *starting at the point* $Z_0 = (0, 6) \in \Sigma$ *and we get the flight time in the region* Σ^-*, namely* $T^- = 10/7(-\pi + \arctan(952/4575) + 2\pi)$*, and the intersection point of this orbit with* Σ *is* $Z_1 = (0, z_1) = (0, y^-(T^-)) = (0, -11)$*. We determine the solution* $(x^+(t), y^+(t))$ *of linear differential system* X^+ *in* (54) *starting at the point* $Z_1 = (0, -11) \in \Sigma$ *and we get the flight time in the region* Σ^+*,* $T^+ = 3.407359..$ *and finally we obtain the intersection point of this orbit with* Σ*,* $Z_2 = (0, z_2) = (0, y^+(T^+)) = (0, 5.799713..)$*, then* $6 = z_0 > z_2 = 5.799713..$. *Therefore obtain that the flow of PWLS* (54) *spirals in the counterclockwise inward for points* $Z_0 = (0, z_0)$*with* $z_0 > y_0$*. Therefore we can conclude that the crossing limit cycle* Γ_1 *is a crossing limit cycle stable.* □

Example 2. *We consider PWLS* (1) *with the configuration* (C^r, S^r) *formed by the linear differential systems* (10) *and* (18)*, with* $A = -7/2$*,* $B = -8/3$*,* $r = 79/100$*,* $d = -28/10$*,* $k = 1$ *and* $s = 0$*, then we obtain the piecewise linear differential system formed by*

(56)

$$X^-(x, y) = \begin{pmatrix} -\dfrac{7}{2} & -\dfrac{8}{3} \\ \dfrac{6027}{800} & \dfrac{7}{2} \end{pmatrix} X + \begin{pmatrix} -\dfrac{7}{2} \\ \dfrac{6027}{800} \end{pmatrix}, \quad X^+(x, y) = \begin{pmatrix} \dfrac{79}{50} & -1 \\ -\dfrac{3759}{10000} & 0 \end{pmatrix} X + \begin{pmatrix} -\dfrac{79}{50} \\ \dfrac{3759}{10000} \end{pmatrix}.$$

For this PWLS it is possible verify computationally that the closing equations (19) *have two real solutions for* $t_1, t_2 \in (0, 2\pi)$*, namely* $(t_1^1, t_2^1, y_0^1) = (1.941361.., 3.063722.., -0.838949..)$ *and* $(t_1^2, t_2^2, y_0^2) = (4.185356.., 3.063722.., -0.838949..)$*. Nevertheless we have that the orbit of linear differential system* X^- *started at point* $(0, y_0^2) = (0, -0.838949..)$ *and with flight time* $t_1^2 = 4.185356..$ *it intersects the region* Σ^- *which cannot happen to obtain a crossing limit cycle of PWLS* (56)*, therefore we have that the unique real solution that generates one crossing limit cycle of the PWLS* (56) *is* $(t_1^1, t_2^1, y_0^1) = (1.941361.., 3.063722.., -0.838949..)$*, and that crossing limit cycle* Γ *starts at the point* $(0, y_0^1) = (0, -0.838949..)$*, enters in the half-plane* Σ^- *and after a time* $t_1^1 = 1.941361..$ *reaches the discontinuity line* Σ *at the point* $(0, y_1^1) = (0, -1.786050..)$*, enters in the half-plane* Σ^+ *and after a time* $t_2^1 = 3.063722..$ *reaches the point* $(0, y_0^1)$*.*

Now we analyze the stability of the crossing limit cycle Γ*. We consider the PWLS* (56) *and we analyze the flow of PWLS around of the crossing limit cycle* Γ *which intersects the discontinuity straight line* Σ *at the points* $y_0 = -0.838949..$ *and* $y_1 = -1.786050..$.

We consider a point $W_0 \in \Sigma$ *and within the region limited by the crossing limit cycle* Γ*, this is,* $W_0 = (0, w_0)$ *with* $-1.786050.. < w_0 < -0.838949..$. *For example we consider that* $w_0 = -9/10$*, then the solution of linear differential system* X^- *in* (56) *starting at the point*

$W_0 = (0, -9/10) \in \Sigma$ *is*

$$x^-(t) = -1 + \cos\left(\frac{14t}{5}\right) - \frac{11}{28}\sin\left(\frac{14t}{5}\right),$$
$$y^-(t) = \frac{3}{320}\left(-96\cos\left(\frac{14t}{5}\right) + 167\sin\left(\frac{14t}{5}\right)\right),$$

and the flight time in the region Σ^- *is*

$$t^- = \frac{5}{14}\left(-\arctan\left(\frac{616}{663}\right) + 2\pi\right),$$

then the intersection point with Σ *is* $W_1 = (0, w_1) = (0, y^-(t^-))$*, where* $y^-(t^-) = -69/40$*. Now the solution of linear differential system* X^+ *in* (56) *starting at the point* $W_1 = (0, -69/40)$ *is*

$$x^+(t) = 1 + \frac{e^{-21t/100}}{400}\left(-387 - 13e^{2t}\right), \quad y^+(t) = \frac{3e^{-21t/100}(-23091 + 91e^{2t})}{40000},$$

the flight time in the region Σ^+ *is* $t^+ = 1.097023..$ *and the intersection point of this orbit with the discontinuity straight line is the point* $W_2 = (0, w_2) = (0, y^+(t^+)) = (0, -1.326846..)$*, then* $-9/10 = w_0 > w_2 = -1.326846...$ *Therefore we obtain that the flow of PWLS* (56) *spirals in the counterclockwise inward for points* $W_0 = (0, w_0)$ *with* $-1.786050.. < w_0 < -0.838949...$ *Now we consider a point on* Σ *and outside the region limited by* Γ*, namely* $Z_0 = (0, z_0)$ *with* $z_0 > y_0$*. We consider* $Z_0 = (0, -209/250)$ *and similarly to above case we determine the solution* $(x^-(t), y^-(t))$ *of linear differential system* X^- *in* (56) *starting at the point* $Z_0 = (0, -209/250) \in \Sigma$ *and we get the flight time in the region* Σ^-*, namely* $T^- = -1.939696..$*, and the intersection point of this orbit with* Σ *is* $Z_1 = (0, z_1) = (0, y^-(T^-)) = (0, -1789/1000)$*. We determine the solution* $(x^+(t), y^+(t))$ *of linear differential system* X^+ *in* (54) *starting at the point* $Z_1 = (0, -1789/1000) \in \Sigma$ *and we get the flight time in the region* Σ^+*,* $T^+ = 3.923945..$ *and finally we obtain the intersection point of this orbit with* Σ*,* $Z_2 = (0, z_2) = (0, y^+(T^+)) = (0, -0.666883)$*, then* $-209/250 = z_0 > z_2 = -0.666883..$ *Therefore obtain that the flow of PWLS* (54) *spirals in the counterclockwise outward for points* $Z_0 = (0, z_0)$*with* $z_0 > y_0$*. Therefore we can conclude that the crossing limit cycle* Γ *is an unstable crossing limit cycle. See Figure* 13. □

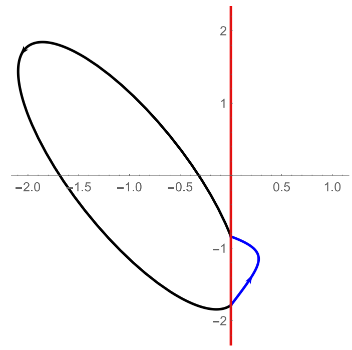

FIGURE 13. The crossing limit cycle of the discontinuous PWLS (56) system with configuration (C^r, S^r).

Example 3. *We consider PWLS* (1) *with the configuration* (C^v, F^v) *formed by the linear differential systems* (21) *and* (22)*, with* $A = -3$*,* $B = -1$*,* $r = 4/5$*,* $d = -4$*,* $k = 1$ *and* $s = 0$*, then we obtain the PWLS formed by*

$$X^-(x,y) = \begin{pmatrix} -3 & -1 \\ 25 & 3 \end{pmatrix} X + \begin{pmatrix} 3 \\ -25 \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \frac{8}{5} & -1 \\ \frac{41}{25} & 0 \end{pmatrix} X + \begin{pmatrix} \frac{8}{5} \\ \frac{41}{25} \end{pmatrix}. \tag{57}$$

For this PWLS it is possible verify computationally that closing equations (23) *have four real solution for* $t_1, t_2 \in (0, 2\pi)$*, namely* (t_1^1, t_2^1, y_0^1)=(0.299957.., 1.862980.., 5.736049..)*,* (t_1^2, t_2^2, y_0^2) = (1.870753.., 1.862980.., 5.736049..)*,* (t_1^3, t_2^3, y_0^3)=(3.441550.., 1.862980.., 5.736049..)*,* (t_1^4, t_2^4, y_0^4)=(5.012346.., 1.862980.., 5.736049..)*. Nevertheless the orbit of the linear differential system* X^- *started at the point* y_0^i *and with flight time* t_1^i *is such that intersects the region* Σ^+ *for* $i = 2, 3, 4$ *which cannot happen to obtain a crossing limit cycle of PWLS* (56)*, therefore we have that the unique real solution that generates one crossing limit cycle* Γ *of the PWLS* (57) *is* $(t_1^1, t_2^1, y_0^1) = (0.299957.., 1.862980.., 5.736049..)$ *which intersects* Σ *in* $(0, y_0^1) = (0, 5.736049..)$ *and* $(0, y_1^1) = (0, 0.263950..)$*. Analogously to above case* (C^r, S^r)*, it is possible verify numerically that* Γ *is an unstable crossing limit cycle. See Figure* 14.

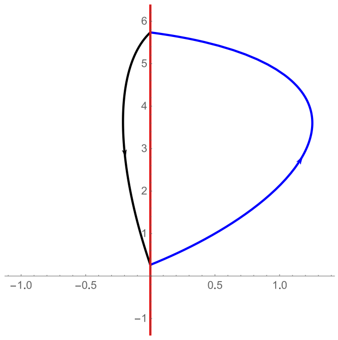

FIGURE 14. The crossing limit cycle of the discontinuous PWLS (57) with configuration (C^v, F^v).

Example 4. *We consider PWLS* (1) *with the configuration* (C^v, N^v) *formed by the linear differential systems* (21) *and* (24)*, with* $A = -5,\ B = -18/10,\ r = 13/10,\ d = -3/2$ *and* $s = 0$*, we obtain the PWLS formed by*

$$(58)\quad X^-(x,y) = \begin{pmatrix} -5 & -\dfrac{9}{5} \\ \dfrac{545}{36} & 5 \end{pmatrix} X + \begin{pmatrix} 5 \\ -\dfrac{545}{36} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{13}{5} & -1 \\ \dfrac{69}{100} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{13}{5} \\ \dfrac{69}{100} \end{pmatrix}.$$

For this PWLS it is possible verify computationally that closing equations (25) *have two real solutions with* $t_1, t_2 \in (0, 2\pi)$*, namely* $(t_1^1, t_2^1, y_0^1) = (0.608026.., 1.109920.., 3.186528..)$*,* $(t_1^2, t_2^2, y_0^2) = (4.796816.., 1.109920.., 3.186528..)$*. But the orbit of the linear differential system* X^- *intersect the region* Σ^+ *when started at the point* $(0, y_0^2) = (0, 3.186528..)$ *with flight time* $t_1^2 = 4.796816..$ *therefore this real solution cannot generates a crossing limit cycle of PWLS* (58) *and we only have one crossing limit cycle* Γ *which intersects* Σ *in* $(0, y_0^1) = (0, 3.186528..)$ *and* $(0, y_1^1) = (0, 2.369026..)$ *with flight times* $t_1^1 = 0.608026..$ *and* $t_2^1 = 1.109920..$ *in the regions* Σ^- *and* Σ^+*, respectively. Analogously to above cases, it is possible verify numerically that* Γ *is an unstable crossing limit cycle. See Figure* 15.

Example 5. *We consider PWLS* (1) *formed by the linear differential systems* (21) *and* (26)*, with* $A = -1/2,\ B = -1/10,\ r = 17/10,\ d = -4/10$ *and* $s = 0$*, we obtain the PWLS formed by*

$$(59)\quad X^-(x,y) = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{1}{10} \\ \dfrac{41}{10} & \dfrac{1}{2} \end{pmatrix} X + \begin{pmatrix} \dfrac{1}{2} \\ -\dfrac{41}{10} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{17}{5} & -1 \\ \dfrac{289}{100} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{17}{5} \\ \dfrac{289}{100} \end{pmatrix}.$$

For this PWLS it is possible verify computationally that closing equations (28) *have one real solution, namely* $(t_1, t_2, y_0) = (2.877804.., 1.249557.., 7.595368..)$*, then the PWLS* (59) *has*

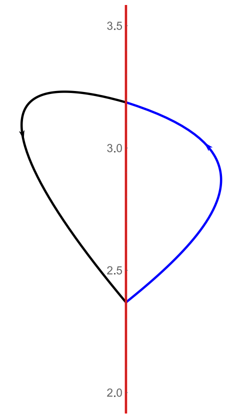

FIGURE 15. The crossing limit cycle of the discontinuous PWLS (58) with configuration (C^v, N^v).

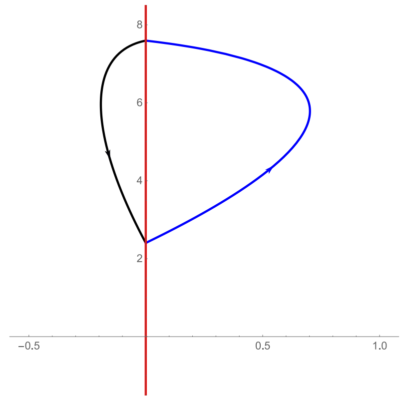

FIGURE 16. The crossing limit cycle of the discontinuous PWLS (59) with configuration (C^v, iN^v).

one crossing limit cycle Γ *which intersects* Σ *in* $(0, 7.595368..)$ *and* $(0, 2.404631..)$. *Analogously to above cases, it is possible verify numerically that* Γ *is an unstable crossing limit cycle. See Figure* 16.

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