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Criteria on the existence of limit cycles in planar polynomial differential systems

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Abstract

We summarize known criteria for the non-existence, existence and on the number of limit cycles of autonomous real planar polynomial differential systems, and also provide new results. We give examples of systems which realize the maximum number of limit cycles provided by each criterion. In particular we consider the class of differential systems of the form $\dot{x} = P_n(x, y) + P_m(x, y)$, $\dot{y} = Q_n(x, y) + Q_m(x, y)$, where n, m are natural numbers with $m > n \geq 1$ and (P_i, Q_i) for $i = n, m$, are quasi-homogeneous vector fields.

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1. Introduction and statement of the main results

Poincaré in [34] defined the concept of limit cycle of a planar differential system and started to study it intensively. Later on the limit cycles were studied by van der Pol [39], Liénard [24], Andronov [3], . . ., and they gave account of how difficult is their control. In fact one of the main problems in the qualitative theory of real planar differential systems

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is to control the existence, non-existence or uniqueness of limit cycles for a given class of differential systems.

In the case of autonomous real planar polynomial differential systems it is known, from the Poincaré–Bendixson Theorem, that in the region bounded by a limit cycle there is at least one equilibrium point of the system. By a translation, if necessary, one can assume without loss of generality that this equilibrium point is the origin of coordinates. We consider an autonomous real planar polynomial differential system of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where x, y are real variables, the dot denotes derivative with respect to an independent real variable and $P(x, y)$ and $Q(x, y)$ are real polynomials such that $P(0, 0) = 0$ and $Q(0, 0) = 0$. A *limit cycle* of the differential system (1) is a periodic solution which is isolated in the set of all periodic solutions of system (1). A limit cycle $(x(t), y(t))$ of system (1) with period T is *hyperbolic* if

$$I = \int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) (x(t), y(t)) dt \neq 0.$$

The hyperbolic limit cycles are either stable if $I < 0$, or unstable if $I > 0$, see more details in [13,32]. Given a limit cycle we can consider the *return map* associated to it, that is, we take a transversal section through a point W of the limit cycle and, for each point in this transversal section and in a neighborhood of the point W we follow its trajectory and we make correspond the first cut with this transversal section. This map is called the first return map and we denote it by h. It is clear that $h(W) = W$ and, thus, W is a zero of the displacement map $\varDelta := h - \text{id}$, where id denotes the identity map. The *multiplicity of a limit cycle* is the multiplicity of W as a zero of the displacement map $\varDelta$. It can be shown that the definition of multiplicity of a limit cycle is independent of the choice of the point W and the transversal section, see for instance [32].

A classical tool for studying the limit cycles that surround the origin of coordinates is to write the differential system in polar coordinates. That is, we consider the change to polar coordinates $x = r \cos\theta$, $y = r \sin\theta$, and the system becomes

$$\dot{r} = R(r, \theta), \quad \dot{\theta} = \varTheta(r, \theta), \tag{2}$$

where $R(r, \theta)$ and $\varTheta(r, \theta)$ are polynomials in r with coefficients trigonometric polynomials in θ and are such that $R(0, \theta) \equiv 0$. System (2) is considered in the region $\{(r, \theta) : r \geq 0\}$. We assume that the region

$$\mathcal{R} = \{(r, \theta) : \varTheta(r, \theta) > 0\} \tag{3}$$

is not empty. In most of our results, $\mathcal{R}$ contains the origin of coordinates $r = 0$. The differential systems (1) or (2) in the region $\mathcal{R}$ are equivalent to the differential equation

$$\frac{dr}{d\theta} = S(r, \theta) = \frac{R(r, \theta)}{\varTheta(r, \theta)}, \tag{4}$$

also defined in $\mathcal{R}$.

In some cases we can apply the Liouville–Cherkas transformation, that we describe below, and transform the system in polar coordinates to an Abel differential equation. There are many papers which provide criteria for the non-existence, existence and on the number of limit cycles for the system in polar coordinates or for the corresponding Abel differential equation. However most of these papers do not give examples of planar polynomial differential systems which realize these criteria, that is, planar polynomial differential systems for which the corresponding system in polar coordinates or in the Abel differential equation satisfies the conditions of the criterion and exhibits the maximum number of limit cycles given by the criterion. In this paper we summarize almost all these criteria (to the best of our knowledge), and provide examples of autonomous real planar polynomial differential systems which realize these criteria.

First we will show a qualitative property of the solutions of system (1), equivalently of system (2), which will be useful for the statement of our results. Let $\mathcal{I}$ denote the positive x-axis: $r \geq 0$, $\theta = 0$. For $x \in \mathcal{I} \cap \mathcal{R}$, we write $\bar{r}(\theta, x)$ for the solution of the differential equation (4) satisfying $\bar{r}(0, r_0) = r_0$.

Lemma 1. *If the region $\mathcal{R}$ defined in (3) contains the origin of coordinates, then the function $\bar{r}(2\pi, r_0)$ is defined for all r_0 in a neighborhood of $r_0 = 0$.*

Lemma 1 is proved in Section 2.

Remark 2. The hypothesis that the region (3) is not empty and contains the origin cannot be easily avoided in order to ensure that the function $\bar{r}(2\pi, r_0)$ is defined for all r_0 in a neighborhood of $r_0 = 0$, as the following example shows. Consider the planar differential system (which is not polynomial)

$$\dot{x} = x\, e^{x^2+y^2} - y(x^2 + y^2), \quad \dot{y} = y\, e^{x^2+y^2} + x(x^2 + y^2).$$

In polar coordinates this system writes as

$$\dot{r} = r e^{r^2}, \quad \dot{\theta} = r^2.$$

Note that this system satisfies that the region $\mathcal{R}$ defined in (3) is the whole plane except the origin of coordinates $r = 0$. The differential equation (4) writes as

$$\frac{dr}{d\theta} = \frac{e^{r^2}}{r},$$

which can be integrated and has the first integral $H(r, \theta) = e^{-r^2} + 2\theta$. The function $\bar{r}(\theta, r_0)$ can be computed and it is

$$\bar{r}(\theta, r_0) = \sqrt{r_0^2 - \ln(1 - 2e^{r_0^2}\theta)}.$$

Given $r_0 > 0$, this function is only defined for θ in the interval

$$\left(\frac{1}{2e^{r_0^2}} - \frac{1}{2}, \frac{1}{2e^{r_0^2}} \right),$$

which does not contain the value 2π for any $r_0 > 0$.

All the known criteria, to the best of our knowledge, on non-existence, existence and on the number of the limit cycles for the differential systems (1), using the associated differential equation (4) come from the seminal paper of Lloyd [29].

The next result is a direct consequence of the results appearing in [29]. Let O denote the origin of coordinates, that is $O = (0, 0)$ in cartesian coordinates and $r = 0$ in polar coordinates. As before, $\mathcal{I}$ denotes the positive x-axis: $r \geq 0$, $\theta = 0$, and for $x \in \mathcal{I} \cap \mathcal{R}$, we write $\bar{r}(\theta, x)$ for the solution of (4) satisfying $\bar{r}(0, r_0) = r_0$.

Theorem 3. *For the differential system (1) the following statements hold.*

(i) *Let U be a simply connected region containing the origin. If $\partial S/\partial r \neq 0$ in $U \setminus \{O\}$, then system (1) has no limit cycles in the region $\mathcal{R} \cap U$.*
(ii) *Let $\mathcal{A}$ be an annular region which encircles the origin. If $\partial S/\partial r \neq 0$ in $\mathcal{A}$, then system (1) has at most one limit cycle in the region $\mathcal{R} \cap \mathcal{A}$.*
(iii) *Let U be a simply connected region containing the origin and suppose that $\bar{r}(2\pi, r_0)$ is defined for some $r_0 \neq 0$. If $\partial^2 S/\partial r^2 \neq 0$ in $U \setminus \{O\}$, then system (1) has at most one limit cycle in the region $\mathcal{R} \cap U$.*
(iv) *Let $\mathcal{A}$ be an annular region which encircles the origin. If $\partial^2 S/\partial r^2 \neq 0$ in $\mathcal{A}$, then system (1) has at most two limit cycles in the region $\mathcal{R} \cap \mathcal{A}$.*
(v) *Let U be a simply connected region containing the origin and suppose that $\bar{r}(2\pi, r_0)$ is defined for some $r_0 \neq 0$. If $\partial^3 S/\partial r^3 > 0$ in $U \setminus \{O\}$, then system (1) has at most two limit cycles in the region $\mathcal{R} \cap U$.*
(vi) *Let $\mathcal{A}$ be an annular region which encircles the origin. If $\partial^3 S/\partial r^3 > 0$ in $\mathcal{A}$, then system (1) has at most three limit cycles in the region $\mathcal{R} \cap \mathcal{A}$.*

We note that all the results of Theorem 3 only provide information on the limit cycles of the differential systems (1) in the region $\mathcal{R}$. But similar results to those of Theorem 3 can be obtained for the limit cycles of the differential systems (1) with $\dot{\theta} < 0$ if the region $\{(r, \theta) : \Theta(r, \theta) < 0\}$ is not empty.

Theorem 3 will be proved in Section 3.

In this work we also consider Abel differential equations of the form

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2 + C(\theta)\rho, \tag{5}$$

where ρ and θ are real variables and θ is 2π-periodic, and $A(\theta)$, $B(\theta)$ and $C(\theta)$ are quotients of two trigonometric polynomials in θ such that the denominators have no real zero. In some cases the systems of the form (2) can be transformed to an Abel differential equation (5) by means of Liouville–Cherkas transformation. In such cases we can apply the criteria for the existence, non-existence and number of limit cycles established for the Abel differential equations (5) for studying the limit cycles of a system (1). Not all the systems of the form (1) can be transformed to an Abel differential equation (5). We consider differential systems defined by the sum of two quasi-homogeneous vector fields as this is the usual setting in which the Liouville–Cherkas transformation can be applied. Other systems that can be transformed to an Abel differential equation (5) see [11,16] and the references therein.

Given $p, q, s \in \mathbb{N}$ we say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is (p, q)-*quasi-homogeneous* of degree s if $f(\lambda^p x, \lambda^q y) = \lambda^s f(x, y)$ for $\lambda \in (0, \infty)$, see [4, page 32]. A vector field $X = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is called (p, q)-*quasi-homogeneous of degree* r if P and Q are (p, q)-quasi-homogeneous functions of degree $p + r - 1$ and $q + r - 1$ respectively, see [7, Chapter 2]. When $p = q = 1$ this definition coincides with the usual definition of *homogeneous vector field of degree* r. Moreover the differential equation $dy/dx = Q/P$ associated to X is invariant by the change of variables $\bar{x} = \lambda^p x$ and $\bar{y} = \lambda^q y$.

Applying the Liouville–Cherkas transformation we can study the limit cycles of the autonomous real planar differential systems of the form

$$\begin{aligned} \dot{x} &= P_n(x, y) + P_m(x, y), \\ \dot{y} &= Q_n(x, y) + Q_m(x, y), \end{aligned} \tag{6}$$

where $m > n \geq 1$ are integers and (P_i, Q_i) for $i = n, m$, is a (p, q)-quasi-homogeneous vector field of degree i.

We recall that when $p = q = 1$ then system (6) is the sum of two homogeneous vector fields of degree n and m respectively. The *quadratic polynomial differential systems with an equilibrium point at the origin of coordinates* are included in systems (6) taking $p = q = n = 1$ and $m = 2$, and the linear differential systems perturbed by homogeneous nonlinearities correspond to the case $p = q = n = 1$ and m arbitrary.

First we introduce some notations and basic results. We take the (p, q)-*polar coordinates* (r, θ) defined by $x = r^p \cos\theta$, $y = r^q \sin\theta$. Note that when $p = q = 1$ we have the usual polar coordinates. System (6) writes as

$$\begin{aligned} \dot{r} &= \alpha(\theta) r^n + \beta(\theta) r^m, \\ \dot{\theta} &= \gamma(\theta) r^{n-1} + \delta(\theta) r^{m-1}, \end{aligned} \tag{7}$$

where we have also done the scaling of time $dt/ds = p\cos^2\theta + q\sin^2\theta$, and

$$\begin{aligned} \alpha(\theta) &= \cos\theta\, P_n(\cos\theta, \sin\theta) + \sin\theta\, Q_n(\cos\theta, \sin\theta), \\ \beta(\theta) &= \cos\theta\, P_m(\cos\theta, \sin\theta) + \sin\theta\, Q_m(\cos\theta, \sin\theta), \\ \gamma(\theta) &= p\cos\theta\, Q_n(\cos\theta, \sin\theta) - q\sin\theta\, P_n(\cos\theta, \sin\theta), \\ \delta(\theta) &= p\cos\theta\, Q_m(\cos\theta, \sin\theta) - q\sin\theta\, P_m(\cos\theta, \sin\theta). \end{aligned}$$

Finally we take the scaling of time $ds/d\tau = r^{n-1}$ and system (7) becomes

$$\begin{aligned} \dot{r} &= \alpha(\theta) r + \beta(\theta) r^{m-n+1}, \\ \dot{\theta} &= \gamma(\theta) + \delta(\theta) r^{m-n}. \end{aligned} \tag{8}$$

We assume that the region

$$\mathcal{R} = \{(r, \theta) : \gamma(\theta) + \delta(\theta) r^{m-n} > 0\}$$

is not empty. In most of the results we also assume that $\mathcal{R}$ contains the origin of coordinates. Also as before, analogous results can be obtained if the region $\{(r, \theta) : \Theta(r, \theta) < 0\}$ is not empty. Then the differential systems (7) or (8) in the region $\mathcal{R}$ are equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\alpha(\theta) r + \beta(\theta) r^{m-n+1}}{\gamma(\theta) + \delta(\theta) r^{m-n}}, \tag{9}$$

also defined in $\mathcal{R}$.

Clearly the periodic orbits of the differential equation (9) provide periodic orbits of the differential system (8) in the region $\mathcal{R}$. By the Poincaré–Bendixson Theorem, there is at least one equilibrium point of the differential system (8) in the interior region bounded by a periodic orbit. Assume that $\mathcal{R}$ contains the origin of coordinates. We remark that the only equilibrium point of the differential system (8) which lies in the region $\mathcal{R}$ is the origin of coordinates and, hence any periodic orbit of the differential system (8) must surround the origin. We shall study these periodic orbits doing the change of variables

$$\rho = \frac{r^{m-n}}{\gamma(\theta) + \delta(\theta) r^{m-n}}, \tag{10}$$

due to Cherkas [9], which in fact goes back to Liouville [26]. In the new variable ρ the differential equation (9) writes

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2 + \left(\frac{(m-n)\alpha - \gamma'}{\gamma}\right)\rho, \tag{11}$$

where

$$A(\theta) := (m-n)\frac{\delta}{\gamma}(\alpha\delta - \beta\gamma), \quad B(\theta) := \frac{1}{\gamma}((m-n)(\beta\gamma - 2\alpha\delta) + \delta\gamma' - \gamma\delta').$$

The change (10) is called the Liouville–Cherkas transformation.

We summarize in Theorem 4, to the best of our knowledge, all the known criteria on non-existence, existence and on the number of the limit cycles for the differential systems (1), using the associated differential equations (9) and (11). We also provide some new results. In the proof of Theorem 4 we give account of the authors who proved them together with the original references. See the works of Gasull and Llibre [17], Coll, Gasull and Prohens [10] and Álvarez, Gasull and Giacomini [2]. We remark that in many of these papers the authors consider directly equations of the form (9) or the Abel differential equation (11). Hence we have shown that this criteria can also be used for planar polynomial differential systems (6).

Let O denote the origin of coordinates, that is $O = (0, 0)$ in cartesian coordinates and $r = 0$ in polar coordinates.

Theorem 4. *Assume that the region $\mathcal{R}$ defined in (3) is not empty. For the differential system (6) the following statements hold.*

(i) *If $\alpha \geq 0$ and $\beta \geq 0$, or $\alpha \leq 0$ and $\beta \leq 0$, then system (6) has no limit cycles in the region $\mathcal{R}$.*
(ii) *If $\alpha\delta - \beta\gamma \equiv 0$, then system (6) has no limit cycles in the region $\mathcal{R}$.*
(iii) *If $\delta \equiv 0$, then system (6) has at most one limit cycle in the region $\mathcal{R}$.*
(iv) *Assume that $O \in \mathcal{R}$. If $A(\theta) \not\equiv 0$ and either $A(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $A(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$, then system (6) has at most two limit cycles surrounding the origin in the region $\mathcal{R}$.*
(v) *Assume that $O \in \mathcal{R}$. If $B(\theta) \not\equiv 0$ and either $B(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $B(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$, then system (6) has at most two limit cycles surrounding the origin in the region $\mathcal{R}$.*
(vi) *Assume that $O \in \mathcal{R}$. If $A(\theta) = 0$ or $B(\theta) = 0$, then system (6) has at most one limit cycle surrounding the origin in the region $\mathcal{R}$.*
(vii) *Let $\Delta_1 := \alpha\delta - \beta\gamma$. If $\Delta_1(\theta) \not\equiv 0$ and either $\Delta_1(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $\Delta_1(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$, then system (6) has at most one limit cycle, and when it exists, it is hyperbolic and it surrounds the origin in the region $\mathcal{R}$.*
(viii) *Let $\Delta_2 := \delta(\alpha\delta - \beta\gamma)$. If $\Delta_2(\theta) \not\equiv 0$ and either $\Delta_2(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $\Delta_2(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$, then system (6) has at most two limit cycles, and*

when they exist, they surround the origin in the region $\mathcal{R}$. Furthermore, if γ does not vanish, the sum of the multiplicities of the limit cycles is at most two.

(ix) *Let $\Delta_3 := \gamma\delta(\alpha\delta - \beta\gamma)$. If $\Delta_3(\theta) \not\equiv 0$ and either $\Delta_3(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $\Delta_3(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$, then for system (6), if there are limit cycles, they surround the origin and the sum of their multiplicities is at most three.*

(x) *Assume that $O \in \mathcal{R}$. If the Abel differential equation (11) is of the form*

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2,$$

and there exist two real numbers a and b such that $\Delta_{(a,b)}(\theta) := aA(\theta) + bB(\theta)$ satisfies that $\Delta_{(a,b)} \not\equiv 0$ and either $\Delta_{(a,b)}(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $\Delta_{(a,b)}(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$. Then system (6) has at most one limit cycle and, when it exists, it is hyperbolic.

(xi) *Assume that $O \in \mathcal{R}$. If the Abel differential equation (11) is such that $\int_0^{2\pi} \frac{\alpha(\theta)}{\gamma(\theta)}\, d\theta = 0$ and there exist two real numbers a and b such that*

$$\tilde{\Delta}_{(a,b)}(\theta) := a\frac{A(\theta)}{\gamma(\theta)} \exp\left((m-n)\int_0^{\theta} \frac{\alpha(s)}{\gamma(s)} ds\right) + bB(\theta)$$

satisfies that $\tilde{\Delta}_{(a,b)} \not\equiv 0$ and either $\tilde{\Delta}_{(a,b)}(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $\tilde{\Delta}_{(a,b)}(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$. Then the differential equation has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.

Theorem 4 will be proved in Section 3.

We list several references which also give criteria on the number of limit cycles of particular planar polynomial differential systems and use some of the ideas used to prove the criteria established in Theorems 3 and 4, but whose results are for particular systems and/or go beyond our settlement. See Pliss [33], section 9 (and in particular Theorem 9.7); Lins Neto [25]; Carbonell and Llibre [8]; Devlin, Lloyd and Pearson [11]; Panov [30]; Gasull, Prohens and Torregrosa [20]; Gasull and Guillamon [16]; Álvarez, Bravo and Fernández [1]; Llibre and Zhang [27,28] and the references therein.

In the works [21,23] the authors consider autonomous real planar polynomial differential systems of the form

$$\dot{x} = ax - y + P_n(x, y), \quad \dot{y} = x + ay + Q_n(x, y),$$

where $P_n(x, y)$ and $Q_n(x, y)$ are real homogeneous polynomials of degree $n \geq 2$ and $a \in \mathbb{R}$, and provide criteria for the non-existence and uniqueness of limit cycles for this class of systems. They also use the change to polar coordinates and the Liouville–Cherkas transformation and the criteria is described in terms of ad hoc functions for these systems. They also give examples of systems with exactly one limit cycle, the maximum obtained by their criteria. Since the criteria that appears in their papers is not general, we do not include it in our results. The same class of systems was studied by Carbonell and Llibre in 1988, see [8].

In the recent work [22] of 2021, Huang and Liang provide a new criterion which can be applied to planar polynomial differential systems of the form (6). They also provide an example of a planar polynomial differential which exhibits one limit cycle and realizes the maximum number described by their criterion. This example also shows that their criterion is different from the classical ones, the ones that we describe in Theorem 4.

We summarize here their main results for the sake of completeness. We use our notation.

Theorem 5. *Consider the differential system (6) and suppose that*

$$\Phi(\theta) := \frac{(m-n)\alpha(\theta)}{\delta(\theta)} - \frac{d}{d\theta}\left(\frac{\gamma(\theta)}{\delta(\theta)}\right) \neq 0 \quad \theta \in [0, 2\pi].$$

Then system (6) has at most one limit cycle. Furthermore, if the limit cycle exists then it is hyperbolic, it surrounds the origin and it is stable (resp. unstable) when $\delta(\theta)\Phi(\theta) > 0$ (resp. < 0) for all $\theta \in [0, 2\pi]$.

In [22] the authors provide the next proposition which ensures the existence of the limit cycle.

Proposition 6. *System (6) has at least 1 limit cycle surrounding the origin if*

$$\gamma(\theta)\delta(\theta) > 0 \text{ for all } \theta \in [0, 2\pi] \text{ and } \quad \int_0^{2\pi} \frac{\alpha(\theta)}{\gamma(\theta)}\, d\theta \cdot \int_0^{2\pi} \frac{\beta(\theta)}{\delta(\theta)} d\theta \; < \; 0.$$

Theorem 5 and Proposition 6 correspond to Theorem 1.1 and Proposition 1.2 of [22]. The following planar polynomial differential system

$$\dot{x} = x - y - x^3 + 5x^2y - xy^2 - y^3, \quad \dot{y} = x + y + 3x^3 - x^2y + 9xy^2 - y^3.$$

appears as Example 2 in [22] and verifies the assumptions of Theorem 5 and also the assumptions of Proposition 6, so that it has one hyperbolic limit cycle surrounding the origin.

There are several generalizations of the criteria described in Theorem 4 but for the generalized Abel equations. We provide three results in this sense that are, as far as we know, all the results in this direction with general application.

The first result is due to Gasull and Guillamon in 2006 [16].

Theorem 7. *Consider the 2π-periodic generalized Abel equation*

$$\frac{d\rho}{d\theta} = a_n(\theta)\rho^n + a_m(\theta)\rho^m + a_1(\theta)\rho, \tag{12}$$

with $n > m > 1$ and a_n, a_m and a_1 being $\mathcal{C}^1$ functions which are 2π-periodic. Assume that $a_n(\theta)$ or $a_m(\theta)$ does not change sign. Then,

(a) *If n is odd, Eq. (12) has at most five limit cycles. Furthermore, apart from the limit cycle $\rho = \rho(\theta) \equiv 0$, in each region $\mathcal{R}^+ := \{\rho > 0\}$ or $\mathcal{R}^- := \{\rho < 0\}$ the equation has at most two limit cycles and the sum of their multiplicities is at most two.*

(b) *If n is even, Eq. (12) has at most four limit cycles. Furthermore, apart from the limit cycle $\rho = \rho(\theta) \equiv 0$, in each region $\mathcal{R}^+ := \{\rho > 0\}$ or $\mathcal{R}^- := \{\rho < 0\}$ the equation has at most two limit cycles and the sum of their multiplicities is at most two, taking into account that never more than four limit cycles can coexist and that a semi-stable limit cycle counts as two limit cycles.*

The previous result is Theorem 3 in [16] written with the notation of the present paper. Note that, as a consequence of Theorem 7, Eq. (12) has at most two positive limit cycles. In [16] another result is provided about generalized Abel equations but it goes beyond the scope of the present article. As far as we know, in the literature, there are no examples of planar polynomial differential systems which realize the criteria provided in Theorem 7. We provide examples in the proof of Theorem 10.

In the work [1] of 2009, Álvarez, Bravo and Fernández provide other criteria for generalized Abel equations. Here we consider 2π-periodic continuous functions in the interval $[0, 2\pi]$. Recall that a function $f(\theta)$ is said to have definite sign if it is not null and either $f(\theta) \geq 0$ or $f(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$.

Theorem 8. *Consider the differential equation*

$$\frac{d\rho}{d\theta} = a(\theta)\rho^{n_a} + b(\theta)\rho^{n_b} + c(\theta)\rho^{n_c} + d(\theta)\rho, \tag{13}$$

where $a(\theta)$, $b(\theta)$, $c(\theta)$ and $d(\theta)$ are 2π-periodic continuous functions and $n_a > n_b > n_c > 1$ are natural numbers.

(1) *Suppose that $a(\theta)$ and $b(\theta)$, or $b(\theta)$ and $c(\theta)$ have the same definite sign, or that $a(\theta)$ and $c(\theta)$ have opposite definite sign. Then (13) has at most two positive limit cycles.*
Moreover, if $d(\theta)$ has null integral over $[0, 2\pi]$, then (13) has at most one positive limit cycle.
(2) *Suppose that $a(\theta)$ and $c(\theta)$ have definite sign, $d(\theta) \equiv 0$, and $b(\theta)$ has null integral over $[0, 2\pi]$. Then, (13) has at most one positive limit cycle.*
(3) *Suppose that $d(\theta) \equiv 0$, $b(\theta) < 0 < c(\theta)$ for all $\theta \in [0, 2\pi]$, and the function*

$$a(\theta)\tilde{\rho}^{n_a}(\theta) + b(\theta)\tilde{\rho}^{n_b}(\theta) + c(\theta)\tilde{\rho}^{n_c}(\theta) - \tilde{\rho}'(\theta)$$

has definite sign, where

$$\tilde{\rho}(\theta) = \left(\frac{(n_a - n_c)c(\theta)}{(n_b - n_a)b(\theta)}\right)^{\frac{1}{n_b - n_c}}.$$

Then (13) has at most two positive limit cycles.

The previous result corresponds to Theorem 3.1, Corollary 2 and Theorem 3.2 of [1] written with the notation of the present paper. As far as we know, in the literature, there are no examples of planar polynomial differential systems which realize the criteria provided in Theorem 8. We provide examples in the proof of Theorem 10.

In the same year 2009, Bravo, Fernández and Gasull [6] provide the following result.

Theorem 9. *Consider the differential equation*

$$\frac{d\rho}{d\theta} = A(\theta)\rho^n + B(\theta)\rho^m + C(\theta)\rho, \tag{14}$$

with $n \neq m$, $n, m \geq 2$ and where $A(\theta)$, $B(\theta)$ and $C(\theta)$ are 2π-periodic continuous functions such that

(1) *$A(\theta)$ has definite sign in $\theta \in [0, \pi]$ and $A(2\pi - \theta) = -A(\theta)$.*
(2) *$B(\theta) + B(2\pi - \theta)$ is not identically null and changes sign at most once in $[0, \pi]$.*
(3) *$C(2\pi - \theta) = -C(\theta)$.*

Then when n is odd (resp. even) it has at most three (resp. two) limit cycles. One of them is $\rho = 0$ and in case of having three limit cycles one is in $\{\rho > 0\}$ and the other one in $\{\rho < 0\}$.

The previous result corresponds to part of Theorem 1.1 of [6] written in our notation. Note that, as a consequence of Theorem 9, Eq. (14) has at most one positive limit cycle. As far as we know, there are no examples in the literature of planar polynomial differential systems which realize the criteria established in Theorem 9. This is an open problem.

The next result shows that the upper bounds for the number of limit cycles provided in Theorems 3, 4, 7, 8 are reached for planar polynomial differential systems as (1).

Theorem 10. *We provide differential systems of the form (1) satisfying each of the statements of Theorems 3, 4 (except statement (ix)), 7 and 8, such that when the statement provides limit cycles, the corresponding example realizes the maximum number of limit cycles allowed.*

Theorem 10 is proved in Section 6.

Statement (ix) of Theorem 4 needs that the function $\gamma(\theta)$ changes sign in order not to be under the assumptions of statement (viii). Thus the origin for a system which would provide an example of statement (ix) needs to have a node at the origin (or a non-monodromic degenerate equilibrium point). It is an open problem to give techniques to study limit cycles which surround such kind of equilibrium points.

The paper is organized as follows. The following section contains several preliminary results and Section 3 contains the proof of Theorems 3 and 4. The proof of Theorem 10 is provided in Section 6, where the particular family studied in Section 4 and the application of Hopf bifurcation described in Section 5 are used. For more details on the Hopf bifurcation see [5,36].

2. Preliminary results

In this section we recall some known results that we shall need for proving Theorems 3 and 4. But first we prove Lemma 1.

Proof of Lemma 1. Given $r_0 \in \mathcal{I} \cap \mathcal{R}$, we define $\tilde{r}(t, r_0)$, $\tilde{\theta}(t, r_0)$ as the solution of system (2) with initial condition $\tilde{r}(0, r_0) = r_0$, $\tilde{\theta}(0, r_0) = 0$. Note that, given a value of θ_*, if t_* is such that $\tilde{\theta}(t_*, r_0) = \theta_*$ then $\bar{r}(\theta_*, r_0) = \tilde{r}(t_*, r_0)$.

Consider the function $F(t, r_0) = \tilde{\theta}(t, r_0) - 2\pi$. Since $r = 0$ is a solution of system (2) we have that there exists a value of time T such that $\tilde{\theta}(T, 0) = 2\pi$. Thus, in the point $t = T$, $r_0 = 0$ we have that $F(T, 0) = 0$. We derive the function $F(t, r_0)$ with respect to t and we obtain

$$\frac{dF}{dt}(t, r_0) = \frac{d\tilde{\theta}}{dt}(t, r_0) = \Theta\left(\tilde{r}(t, r_0), \tilde{\theta}(t, r_0)\right).$$

Therefore this function in the point $t = T$, $r_0 = 0$ takes the value

$$\frac{dF}{dt}(T, 0) \;=\; \Theta\,(0, 2\pi) \;>\; 0,$$

because we are under the assumption that the region $\mathcal{R}$ is not empty and contains $r = 0$. Hence, by the Implicit Function Theorem, there exists a continuously differentiable function $\tilde{t}(r_0)$ defined in a neighborhood of $r_0 = 0$ such that $\tilde{t}(0) = T$ and $F(\tilde{t}(r_0), r_0) \equiv 0$ for all r_0 in this neighborhood of $r_0 = 0$. Since $\tilde{t}(r_0)$ is such that $\tilde{\theta}(\tilde{t}(r_0), r_0) = 2\pi$, we have that $\bar{r}(2\pi, r_0) = \tilde{r}(\tilde{t}(r_0), r_0)$ and, thus, the function $\bar{r}(2\pi, r_0)$ is defined for all r_0 in a neighborhood of $r_0 = 0$. $\square$

In [29] Lloyd studied the two-dimensional differential systems (1) transformed in an ordinary differential equation (4) by using polar coordinates. In this setting he provided several results on the existence of limit cycles for ordinary differential equations of the form (4) that we describe below.

Let O denote the origin of coordinates, that is $O = (0, 0)$ in cartesian coordinates and $r = 0$ in polar coordinates. Let $\mathcal{I}$ denote the positive x-axis: $r \geq 0$, $\theta = 0$.

The next result corresponds to Theorems 3, 5, 6 and 9 of Lloyd [29].

Theorem 11. *We have a differential system in polar coordinates*

$$\dot{r} = F(r, \theta), \qquad \dot{\theta} = G(r, \theta), \tag{15}$$

defined in a simply connected region U containing the origin O, where F and G are C^1 2π-periodic functions such that $F(0, \theta) = 0$ for all θ, and $G(r, \theta) > 0$ in U. Then in U the differential system (15) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{F(r, \theta)}{G(r, \theta)} = S(r, \theta). \tag{16}$$

For $r_0 \in \mathcal{I} \cap U$, we write $\bar{r}(\theta, r_0)$ for the solution of (16) satisfying $\bar{r}(0, r_0) = r_0$.

(i) *If*

$$\text{either } \frac{\partial S}{\partial r} > 0, \text{ or } \frac{\partial S}{\partial r} < 0 \text{ in } U \setminus \{O\},$$

then the differential system (15) has no limit cycles in U.

(ii) *Suppose that $\bar{r}(2\pi, r_0)$ is defined for some $r_0 \neq 0$. If*

$$\text{either } \frac{\partial^2 S}{\partial r^2} > 0, \text{ or } \frac{\partial^2 S}{\partial r^2} < 0 \text{ in } U \setminus \{O\},$$

then the differential system (15) has at most one limit cycle in U.

(iii) *Suppose that $\bar{r}(2\pi, r_0)$ is defined for some $r_0 \neq 0$. If*

$$\frac{\partial^3 S}{\partial r^3} > 0 \text{ in } U \setminus \{O\},$$

then the differential system (15) has at most two limit cycles in U.

The next result corresponds to Theorems 2, 4 and 8 of Lloyd [29].

Theorem 12. *Consider the differential system (15) defined in an annular region $\mathcal{A}$ which encircles the origin O and where $G(r,\theta) > 0$. Then, in $\mathcal{A}$ the differential system (15) is equivalent to the differential equation (16).*

(i) *If*

$$\frac{\partial S}{\partial r} \neq 0 \ \textit{in}\ \mathcal{A},$$

then the differential system (15) has at most one limit cycle entirely contained in $\mathcal{A}$.

(ii) *If*

$$\frac{\partial^2 S}{\partial r^2} \neq 0 \ \textit{in}\ \mathcal{A},$$

then the differential system (15) has at most two limit cycles entirely contained in $\mathcal{A}$.

(iii) *If*

$$\frac{\partial^3 S}{\partial r^3} > 0 \ \textit{in}\ \mathcal{A},$$

then the differential system (15) has at most three limit cycles entirely contained in $\mathcal{A}$.

The following results have been developed by several authors based on the seminal paper of Gasull and Llibre [17].

In [17] Gasull and Llibre studied the Abel differential equations (5) where $A(\theta)$, $B(\theta)$ and $C(\theta)$ are smooth functions of θ and periodic in θ. The authors gave several criteria for the existence of limit cycles depending on the signs of the functions $A(\theta)$ and $B(\theta)$, that we summarize below.

The next result is a consequence of Theorem C of [17]. There this result is proved for sum of two homogeneous systems, but the proof extends to the sum of two quasi-homogeneous systems.

Theorem 13. *Let $A(\theta)$ and $B(\theta)$ be the functions associated to the Abel differential equation (5).*

(a) Suppose that $A(\theta) \not\equiv 0$, $B(\theta) \not\equiv 0$ and the function either $A(\theta)$, or $B(\theta)$ does not change sign. Then this system has at most two limit cycles surrounding the origin in the region $\mathcal{R}$.

(b) Suppose that either $A(\theta) \equiv 0$, or $B(\theta) \equiv 0$, then this system has at most one limit cycle surrounding the origin in the region $\mathcal{R}$.

Remark 14. In statement (b) of the previous Theorem 13 we have that the hypothesis either $A(\theta)$ or $B(\theta)$ is zero. We will see that these two hypothesis are equivalent. Consider an Abel differential equation (5) such that $B(\theta)$ is null, that is, an equation of the form

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + C(\theta)\rho.$$

Consider the change of the dependent variable $\bar{\rho} = \rho^2$. Then, the differential equation becomes

$$\frac{d\bar{\rho}}{d\theta} = 2A(\theta)\bar{\rho}^2 + 2C(\theta)\bar{\rho},$$

which is an Abel differential equation (5) with a zero coefficient in $\bar{\rho}^3$. In case of having an Abel differential equation (5) such that $A(\theta)$ is null, the inverse change $\bar{\rho} = \sqrt{\rho}$ provides and Abel differential equation (5) with $B(\theta)$ null.

In [10] Coll, Gasull and Prohens provide several results on the existence of limit cycles for the systems of the form (6). We summarize some statements of their Theorems A, B and C in the following result.

Theorem 15.

(a) *Assume that the function $\alpha\delta - \beta\gamma \not\equiv 0$ associated to the differential system (6) does not change sign. Then this system has at most one limit cycle and, when it exists, it is hyperbolic and it surrounds the origin.*
(b) *Assume that the function $\delta(\alpha\delta - \beta\gamma) \not\equiv 0$ associated to the differential system (6) does not change sign. Then this system has at most two limit cycles and, when they exist, they surround the origin. Furthermore, if γ does not vanish, the sum of the multiplicities of the limit cycles is at most two.*
(c) *Assume that the function $\gamma\delta(\alpha\delta - \beta\gamma) \not\equiv 0$ associated to the differential system (6) does not change sign. Then, for this system, if there are limit cycles, they surround the origin and the sum of their multiplicities is at most three.*

In the work [10] the authors provide several examples for statements (a) and (b), but they do not analyze the conditions which appear in [19] to ensure the presence of limit cycles.

In [2] Álvarez, Gasull and Giacomini establish a uniqueness criterion for the number of periodic orbits of some Abel equations. We summarize their main results in the following statement, which corresponds to Theorems A and 15 in [2].

Theorem 16.

(a) *Consider an Abel equation (5) of the form*

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2,$$

and assume that there exist two real numbers a and b such that $aA(\theta)+bB(\theta)$ does not vanish identically and does not change sign in $[0, 2\pi]$. Then the differential equation has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.
(b) *Consider an Abel equation (5) satisfying $\int_0^{2\pi} C(\theta)d\theta = 0$. Assume that there exist two real numbers a and b such that*

$$aA(\theta)\exp\left(\int_0^{\theta} C(s)ds\right) + bB(\theta)$$

does not vanish identically and does not change sign in $[0, 2\pi]$. *Then the differential equation has at most one non-zero periodic orbit. Furthermore, when this periodic orbit exists, it is hyperbolic.*

We note that statement (b) implies statement (a) in Theorem 16.

3. Proof of Theorems 3 and 4

Proof of Theorem 3. We can apply Theorems 11 and 12 to the differential equation (9), and we obtain the proof of the theorem. □

We prove Theorem 4 statement by statement.

Proof of statement (i) **of Theorem 4.** Since $\alpha \geq 0$ and $\beta \geq 0$, or $\alpha \leq 0$ and $\beta \leq 0$, then $dr/d\theta$ does not change sign in $\mathcal{R}$. Therefore a solution $r(\theta)$ of (8) increases, decreases, or is constant if $\alpha = \beta = 0$. So, in the first two cases these solutions cannot be periodic, and in the third case there is a continuum of periodic solutions. Consequently the differential system (6) has no limit cycles in the region $\mathcal{R}$. □

Proof of statement (ii) **of Theorem 4.** This statement is proved in Proposition 26 of [10], where slight modifications in the hypothesis are considered. We include here a proof for the sake of completeness.

The region $\mathcal{R}$ is not empty and, hence, either γ or δ is a trigonometric polynomial which is not null. Assume that γ is not null, since $\alpha\delta - \beta\gamma \equiv 0$ we can isolate $\beta = \alpha\delta/\gamma$ and we have that $dr/d\theta = \alpha r/\gamma$. Now $r(\theta) = r_0 \exp\left(\int_0^\theta \alpha/\gamma \, ds\right)$. In order that this function be periodic we need that $r(2\pi) = r_0$ which implies that $\int_0^{2\pi} \alpha/\gamma ds = 0$. In such a case all the orbits are periodic, because this condition does not depend on r_0. Consequently the differential system (6) has no limit cycles surrounding the origin in the region $\mathcal{R}$.

In the case that γ is null, the assumption $\alpha\delta - \beta\gamma \equiv 0$ gives that α is null in the region $\mathcal{R}$. Then we have $dr/d\theta = \beta r/\delta$ and the same argument as in the previous paragraph gives the desired result. □

Proof of statement (iii) **of Theorem 4.** This statement is proved in Proposition 26 of [10], where slight modifications in the hypothesis are considered. We include here a proof for the sake of completeness.

If $\delta \equiv 0$ the differential equation (9) becomes

$$\frac{dr}{d\theta} = \frac{\alpha(\theta)}{\gamma(\theta)} r + \frac{\beta(\theta)}{\gamma(\theta)} r^{m-n+1}, \tag{17}$$

which is a Bernoulli differential equation. By the change $v = r^{n-m}$ this differential equation can be transformed into a linear differential equation that can have at most one limit cycle, see for instance [15]. This completes the proof of statement (iii). □

Proof of Statements (iv), (v) **and** (vi) **of Theorem 4.** We can apply Theorem 13 and these proofs follow. □

Proof of statements (vii), $(viii)$ and (ix) of Theorem 4. We can apply Theorem 15 and these proofs follow. □

Proof of statements (x) and (xi) of Theorem 4. We can apply Theorem 16 and these proofs follow. □

4. A differential equation in polar coordinates

In this section we are concerned with a particular equation which will allow to give examples in order to prove Theorem 10. We consider an ordinary differential equation of the form

$$\frac{dr}{d\theta} = \mathcal{P}(r), \tag{18}$$

where (r, θ) are real variables, θ is 2π-periodic, $\mathcal{P}(0) = 0$ and $\mathcal{P}(r)$ is a real rational function which is not identically zero. With some additional hypothesis, this is a particular case of Eq. (4). The following result is well-known but we prove it for the sake of completeness.

Lemma 17. *Let $\xi \in \mathbb{R}$. The value ξ is an isolated zero of the function $\mathcal{P}(r)$ if and only if $r = \xi$ is a limit cycle of the differential equation (18).*

Proof. It is clear that if ξ is a zero of $\mathcal{P}(r)$ then $r = \xi$ is an orbit of the differential equation (18). The zeros of a rational function as $\mathcal{P}(r)$, which is not identically zero by assumption, are all isolated. The orbit $r = \xi$ is an isolated orbit because the neighboring orbits increase or decrease in r with respect to θ depending if $\mathcal{P}(r)$ is positive or negative. By the same argument any periodic orbit has to be a root of $\mathcal{P}(r)$. □

We are interested in those Eqs. (18) which come from an autonomous real planar differential system (1) by means of a change to polar coordinates. Since $\mathcal{P}(0) = 0$ and the change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ is invariant under the change $(r, \theta) \to (-r, \theta + \pi)$, we also have that $\mathcal{P}(r)$ needs to be an odd function, that is, $\mathcal{P}(r) = r\tilde{\mathcal{P}}(r^2)$, where $\tilde{\mathcal{P}}(z)$ is a real rational function in z. We consider the associated system

$$\dot{r} = r\tilde{\mathcal{P}}(r^2), \quad \dot{\theta} = 1, \tag{19}$$

and we undo the change to polar coordinates to get the autonomous real planar differential system

$$\dot{x} = -y + x\tilde{\mathcal{P}}\left(x^2 + y^2\right), \quad \dot{y} = x + y\tilde{\mathcal{P}}\left(x^2 + y^2\right). \tag{20}$$

In the particular case that $\mathcal{P}$ is a polynomial, we have a planar polynomial differential system.

Planar systems whose angular speed is constant are usually called rigid or uniformly isochronous, see [20] and the references therein. We consider planar polynomial differential systems of the form

$$\dot{x} = -y + xF(x, y), \quad \dot{y} = x + yF(x, y), \tag{21}$$

where $F(x, y)$ is a real polynomial. These systems in polar coordinates write as

$$\dot{r} = rF(r\cos\theta, r\sin\theta), \quad \dot{\theta} = 1.$$

We remark that systems of the form (21) are rigid systems.

The following examples illustrate some of the criteria stated in Theorems 3 and 4.

Example 18. We consider the differential system (20) such that $\tilde{\mathcal{P}}(z) = z - 1$. In polar coordinates the system writes as

$$\frac{dr}{d\theta} = r(r^2 - 1).$$

We could have chosen any polynomial $\tilde{\mathcal{P}}(z) = z - \xi$ with a positive value ξ and a rescaling of the variables enables to choose $\xi = 1$ without loss of generality. By Lemma 17 we have that the system has a unique limit cycle at $r = 1$. The function $S(r, \theta)$ which appears in Theorem 3 is $r(r^2 - 1)$ and its derivative is

$$\frac{\partial S}{\partial r} = 3r^2 - 1.$$

In the simply connected region containing the origin $U = \{(r, \theta) : 0 \leq r < 1/\sqrt{3}\}$ there is no limit cycle of the system. This provides an example of criterion (i) of Theorem 3. And in the annular region encircling the origin $\mathcal{A} = \{(r, \theta) : 1/\sqrt{3} < r < 2\}$ there is exactly one limit cycle of the system. This provides an example of criterion (ii) of Theorem 3. The second derivative of the function S with respect to r is

$$\frac{\partial^2 S}{\partial r^2} = 6r.$$

In any simply connected region containing the origin and the orbit $r = 1$ it is verified that $\partial^2 S/\partial r^2 \neq 0$ (except the origin) and the region contains one limit cycle. This is an example of criterion (iii) of Theorem 3 where the maximum number of limit cycles is achieved. It is also an example of criterion (vi) of Theorem 4 because when $\delta \equiv 0$ we have that the function $A(\theta)$ which appears in the Abel differential equation (11) is equivalently zero. On the other hand we can consider the differential equation

$$\frac{dr}{d\theta} = r(r^2 - 1),$$

as an Abel differential equation (5) with constant coefficients and $B(\theta) \equiv 0$.

Example 19. Consider the differential equation

$$\frac{dr}{d\theta} = r^5 - r^3, \tag{22}$$

defined in the region $r \geq 0$. This equation has $r = 1$ as the unique limit cycle in the considered region by virtue of Lemma 17. The change to cartesian coordinates $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ provides the following planar polynomial differential system

$$\dot{x} = -y + x\left((x^2 + y^2)^4 - (x^2 + y^2)^2\right), \dot{y} = x + y\left((x^2 + y^2)^4 - (x^2 + y^2)^2\right).$$

We take the following change to the differential equation (22), $\rho = r^2$ which leads to the Abel differential equation

$$\frac{d\rho}{d\theta} = 2\rho^3 - 2\rho^2.$$

This is an example of criterion (x) of Theorem 4 where the maximum number of limit cycles is achieved.

Example 20. We consider the differential system (20) such that $\tilde{\mathcal{P}}(z) = (z-1)(z-a)$, where a is a real value with $a > 1$. In polar coordinates the system writes as

$$\frac{dr}{d\theta} = r(r^2-1)(r^2-a). \tag{23}$$

We could have chosen any polynomial $\tilde{\mathcal{P}}(z) = (z-\xi_1)(z-\xi_2)$ with positive values ξ_1 and ξ_2 and a rescaling of the variables enables to choose the lowest root equal to 1 without loss of generality. By Lemma 17 we have that the system has two limit cycles: one at $r = 1$ and the other one at $r = \sqrt{a}$. The function $S(r,\theta)$ which appears in Theorem 3 is $r(r^2-1)(r^2-a)$ and its second derivative with respect to r is

$$\frac{\partial^2 S}{\partial r^2} = 2r(10r^2 - 3 - 3a).$$

If a is any value with $1 < a < 7/3$, then $\sqrt{3(1+a)/10} < 1$. Take values b_1 and b_2 such that $\sqrt{3(1+a)/10} < b_1 < 1 < a < b_2$ and in the annular region encircling the origin $\mathcal{A} = \{(r,\theta) : b_1 < r < b_2\}$ there are exactly two limit cycles of the system. This provides an example of criterion (iv) of Theorem 3 where the maximum number of limit cycles is achieved.

We consider now the differential equation (23) where a is any real value with $a > 1$. We consider only the region $r \geq 0$ and we take the change $\rho = r^2$ which leads to the Abel differential equation

$$\frac{d\rho}{d\theta} = 2\rho^3 - 2(a+1)\rho^2 + 2a\rho.$$

This is an example of criteria (iv) and (v) of Theorem 4 where the maximum number of limit cycles is achieved.

We consider again the differential equation (23) where a is any real value with $a > 1$. This equation illustrates an application of Theorem 7 where the maximum number of limit cycles provided by the criterion is achieved (in the case n odd, for n defined in Theorem 7). We can generalize Eq. (23) to

$$\frac{dr}{d\theta} = r(r^{2k}-1)(r^{2k}-a),$$

with $a > 1$ and $k \in \mathbb{N}$ and the corresponding system in cartesian coordinates provides an example of Theorem 7 where the maximum number of limit cycles provided by the criterion is achieved.

Example 21. We consider the differential system (20) such that $\tilde{\mathcal{P}}(z) = (z-1)(z-a)(z-b)$, where a and b are real values with $1 < a < b$. In polar coordinates the system writes as

$$\frac{dr}{d\theta} = r(r^2-1)(r^2-a)(r^2-b).$$

We could have chosen any polynomial $\tilde{\mathcal{P}}(z) = (z-\xi_1)(z-\xi_2)(z-\xi_3)$ with positive values ξ_1, ξ_2 and ξ_3 and a rescaling of the variables enables to choose the lowest root equal to 1 without loss of generality. By Lemma 17 we have that the system has three limit cycles: one at $r = 1$, another one at $r = \sqrt{a}$ and the third one at $r = \sqrt{b}$. The function $S(r, \theta)$ which appears in Theorem 3 is $r(r^2-1)(r^2-a)(r^2-b)$ and its second derivative with respect to r is

$$\frac{\partial^2 S}{\partial r^2} = 2r(21r^4 - 10(1+a+b)r^2 + 3(a+b+ab)).$$

We consider the polynomial $P_0(z) = 21z^2 - 10(1+a+b)z + 3(a+b+ab)$ which satisfies that $(\partial^2 S)/(\partial r^2) = 2r P_0(r^2)$. In order to analyze the number of zeros of $P_0(z)$ we use the classical Sturm's theorem, see [38]. We consider the Sturm sequence associated to $P_0(z)$ with respect to z and we have that

$$P_1(z) = \frac{dP_0}{dz} = 2(21z - 5(1+a+b)),$$

and P_2 is defined as the remainder of the polynomial division of P_0 quotient P_1 with respect to z, multiplied by -1. It turns out that

$$P_2(z) = \frac{1}{21}(25(1+a^2+b^2) - 13(a+b+ab)).$$

This polynomial in (a, b) is positive for any values of a and b because it has a unique extremum in the point $(a, b) = (13/37, 13/37)$ and in this point it takes the value $36/37$ which is positive. The Sturm sequence (P_0, P_1, P_2) in $z = 0$ takes the values $(3(a+b+ab), -10(1+a+b), P_2)$. Since $1 < a < b$ we have that its sequence of signs is $(+, -, +)$ and, therefore, there are two changes of signs. The Sturm sequence (P_0, P_1, P_2) in $z = 1$ is $(11 - 7a - 7b + 3ab, 2(16 - 5a - 5b), P_2)$. There are three different possibilities:

In the region $\Omega_1 = \{(a, b) \in \mathbb{R}^2 : 1 < a < b \text{ and } 5(a+b) - 16 \le 0\}$ we have that the sequence of signs is $(-, +, +)$, so there is one change of signs.

In the region $\Omega_2 = \{(a, b) \in \mathbb{R}^2 : 1 < a < b,\ 5(a+b) - 16 > 0 \text{ and } 11 - 7(a+b) + 3ab \le 0\}$ we have that the sequence of signs is $(-, -, +)$, so there is one change of signs.

In the region $\Omega_3 = \{(a, b) \in \mathbb{R}^2 : 1 < a < b \text{ and } 11 - 7(a+b) + 3ab > 0\}$ we have that the sequence of signs is $(+, -, +)$, so there is two changes of signs.

It is easy to show that $\{(a, b) \in \mathbb{R}^2 : 1 < a < b\} = \Omega_1 \cup \Omega_2 \cup \Omega_3$. By the Sturm theorem, we have that if $(a, b) \in \Omega_1 \cup \Omega_2$, then the number of different real roots of the polynomial P_0 in the interval $z \in (0, 1]$ is 1, and if $(a, b) \in \Omega_3$ then the number of real roots of the polynomial P_0 in the interval $z \in (0, 1]$ is 0.

Assume that there is one root of the polynomial P_0 in $z \in (0, 1)$. If the other root is in $z > b$, then there is a contradiction with statement (iv) of Theorem 3. Thus, the other root of P_0 is in the interval $z \in [1, a)$ or in the interval $z \in (a, b]$ or in $z = a$. In the first two cases, we have an example of application of statement (iv) of Theorem 3 where the maximum number of limit cycles is achieved. For instance, if $a = 3/2$ and $b = 2$ we are in such a case.

Assume that there are no roots of the polynomial P_0 in $z \in (0, 1]$. If the polynomial P_0 has no real roots or they are in $z > a$, we are in contradiction with statement (iii) of Theorem 3. Thus, this polynomial has a root in the interval $z \in (1, a]$ and we have an example of statement (iii) of Theorem 3 where the maximum number of limit cycles is achieved. For instance, if $a = 7/2$ and $b = 4$ we are in such a case.

Example 22. We take the same differential equation as in the previous Example 21 and we compute the function

$$\frac{\partial^3 S}{\partial r^3} = 6(35r^4 - 10(1+a+b)r^2 + a + b + ab) = 6Q_0(r^2),$$

with $Q_0(z) = 35z^2 - 10(1+a+b)z + a + b + ab$. We can compute the Sturm sequence of Q_0 with respect to z and perform a thorough analysis of the changes of signs as we have described in the previous example. We get the following conclusions:

If the values (a, b) belong to the region $\{(a, b) \in \mathbb{R}^2 : 1 < a < b,\ 25-9(a+b)+ab > 0 \text{ and } a + b - 6 < 0\}$, then the number of different roots of Q_0 in the interval $z \in (0, 1)$ is 2. In such a case, we have an example of statement (vi) of Theorem 3 where the maximum number of limit cycles is achieved. For instance, if $a = 5/4$ and $b = 3/2$, we are in such a case.

If the values (a, b) are such that $1 < a < b$ and do not belong to the previous region, we have no example of statements (v) or (vi) of Theorem 3 where the maximum number of limit cycles is achieved.

Example 23. It is not possible to find an example of the criteria given by statement (v) of Theorem 3 using planar polynomial differential systems coming from system (19) by a change from polar coordinates to cartesian coordinates. The reason is that the parity of the corresponding function $S(r, \theta) = r\tilde{\mathcal{P}}(r^2)$ implies that $\partial^2 S/\partial r^2$ vanishes at $r = 0$. Moreover if $r\tilde{\mathcal{P}}(r^2)$ has two positive real roots at ξ_1 and ξ_2, that is, the system has two limit cycles, we can assume that $0 < \xi_1 < \xi_2$ and that there is no other real root, nor a pole, of $S(r, \theta) = r\tilde{\mathcal{P}}(r^2)$ in the interval $r \in [0, \xi_2]$. Therefore, by Rolle's Theorem there are two roots of $\partial S/\partial r$ in the interval $r \in (0, \xi_2)$ and, thus, one real root of $\partial^2 S/\partial r^2$ in the interval $r \in (0, \xi_2)$, say at the value ξ_*. Since the function $\partial^2 S/\partial r^2$ vanishes at $r = 0$ and at $r = \xi_*$, again by Rolle's Theorem, we can deduce that the function $\partial^3 S/\partial r^3$ has at least one zero at the interval $r \in (0, \xi_*)$, which belongs to the interval $r \in (0, \xi_2)$. Hence, the assumptions of criterion (v) of Theorem 3 cannot be satisfied.

However, we can consider a variation of an equation of the form (18) such that the origin is a pole. Take the differential equation

$$\frac{dr}{d\theta} = \frac{(1-r^2)(r^2-a)}{r^3} \tag{24}$$

where $a \in (1, 9)$ is a real value. Note that this equation has two positive limit cycles, one in $r = 1$ and the other one in $r = \sqrt{a}$. The third derivative with respect to r of the function $S(r, \theta) = (1 - r^2)(r^2 - a)/r^3$ is

$$\frac{\partial^3 S}{\partial r^3} = \frac{6(10a - (1+a)r^2)}{r^6}.$$

Since $a \in (1, 9)$ we have that this function is strictly positive in the interval $r \in (0, \sqrt{a})$. Thus, we are under the assumptions of the third statement of Theorem 11. Note that the region defined in (3) does not contain the origin but the displacement map is well-defined. The differential equation (24) in polar coordinates becomes

$$\dot{r} = r(1-r^2)(r^2-a), \quad \dot{\theta} = r^4,$$

and the change from polar coordinates to cartesian coordinates leads to the differential system

$$\begin{aligned} \dot{x} &= x(1-x^2-y^2)(x^2+y^2-a) - y(x^2+y^2)^2, \\ \dot{y} &= y(1-x^2-y^2)(x^2+y^2-a) + x(x^2+y^2)^2. \end{aligned} \tag{25}$$

Note that system (25) has a star node at the origin of coordinates.

Example 24. Consider the planar differential system (20) such that $\tilde{\mathcal{P}}(z) = (1-z)(a-z)(1+a+z)$, where $a \in \mathbb{R}$ with $a > 1$. We take polar coordinates to get the differential equation

$$\frac{dr}{d\theta} = r(1-r^2)(a-r^2)(1+a+r^2).$$

By Lemma 17, in the region $r > 0$ there are exactly two limit cycles. We take the change $\rho = r^2$ and we get the ordinary differential equation

$$\frac{d\rho}{d\theta} = 2\rho(1-\rho)(a-\rho)(1+a+\rho) = 2a(1+a)\rho - 2(1+a+a^2)\rho^2 + 2\rho^4.$$

This equation provides an example of application of the criterion described in Theorem 7 in the case that the value of n defined in the theorem is even. Moreover, it gives the maximum number of limit cycles provided by the criterion.

Example 25. Let us consider the differential system (20) such that $\tilde{\mathcal{P}}(z) = (z-1)(z-a)(z+b)$, where a and b are real values with $1 < a$ and $b > 0$. In polar coordinates the system writes as

$$\frac{dr}{d\theta} = r(r^2-1)(r^2-a)(r^2+b) = r^7 - (1+a-b)r^5 + (a-b-ab)r^3 + abr.$$

This system has exactly two hyperbolic limit cycles in the region $r > 0$ by Lemma 17. It is easy to find values of a and b which satisfy the conditions of statement (1) of Theorem 8. For instance, if $b > 1 + a$ then the functions $a(\theta)$ and $b(\theta)$ defined in the theorem have the same definite sign. If $a = 3$ and $b = 1$, then the functions $b(\theta)$ and $c(\theta)$ have the same definite sign or $a(\theta)$ and $c(\theta)$ have opposite definite sign. This example provides an illustration of statement (1) of Theorem 8 where the maximum number of limit cycles is achieved. If we choose $a = 0$ in the previous expression, we have an example of the second part of statement (1) of Theorem 8. If, moreover, we choose $b = 1$ in the previous expression, we have an example of statement (2) of Theorem 8. In the case that we consider $a > 1$ and $b = 0$ in the previous expression, we have an example of statement (3) of Theorem 8.

5. Application of Hopf bifurcation

We provide in this section some examples where the limit cycles are encountered using the Hopf bifurcation, see [5,36]. The same references provide the definition of the Poincaré–Lyapunov quantities and how to use them to prove Hopf bifurcation.

As it was proved by Sibirsky [37], see also [35] and the references therein, by an affine change of coordinates, any planar polynomial differential system of the form

$$\dot{x} = -y + \lambda x + P_3(x, y), \quad \dot{y} = x + \lambda y + Q_3(x, y),$$

where λ is real and $P_3(x, y)$ and $Q_3(x.y)$ are real cubic homogeneous polynomials can be written

$$\begin{aligned} \dot{x} &= -y + \lambda x - (\omega + \vartheta - a)x^3 - (\eta - 3\mu)x^2 y \\ &\quad -(3\omega - 3\vartheta + 2a - \xi)xy^2 - (\mu - \nu)y^3, \\ \dot{y} &= x + \lambda y + (\mu + \nu)x^3 + (3\omega + 3\vartheta + 2a)x^2 y \\ &\quad +(\eta - 3\mu)xy^2 + (\omega - \vartheta - a)y^3, \end{aligned} \tag{26}$$

where λ, ω, ϑ, a, η, μ, ξ, ν are real parameters. Note that the region $\mathcal{R}$, defined in (3), corresponding to the previous system is not empty and contains the origin of coordinates. For convenience of the readers and completeness of the paper, we state below the expressions of the first three Poincaré–Lyapunov quantities corresponding to system (26) with $\lambda = 0$. These expressions were first computed by Sibirsky [37]:

$$v_3 = \pi\xi/4, \quad v_5 = -5\pi a\nu/4, \quad v_7 = 25\pi a\omega\vartheta/8.$$

Example 26. We take system (26) and we impose that the corresponding function $A(\theta)$ is not null and does not change sign together with the fact that we can provide two limit cycles which have bifurcated from the origin by a Hopf bifurcation. The following system realizes this search. We take the following values of the parameters:

$$\lambda = \frac{1}{1000}, \ \omega = -\frac{13}{32}, \ \vartheta = \frac{1}{16}, \ a = -1, \ \eta = \frac{1}{2},$$
$$\mu = \frac{1}{8}, \ \xi = -\frac{1}{2}, \ \nu = \frac{19}{16}.$$

Then we obtain the following planar polynomial differential system:

$$\begin{aligned} \dot{x} &= -y + \frac{x}{1000} - \frac{21x^3}{32} - \frac{x^2 y}{8} + \frac{93xy^2}{32} + \frac{17y^3}{16}, \\ \dot{y} &= x + \frac{y}{1000} + \frac{21x^3}{16} - \frac{97x^2 y}{32} + \frac{xy^2}{8} + \frac{17y^3}{32}. \end{aligned} \tag{27}$$

This system exhibits the function

$$A(\theta) = \frac{501}{128000}\left(2 + 19\cos(2\theta) - 19\sin(2\theta)\right)^2,$$

and has two limit cycles inside the region $\mathcal{R}$ defined in (3). Note that this region contains the origin of coordinates. Each limit cycle passes through a point $(x_{0i}, 0)$, $i = 1, 2$, with $x_{01} \approx 0.25633$ and $x_{02} \approx 0.1437$. Fig. 1 is a representation of the limit cycles, numerically found, together with part of the boundary of the region $\mathcal{R}$ defined in (3).

This example provides a system which illustrates statement (iv) of Theorem 4.

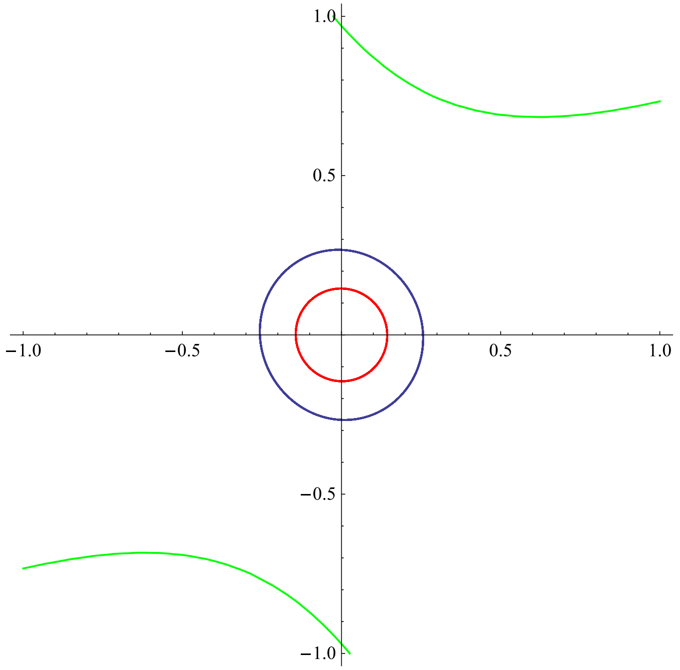

Fig. 1. The two limit cycles of system (27) and the boundary of the region $\mathcal{R}$.

Example 27. We take system (26) and we impose that the corresponding function $\delta(\alpha\delta-\beta\gamma)$ be nonzero and does not change sign together with the fact that we can provide two limit cycles which have bifurcated from the origin by a Hopf bifurcation. The following system realizes this search. We take the following values of the parameters:

$$\lambda = \frac{1}{1250}, \ \omega = -\frac{11}{15}, \ \vartheta = \frac{11}{240}, \ a = -1,$$
$$\eta = \frac{11}{16}, \ \mu = \frac{11}{64}, \ \xi = -\frac{11}{30}, \ \nu = 1.$$

We get the following planar polynomial differential system:

$$\begin{aligned} \dot{x} &= -y + \frac{x}{1250} - \frac{5x^3}{16} - \frac{11x^2y}{64} + \frac{953xy^2}{240} + \frac{53y^3}{64}, \\ \dot{y} &= x + \frac{y}{1250} + \frac{75x^3}{64} - \frac{65x^2y}{16} + \frac{11xy^2}{64} + \frac{53y^3}{240}. \end{aligned} \tag{28}$$

This system exhibits the functions

$$\delta(\theta) = \frac{11 + 64\cos(2\theta) - 120\sin(2\theta)}{64}$$

and

$$(\delta(\alpha\delta-\beta\gamma))(\theta) = \frac{1003(11 + 64\cos(2\theta) - 120\sin(2\theta))^2}{15360000},$$

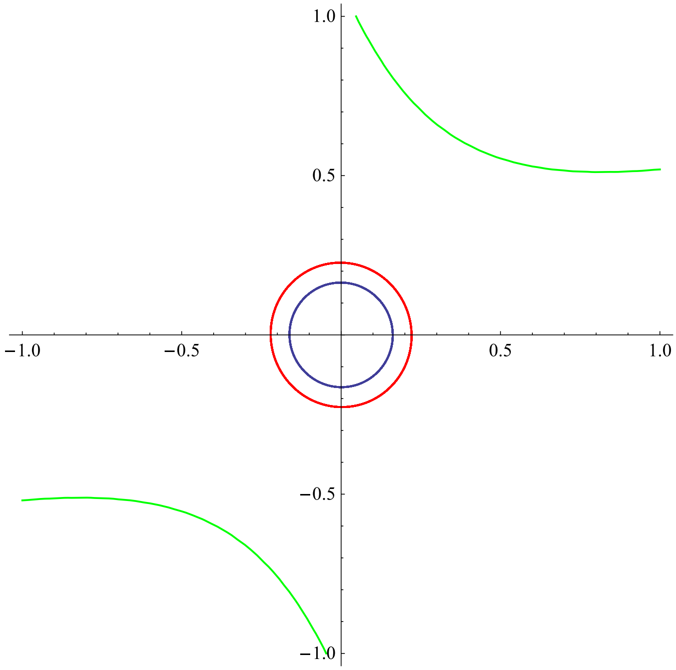

Fig. 2. The two limit cycles of system (28) and the boundary of the region $\mathcal{R}$.

and has two limit cycles inside the region $\mathcal{R}$ defined in (3). Note that this region contains the origin of coordinates. Each limit cycle pass through a point $(x_{0i}, 0)$, $i = 1, 2$, with $x_{01} \approx 0.162$ and $x_{02} \approx 0.221$. Fig. 2 is a representation of the limit cycles, numerically found, together with part of the boundary of the region $\mathcal{R}$ defined in (3).

This example provides a system which illustrates statement (iv) of Theorem 4.

Example 28. We take system (26) with $\lambda = 0$, so that the function $\alpha(\theta)$ becomes zero and the function $\gamma(\theta) \equiv 1$. We also impose that the corresponding function $aA(\theta) + bB(\theta)$, where a and b are real parameters to be chosen, is not zero and does not change sign together with the fact that we can provide one limit cycle which has bifurcated from the origin by a Hopf bifurcation. The following system realizes this search. We take the following values of the parameters:

$$\lambda = 0, \ \omega = -\frac{13}{18}, \ \vartheta = \frac{13}{180}, \ a = -1, \ \eta = \frac{26}{25}, \ \mu = \frac{13}{50}, \ \xi = -\frac{26}{45}, \ \nu = 1,$$

which provides the following planar polynomial differential system

$$\begin{aligned} \dot{x} &= -y - \frac{7x^3}{20} - \frac{13x^2y}{50} + \frac{137xy^2}{36} + \frac{37y^3}{50}, \\ \dot{y} &= x + \frac{63x^3}{50} - \frac{79x^2y}{20} + \frac{13xy^2}{50} + \frac{37y^3}{180}. \end{aligned} \tag{29}$$

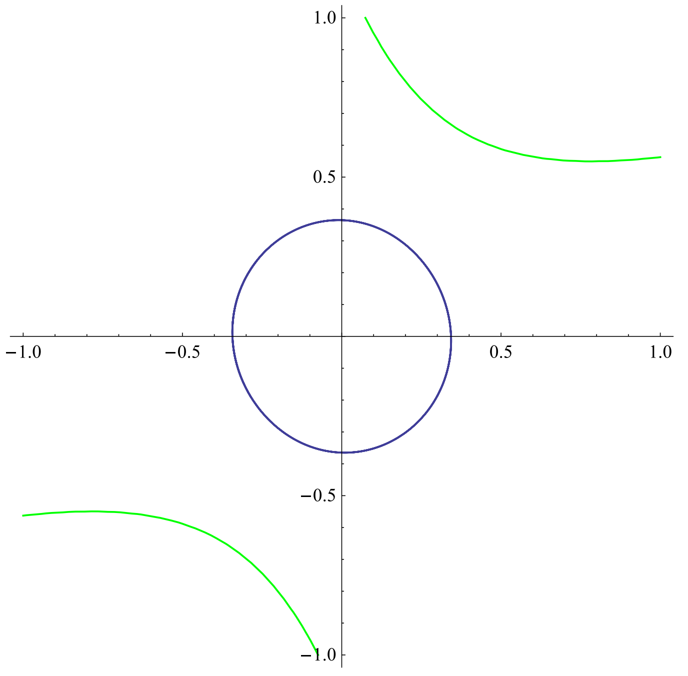

Fig. 3. The limit cycle of system (29) and the boundary of the region $\mathcal{R}$.

The corresponding Abel equation (11) for this system is

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2,$$

where

$$A(\theta) = \frac{(13 - 90\sin(2\theta) + 50\cos(2\theta))^2}{4500}$$

and

$$B(\theta) = 3\sin(2\theta) + \frac{137}{45}\cos(2\theta) - \frac{13}{90}.$$

Thus the expression $aA(\theta) + bB(\theta)$ is not zero and does not change sign if $a \neq 0$ and $b = 0$. System (29) has one limit cycle inside the region $\mathcal{R}$ defined in (3). Note that this region contains the origin of coordinates. The limit cycle passes through the point $(x_0, 0)$ with $x_0 \approx 0.343$. Fig. 3 is a representation of the limit cycle, numerically found, together with part of the boundary of the region $\mathcal{R}$ defined in (3).

This example provides a system which illustrates statement (x) of Theorem 4.

6. Proof of Theorem 10

We give a set of autonomous real planar polynomial differential systems of the form (1) which satisfy the criteria established in the statements of Theorems 3 and 4 (except statement (ix)), 7 and 8 and realize the maximum number of limit cycles provided in these statements.

Example 29. Consider the differential system

$$\dot{x} = -y + \lambda\left(\mu x^3 + 2x^2 y + \mu x y^2\right), \qquad \dot{y} = x + \lambda\left(\mu x^2 y + 2xy^2 + \mu y^3\right), \tag{30}$$

where λ and μ are real parameters with $\mu > 1$. If we take polar coordinates system (30) is transformed to

$$\dot{r} = \lambda r^3 (\mu + \sin(2\theta)), \qquad \dot{\theta} = 1.$$

Note that the region $\mathcal{R}$ is the whole plane. We have that $\alpha(\theta) = 0$ and $\beta(\theta) = \lambda(\mu + \sin(2\theta))$. Note that this last function is always positive or negative, depending on the sign of λ, because we are under the assumption that $\mu > 1$. Applying statement (i) of Theorem 4 we obtain that system (30) has no limit cycles in the real plane. This example provides a system which illustrates statement (i) of Theorem 4.

Example 30. Assume that we have an autonomous real planar polynomial differential system defined by the sum of two quasi-homogeneous vector fields as in (6) and we do the transformation to system (7). Assume that the functions α, β, γ and δ are such that $\alpha\delta - \beta\gamma \equiv 0$. From the definition of these functions, this condition is equivalent to

$$P_n(\cos\theta, \sin\theta)\, Q_m(\cos\theta, \sin\theta) - P_m(\cos\theta, \sin\theta)\, Q_n(\cos\theta, \sin\theta) \equiv 0.$$

Since P_n and Q_m are (p, q)-quasi-homogeneous functions we deduce the identity

$$P_n(x, y)\, Q_m(x, y) - P_m(x, y) Q_n(x, y) \equiv 0, \tag{31}$$

which is satisfied for any $(x, y) \in \mathbb{R}^2$. If one of these four polynomials P_n, P_m, Q_n, Q_m is identically zero, we get that there is at least another polynomial which is identically zero among the other three. Thus we have two possibilities: either a system in which $\dot{x} \equiv 0$ or $\dot{y} \equiv 0$, which has no limit cycles because there is a linear first integral, or we have a system which is (p, q)-quasi-homogeneous and it has no limit cycles as it is proved, for instance, in [14]. If the polynomial $P_n P_m Q_n Q_m \not\equiv 0$, then condition (31) implies a divisibility condition between the polynomials and we can deduce that system (6) writes as

$$\dot{x} = P_r(x, y)\, A(x, y), \qquad \dot{y} = Q_r(x, y)\, A(x, y),$$

where P_r and Q_r are (p, q)-quasi-homogeneous polynomials of degrees $p + r - 1$ and $q + r - 1$ respectively and $A(x, y)$ is a real polynomial defined by the sum of two (p, q)-quasi-homogeneous polynomials of degrees $n - r$ and $m - r$. Therefore the system has no limit cycles as a consequence of the results provided in [14]. This analysis shows all the possible examples of systems of the form (6) which satisfy statement (ii) of Theorem 4.

Example 31. Consider the differential system

$$\begin{aligned} \dot{x} &= (x - y)(x^2 - xy + y^2) - x(x^4 + 3x^2y^2 + 2y^4), \\ \dot{y} &= (x + y)(2x^2 - xy + 2y^2) - y(x^4 + 3x^2y^2 + 2y^4), \end{aligned} \tag{32}$$

Note that this system is the sum of two homogeneous vector fields, one of degree 3 and the other of degree 5. We take the polar coordinates and system (32) writes as

$$\dot{r} = r^3(\cos^2\theta + 2\sin^2\theta)(1-r^2), \quad \dot{\theta} = r^2(2\cos^2\theta + \sin^2\theta).$$

We can reparameterize the independent variable and we obtain the system

$$\dot{r} = r(\cos^2\theta + 2\sin^2\theta)(1-r^2), \quad \dot{\theta} = 2\cos^2\theta + \sin^2\theta. \tag{33}$$

The region $\mathcal{R}$ is the whole plane as $2\cos^2\theta + \sin^2\theta > 0$ for all $\theta \in \mathbb{R}$. In this system we have that $\delta(\theta) = 0$. Hence we can apply statement (iii) of Theorem 4 and system (32) has at most one limit cycle. In fact system (32) has the circle $x^2 + y^2 = 1$ as limit cycle as we can observe from its expression in polar coordinates and this is its unique limit cycle. This example provides a system which illustrates statement (iii) of Theorem 4.

Example 32. Consider system (32) of the previous example. In polar coordinates (after several reparameterizations) we obtain system (33), which can be written as the ordinary differential equation

$$\frac{dr}{d\theta} = \frac{r(\cos^2\theta + 2\sin^2\theta)(1-r^2)}{2\cos^2\theta + \sin^2\theta} = S(r,\theta).$$

Then we have that

$$\frac{\partial S}{\partial r} = \left(\frac{\cos^2\theta + 2\sin^2\theta}{2\cos^2\theta + \sin^2\theta}\right)(1-3r^2).$$

Thus in the simply connected region $U = \left\{(r,\theta) : 0 \le r < 1/\sqrt{3}\right\}$ we have that there are no limit cycles. So we have an example which illustrates statement (i) of Theorem 3.

Moreover in the annular region $\mathcal{A} = \left\{(r,\theta) : 1/\sqrt{3} < r < 2\right\}$, by statement (ii) of Theorem 3 we have that there is at most one limit cycle, and this is the case because system (32) has the circle $x^2 + y^2 = 1$ as limit cycle as we have previously stated. This example provides a system which also illustrates statement (ii) of Theorem 3. Moreover, we can compute the function

$$\frac{\partial^2 S}{\partial r^2} = -6r\left(\frac{\cos^2\theta + 2\sin^2\theta}{2\cos^2\theta + \sin^2\theta}\right).$$

It is clear that this function is negative in a punctured neighborhood of the origin and, therefore, as an application of criterion (iii) of Theorem 3 we have that there is at most one limit cycle.

Example 33. Consider the following system

$$\dot{x} = \lambda x - y + x^3, \qquad \dot{y} = x + \lambda y + y^3, \tag{34}$$

where λ is a real parameter. In polar coordinates this system writes as

$$\dot{r} = \lambda r + r^3\left(\cos^4(\theta) + \sin^4(\theta)\right), \quad \dot{\theta} = 1 - r^2\,\frac{\sin(4\theta)}{4}.$$

We consider this system in the region

$$\mathcal{R} = \left\{(r,\theta) : 1 - r^2 \frac{\sin(4\theta)}{4} > 0\right\}.$$

Note that the region $\mathcal{R}$ contains the simply connected region $U = \{(r,\theta) : r^2 < 4\}$, which is a neighborhood of the origin. Note also that in the region $\mathcal{R}$, system (34) contains no equilibrium point but the origin of coordinates. We constraint to study the limit cycles entirely contained in the region $\mathcal{R}$.

By the computation of the Poincaré–Liapunov quantities, it is easy to show that when $\lambda < 0$ is small enough there is a limit cycle in $\mathcal{R}$ which surrounds the origin and appears by a Hopf bifurcation. Easy computations show that system (34) defines a rotated family of vector fields in the region $\mathcal{R}$, see [12,13,31]. Thus the limit cycle born in a Hopf bifurcation grows as λ decreases until it meets the boundary of the region $\mathcal{R}$. We will use the criteria given in statements (i) and (ii) of Theorem 3 in order to locate this limit cycle and to give values of λ for its existence.

In the region $\mathcal{R}$ system (34) is equivalent to the ordinary differential equation

$$\frac{dr}{d\theta} = S(r,\theta) = \frac{4\lambda r - r^3(3+\cos(4\theta))}{4 - r^2\sin(4\theta)}.$$

The derivative of this function with respect to r gives

$$\frac{\partial S}{\partial r} = \frac{32\lambda + 8r^2\left(9 + 3\cos(4\theta) + \lambda\sin(4\theta)\right) - r^4\left(6\sin(4\theta) + \sin(8\theta)\right)}{2(4 - r^2\sin(4\theta))^2}.$$

Let

$$\Sigma(\lambda, r, \theta) = 32\lambda + 8r^2\left(9 + 3\cos(4\theta) + \lambda\sin(4\theta)\right) - r^4\left(6\sin(4\theta) + \sin(8\theta)\right).$$

We can solve the equation $\Sigma(\lambda, r, \theta) = 0$ with respect to r, and we see that when we fix λ small, there is a solution curve which is close to $r = 0$ and it belongs to the region $\mathcal{R}$. This curve is an oval surrounding the origin in the region $\mathcal{R}$ and we denote it by σ_λ. By applying statements (i) and (ii) of Theorem 3, we have that σ_λ defines an inner boundary for the limit cycle because inside this oval there is no limit cycle as an application of statement (i) of Theorem 3, and in the region outside this oval (inside the region $\mathcal{R}$) there can be at most one limit cycle as an application of statement (ii) of Theorem 3. For the application of statement (ii) of Theorem 3, we can take as annular region the one whose inner boundary is σ_λ and whose outer boundary is $r = 2$. If the limit cycle exists for some value $\lambda^* < 0$, which is the case by the Hopf bifurcation, then it exists by some value slightly smaller than λ^* by the properties of a rotated family of vector fields. Moreover, the oval σ_λ keeps growing when λ decreases. Thus it provides an inner boundary for the limit cycle which grows with it. We are going to study until which value of λ we can ensure the existence of such limit cycle. We solve the following optimization problem with constraints: find the minimum λ such that $\Sigma(\lambda, r, \theta) = 0$. We construct the Lagrangian function associated to this problem and we study its equilibrium points, assuming $\lambda < 0$. We get that there is a minimum when $\lambda = -2\sqrt{2}$. We have checked that when $0 > \lambda > -2\sqrt{2}$, the oval σ_λ is contained in the region $\mathcal{R}$, so we can ensure the existence of exactly one limit cycle in the region $\mathcal{R}$ for any value of λ such that $0 > \lambda > -2\sqrt{2}$, and it must be located outside the oval σ_λ. We provide a

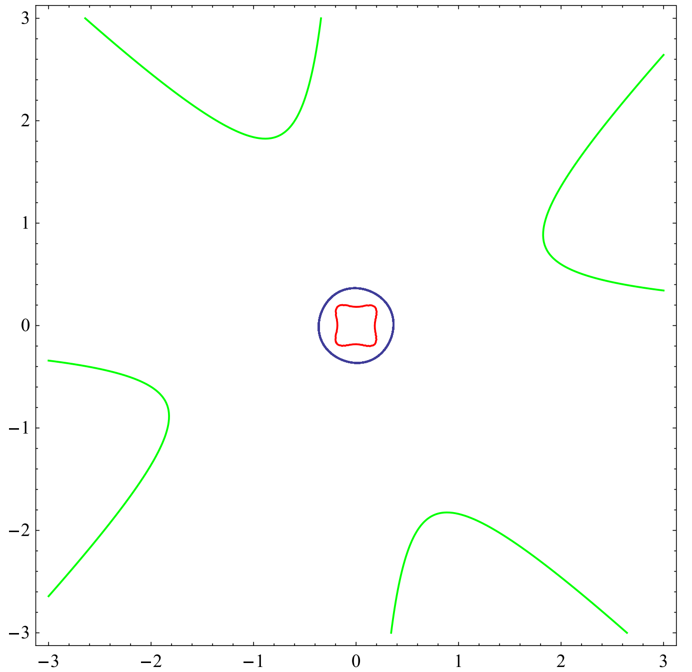

Fig. 4. The limit cycle of system (34), the oval σ_λ and the boundary of the region $\mathcal{R}$ when $\lambda = -1/10$.

couple of figures of the limit cycle numerically found, see Figs. 4–7. We note that the oval σ_λ touches the boundary of the region $\mathcal{R}$ exactly when λ decreases from 0 and gets to $\lambda = -2\sqrt{2}$.

Example 34. We consider the following planar polynomial differential system

$$\dot{x} = y(1 + 2x^2 + 3xy), \quad \dot{y} = -x(3 + 3xy - 2y^2), \tag{35}$$

which in polar coordinates writes as

$$\dot{r} = 2\cos\theta \sin\theta r(r^2 - 1), \quad \dot{\theta} = -3 + 2\sin^2\theta - 3\cos\theta\sin\theta\, r^2.$$

Note that the region

$$\mathcal{R} := \left\{(r,\theta) : -3 + 2\sin^2\theta - 3\cos\theta\sin\theta\, r^2 < 0\right\},$$

is not empty and contains the origin of coordinates. We remark that $r = 1$ (the unit circumference in cartesian coordinates) is an orbit of the system which contains no equilibrium points because over $r = 1$ we have that $\dot{\theta}$ is $-3+\sin^2\theta - 3\cos\theta\sin\theta$ which is strictly negative for all $\theta \in [0, 2\pi]$. Thus $r = 1$ is a periodic orbit. The divergence of system (35) is $\mathrm{div}(x, y) = (3x + y)(-x + 3y)$. We want to compute the value of the integral of the divergence over the orbit given by the unit circumference in order to prove that this periodic orbit is a hyperbolic limit cycle. We denote by $x(t)$, $y(t)$ the periodic

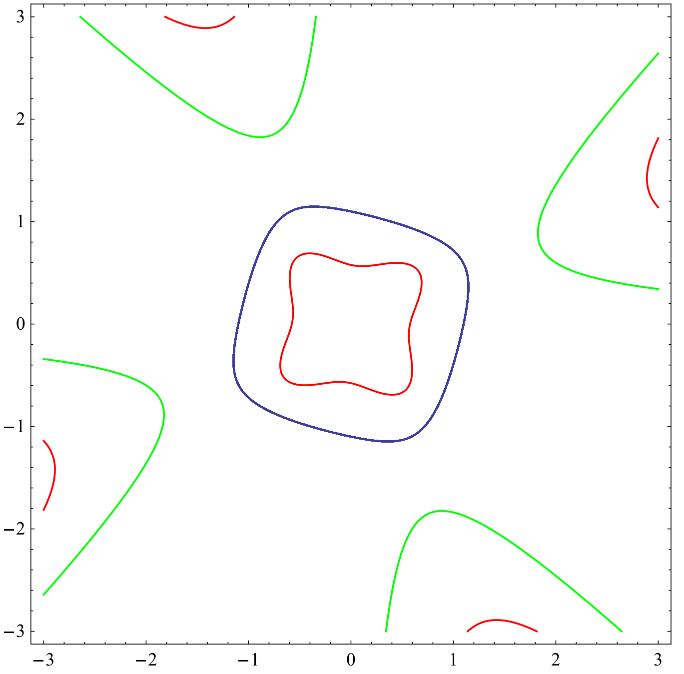

Fig. 5. The limit cycle of system (34), the oval σ_λ and the boundary of the region $\mathcal{R}$ when $\lambda = -1$.

orbit of system (35) contained in the unit circumference and we denote by T its period. We consider the change of the time t to the variable τ provided by $x(t) = \cos\tau$ and we have that

$$\begin{aligned} \int_0^T \operatorname{div}(x(t), y(t))\, dt &= \int_0^{2\pi} \left. \frac{(3x+y)(-x+3y)}{y(1+2x^2+3xy)} \right|_{\left\{ \begin{array}{l} x=\cos\tau \\ y=\sin\tau \end{array} \right\}} (-\sin\tau) d\tau \\ &= \int_0^{2\pi} \frac{(\cos\tau - 3\sin\tau)(\sin\tau + 3\cos\tau)}{2\cos^2\tau + 3\sin\tau\cos\tau + 1}\, d\tau \\ &= \frac{8}{13}\left(4\sqrt{3} - 3\right)\pi \neq 0. \end{aligned}$$

Thus, the unit circumference is a hyperbolic limit cycle of system (35).

We consider the functions $\alpha(\theta)$, $\beta(\theta)$, $\gamma(\theta)$ and $\delta(\theta)$ defined in (8) and we have that

$$\int_0^{2\pi} \frac{\alpha(\theta)}{\gamma(\theta)} d\theta = \int_0^{2\pi} \frac{2\cos\theta\sin\theta}{3 - 2\sin^2\theta}\, d\theta = 0.$$

On the other hand, we remark that system (35) is the sum of a linear system plus a cubic homogeneous system. Thus the expression

$$\exp\left((m-n) \int_0^\theta \frac{\alpha(s)}{\gamma(s)} ds \right),$$

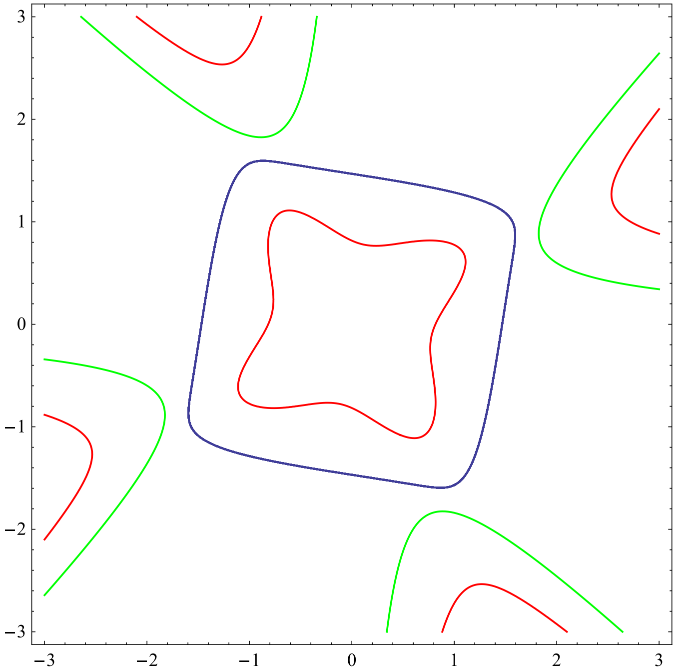

Fig. 6. The limit cycle of system (34), the oval σ_λ and the boundary of the region $\mathcal{R}$ when $\lambda = -2$.

which appears in statement (xi) of Theorem 4 is

$$\exp\left(2\int_0^\theta \frac{2\cos s\sin s}{3-2\sin^2 s}\,ds\right) = \frac{3}{3-2\sin^2\theta}.$$

This expression is strictly positive for all $\theta \in [0, 2\pi]$. The expression of the function $A(\theta)$ defined in (11) is

$$A(\theta) = \frac{12\cos^2\theta\sin^2\theta\,(3\cos^2\theta + 3\cos\theta\sin\theta + 2\sin^2\theta)}{3\cos^\theta + \sin^2\theta},$$

which does not vanish identically and does not change sign in $[0, 2\pi]$. Therefore system (35) illustrates statement (xi) of Theorem 4 (taking $a \neq 0$ and $b = 0$) and it has the maximum number of limit cycles given by the statement.

Example 35. In the works [18,19] Gasull, Llibre and Sotomayor study planar vector fields of the form $X(v) = Av + f(v)Bv$ where A and B are 2×2 real matrices which satisfy certain hypothesis about its eigenvalues, $\det A \neq 0$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth real function such that its expression in polar coordinates is $f(r\cos\theta, r\sin\theta) = r^D \bar{f}(\theta)$ with $D \geq 1$. For this class of systems, the authors prove several results on the existence of limit cycles. In Proposition 6.3 of [19] it is proved that the following planar polynomial

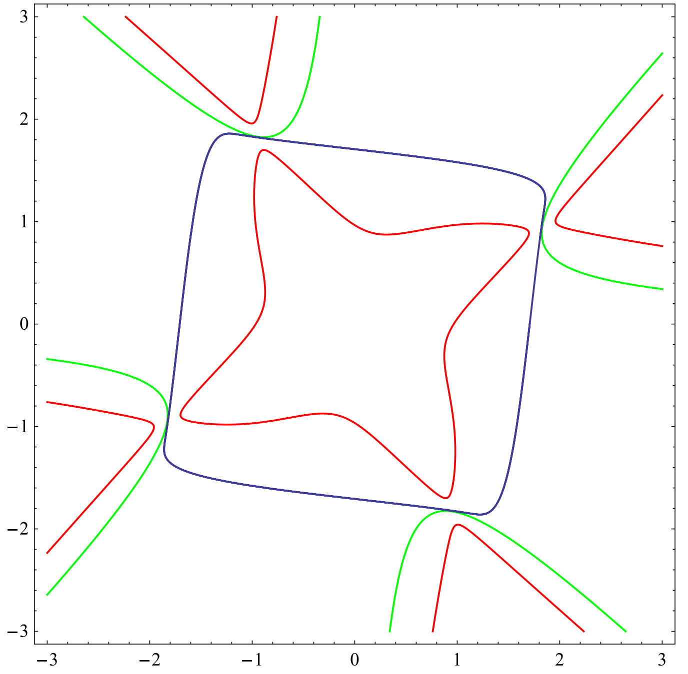

Fig. 7. The limit cycle of system (34), the oval σ_λ and the boundary of the region $\mathcal{R}$ when $\lambda = -2.8$.

differential system

$$\dot{x} = y + ax - (x+y)(ax^2 + xy + ay^2), \quad \dot{y} = -ay + (x-y)(ax^2 + xy + ay^2), \tag{36}$$

where $a \in (-1/2, (1-\sqrt{2})/2)$ satisfies the hypothesis stated in the work and

(i) if $a \in (-1/2, -1/4) \cup (-1/4, (1-\sqrt{2})/2)$, then system (36) has exactly two hyperbolic limit cycles;
(ii) if $a = -1/4$, then system (36) has exactly one semistable limit cycle.

In polar coordinates system (36) writes as

$$\dot{r} = (a + \cos\theta \sin\theta)(r - r^3), \quad \dot{\theta} = -\sin^2\theta + r^2(a + \cos\theta\sin\theta).$$

It is easy to show that the functions α, β, γ and δ defined in (8) satisfy $\delta(\alpha\delta - \beta\gamma)$ does not change sign. This is an example of application of criterion (viii) of Theorem 4 with the maximum number of limit cycles stated by the criterion.

Example 36. We consider the van der Pol system

$$\dot{x} = y - \varepsilon\left(\frac{x^3}{3} - x\right), \quad \dot{y} = -x, \tag{37}$$

with $\varepsilon > 0$.

The origin is the only finite equilibrium point of the system and it is an unstable focus. This system was presented and studied in [39]. It is known that system (37) has a unique stable and hyperbolic limit cycle for all $\varepsilon > 0$ which bifurcates from the circle of radius 2 when $\varepsilon = 0$ and which disappears to a slow-fast periodic limit set when $\varepsilon \to +\infty$. See also the book [32].

In polar coordinates the system becomes

$$\dot{r} = \varepsilon \cos^2(\theta) r^2 - \frac{\varepsilon}{3} \cos^4(\theta) r^3, \quad \dot{\theta} = -1 - \varepsilon \cos(\theta)\sin(\theta) + \frac{\varepsilon}{3} \cos^3(\theta)\sin(\theta) r^2.$$

The functions $\alpha(\theta)$, $\beta(\theta)$, $\gamma(\theta)$ and $\delta(\theta)$ are defined as in (7) and it is easy to see that

$$\alpha\delta - \beta\gamma = -\frac{\varepsilon}{3} \cos^4(\theta),$$

which does not change sign for $\theta \in [0, 2\pi]$. When $0 < \varepsilon < 2$, we are under the analogous hypothesis to (3) that

$$\mathcal{R} = \left\{ (r, \theta) \, : \, -1 - \varepsilon \cos(\theta)\sin(\theta) + \frac{\varepsilon}{3} \cos^3(\theta)\sin(\theta) r^2 < 0 \right\},$$

is not empty and contains the origin of coordinates $r = 0$. For instance, when $\varepsilon = 1$ it can be shown that the limit cycle of (37) is inside the region $\mathcal{R}$. Thus this system illustrates an application of criterion (vii) of Theorem 4.

Example 37. In the work of 1960 [40] Vorob'ev gives the following planar polynomial differential system

$$\dot{x} = -y + ax(x^2 + y^2 - 1), \quad \dot{y} = x + by(x^2 + y^2 - 1), \tag{38}$$

where a and b are real parameters such that $ab > -1$ and $(a - b)^2 > 4$. This system has a node at the origin and one limit cycle surrounding the origin. In polar coordinates, it writes as

$$\dot{r} = (r^3 - r)(a \cos^2\theta + b \sin^2\theta), \; \dot{\theta} = 1 + (r^2 - 1)(b - a)\cos\theta \sin\theta.$$

The functions α, β, γ and δ defined in (8) satisfy $(\alpha\delta - \beta\gamma) = -(a \cos^2 t + b \sin^2 t)$ which does not change sign if $ab > 0$. This is an example of application of criterion (vii) of Theorem 4.

We summarize below how each statement of Theorems 3, 4, 7 and 8 has been illustrated by an example which exhibits the maximum number of limit cycles given by the criterion.

- Criterion (i) of Theorem 3 is illustrated in Examples 18, 32 and 33.
- Criterion (ii) of Theorem 3 is illustrated in Examples 18, 32 and 33.
- Criterion (iii) of Theorem 3 is illustrated in Examples 18, 21 and 32.
- Criterion (iv) of Theorem 3 is illustrated in Examples 20 and 21.
- Criterion (v) of Theorem 3 is illustrated in Example 23.
- Criterion (vi) of Theorem 3 is illustrated in Example 22.
- Criterion (i) of Theorem 4 is illustrated in Example 29.
- Criterion (ii) of Theorem 4 is illustrated in Example 30.
- Criterion (iii) of Theorem 4 is illustrated in Example 31.

- Criterion (iv) of Theorem 4 is illustrated in Examples 20 and 26.
- Criterion (v) of Theorem 4 is illustrated in Example 20.
- Criterion (vi) of Theorem 4 is illustrated in all the examples where criterion (iii) of Theorem 3 is applied because $\delta \equiv 0$ implies that $A(\theta) \equiv 0$, where $A(\theta)$ is the function that appears in (11). Indeed, by virtue of Remark 14, we have that the case $A(\theta) \equiv 0$ and $B(\theta) \equiv 0$ are equivalent.
- Criterion (vii) of Theorem 4 is illustrated in Examples 36 and 37.
- Criterion (viii) of Theorem 4 is illustrated in Examples 27 and 35.
- Criterion (x) of Theorem 4 is illustrated in Examples 19 and 28.
- Criterion (xi) of Theorem 4 is illustrated in Example 34. All the examples which illustrate criterion (x) of Theorem 4 also illustrate this criterion since criterion (xi) implies criterion (x) in Theorem 4.
- Theorem 7 is illustrated in Examples 20 and 24.
- Theorem 8 is illustrated in Example 25.

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Data availability

No data was used for the research described in the article.

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