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# The criticality of reversible quadratic centers at the outer boundary of its period annulus ☆

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**Abstract**

This paper deals with the period function of the reversible quadratic centers

$$X_\nu = -y(1-x)\partial_x + (x + Dx^2 + Fy^2)\partial_y,$$

where $\nu = (D, F) \in \mathbb{R}^2$. Compactifying the vector field to $\mathbb{S}^2$, the boundary of the period annulus has two connected components, the center itself and a polycycle. We call them the inner and outer boundary of the period annulus, respectively. We are interested in the bifurcation of critical periodic orbits from the polycycle $\Pi_\nu$ at the outer boundary. A critical period is an isolated critical point of the period function. The criticality of the period function at the outer boundary is the maximal number of critical periodic orbits of $X_\nu$ that tend to $\Pi_{\nu_0}$ in the Hausdorff sense as $\nu \to \nu_0$. This notion is akin to the cyclicity in Hilbert's 16th Problem. Our main result (Theorem A) shows that the criticality at the outer boundary is at most 2 for all $\nu = (D, F) \in \mathbb{R}^2$ outside the segments $\{-1\} \times [0, 1]$ and $\{0\} \times [0, 2]$. With regard to the bifurcation from the inner boundary, Chicone and Jacobs proved in their seminal paper on the issue that the upper bound is 2 for all $\nu \in \mathbb{R}^2$. In this paper the techniques are different because, while the period function extends analytically to the center, it has no smooth extension to the polycycle. We show that the period function has an asymptotic expansion near the polycycle with the remainder being uniformly flat with respect to $\nu$

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and where the principal part is given in a monomial scale containing a deformation of the logarithm, the so-called Écalle-Roussarie compensator. More precisely, Theorem A follows by obtaining the asymptotic expansion to fourth order and computing its coefficients, which are not polynomial in $\nu$ but transcendental. Theorem A covers two of the four quadratic isochrones, which are the most delicate parameters to study because its period function is constant. The criticality at the inner boundary in the isochronous case is bounded by the number of generators of the ideal of all the period constants but there is no such approach for the criticality at the outer boundary. A crucial point to study it in the isochronous case is that the flatness of the remainder in the asymptotic expansion is preserved after the derivation with respect to parameters. We think that this constitutes a novelty that is of particular interest also in the study of similar problems for limit cycles in the context of Hilbert's 16th Problem. Theorem A also reinforces the validity of a long standing conjecture by Chicone claiming that the quadratic centers have at most two critical periodic orbits. A less ambitious goal is to prove the existence of a uniform upper bound for the number of critical periodic orbits in the family of quadratic centers. By a compactness argument this would follow if one can prove that the criticality of the period function at the outer boundary of any quadratic center is finite. Theorem A leaves us very close to this existential result.


*MSC:* 34C07; 34C20; 34C23

*Keywords:* Criticality; Period function; Asymptotic expansion; Critical periodic orbit; Hilbert's 16th problem

**Contents**



## 1. Introduction and main results

A singular point $p\in\mathbb{R}^2$ of a planar differential system

$$\begin{cases}\dot{x}=f(x,y),\\ \dot{y}=g(x,y),\end{cases}$$

is a *center* if it has a punctured neighborhood that consists entirely of periodic orbits surrounding $p$. The *period annulus* of the center is the largest punctured neighborhood with this property and we denote it by $\mathscr{P}$. The period annulus is an open subset of $\mathbb{R}^2$ that may be unbounded. For this reason we embed $\mathscr{P}$ in $\mathbb{RP}^2$ and, abusing notation, we denote the boundary of the resulting set by $\partial\mathscr{P}$. Clearly the center $p$ belongs to $\partial\mathscr{P}$ and in what follows we call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi := \partial\mathscr{P} \setminus \{p\}$, which is a nonempty compact subset of $\mathbb{RP}^2$. The subject of our study is the *period function* of the center, that assigns to each periodic orbit in $\mathscr{P}$ its period. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties we need to parametrize this set. This can be done by taking a transverse section to the vector field $X = f(x, y)\partial_x + g(x, y)\partial_y$ on $\mathscr{P}$, for instance an orbit of the orthogonal vector field $X^\perp$. To fix ideas let us suppose that $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization where $s \approx 0$ corresponds to the periodic orbits near $p$ and $s \approx 1$ to the ones near $\Pi$. Then the map $P\colon (0, 1) \longrightarrow (0, +\infty)$ defined by $P(s) := \{\text{period of } \gamma_s\}$ provides the qualitative properties of the period function that we are concerned with and one can readily show by using the Implicit Function Theorem that it is as smooth as $X$. It is also well-known that if $X$ is analytic and the center $p$ is non-degenerate then $P$ extends analytically to $s = 0$. Let us advance that, on the contrary, $P$ does not extend smoothly to $s = 1$. The *critical periods* are the isolated critical points of $P$, i.e. $\hat{s} \in (0, 1)$ such that $P'(\hat{s}) = 0$ and $P'(s) \neq 0$ if $0 < |s - \hat{s}| < \varepsilon$. In this case, more geometrically, we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of $X$. One can easily see that the property of being a critical periodic orbit does not depend on the particular parametrization of the set of periodic orbits used, see Remark 2.2. The study of the critical periodic orbits is another issue arising from the famous Hilbert's 16th Problem and it has strong parallelisms with the research on limit cycles, from both the conceptual and technical point of views. In this regard we can mention for instance that the isochronicity problem (i.e., to decide whether a center has a constant period function) is the counterpart of the center-focus problem. The renowned conjecture claiming that a quadratic differential system can have at most four limit cycles has also an analogue in the context of the period function and it was posed by C. Chicone [5]. More specifically this conjecture asserts that if a quadratic center has some critical periodic orbit then by an affine transformation and a constant rescaling of time it can be brought to Loud normal form

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases} \tag{1}$$

and that this center has at most two critical periodic orbits for any $(B, D, F) \in \mathbb{R}^3$. In fact there is much analytic evidence that this conjecture is true (see [7,40,41] for instance).

The problems that we are interested in take place when the vector field $X$ depends on parameters. To fix notation, let $U$ be an open subset of $\mathbb{R}^N$ and consider a family of planar vector fields $\{X_\mu, \mu \in U\}$ such that each $X_\mu$ has a center $p_\mu$ with period annulus $\mathscr{P}_\mu$. Let us denote the period function of the center $p_\mu$ by $P(\,\cdot\,;\mu)$ and observe that, given some $\mu_0 \in U$, the number of critical periodic orbits of $X_\mu$ can vary as we perturb $\mu \approx \mu_0$. Under some regularity assumptions on the dependence of $\mathscr{P}_\mu$ with respect to $\mu$ it can be proved (see Lemma 2.12) that the emergence/disappearance of critical periodic orbits can only occur from three different places:

($a$) Bifurcations at the inner boundary of the period annulus (i.e., the center $p_\mu$).
($b$) Bifurcations at the outer boundary of the period annulus (i.e., the polycycle $\Pi_\mu$).
($c$) Bifurcations at the interior of the period annulus $\mathscr{P}_\mu$.

Chicone and Jacobs give in their seminal paper [6] a complete description of the bifurcations from the inner boundary for the whole family of quadratic centers. In this case the parameter $\mu$ are the coefficients of the vector field and since the center is non-degenerate $P(s;\mu)$ extends analytically to $s=0$, so that one can consider its Taylor series $P(s;\mu)=\sum_{i=0}^{\infty}a_i(\mu)s^i$ at $s=0$, whose coefficients $a_i$ belong to the polynomial ring $\mathbb{R}[\mu]$. On account of this the result about the bifurcations from the isochronous centers (see [6, Theorem 2.2]), which are the most difficult ones to study, follows by analyzing the ideal $(a_1,a_2,\ldots)$ of all Taylor coefficients exactly as N. Bautin does in [4] to study the bifurcations of limit cycles from the quadratic centers. In the present paper we resume our study of the bifurcations from the outer boundary that we initiated in [22,23]. Let us recall that the differential system (1) has no critical periodic orbits if $B=0$, see [10, Theorem 1]. By means of a rescaling the case $B\neq 0$ can be brought to $B=1$, i.e.,

$$X_\nu := -y(1-x)\partial_x + (x+Dx^2+Fy^2)\partial_y \text{ with } \nu := (D,F). \tag{2}$$

Here we already adopt the parameter notation that we shall use throughout the paper, which is devoted to the bifurcation of critical periodic orbits from the outer boundary in the family $\{X_\nu,\nu\in\mathbb{R}^2\}$. Since each vector field $X_\nu$ is polynomial we can consider its Poincaré compactification $p(X_\nu)$, see [3, §5], which is an analytic vector field on the sphere $\mathbb{S}^2$ topologically equivalent to $X_\nu$. The outer boundary $\Pi_\nu$ becomes then a polycycle of $p(X_\nu)$ that can be studied using local charts of $\mathbb{S}^2$, but even so the period function $P(s;\nu)$ cannot be smoothly extended to $s=1$. For the family under consideration we show that $P(s;\nu)$ has an asymptotic expansion at $s=1$ with the remainder being uniformly flat with respect to $\nu$ and where the principal part is given in a monomial scale containing a deformation of the logarithm, the so-called Écalle-Roussarie compensator. Our main theorem follows by obtaining the asymptotic expansion to fourth order and computing its coefficients, which are not polynomial in $\nu$ but transcendental (more concretely, they are hypergeometric functions). To this end we strongly rely on the tools that we develop in our recent papers [29–31]. The results that we obtain in the present paper can be viewed conceptually as the analogue for the outer boundary of the work carried out by Chicone and Jacobs in [6] on the bifurcation of critical periodic orbits from the inner boundary of the quadratic centers. That being said, the proofs of the results on the outer boundary are technically tougher than the ones on the inner boundary because $\Pi_\nu$ is a polycycle and the period function $P(s;\nu)$ cannot be analytically extended there. By way of example, to determine the parameters that vanish simultaneously two coefficients in the asymptotic expansion at $s=1$ takes 5 pages of computations dealing with a hypergeometric function (see Appendix C), whereas the same problem for the Taylor series at $s=0$ can be solved readily by taking resultants because the coefficients are polynomials.

In this paper we use the notion of *criticality* of the period function at the outer boundary which, roughly speaking, is the number of critical periodic orbits that can emerge or disappear from $\Pi_\nu$ as we perturb $\nu$ slightly. It is defined in exactly the same way as the notion of *cyclicity* of a limit periodic set, which is used to study the bifurcation of limit cycles in the context of Hilbert's 16th Problem, see [36] for instance. Before giving its precise definition, and the statement of our main contribution, we enumerate the previous results about the bifurcation of critical periodic orbits from the outer boundary $\Pi_\nu$ for the family $\{X_\nu,\nu\in\mathbb{R}^2\}$. In this regard we stress that these results are given according to the dichotomy between local regular value and local bifurcation value (of the period function at the outer boundary) that we introduce in our early paper [23]. This notion (see Definition 2.10) enables to obtain a structure theorem for the bifurcation diagram of the period function in its full domain (see Lemma 2.12), but it has the inconvenience of not being so

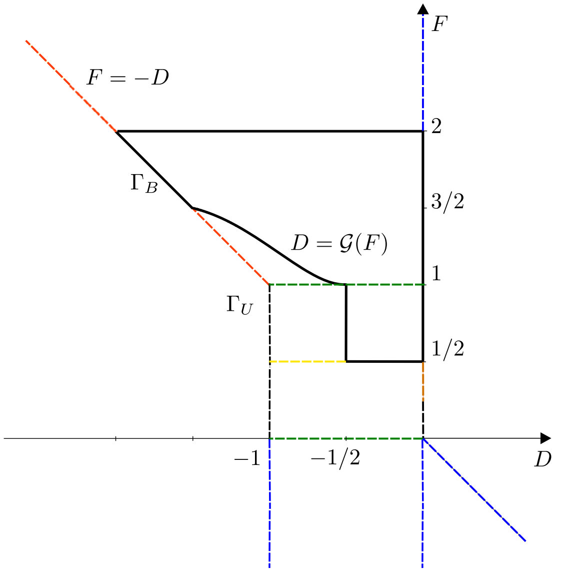


Fig. 1. The thick (closed) curve $\Gamma_B$ consists of local bifurcation values of the period function at the outer boundary according to [23], where the curve that joins $\left(-\frac{3}{2}, \frac{3}{2}\right)$ and $\left(-\frac{1}{2}, 1\right)$ is the graphic of an analytic function $D = \mathcal{G}(F)$, see Remark 3.4. The dotted lines $\Gamma_U$ correspond to parameters that remained unspecified in that paper and we color the subsequent improvements obtained in [19,24,25,27,28,38,39]. The parameters outside $\Gamma_B \cup \Gamma_U$ are local regular values by the result in [23]. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

quantitative and geometric as the criticality. In order to simplify the exposition for the moment we can think that $\nu_0 \in \mathbb{R}^2$ is a local regular value if and only if the criticality of the period function at $\Pi_{\nu_0}$ is zero (i.e., no critical periodic orbit bifurcates from $\Pi_{\nu_0}$ as we perturb $\nu \approx \nu_0$). That said, let $\Gamma_U$ be the union of dotted straight lines in Fig. 1, whatever its color is. Consider also the thick curve $\Gamma_B$. (Here the subscripts $B$ and $U$ stand for bifurcation and unspecified respectively.) Then according to [23, Theorem A] the open set $\mathbb{R}^2 \setminus (\Gamma_B \cup \Gamma_U)$ corresponds to local regular values and $\Gamma_B$ consists of local bifurcation values (of the period function at the outer boundary). In that paper we also conjecture that any parameter in $\Gamma_U$ is regular, except for the segment $\{0\} \times \left[0, \frac{1}{2}\right]$ in the vertical axis, that should consist of bifurcation values. Since the formulation of this conjecture there has been some progress in the study of the parameters in $\Gamma_U$:

- From the results in [19,39] it follows that the parameters in blue are regular. In these papers the authors determine a region $M$ in the parameter plane for which the corresponding center has a globally monotonous period function (i.e., it has no critical periodic orbits). The parameters that we draw in blue are inside the interior of $M$, which prevents the bifurcation of critical periodic orbits.
- Along the straight line $F = -D$ there is a breaking of a heteroclinic connection between two hyperbolic saddles at the outer boundary. From the results in [24] it follows that the parameters in red are regular.
- Along the two segments in green it occurs a saddle-node bifurcation at the outer boundary of the period annulus. An asymptotic expansion of the Dulac time of this type of unfolding is obtained in [25] and as an application it is proved that the parameters in the segment

$(-1,0)\times\{1\}$, with the exception of $(-\frac{1}{2},1)$, are local regular values. A subsequent refinement of this approach shows in [27] that the segment $(-1,0)\times\{0\}$ also consists of local regular values.

- By [28, Theorem B] the parameters in brown, more precisely the segment $\{0\}\times\left[\frac{1}{4},\frac{1}{2}\right]$, are local bifurcation values of the period function at the outer boundary.
- Along the segment $(-1,0)\times\{\frac{1}{2}\}$ there is a resonant saddle at $\Pi_\nu$ and the parameters in yellow are local regular values at the outer boundary of the period annulus according to [38, Corollary B].

As we already explained, these results are addressed to solve the dichotomy between local regular value and local bifurcation value (of the period function at the outer boundary). Beyond this dichotomy a challenging problem is the computation of the exact number of critical periodic orbits that can bifurcate from the outer boundary, which constitutes the counterpart of the result by Chicone and Jacobs [6] about the bifurcation from the inner boundary. The following is the precise definition of the number that we aim to compute for the quadratic centers, where $d_H$ stands for the Hausdorff distance between compact sets of $\mathbb{RP}^2$.

**Definition 1.1.** Consider a $\mathscr{C}^\infty$ family $\{X_\mu,\mu\in U\}$ of planar vector fields with a center and fix some $\mu_0\in U$. Suppose that the outer boundary of the period annulus varies continuously at $\mu_0\in U$, meaning that $d_H(\Pi_\mu,\Pi_{\mu_0})$ tends to zero as $\mu\to\mu_0$. Then, setting

$$N(\delta,\varepsilon)=\sup\Big\{\#\text{ critical periodic orbits }\gamma\text{ of }X_\mu\text{ in }\mathscr{P}_\mu\text{ with }d_H(\gamma,\Pi_{\mu_0})\leqslant\varepsilon\text{ and }\|\mu-\mu_0\|\leqslant\delta\Big\},$$

the *criticality* of $(\Pi_{\mu_0},X_{\mu_0})$ w.r.t. $X_\mu$ is $\mathrm{Crit}\big((\Pi_{\mu_0},X_{\mu_0}),X_\mu\big):=\inf_{\delta,\varepsilon}N(\delta,\varepsilon)$. □

We stress that in this definition the vector field $X_\mu$ is not required to be polynomial but $\mathscr{C}^\infty$. This is so because in order to define the outer boundary $\Pi_\mu$ of the period annulus $\mathscr{P}_\mu$ of $X_\mu$ we do not compactify the vector field but only the set $\mathscr{P}_\mu$ and to this end there is no need that $X_\mu$ is polynomial. Certainly $\mathrm{Crit}\big((\Pi_{\mu_0},X_{\mu_0}),X_\mu\big)$ may be infinite but, if it is not, then it gives the maximal number of critical periodic orbits of $X_\mu$ that tend to $\Pi_{\mu_0}$ in the Hausdorff sense as $\mu\to\mu_0$. Related with this issue we point out that the contour of the period annulus $\mathscr{P}_{\mu_0}$ may change for $\mu\approx\mu_0$. The assumption that the period annulus varies continuously ensures that this change does not occur abruptly. In this regard note that $X_\mu=-y\partial_x+(x+\mu x^3+x^5)\partial_y$, with $\mu\in\mathbb{R}$, is a polynomial family of vector fields with a center at the origin for which the outer boundary does not vary continuously at $\mu=2$. This is so because the period annulus $\mathscr{P}_\mu$ is the whole plane for $\mu<2$, whereas it is bounded for $\mu=2$ (see [20] for details). In this example $\mathrm{Crit}\big((\Pi_{\mu_0},X_{\mu_0}),X_\mu\big)$, as introduced in Definition 1.1, does not give the number of critical periodic orbits bifurcating from $\Pi_\mu$ as $\mu\to\mu_0$. Let us mention that this assumption is also required in [17,18], where the authors obtain several results addressed to bound the criticality at the outer boundary of families of vector fields of potential type, i.e., $-y\partial_x+V'(x)\partial_y$.

Let us remark at this point that if Chicone's conjecture about the number of critical periodic orbits of the quadratic centers is true then $\mathrm{Crit}\big((\Pi_{\nu_0},X_{\nu_0}),X_\nu\big)\leqslant 2$ for all $\nu\in\mathbb{R}^2$, see (2). In this paper, by applying our recent results from [29–31], we prove the following (see Fig. 2):

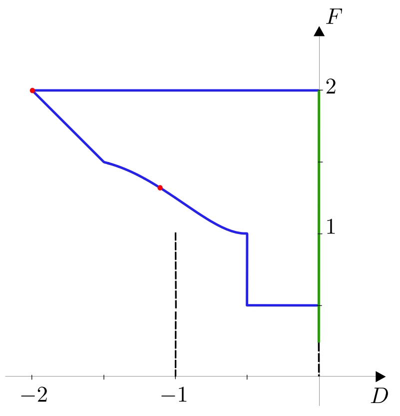


Fig. 2. Overview of the criticality results in Theorem A. The blue curves have criticality 1, the green segment has criticality greater or equal than 1 and the two points in red have criticality 2. Any other parameter, except the two dotted segments in black, has criticality 0.

**Theorem A.** *Let $\{X_\nu, \nu \in \mathbb{R}^2\}$ be the family of quadratic vector fields given in (2) and consider the period function of the center at the origin. Then the following assertions hold:*

(*a*) $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$ *for* $\nu_0 \notin \Gamma_B \cup \big\{D = -1, F \in [0,1]\big\} \cup \big\{D = 0, F \in [0, \frac{1}{2}]\big\}$.
(*b*) $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$ *for* $\nu_0 \in \Gamma_B \setminus \big(\{D = 0\} \cup \big\{(-2,2), (\mathcal{G}(\frac{4}{3}), \frac{4}{3})\big\}\big)$.
(*c*) $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \geqslant 1$ *for* $\nu_0 \in \big\{D = 0, F \in [\frac{1}{4}, 2]\big\}$.
(*d*) $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 2$ *for* $\nu_0 \in \big\{(-2,2), (\mathcal{G}(\frac{4}{3}), \frac{4}{3})\big\}$.
(*e*) *There is a $\mathscr{C}^1$ curve arriving at $\nu = (\mathcal{G}(\frac{4}{3}), \frac{4}{3})$ tangent to $\Gamma_B$ and there is a $\mathscr{C}^0$ curve with a exponential flat contact with $\{F = 2\}$ at $\nu = (-2, 2)$, consisting both of local bifurcation values of the period function at the interior.*

There are some papers containing results related with assertion (*b*) in Theorem A to be referred. Thus, by [33, Theorem A], $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$ for any $\nu_0 = (\mathcal{G}(F_0), F_0)$ with $F_0 \in (\frac{4}{3}, \frac{3}{2})$. This is a piece of the curve that joins $\big(-\frac{3}{2}, \frac{3}{2}\big)$ and $\big(-\frac{1}{2}, 1\big)$, see Fig. 1, and in this regard observe that the criticality is 2 for $\nu_0 = (\mathcal{G}(\frac{4}{3}), \frac{4}{3})$. Furthermore, it is proved in [32, Theorem B] that if $\nu_0 = (D_0, 2)$ with $D_0 \in (-2, 0) \setminus \{-\frac{1}{2}\}$ then $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$. The same conclusion is true for any $\nu_0 = (-F_0, F_0)$ with $F_0 \in [\frac{3}{2}, 2)$ thanks to [24, Theorem C]. In that paper it is also partially proved the claim about the parameter $\nu_0 = (-2, 2)$ in assertion (*e*) of Theorem A. Apart from these references to previous results we also want to point out the following issues with regard to the statement and proof of Theorem A:

- As expected, the study of the bifurcation of critical periodic orbits, either from the inner or the outer boundary, is much more delicate when we perturb an isochronous center. By the result of W.S. Loud, see [15], we know that there are four nonlinear quadratic isochrones,

$$\nu_1 = (0, 1),\ \ \nu_2 = (-1/2, 2),\ \ \nu_3 = (-1/2, 1/2) \text{ and } \nu_4 = (0, 1/4), \tag{3}$$

  which are located in $\Gamma_B$. Chicone and Jacobs prove, see [6, Theorem 3.1], that the criticality of each isochrone $\nu_i$ at the inner boundary of its period annulus (i.e., the center itself) is one. The proof of this follows by finding a finite set of generators for the ideal of *all* the coefficients of the Taylor series of $P(s;\nu)$ at $s=0$. In the present paper we are able to show that $\nu_2$ and $\nu_3$ have criticality one also at the outer boundary (i.e., the polycycle), see Propositions 4.2 and 4.3 respectively. A crucial point to see this is that, as we prove in [30], the flatness of the remainder in the asymptotic expansion at $s=1$ is preserved after the derivation with respect to parameters. This constitutes the cornerstone to obtain Lemma 4.1, which enables us to perform a convenient division in the space of coefficients and proceed then as in the proof of Bautin [4, §3] for the analogous result about the bifurcation of limit cycles from the center. The isochrones $\nu_1$ and $\nu_4$ cannot be analyzed following this approach because the polycycle at the outer boundary is not hyperbolic.

- It is well-known, see [6, Theorem 3.2], that the criticality at the inner boundary of any quadratic center is at most two and that this maximum criticality is achieved at three parameter values, the so-called Loud points, which we give in (13). For consistency with Chicone's conjecture, each one of these three parameters should have a "twin" where the maximum criticality at the outer boundary is attained. In this paper we identify two of these twin parameters, see assertion $(d)$ in Theorem A. We conjecture that each pair of twins is connected by a curve that consists of local bifurcation values at the interior, see Remarks 4.5 and 5.1.
- The local bifurcation values of the period function can only occur at the inner boundary (i.e., the center), at the outer boundary (i.e., the polycycle) or at the interior of the period annulus, see Lemma 2.12. (With regard to the latter, its counterpart in the context of Hilbert's 16th Problem is the bifurcation from a semi-stable limit cycle, which is characterized by the sudden emergence of a double limit cycle that gives rises to two hyperbolic limit cycles with different stability, see [12, §13.3] for instance). As occurs with limit cycles, the identification of this third type of local bifurcation value is out of reach for the moment and only partial results have been obtained. Thus, in a joint paper with P. Mardešić we prove (see [23, Theorem 4.3]) that at each Loud point there exists a germ of analytic curve that consists of local bifurcation values at the interior. Since $P(s;\nu)$ extends analytically to $s=0$, this follows readily by applying the Weierstrass Preparation Theorem. In the present paper, see assertion $(e)$ in Theorem A, we show the existence of two germs of curve which also consists of local bifurcation values at the interior and that are the mirror image at the outer boundary (i.e., at $s=1$) of those previously obtained in [23], see Fig. 6.

In another vein it is well-known (see [36, §2.2] for details) that the problem of proving the *existence* of a uniform bound for the number of limit cycles in a given family, for instance Hilbert's 16th Problem, can be replaced by a local problem that consists in showing that the cyclicity of each limit periodic set within the family is finite. The proof of this is by a compactness argument and it does not provide an algorithm to compute an explicit upper bound even if we had an explicit bound for the cyclicity of every limit periodic set. In any case this gives a program for solving the existential Hilbert's 16th Problem that has been posed and implemented for the quadratic vector fields by R. Roussarie and his collaborators (see [9,34]). One can of course transfer this problem to the period function and ask for the existence of a uniform bound for the number of critical periodic orbits in the family of quadratic centers. Similarly as it occurs in the context of limit cycles, an affirmative answer would follow if one can prove that the criticality of the period function at the outer boundary of any quadratic center is finite, cf. Lemma 2.17. On account of Theorem A we are not very far from proving the existence of this uniform bound for the family of reversible quadratic centers. It will follow in particular if one can prove the validity of the following conjecture (see Figs. 2 and 7):

**Conjecture.** *Let $\{X_\nu, \nu \in \mathbb{R}^2\}$ be the family of quadratic vector fields given in (2) and consider the period function of the center at the origin. Then the following assertions are true:*

$(a)$ $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$ *for* $\nu_0 \in \{D = -1, F \in [0,1]\}$.
$(b)$ $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$ *for* $\nu_0 \in \{D = 0, F \in (0,2]\}$.
$(c)$ $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 2$ *for* $\nu_0 = (0,0)$.
$(d)$ *There is a curve of local bifurcation values of the period function at the interior arriving to* $\nu = (0,0)$ *tangent to* $D = 0$.

As a matter of fact to show the existence of a uniform bound for the number of critical periodic orbits of the reversible quadratic centers it suffices to verify that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big)$ is finite for all $\nu_0 = (D_0, F_0)$ inside the segments $\{-1\} \times [0,1]$ and $\{0\} \times [0,2]$. To put this into context let us recall that the differential system (1) has no critical periodic orbits if $B = 0$ by [10, Theorem 1]. On the other hand, apart from the reversible one, there are essentially three other families of quadratic centers: the Hamiltonian, the codimension four $Q_4$ and the generalized Lotka-Volterra systems $Q_3^{LV}$. According to Chicone's conjecture the number of critical periodic orbits should be zero for the centers in these three families. This is known to be true for the Hamiltonian and $Q_4$ families thanks to the results of Coppel and Gavrilov [7] and Zhao [41], respectively. With regard to the family $Q_3^{LV}$ it is proved in [40] that, except for a subset of codimension one in the parameter plane, the criticality at the outer boundary is zero. It is clear then that any contribution to the proof of the above conjecture will constitute a very significant step forward to the existence of a uniform bound for the number of critical periodic orbits in the *whole* family of quadratic centers. Let us mention in this respect that the singularity at the outer boundary of the period annulus is nilpotent along $D = -1$ and $D = 0$. In this situation the results of [30,31] do not apply and new techniques must be developed.

The paper is organized in the following way. In Section 2 we recall the definition of local bifurcation value at the outer boundary, that we introduce in our early paper [23] to study the bifurcation diagram of the period function of the family $\{X_\nu, \nu \in \mathbb{R}^2\}$, and we prove several results that relate it with the criticality. We also show how to study the criticality by means of a suitable parametrization of the set of periodic orbits near the outer boundary. Section 3 is devoted to the asymptotic expansion of the period function near the outer boundary, which is the cornerstone in the proof of Theorem A. To this end we prove three results that are addressed to three different parameter subsets according to the contour of the period annulus. As one might expect the proofs of these results are rather long and technical. Furthermore they are based on previous tools developed in [29–31] that need to be introduced appropriately. For these reasons, to ease the paper's readability we defer some proofs to Appendix A. In Section 4 we study three distinguished parameters. On one hand the two isochrones for which we succeed in proving that the criticality is one (see Propositions 4.2 and 4.3) and, on the other hand, the parameter $\nu = (\mathcal{G}(\frac{4}{3}), \frac{4}{3})$, which is also rather special because it has criticality two (see Proposition 4.4). Due to the novel approach of its proof we think that each one of these results is of particular interest in the context of Hilbert's 16th Problem. Section 5 is entirely devoted to the proof of Theorem A. Next, in Appendix A we prove the results stated in Section 3 that we mentioned before and in Appendix B we are concerned with the integral representation of the Beta and hypergeometric functions, which usually appear as coefficients in the asymptotic expansions that we obtain. Finally Appendix C is addressed to prove a technical result that is used to study the vanishing set of two coefficients.

## 2. Criticality vs bifurcation

In this section we recap the notion of local bifurcation value of the period function at the outer boundary as we introduced in our early paper [23]. We relate it with the criticality, which is a more quantitative and geometric definition, and prove a general result connecting both notions, see Lemma 2.16. More specifically our aim is to take advantage in the present paper of the results that we obtained in [23] with regard to the period function of Loud's centers (2) and that are not stated using the notion of criticality. Related with this issue, our goal in this section is also to clarify the usage of a parametrization of the period function near the outer boundary to compute its criticality, see Lemma 2.4. Finally we give a sufficient condition in order that a parameter is a local bifurcation value of the period function at the interior, see Lemma 2.15.

Several results in this section are equally valid in the finitely smooth class $\mathscr{C}^k$, $k \in \mathbb{N}$, the infinitely smooth class $\mathscr{C}^\infty$ and the analytic class $\mathscr{C}^\omega$. For simplicity in the exposition we write $\mathscr{C}^\varpi$ with the wild card $\varpi \in \mathbb{N} \cup \{\infty, \omega\}$. Our first result is addressed to the regularity properties of the map $(p, \mu) \mapsto \hat{P}(p; \mu)$ that assigns to each $\mu \in U$ and $p \in \mathscr{P}_\mu$ the period $\hat{P}$ of the periodic orbit of $X_\mu$ passing through the point $p$. The result is given under a technical assumption concerning the existence of a continuous parametrization $\sigma(s; \mu)$ of the period annulus $\mathscr{P}_\mu$ near its outer boundary $\Pi_\mu$. We point out that from now on, in contrast with the notation used in the introduction, for the sake of convenience $s = 0$ corresponds to $\Pi_\mu$ and $s = 1$ to the center.

**Lemma 2.1.** *Let us fix $\varpi \in \mathbb{N} \cup \{\infty, \omega\}$ and consider a $\mathscr{C}^\varpi$ family of planar vector fields $\{X_\mu\}_{\mu \in U}$ such that, for each $\mu \in U$, $X_\mu$ has a center $p_\mu \in \mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$. Suppose that there exists a continuous map $\sigma : (0, \delta) \times U \to \mathbb{R}^2$ verifying, for each fixed $\mu \in U$, that*

(*a*) *the map $\sigma(\,\cdot\,; \mu) : (0, \delta) \to \mathbb{R}^2$ is $\mathscr{C}^1$,*
(*b*) *the vectors $\partial_s \sigma(s; \mu)$ and $X_\mu(\sigma(s; \mu))$ are linearly independent for all $s \in (0, \delta)$, and*
(*c*) *for each compact set $K \subset \mathscr{P}_\mu \cup \{p_\mu\}$ there exists $s_K > 0$ such that $\sigma(s; \mu) \in \mathscr{P}_\mu \setminus K$ for all $s \in (0, s_K)$.*

*Then the following assertions hold:*

1. $\mathscr{U} = \bigcup_{\mu \in U} \mathscr{P}_\mu \times \{\mu\}$ *is an open subset of $\mathbb{R}^2 \times U$, and*
2. *the map $(p, \mu) \mapsto \hat{P}(p; \mu) = \{$period of the periodic orbit of $X_\mu$ passing through $p\}$ is $\mathscr{C}^\varpi$ on $\mathscr{U}$.*

**Proof.** We consider the family $\{X_\mu\}_{\mu \in U}$ as a single $\mathscr{C}^\varpi$ vector field $Y$ on $\mathbb{R}^2 \times U$ whose trajectories are contained in the submanifolds $\mu =$ constant. Denote the flow of $Y$ by $\phi(t; p, \mu) = (\varphi(t; p, \mu), \mu)$. In order to prove the first assertion, for a given $(p_0, \mu_0) \in \mathscr{U}$ we must show that there is an open subset $V$ of $\mathbb{R}^2 \times U$ such that $(p_0, \mu_0) \in V \subset \mathscr{U}$. We claim that this is true in the particular case that there exists $s_0 \in (0, \delta)$ such that $\sigma(s_0; \mu_0) = p_0$. Indeed, due to the assumption in (*b*), note that $Y$ is transverse to

$$\Sigma_\varepsilon := \{(\sigma(s; \mu), \mu);\ |s - s_0| < \varepsilon \text{ and } \|\mu - \mu_0\| < \varepsilon\}$$

for all $\varepsilon > 0$ small enough and that $(p_0, \mu_0) \in \Sigma_\varepsilon$. Then, since $\sigma : (0, \delta) \times U \to \mathbb{R}^2$ is continuous, by the flow box theorem (and shrinking $\varepsilon > 0$ if necessary) it follows that

$$V := \bigcup_{t\in(-\varepsilon,\varepsilon)} \phi(t;\Sigma_\varepsilon)$$

is an open subset of $\mathbb{R}^2 \times U$. Furthermore, since $\mathscr{U}$ is invariant by $\phi$ and $\Sigma_\varepsilon \subset \mathscr{U}$ by construction, we have that $(p_0, \mu_0) \in V \subset \mathscr{U}$ and this proves the claim. Let us consider now an arbitrary $p_0 \in \mathscr{P}_{\mu_0}$. Denote the periodic orbits of $X_{\mu_0}$ passing through $q := \sigma(\delta/2; \mu_0)$ and $p_0$ by $\gamma_q$ and $\gamma_{p_0}$, respectively. For each $\mu \in U$ we take the orthogonal vector field to $X_\mu$, say $X_\mu^\perp$, pointing inward the periodic orbits in $\mathscr{P}_\mu$. We consider the family $\{X_\mu^\perp\}_{\mu\in U}$ as a single $\mathscr{C}^\varpi$ vector field $\hat{Y}$ on $\mathbb{R}^2 \times U$ and denote its flow by $\hat{\phi}(t; p, \mu) = (\hat{\varphi}(t; p, \mu), \mu)$. Note that $p_\mu$ is also a singular point for $X_\mu^\perp$ that, by applying the Poincaré-Bendixson Theorem (see for instance [3]), it is easy to show to be asymptotically stable. Observe moreover that $\hat{\phi}(t; \mathscr{U}) \subset \mathscr{U}$ for all $t \geqslant 0$. We define $\Gamma := \{\hat{\varphi}(t; q, \mu_0);\ t \geqslant 0\} \subset \mathscr{P}_{\mu_0}$, which is clearly a transverse section for $X_{\mu_0}$, and distinguish two cases:

- Case 1: $\Gamma \cap \gamma_{p_0} \neq \emptyset$. In this case there exist $t_1, t_2 \geqslant 0$ such that $\hat{\varphi}(t_2; q, \mu_0) = \varphi(t_1; p_0, \mu_0) \in \Gamma$. Since $q = \sigma(\delta/2; \mu_0)$, on account of the claim we can take an open neighborhood $V_1$ of $(q, \mu_0)$ in $\mathbb{R}^2 \times U$ with $V_1 \subset \mathscr{U}$. Then, by the continuity of solutions with respect to initial conditions, there exists an open neighborhood $V_2$ of $(p_0, \mu_0)$ such that $\hat{\phi}\big(-t_2; \phi(t_1; V_2)\big) \subset V_1$. Thus $V_2 \subset \phi\big(-t_1; \hat{\phi}(t_2; V_1)\big) \subset \mathscr{U}$, where the second inclusion follows due to the $\hat{\phi}(t; \mathscr{U}) \subset \mathscr{U}$ for all $t \geqslant 0$ and $\phi(t; \mathscr{U}) = \mathscr{U}$ for all $t \in \mathbb{R}$, together with the fact that $V_1 \subset \mathscr{U}$.
- Case 2: $\Gamma \cap \gamma_{p_0} = \emptyset$. Note that in this case $\mathrm{Int}(\gamma_q) \subset \mathrm{Int}(\gamma_{p_0})$. (Here, given a Jordan curve $\gamma \subset \mathbb{R}^2$, $\mathrm{Int}(\gamma)$ denotes the bounded connected component of $\mathbb{R}^2 \setminus \{\gamma\}$.) Moreover, by the assumption in $(c)$ and taking $K = \overline{\mathrm{Int}(\gamma_{p_0})}$, there exists $s_1 \in (0, \delta/2)$ satisfying that $\sigma(s_1; \mu_0) \in \mathscr{P}_{\mu_0} \setminus K$. Therefore, since $q = \sigma(\delta/2; \mu_0) \in \mathrm{Int}(\gamma_{p_0})$, by continuity there exists $s_2 \in (s_1, \delta/2)$ such that $\sigma(s_2; \mu_0) \in \gamma_{p_0}$. Consequently $\sigma(s_2; \mu_0) = \varphi(t_3; p_0, \mu_0)$ for some $t_3 \in \mathbb{R}$ and on the other hand, again on account of the claim, there exists an open neighborhood $V_3$ of $(\sigma(s_2; \mu_0), \mu_0)$ in $\mathbb{R}^2 \times U$ with $V_3 \subset \mathscr{U}$. Thus, exactly as before, by continuity of solutions with respect to initial conditions, there is an open neighborhood $V_4$ of $(p_0, \mu_0)$ such that $V_4 \subset \phi(-t_3; V_3) \subset \mathscr{U}$.

This proves the validity of the first assertion.

Let us prove now that the function $\hat{P}\colon \mathscr{U} \longrightarrow (0, +\infty)$ defined by

$$(p, \mu) \mapsto \hat{P}(p; \mu) = \{\text{period of the periodic orbit of } X_\mu \text{ passing through } p\}$$

is $\mathscr{C}^\varpi$. In what follows we shall use the notation $p = (x, y)$ for the components of a point of $\mathbb{R}^2$. We fix $(\hat{p}, \hat{\mu}) \in \mathscr{U}$ and suppose that the period of the periodic orbit of $X_{\hat{\mu}}$ passing through $\hat{p} = (\hat{x}, \hat{y}) \in \mathscr{P}_{\hat{\mu}}$ is $\hat{\tau} > 0$. Then, due to $X(\hat{p}; \hat{\mu}) := X_{\hat{\mu}}(\hat{p}) \neq (0, 0)$, there is $i \in \{1, 2\}$ such that

$$\partial_t \varphi_i(\hat{\tau}; \hat{p}, \hat{\mu}) = \partial_t \varphi_i(0; \hat{p}, \hat{\mu}) = X_i(\hat{p}; \hat{\mu}) \neq 0.$$

For simplicity in the exposition let us suppose that $X_1(\hat{p}; \hat{\mu}) > 0$. In this case we can apply the Implicit Function Theorem to the equation $\varphi_1(t; p, \mu) = x$ at $(t, p, \mu) = (\hat{\tau}, \hat{p}, \hat{\mu})$ in order to obtain a $\mathscr{C}^\varpi$ positive function $S(p; \mu)$ in a open neighborhood $W \subset \mathscr{U}$ of $(\hat{p}, \hat{\mu})$ verifying $S(\hat{p}; \hat{\mu}) = \hat{\tau}$ and

$$\varphi_1(t;p,\mu)|_{t=S(p;\mu)} = x \text{ for all } (p,\mu) \in W. \tag{4}$$

Clearly we can assume that $W$ is a cube $Q(\varepsilon_1)$ with center $(\hat{p},\hat{\mu})$ and edge length $\varepsilon_1 > 0$. We diminish $\varepsilon_1$ if necessary so that $X_1(p;\mu) > 0$ for all $(p,\mu) \in Q(\varepsilon_1)$. Furthermore, thanks to $S(\hat{p};\hat{\mu}) = \hat{\tau}$ together with the continuity of $S$ and $\phi$, we can take $\varepsilon_2 \in (0,\varepsilon_1)$ such that

$$\phi(t;p,\mu)|_{t=S(p;\mu)} \in Q(\varepsilon_1) \text{ for all } (p,\mu) \in Q(\varepsilon_2).$$

We claim that $\hat{P} = S$ on $Q(\varepsilon_2)$. Clearly the claim will follow once we show that

$$\varphi_2(t;p,\mu)|_{t=S(p;\mu)} = y \text{ for all } (p,\mu) \in Q(\varepsilon_2).$$

By contradiction, suppose that there exists $(\bar{p},\bar{\mu}) \in Q(\varepsilon_2)$ such that $\varphi_2\big(t;\bar{p},\bar{\mu}\big)|_{t=S(\bar{p};\bar{\mu})} \neq \bar{y}$. Due to $Q(\varepsilon_2) \subset \mathscr{U}$, the trajectory of $X_{\bar{\mu}}$ passing through $\bar{p}$ is a periodic orbit which, for simplicity in the exposition, we assume to travel clockwise around the center $p_{\bar{\mu}}$ (the other case follows verbatim). That being said we consider the piece of trajectory

$$\ell := \{\varphi(t;\bar{p},\bar{\mu});\ t \in [0, S(\bar{p},\bar{\mu})]\}$$

and the vertical segment, recall (4),

$$\Gamma := \big\{(1-s)\bar{p} + s\varphi\big(S(\bar{p},\bar{\mu});\bar{p},\bar{\mu}\big);\ s \in (0,1)\big\} \subset \{x = \bar{x}\}.$$

Arguing on the phase portrait of $X_{\bar{\mu}}$, due to $X_1(p;\bar{\mu}) > 0$ for all $p \in \Gamma$, if $\varphi_2\big(S(\bar{p},\bar{\mu});\bar{p},\bar{\mu}\big) < \bar{y}$ then interior of the Jordan curve $\ell \cup \Gamma$ is a positively but not negatively invariant subset of $\mathscr{P}_{\bar{\mu}}$. Similarly, if $\varphi_2\big(S(\bar{p},\bar{\mu});\bar{p},\bar{\mu}\big) > \bar{y}$ then we obtain a negatively invariant subset of $\mathscr{P}_{\bar{\mu}}$ which is not positively invariant. In both cases we get a contradiction with the fact that $\mathscr{P}_{\bar{\mu}}$ is foliated by periodic orbits of $X_{\bar{\mu}}$ and $Q(\varepsilon_2) \subset \mathscr{U}$. Consequently $\varphi_2\big(S(p,\mu);p,\mu\big) = y$ for all $(p,\mu) \in Q(\varepsilon_2)$ and so the validity of the claim follows. Since $Q(\varepsilon_2)$ is an open neighborhood of an arbitrary point of $\mathscr{U}$ and $S$ is $\mathscr{C}^{\varpi}$ in $Q(\varepsilon_2)$, the claim implies the second assertion in the statement. ■

The previous result is addressed to a family $\{X_\mu\}_{\mu\in U}$ of vector fields and this is the reason why we require the existence of a local transverse section near the outer boundary of the period annulus $\Pi_\mu$ that behaves well with respect to parameters. That being said, Lemma 2.1 can be applied to a single vector field $X$ without this requirement because a trajectory of the orthogonal vector field $X^{\perp}$ already provides a transverse section in the whole period annulus. Thus in order to assert that $p \mapsto \hat{P}(p;\mu)$ is $\mathscr{C}^{\varpi}$ on $\mathscr{P}_\mu$ for each fixed $\mu \in U$, it is not necessary to verify the existence of a continuous map $\sigma : (0,\delta) \times U \to \mathbb{R}^2$ satisfying $(a)$, $(b)$ and $(c)$.

**Remark 2.2.** If $X$ is a $\mathscr{C}^{\varpi}$ vector field, $\varpi \in \mathbb{N} \cup \{\infty,\omega\}$, with a center then the period function $\hat{P}$ is a first integral for the flow of $X$ on the period annulus $\mathscr{P}$ that, by Lemma 2.1, is $\mathscr{C}^{\varpi}$. Consequently the scalar product $\nabla\hat{P}(p)\cdot X(p)$ is zero for all $p \in \mathscr{P}$. This implies that if $\gamma$ is a critical periodic orbit of $X$ then the gradient $\nabla\hat{P}$ vanishes on $\gamma$. Indeed, if $\sigma : (0,1) \longrightarrow \mathscr{P}$ is a $\mathscr{C}^{\varpi}$ transverse section to $X$ on $\mathscr{P}$ and $P(s) := \hat{P}(\sigma(s))$ then $P'(s) = \nabla\hat{P}(\sigma(s))\cdot\sigma'(s)$. Thus, since $\nabla\hat{P}(\sigma(s))\cdot X(\sigma(s)) = 0$, the transversality of $\sigma$ implies that $P'(s) = 0$ if, and only if, $\nabla\hat{P}(\sigma(s)) = (0,0)$. This shows in particular that the condition for $\gamma$ to be a critical periodic orbit is local and independent of the particular transverse section used to parametrize the set of critical periodic orbits near $\gamma$. □

We define next the notion that enable us to study the criticality at the outer boundary.

**Definition 2.3.** Let $U$ be an open set of $\mathbb{R}^N$ and consider a family of functions $\{h(\,\cdot\,;\mu)\}_{\mu\in U}$ on $(0,\varepsilon)$. Given any $\mu_\star\in U$ we define $\mathcal{Z}_0(h(\,\cdot\,;\mu),\mu_\star)$ to be the smallest integer $n$ having the property that there exist $\delta>0$ and a neighborhood $V$ of $\mu_\star$ such that for every $\mu\in V$ the function $h(s;\mu)$ has no more than $n$ isolated zeros on $(0,\delta)$ counted with multiplicities. □

The hypothesis with regard to the local transverse section in our next result are slightly stronger than in the previous one because we require the continuity at $s=0$ and that $\sigma(0;\mu)$ belongs to the outer boundary $\Pi_\mu$ for all $\mu\in U$, cf. assumption $(c)$ in Lemma 2.1. We also remark that in the statement $\hat{P}(p;\mu)$ stands for the period of the periodic orbit of $X_\mu$ passing through $p\in\mathscr{P}_\mu$.

**Lemma 2.4.** *Let us consider a $\mathscr{C}^\omega$ family $\{X_\mu\}_{\mu\in U}$ of planar polynomial vector fields such that, for each $\mu\in U$, $X_\mu$ has a center $p_\mu\in\mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$. Let $\Pi_\mu\subset\mathbb{RP}^2$ be the outer boundary of $\mathscr{P}_\mu$. Suppose there exists a continuous map $\sigma:[0,\delta)\times U\to\mathbb{RP}^2$ verifying that, for each $\mu\in U$,*

- $(a)$ *the map $\sigma(\,\cdot\,;\mu):(0,\delta)\to\mathbb{R}^2$ is $\mathscr{C}^1$,*
- $(b)$ *the vectors $\partial_s\sigma(s;\mu)$ and $X_\mu(\sigma(s;\mu))$ are linearly independent for all $s\in(0,\delta)$,*
- $(c)$ *$\sigma(s;\mu)\in\mathscr{P}_\mu$ for all $s\in(0,\delta)$ and $\sigma(0;\mu)\in\Pi_\mu$.*

*Then, for each fixed $\mu_\star\in U$, the following assertions hold:*

1. *The Hausdorff distance between the outer boundaries $\Pi_\mu$ and $\Pi_{\mu_\star}$ tends to zero as $\mu\to\mu_\star$.*
2. *If $P(s;\mu):=\hat{P}(\sigma(s;\mu);\mu)$ for all $(s,\mu)\in(0,\delta)\times U$, then*

   - $(2a)$ $\mathrm{Crit}\big((\Pi_{\mu_\star},X_{\mu_\star}),X_\mu\big)\leqslant\mathcal{Z}_0(P'(\,\cdot\,;\mu),\mu_\star)$.
   - $(2b)$ $\mathrm{Crit}\big((\Pi_{\mu_\star},X_{\mu_\star}),X_\mu\big)\geqslant n$ *if for each open neighborhood $V$ of $\mu_\star$ and $\delta>0$ there exist $n$ different numbers $s_1,s_2\ldots,s_n\in(0,\delta)$ and $\hat{\mu}\in V$ such that $P'(s_i;\hat{\mu})=0$ for $i=1,2,\ldots,n$.*
   - $(2c)$ $\mathrm{Crit}\big((\Pi_{\mu_\star},X_{\mu_\star}),X_\mu\big)=0$ *if, and only if, $\mathcal{Z}_0(P'(\,\cdot\,;\mu),\mu_\star)=0$.*

**Proof.** To show the first assertion note that, since $X_\mu$ is polynomial, we can consider its Poincaré compactification $p(X_\mu)$, see [3, §5] for details, which is an analytic vector field on the sphere $\mathbb{S}^2$ topologically equivalent to $X_\mu$. The outer boundary $\Pi_\mu$ becomes then a polycycle of $p(X_\mu)$ that can be studied using local charts of $\mathbb{S}^2$. On account of this, the fact that $d_H(\Pi_\mu,\Pi_{\mu_\star})\to 0$ as $\mu\to\mu_\star$ follows by the continuity of $\mu\mapsto\sigma(0;\mu)\in\Pi_\mu$ together with the continuity with respect to initial conditions and parameters of the trajectories of $p(X_\mu)$. The interested reader is referred to [36, Lemma 22, p. 110] for a related result for limit periodic sets.

With regard to the upper bound in $(2a)$ it is clear that if $\mathcal{Z}_0(P'(\,\cdot\,;\mu),\mu_\star)=+\infty$ then there is nothing to be proved. So let us assume that $\mathcal{Z}_0(P'(\,\cdot\,;\mu),\mu_\star)=\ell\in\mathbb{Z}_{\geq 0}$ and argue by contradiction. If $\mathrm{Crit}\big((\Pi_{\mu_\star},X_{\mu_\star}),X_\mu\big)\geqslant\ell+1$ then there exist $\ell+1$ sequences $\{\gamma_{\mu_i}^k\}_{i\in\mathbb{N}}$, $k=1,2,\ldots,\ell+1$, where $\gamma_{\mu_i}^1,\gamma_{\mu_i}^2,\ldots,\gamma_{\mu_i}^{\ell+1}$ are different critical periodic orbits of $X_{\mu_i}$ for each

$i \in \mathbb{N}$, such that $\mu_i \to \mu_\star$ and $d_H(\gamma^k_{\mu_i}, \Pi_{\mu_\star}) \to 0$ as $i \to +\infty$. Then, due to $d_H(\Pi_\mu, \Pi_{\mu_\star}) \to 0$ as $\mu \to \mu_\star$ and

$$d_H(\gamma^k_{\mu_i}, \Pi_{\mu_i}) \leqslant d_H(\gamma^k_{\mu_i}, \Pi_{\mu_\star}) + d_H(\Pi_{\mu_i}, \Pi_{\mu_\star}),$$

we have $d_H(\gamma^k_{\mu_i}, \Pi_{\mu_i}) \to 0$ as $i \to +\infty$ for each $k = 1, 2, \ldots, \ell+1$. Since $\sigma(0; \mu_i) \in \Pi_{\mu_i}$ and there is a one-to-one correspondence between zeros of $P'(s; \mu_i)$ arbitrarily near $s = 0$ and critical periodic orbits of $X_{\mu_i}$ arbitrarily close to $\Pi_{\mu_i}$ (cf. [36, Lemma 22]), this implies that there exist $\ell+1$ sequences of positive numbers $\{s^k_i\}_{i\in\mathbb{N}}$, $k = 1, 2, \ldots, \ell+1$, such that $P'(s^k_i; \mu_i) = 0$ and $\#\{s^1_i, s^2_i, \ldots, s^{\ell+1}_i\} = \ell+1$ for each $i \in \mathbb{N}$, and $\lim_{i\to+\infty} s^k_i = 0$ for each $k = 1, 2, \ldots, \ell+1$. This clearly contradicts that $\mathcal{Z}_0(P'(\,\cdot\,; \mu), \mu_\star) = \ell$, see Definition 2.3. The assertion in (2$b$) follows similarly. Indeed, on account of the assumption and the above mentioned one-to-one correspondence between zeros of $P'(s; \mu)$ near $s = 0$ and critical periodic orbits of $X_\mu$ close to $\Pi_{\mu_i}$, we can construct $n$ sequences $\{\gamma^k_{\mu_i}\}_{i\in\mathbb{N}}$, $k = 1, 2, \ldots, n$, where $\gamma^1_{\mu_i}, \gamma^2_{\mu_i}, \ldots, \gamma^n_{\mu_i}$ are different critical periodic orbits of $X_{\mu_i}$ for each $i \in \mathbb{N}$, such that $\mu_i \to \mu_\star$ and $d_H(\gamma^k_{\mu_i}, \Pi_{\mu_i}) \to 0$ as $i \to +\infty$. Then, using that $d_H(\Pi_\mu, \Pi_{\mu_\star}) \to 0$ as $\mu \to \mu_\star$, we can assert that $\lim_{i\to+\infty} d_H(\gamma^k_{\mu_i}, \Pi_{\mu_0}) = 0$ for each $k = 1, 2, \ldots, n$, which implies $\mathrm{Crit}\big((\Pi_{\mu_\star}, X_{\mu_\star}), X_\mu\big) \geqslant n$, as desired. Finally the assertion in (2$c$) follows easily from the ones in (2$a$) and (2$b$). This completes the proof of the result. ■

Next we introduce the notion of global transverse section for a family of period annuli. Roughly speaking it is a transverse section, joining the center with some point at the outer boundary of the period annulus, that behaves well with the parameters.

**Definition 2.5.** Let us fix $\varpi \in \mathbb{N} \cup \{\infty, \omega\}$ and consider a $\mathscr{C}^\varpi$ family $\{X_\mu\}_{\mu\in U}$ of planar vector fields such that, for each $\mu \in U$, $X_\mu$ has a center $p_\mu \in \mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$. Let $\Pi_\mu \subset \mathbb{RP}^2$ be the outer boundary of $\mathscr{P}_\mu$. A *global transverse section* for the family of period annuli $\{\mathscr{P}_\mu\}_{\mu\in U}$ is a continuous map $\sigma : [0, 1] \times U \to \mathbb{RP}^2$ verifying that

($a$) the map $\sigma(\,\cdot\,; \mu) : [0, 1] \to \mathbb{RP}^2$ is $\mathscr{C}^\varpi$ for each $\mu \in U$,
($b$) the vectors $\partial_s\sigma(s; \mu)$ and $X_\mu(\sigma(s; \mu))$ are linearly independent for all $(s, \mu) \in (0, 1) \times U$ and the map $\partial_s\sigma : (0, 1) \times U \to \mathbb{R}^2$ is continuous,
($c$) $\sigma(s; \mu) \in \mathscr{P}_\mu$ for all $s \in (0, 1)$, $\sigma(0; \mu) \in \Pi_\mu$ and $\sigma(1; \mu) = p_\mu$.

When such a global transverse section exists we say that the family of period annuli $\{\mathscr{P}_\mu\}_{\mu\in U}$ varies continuously. □

**Remark 2.6.** The period annulus of the family of Loud's quadratic centers given in (2) varies continuously in the sense of Definition 2.5. Indeed, it follows from the proof of [23, Lemma 3.2] that

$$\mu = (D, F) \mapsto \xi_\mu := \sup\{t > 0;\ (s, 0) \in \mathscr{P}_\mu \text{ for all } s \in (0, t)\}$$

is a well-defined continuous function on $\mathbb{R}^2$. Moreover the point $(\xi_\mu, 0)$ belongs to $\Pi_\mu$ and $0 < \xi_\mu \leqslant 1$ for all $\mu \in \mathbb{R}^2$. Then $\sigma(s; \mu) = ((1-s)\xi_\mu, 0)$ for $(s, \mu) \in [0, 1] \times \mathbb{R}^2$ is clearly a global transverse section. In particular, since the Loud's system is polynomial, the outer boundary of the period annulus varies continuously in the Hausdorff sense by the first assertion in Lemma 2.4. □

Note, see $(b)$ in Definition 2.5, that we also require $(s,\mu)\mapsto \partial_s\sigma(s;\mu)$ to be continuous. The reason for this is because if we define $P(s;\mu)=\hat{P}(\sigma(s;\mu);\mu)$ then $(s,\mu)\mapsto \partial_s P(s;\mu)$ is a continuous function by Lemma 2.1. This continuity is a key point in the forthcoming results. Before that we summarize in the next statement the properties that we get for $P(s;\mu)$ as a consequence of Lemma 2.1 and Definition 2.5.

**Corollary 2.7.** *Let us fix $\varpi\in\mathbb{N}\cup\{\infty,\omega\}$ and consider a $\mathscr{C}^{\varpi}$ family of planar vector fields $\{X_\mu\}_{\mu\in U}$ such that, for each $\mu\in U$, $X_\mu$ has a center $p_\mu\in\mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$. Assume that the family of period annuli varies continuously and let $\sigma:[0,1]\times U\to\mathbb{RP}^2$ be a global transverse section for $\{\mathscr{P}_\mu\}_{\mu\in U}$. If $P(s;\mu):=\hat{P}(\sigma(s;\mu);\mu)$ for all $(s,\mu)\in(0,1)\times U$ then the following holds:*

- $(a)$ *$P(\,\cdot\,;\mu)\in\mathscr{C}^{\varpi}((0,1))$ for each $\mu\in U$, and*
- $(b)$ *$P$ and $\partial_s P$ are continuous functions on $(0,1)\times U$.*

**Definition 2.8.** Under the assumptions of Corollary 2.7, we say that $P(s;\mu)=\hat{P}(\sigma(s;\mu);\mu)$, which is defined for $(s,\mu)\in(0,1)\times U$, is a *global parametrization of the period function*. In contrast,

$$(p,\mu)\mapsto \hat{P}(p;\mu)=\{\text{period of the periodic orbit of } X_\mu \text{ passing through } p\}$$

is defined on $\bigcup_{\mu\in U}\mathscr{P}_\mu\times\{\mu\}$, which is not so easy to handle. □

One of the main goals in the present section is to relate the concept of local bifurcation value of the period function, as introduced in [23], with the notion of criticality, see Definition 1.1. As we will see the first one concerns with the qualitative properties of the period function, whereas the second is more geometric and quantitative. In doing so we will be able to take advantage of the results about the bifurcation diagram of the period function of the Loud's centers that we obtained in [23]. For reader's convenience we next recall the definition of local bifurcation value of the period function.

**Definition 2.9.** Let $\{I_\mu\}_{\mu\in U}$ be a continuous family of intervals, i.e., such that $I_\mu=(\ell(\mu),r(\mu))$ with $\ell,r\in\mathscr{C}^0(U)$, and consider a continuous family of functions $\{F_\mu\colon I_\mu\longrightarrow\mathbb{R}\}_{\mu\in U}$. We say that $\mu_0\in U$ is a *regular value of the family* $\{F_\mu\colon I_\mu\longrightarrow\mathbb{R}\}_{\mu\in U}$ if there exist a neighborhood $V$ of $\mu_0$ and an isotopy $\{h_\mu\colon I_\mu\longrightarrow I_{\mu_0}\}_{\mu\in V}$, with $h_{\mu_0}=id$, such that

$$\operatorname{sgn}\Big(F_\mu(s)\Big)=\operatorname{sgn}\Big(F_{\mu_0}\big(h_\mu(s)\big)\Big) \text{ for all } s\in I_\mu \text{ and } \mu\in V, \tag{5}$$

where $\operatorname{sgn}:\mathbb{R}\to\{-1,0,1\}$ is the extended sign function. A parameter $\mu_0$ which is not regular is called a *bifurcation value*. □

The endpoints of $I_\mu$, the domain of definition of $F_\mu$, depend continuously on $\mu$, so that $\cup_{\mu\in U} I_\mu \times \{\mu\}$ is an open subset of $\mathbb{R}\times U$. Thus, by a continuous family of functions $\{F_\mu : I_\mu \longrightarrow \mathbb{R}\}_{\mu\in U}$, we mean that the map $(s,\mu)\mapsto F_\mu(s)$ is continuous on $\cup_{\mu\in U} I_\mu \times \{\mu\}$. Next we particularize the previous definition to study the period function. To this aim note that, by Corollary 2.7, if $\{X_\mu\}_{\mu\in U}$ is a $\mathscr{C}^1$ family of vector fields with a center such that the corresponding family of period annuli varies continuously, and we set $P(s;\mu)=\hat{P}(\sigma(s;\mu);\mu)$, then $\{\partial_s P(\,\cdot\,;\mu)\}_{\mu\in U}$ is a continuous family of functions on $(0,1)$.

**Definition 2.10.** Consider a $\mathscr{C}^1$ family of planar vector fields $\{X_\mu\}_{\mu\in U}$ such that, for each $\mu\in U$, $X_\mu$ has a center $p_\mu\in\mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$, that we suppose to vary continuously.

($a$) We say that $\mu_0\in U$ is a *regular* (respectively, *bifurcation*) *value of the period function* if for some global parametrization of the period function $P\colon(0,1)\times U\longrightarrow(0,+\infty)$ we have that $\mu_0$ is a regular (respectively, bifurcation) value of the family $\{P'(\,\cdot\,;\mu):(0,1)\to\mathbb{R}\}_{\mu\in U}$.
($b$) We say that $\mu_0\in U$ is a *local regular value of the period function at the interior* if there is some global parametrization of the period function $P\colon(0,1)\times U\longrightarrow(0,+\infty)$ satisfying that for each $c\in(0,1)$ there exists a continuously varying neighborhood $I_\mu(c)$ of $c$ in $(0,1)$ such that $\mu_0$ is a regular value of the family $\{P'(\,\cdot\,;\mu):I_\mu(c)\to\mathbb{R}\}_{\mu\in U}$. A parameter which is not a local regular value at the interior is called a *local bifurcation value at the interior.*
($c$) We say that $\mu_0\in U$ is a *local regular value of the period function at the outer* (respectively, *inner*) *boundary* if for some global parametrization of the period function $P\colon(0,1)\times U\longrightarrow(0,+\infty)$ there exists a continuously varying neighborhood $I_\mu(c)$ of $c=0$ (respectively, $c=1$) such that $\mu_0$ is a regular value of the family $\{P'(\,\cdot\,;\mu)\colon I_\mu(c)\cap(0,1)\longrightarrow\mathbb{R}\}_{\mu\in U}$. A parameter which is not a local regular value at the outer (respectively, inner) boundary is called a *local bifurcation value at the outer* (respectively, *inner*) *boundary.* □

**Remark 2.11.** Let us make the following easy observations with regard to the previous definitions:

($a$) One can replace "some global parametrization" by "any global parametrization". Indeed, suppose that $\mu_0\in U$ is a regular value for $\{P'(\,\cdot\,;\mu)\}_{\mu\in U}$ where $P(s;\mu)=\hat{P}(\sigma(s;\mu);\mu)$ and consider another global parametrization $\bar{P}(s;\mu)=\hat{P}(\bar{\sigma}(s;\mu);\mu)$ of the period function, see Definition 2.8. If we denote by $\tau_\mu(s)$ the Poincaré map from the transverse section $\Sigma$ given by $s\mapsto\sigma(s;\mu)$ to the transverse section $\bar{\Sigma}$ given by $s\mapsto\bar{\sigma}(s;\mu)$ then $\tau_\mu$ is an increasing diffeomorphism and $P(s;\mu)=\bar{P}(\tau_\mu(s);\mu)$, so that $P'(s;\mu)=\bar{P}'(\tau_\mu(s);\mu)\tau_\mu'(s)$. On account of this and following the notation in Definition 2.9, $\bar{h}_\mu:=\tau_{\mu_0}\circ h_\mu\circ\tau_\mu^{-1}$ is a suitable isotopy in order to show that $\mu_0$ is a regular value for the family $\{\bar{P}'(\,\cdot\,;\mu\}_{\mu\in U}$ because

$$\operatorname{sgn}\big(\bar{P}'(s;\mu)\big)=\operatorname{sgn}\big(P'(\tau_\mu^{-1}(s);\mu)\big)=\operatorname{sgn}\big(P'\big((h_\mu\circ\tau_\mu^{-1})(s);\mu_0\big)\big)=\operatorname{sgn}\big(\bar{P}'\big(\bar{h}_\mu(s);\mu_0\big)\big),$$

where we use that $\tau'(s)>0$.
($b$) In order to study if a parameter is a local regular value at the outer boundary it is not necessary to consider a global transverse section $\sigma\colon[0,1]\times U\longrightarrow\mathbb{RP}^2$ for the family of period annuli. Indeed, see point ($c$) in Definition 2.10, it suffices to take a local parametrization

$\sigma : [0, \delta) \times U \longrightarrow \mathbb{RP}^2$. Similarly, to study the local regular values at the inner boundary it suffices to take a local parametrization $\sigma : (1-\delta, 1] \times U \longrightarrow \mathbb{R}^2$. □

As expected, $\mu_0$ is a bifurcation value of the period function if, and only if, $\mu_0$ is either a local bifurcation value at the inner boundary, at the outer boundary or at the interior. This is stated in the following result and the interested reader is referred to [23, Lemma 2.7] for the proof.

**Lemma 2.12.** *Let us consider a $\mathscr{C}^1$ family of analytic planar vector fields $\{X_\mu\}_{\mu\in U}$ such that, for each $\mu \in U$, $X_\mu$ has a center $p_\mu \in \mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$, that we suppose to vary continuously. Then the bifurcation diagram of the period function is the union of the local bifurcation diagrams at the inner and outer boundary and in the interior.*

Under the assumptions and notation in Corollary 2.7, a sufficient condition for $\mu_\star \in U$ to be a local regular value of the period function at the interior is that $P'(s; \mu_\star) \neq 0$ for all $s \in (0, 1)$. This follows easily by the continuity of $(s, \mu) \mapsto P'(s; \mu)$ on $(0, 1) \times U$ and a compactness argument. In case that this function is $\mathscr{C}^1$ then another sufficient condition is that $P'(\,\cdot\,; \mu_\star)$ has only simple zeros because the application of the Implicit Function Theorem provides the appropriate isotopies. Hence, in this context, the set of local bifurcation values of the period function at the interior is contained in

$$\Delta := \{\mu \in U;\ \text{there exists } s \in (0, 1) \text{ such that } P'(s; \mu) = P''(s; \mu) = 0\}.$$

If $P'(s; \mu)$ was polynomial in $s$ (which is certainly not true) then $\Delta$ would consist of those parameters $\mu_\star$ such that the *discriminant* of $P'(s; \mu_\star)$ is equal to zero. (Recall that the discriminant of $q \in \mathbb{R}[x]$ is the resultant between $q(x)$ and $q'(x)$, see for instance [8].) One may expect on the other hand that the parameters in $\Delta$ are always local bifurcation values of the period function at the interior. However this is not always the case and the following toy models show that some additional assumptions are needed to this end.

**Example 2.13.** Setting $N = 1$, we take $P'$ to be $F(s; \mu) = (s - \mu)^2$ and $U = (0, 1)$. Then it is clear that any $\mu \in U$ is a local regular value of $F$ at the interior (i.e., there are no local bifurcation values) but we have that $\Delta = U$. Note that in this case the interior of $\Delta$ is non-empty. □

**Example 2.14.** Setting $N = 2$, we take $P'$ to be $F(s; \mu) = (s - \mu_1)^3 - \mu_2$ and $U = (0, 1)^2$. Then again it turns out that any $\mu = (\mu_1, \mu_2) \in U$ is a local regular value of $F$ at the interior, whereas $\Delta = \{\mu \in U : \mu_2 = 0\}$. Observe that in this case the interior of $\Delta$ is empty but $F(\,\cdot\,; \mu)$ has zeroes of multiplicity 3. □

The following result provides us with an analytical tool to study the local bifurcation values of the period function at the interior. We emphasize that it has the natural hypothesis in view of the previous discussion.

**Lemma 2.15.** *Let $\{X_\mu\}_{\mu\in U}$ be an analytic family of planar vector fields such that, for each $\mu \in U$, $X_\mu$ has a center $p_\mu \in \mathbb{R}^2$ with period annulus $\mathscr{P}_\mu$. Assume that the family of period annuli varies continuously and let $\sigma : [0, 1] \times U \to \mathbb{RP}^2$ be a global transverse section for $\{\mathscr{P}_\mu\}_{\mu\in U}$. Setting $P(s; \mu) = \hat{P}(\sigma(s; \mu); \mu)$ for all $(s, \mu) \in (0, 1) \times U$, suppose additionally that*

(*a*) *the interior of* $\Delta$ (*as a subset of* $U \subset \mathbb{R}^N$) *is empty, and*
(*b*) *for each* $\mu \in U$*, the zeros of* $P'(\,\cdot\,;\mu)$ *have at most multiplicity 2.*

*Then each* $\mu \in \Delta$ *is a local bifurcation value of the period function at the interior.*

**Proof.** Note first that, by Corollary 2.7, the function $P(\,\cdot\,;\mu)$ is analytic on $(0,1)$ for each $\mu \in U$. Let us take any $\mu_0 \in \Delta$. Then there exists $s_0 \in (0,1)$ such that $P'(s_0;\mu_0) = P''(s_0;\mu_0) = 0$ and, by the hypothesis in $(b)$, $P'''(s_0;\mu_0) \neq 0$. Consequently $P'(\,\cdot\,;\mu_0)$ has a local extremum at $s = s_0$ and so there exists $\varepsilon > 0$ small enough such that $P'(\,\cdot\,;\mu_0)$ has the same sign $+1$ or $-1$ on $(s_0-\varepsilon, s_0+\varepsilon) \setminus \{s_0\}$. Assume by contradiction that $\mu_0$ is a local regular value of the period function at the interior. Then, taking $c = s_0$ in $(b)$ of Definition 2.10, we can consider a neighborhood $V$ of $\mu_0$, a continuously varying neighborhood $I_\mu$ of $s_0$ in $(0,1)$ and an isotopy $h_\mu : I_\mu \to I_{\mu_0}$ for $\mu \in V$, with $h_{\mu_0} = \text{id}$, verifying the equality in (5). Since $\Delta$ has empty interior we can take $\hat{\mu} \in V \setminus \Delta$ and define $\hat{s} := h_{\hat{\mu}}^{-1}(s_0) \in I_{\hat{\mu}}$. On account of this, particularizing (5) with $\mu = \hat{\mu}$ and $s = \hat{s}$ we deduce that $P'(\hat{s};\hat{\mu}) = 0$. Accordingly, due to $\hat{\mu} \notin \Delta$, it follows that $P''(\hat{s};\hat{\mu}) \neq 0$. Therefore the function $s \mapsto P'(s;\hat{\mu})$ changes sign at $s = \hat{s}$. This contradicts (5) taking $\mu = \hat{\mu}$ and $s \approx \hat{s}$ because $P'(\,\cdot\,;\mu_0)$ has the same sign on $(s_0-\varepsilon, s_0+\varepsilon) \setminus \{s_0\}$. ■

In the statement of our next result $p(X)$ stands for the Poincaré compactification in $\mathbb{S}^2$ of a planar polynomial vector field $X$, see [3, §5] for details. Recall also that any polycycle of an analytic vector field can be desingularized giving a polycycle with only hyperbolic or semi-hyperbolic vertices. By a hyperbolic polycycle we mean that its desingularization does not have semi-hyperbolic vertices (i.e., saddle-nodes).

**Lemma 2.16.** *Consider a* $\mathscr{C}^\omega$ *family of planar polynomial vector fields* $\{X_\mu\}_{\mu \in U}$ *such that, for each* $\mu \in U$*,* $X_\mu$ *has a center* $p_\mu \in \mathbb{R}^2$ *with period annulus* $\mathscr{P}_\mu$*, that we suppose to vary continuously. Then the following assertions hold for any given* $\mu_\star \in U$*:*

(*a*) *If* $\text{Crit}\big((\Pi_{\mu_\star}, X_{\mu_\star}), X_\mu\big) = 0$ *then* $\mu_\star$ *is a local regular value of the period function at the outer boundary.*
(*b*) *Assuming additionally that the outer boundary* $\Pi_{\mu_\star}$ *is a hyperbolic polycycle of* $p(X_{\mu_\star})$*, if* $\mu_\star$ *is a local regular value of the period function at the outer boundary then* $\text{Crit}\big((\Pi_{\mu_\star}, X_{\mu_\star}), X_\mu\big) = 0$.

**Proof.** Since the family of period annuli varies continuously, see Definition 2.5, we can take a global transverse section $\sigma : [0,1] \times U \to \mathbb{RP}^2$ and consider the global parametrization of the period function given by $P(s;\mu) := \hat{P}(\sigma(s;\mu);\mu)$ for $(s,\mu) \in (0,1) \times U$, see Corollary 2.7.

In order to show $(a)$ note that if $\text{Crit}\big((\Pi_{\mu_\star}, X_{\mu_\star}), X_\mu\big) = 0$ then $\mathscr{Z}_0(P'(\,\cdot\,,\mu), \mu_\star) = 0$ by assertion $(2c)$ in Lemma 2.4. This implies, see Definition 2.3, the existence of $\delta > 0$ and a neighborhood $V$ of $\mu_\star$ such that $P'(s;\mu)$ does not vanish on $(0,\delta) \times V$. Hence, since $(s,\mu) \mapsto P'(s;\mu)$ is continuous thanks to $(b)$ in Corollary 2.7, the function $P'(s;\mu)$ has constant sign on $(0,\delta) \times V$. Thus, see Definitions 2.9 and 2.10, taking $I_\mu = (0,\delta)$ and $h_\mu = id$ it follows that $\mu_\star$ is a regular value of the family $\{P'(\,\cdot\,;\mu) : I_\mu \to \mathbb{R}\}_{\mu \in U}$ as desired. This shows the validity of the assertion in $(a)$.

Let us turn next to the assertion in $(b)$. If $\mu_\star$ is a local regular value of the period function at the outer boundary then there exist a neighborhood $V$ of $\mu_\star$, a continuous strictly positive function $\mu \mapsto \varepsilon_\mu$ on $V$ and an isotopy $\{h_\mu : (0, \varepsilon_\mu) \longrightarrow (0, \varepsilon_{\mu_\star})\}_{\mu \in V}$ such that $\operatorname{sgn}(P'(s; \mu)) = \operatorname{sgn}(P'(h_\mu(s)); \mu_\star)$ for all $s \in (0, \varepsilon_\mu)$ and $\mu \in V$. From this point we distinguish two cases:

1. If the center of $X_{\mu_\star}$ is not isochronous then, by applying [26, Theorem 1.1], the zeros of $P'(s; \mu_\star)$ do not accumulate to $s = 0$. Let us remark that to apply this result we take into account that the transverse section $\sigma(\,\cdot\,; \mu_\star)$ is analytic at $s = 0$, see Definition 2.5, and the hypothesis that $\Pi_{\mu_\star}$ is a hyperbolic polycycle of $p(X_{\mu_\star})$. Hence there exists $\rho > 0$ such that $P'(s; \mu_\star) \neq 0$ for all $s \in (0, \rho)$. Thus, since we can suppose without loss of generality that $\varepsilon_{\mu_\star} \in (0, \rho)$ and $\delta := \inf\{\varepsilon_\mu : \mu \in V\} > 0$, it follows that $P'(s; \mu) \neq 0$ on $(0, \delta) \times V$, which implies (see Definition 2.3) that $\mathscr{Z}_0(P'(\,\cdot\,, \mu), \mu_\star) = 0$. Therefore, by assertion $(2c)$ in Lemma 2.4, $\operatorname{Crit}\big((\Pi_{\mu_\star}, X_{\mu_\star}), X_\mu\big) = 0$.
2. If the center of $X_{\mu_\star}$ is isochronous then $P'(\,\cdot\,; \mu_\star) \equiv 0$. Hence $\operatorname{sgn}(P'(s; \mu)) = \operatorname{sgn}(P'(h_\mu(s)); \mu_\star) = 0$ for all $s \in (0, \varepsilon_\mu)$ and $\mu \in V$. Thus $P'(\,\cdot\,; \mu)$ has not isolated zeros for all $\mu \in V$ and consequently, see Definition 2.3, $\mathscr{Z}_0(P'(\,\cdot\,, \mu), \mu_\star) = 0$. Then $\operatorname{Crit}\big((\Pi_{\mu_\star}, X_{\mu_\star}), X_\mu\big) = 0$ by $(2c)$ in Lemma 2.4.

This shows $(b)$ and completes the proof of the result. ■

We conclude this section by showing that, as we explain in the introduction, Theorem A leaves us very close to the proof of the existence of an upper bound for the number of critical periodic orbits in the family $\{X_\nu, \nu \in \mathbb{R}^2\}$. In this respect we note that there are parameter values $\nu \in \mathbb{R}^2$ for which $X_\nu$ has another center $p_\nu$ apart from the one at the origin (see for instance [23, Figure 4]). The bound also holds for the critical periodic orbits of this second center because one can always find an invertible affine transformation $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $g(p_\nu) = (0, 0)$ such that the push-forward of $X_\nu$ by $g$ verifies $g_*(X_\nu) = \beta X_{\hat{\nu}}$ for some $\hat{\nu} \in \mathbb{R}^2$ and $\beta \neq 0$.

**Lemma 2.17.** *Consider the family of vector fields $\{X_\nu, \nu \in \mathbb{R}^2\}$ given in (2). If $\operatorname{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big)$ is finite for every $\nu_0 \in \mathbb{R}^2$ then there exists $N \in \mathbb{N}$ such that the center at the origin of $X_\nu$ has at most $N$ critical periodic orbits for all $\nu \in \mathbb{R}^2$.*

**Proof.** By Lemma 2.1, $\mathscr{U} := \bigcup_{\nu \in \mathbb{R}^2} \mathscr{P}_\nu \times \{\nu\}$ is an open subset of $\mathbb{R}^2 \times \mathbb{R}^2$ and the map

$$(p, \nu) \mapsto \hat{P}(p; \nu) = \{\text{period of the periodic orbit of } X_\nu \text{ passing through } p\}$$

is analytic on $\mathscr{U}$. We define $P(s; \nu) := \hat{P}\big(((1 - s)\xi_\nu, 0); \nu\big)$ for each $(s, \nu) \in (0, 1) \times \mathbb{R}^2$, see Remark 2.6, which provides us with a suitable global parametrization of the period function. Let us note in particular that $\partial_s^k P(s; \nu)$ is a continuous function on $(0, 1) \times \mathbb{R}^2$ for each $k \in \mathbb{N}$. Moreover, by [19, Theorem A], we know that if $\nu = (D, F) \notin K := [-7, 2] \times [0, 4]$ then the center at the origin of $X_\nu$ has no critical periodic orbits. Consequently, if for each fixed $\nu \in \mathbb{R}^2$ we define $N_\nu$ to be the number of isolated zeros of $s \mapsto P'(s; \nu)$ on the interval $(0, 1)$ counted without multiplicities, the result will follow once we prove that

$$\sup_{\nu \in K} (N_\nu) < +\infty.$$

Let us advance that this will be a consequence of the compactness of $[0, 1] \times K$. With this end in view we fix any $(s_\star, \nu_\star) \in [0, 1] \times K$ and observe that three different situations may occur:

$(a)$ **Case $\mathbf{s}_\star = \mathbf{1}$.** As a consequence of the result of Chicone and Jacobs, see [6, Theorem 3.1], there exist $\varepsilon, \delta > 0$, depending on $\nu_\star$, such if $\nu \in B_\varepsilon(\nu_\star) := \{\nu \in \mathbb{R}^2 : \|\nu - \nu_\star\| < \varepsilon\}$ then the number of isolated roots of $P'(s;\nu) = 0$ with $s \in (1-\delta, 1)$ is at most 2 (counted with multiplicities).

$(b)$ **Case $\mathbf{s}_\star = \mathbf{0}$.** Since $\ell := \mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big) < +\infty$ by assumption, $(2b)$ in Lemma 2.4 implies that there exist $\varepsilon, \delta > 0$ (depending on $\nu_\star$ again) such if $\nu \in B_\varepsilon(\nu_\star)$ then the number of isolated roots of $P'(s;\nu) = 0$ with $s \in (0, \delta)$ is at most $\ell$ (counted without multiplicities).

$(c)$ **Case $\mathbf{s}_\star \in (\mathbf{0}, \mathbf{1})$.**

$(c1)$ If the center of $X_{\nu_\star}$ is not isochronous then there exists $k \in \mathbb{N}$, depending on $(s_\star, \nu_\star)$, such that $\partial_s^k P(s_\star; \nu_\star) \neq 0$. By continuity there is a neighborhood $V$ of $(s_\star, \nu_\star)$ such that $\partial_s^k P(s;\nu) \neq 0$ for all $(s,\nu) \in V$. Hence the application of Rolle's Theorem shows that there exist $\varepsilon, \delta > 0$ such if $\nu \in B_\varepsilon(\nu_\star)$ then the number of roots of $P'(s;\nu) = 0$ with $s \in (s_\star - \delta, s_\star + \delta)$ is at most $k$ (counted with multiplicities).

$(c2)$ Let us suppose finally that the center of $X_{\nu_\star}$ is isochronous. Since $\big(((1-s)\xi_{\nu_\star}, 0), \nu_\star\big) \in \mathscr{U}$, and by taking for instance the flow of the orthogonal vector field $X_\nu^\perp$, there exists a transverse section $\bar{s} \mapsto \sigma(\bar{s};\nu)$ given by an analytic map

$$\sigma : (-\delta_1, \delta_1) \times B_{\varepsilon_1}(\nu_\star) \longrightarrow \mathscr{U}$$

and such that $\sigma(0;\nu_\star) = \big(((1-s_\star)\xi_{\nu_\star}, 0), \nu_\star\big)$. We then define $\bar{P}(\bar{s};\nu) := \hat{P}(\sigma(\bar{s};\nu))$, which is clearly analytic on $(-\delta_1, \delta_1) \times B_{\varepsilon_1}(\nu_\star)$. We can thus compute its Taylor's series at $\bar{s} = 0$,

$$\bar{P}(\bar{s};\nu) = \sum_{i=0}^{\infty} a_i(\nu)\bar{s}^i,$$

where each $a_i$ is an analytic function on $B_{\varepsilon_1}(\nu_\star)$ with $a_i(\nu_\star) = 0$. Working in the local ring $\mathbb{R}\{\nu\}_{\nu_\star}$ of convergent power series at $\nu_\star$, we consider the ideal $\mathfrak{B} := \big(a_i, i \in \mathbb{N}\big)$. The ring is Noetherian and so there exists $\ell \in \mathbb{N}$ such that $\mathfrak{B} = (a_1, a_2, \ldots, a_\ell)$. Verbatim the proof of Chicone and Jacobs for [6, Theorem 2.2] (see also the result of Roussarie in [36, §4.3.1] for a similar result for the displacement map), there exist analytic functions $h_i(\bar{s};\nu)$ in a neighborhood of $(0,\nu_\star)$ with $h_i(0;\nu) \equiv 1$ for $i = 1, 2, \ldots, \ell$ such that we can write

$$\bar{P}'(\bar{s};\nu) = \sum_{i=1}^{\ell} a_i(\nu)\bar{s}^{i-1} h_i(\bar{s};\nu).$$

Now, setting $\psi_i(\bar{s};\nu) := \bar{s}^{i-1} h_i(\bar{s};\nu)$ and proceeding just like the proof of [6, Theorem 2.2], one can apply the well-known derivation-division algorithm and use recursively Rolle's Theorem to show that there exist $\delta_2, \varepsilon_2 > 0$ small enough such that if $\nu \in B_{\varepsilon_2}(\nu_\star)$ then the ordered set $(\psi_1, \psi_2, \ldots, \psi_\ell)$ is an extended complete Chebyshev system for $\bar{s} \in (-\delta_2, \delta_2)$, see [13] for a definition. Accordingly if $\nu \in B_{\varepsilon_2}(\nu_\star)$ then either $\bar{P}'(\cdot\,;\nu) \equiv 0$ or $P'(\bar{s};\nu) = 0$ has at most $\ell - 1$ roots with $\bar{s} \in (-\delta_2, \delta_2)$ counted with multiplicities. Using the original parametrization of the period function, this shows the existence of $\delta_3, \varepsilon_3 > 0$ small enough such that if $\nu \in B_{\varepsilon_3}(\nu_\star)$ then the number of isolated roots of $P'(s;\nu)=0$ with $s\in(s_\star-\delta_3, s_\star+\delta_3)$ is at most $\ell-1$ taking multiplicities into account.

Since in each one of the possible cases there is a neighborhood of $(s_\star,\nu_\star)$ where the number of critical periods is finite, the result follows by taking a finite subcover of $[0,1]\times K$. ■

## 3. Asymptotic expansion of the period function

From now on we focus on the quadratic family $\{X_\nu, \nu\in\mathbb{R}^2\}$ given in (2) and study the period function of the center at the origin. In this section we give its asymptotic expansion near the outer boundary $\Pi_\nu$ for parameters $\nu$ inside three specific sets (see Fig. 1):

$$\Gamma_1 = \big\{D=-\tfrac{1}{2}, F\in(\tfrac{1}{2},1)\big\}\cup\big\{F=\tfrac{1}{2}, D\in(-1,0)\big\},$$
$$\Gamma_2 = \big\{F=2, D\in(-2,0)\big\}\cup\big\{D=\mathcal{G}(F): F\in(1,\tfrac{3}{2})\big\}$$

and

$$\Gamma_3 = \big\{F=1, D\in(-1,0)\big\}.$$

In all the cases the period annulus $\mathscr{P}_\nu$ is unbounded. Since the vector field $X_\nu$ is polynomial, in order to study the behavior of the trajectories near infinity one can use its Poincaré compactification $p(X_\nu)$, which is an analytic vector field on the sphere $\mathbb{S}^2$ topologically equivalent to $X_\nu$, see [3, §5] for details. The outer boundary $\Pi_\nu$ is a polycycle of $p(X_\nu)$ that can be studied using local charts of $\mathbb{S}^2$. In doing so one obtains (see [23, Figure 4]) the different phase portraits in the dehomogeneized Loud's family $\{X_\nu, \nu\in\mathbb{R}^2\}$. For the parameter values studied in this section it occurs that the polycycle $\Pi_\nu$ of $p(X_\nu)$ is hyperbolic if $\nu\in\Gamma_1\cup\Gamma_2$ and has a saddle-node singularity if $\nu\in\Gamma_3$. With regard to the phase portrait, it happens that the affine part of $\Pi_\nu$ is a straight line for $\nu\in\Gamma_1$, whereas it is a branch of a hyperbola for $\nu\in\Gamma_2$. These are the reasons why we split the parameters under consideration in these three subsets, which are studied in the forthcoming subsections. Concerning the behavior of the period function near $\Pi_\nu$, the dichotomy between local regular value and local bifurcation value (see Definition 2.10) is solved for any $\nu\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$ thanks to the results in [22,23,25,38]. In these papers it is computed the asymptotic expansion of the period function to second order, which usually suffices to tackle the regular/bifurcation dichotomy. However in order to study the criticality we need here to go further and compute the third, and even the fourth, order expansion. Let us advance that the asymptotic expansions for $\nu\in\Gamma_1$ are given in Proposition 3.2 and the ones for $\nu\in\Gamma_2$ in Proposition 3.3. Being the proof of both results rather long and technical, for the sake of paper's readability we postpone them to Appendix A, where we also summarize the fundamental results and definitions from [29–31] that we shall use here. Among them we point out the notion of $L$-flatness satisfied by the remainder and that we advance now for reader's convenience (see Definition A.2). Given $L\in\mathbb{R}$ we say that a $\mathscr{C}^\infty$ function $\psi(s;\nu)$ defined for $s>0$ small enough and $\nu\in U\subset\mathbb{R}^2$ is $L$-flat at $\nu_\star\in U$ if for each $\ell=(\ell_0,\ell_1,\ell_2)\in\mathbb{Z}^3_{\geq 0}$ there exist a neighborhood $V$ of $\nu_\star$ and constants $C, s_0>0$ such that

$$\left|\frac{\partial^{|\ell|}\psi(s;\nu)}{\partial s^{\ell_0}\partial_{\nu_1}^{\ell_1}\partial_{\nu_2}^{\ell_2}}\right| \leqslant C s^{L-\ell_0} \text{ for all } s\in(0,s_0) \text{ and } \nu\in V.$$

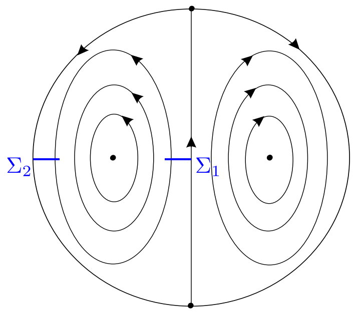


Fig. 3. Phase portrait in the Poincaré disc of $X_\nu$ for $\nu \in V$, where for convenience the center at $(0,0)$ is shifted to the left and the vertical invariant line is $\{x=1\}$.

In this case we write $\psi \in \mathcal{F}_L^\infty(\nu_\star)$. We also consider the Écalle-Roussarie compensator $\omega(s;\alpha)$, that is a deformation of the logarithm used in the monomial scale in which the asymptotic expansion is given.

**Definition 3.1.** The function defined for $s>0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) := \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log s & \text{if } \alpha = 0, \end{cases}$$

is called the *Écalle-Roussarie compensator*. In the sequel we shall also use the notation $\omega_\alpha(s) = \omega(s;\alpha)$. □

The asymptotic expansion for $\nu \in \Gamma_3$ is given in Proposition 3.6 and its proof is of a different nature due to the occurrence of a saddle-node bifurcation at the polycycle.

### 3.1. *Study of* $\{D=-1/2,\ F\in(1/2,1)\}$ *and* $\{F=1/2,\ D\in(-1,0)\}$

Fig. 3 shows the phase portrait in the Poincaré disc of the vector field $X_\nu$ in (2) for $\nu$ varying inside

$$V := \Big\{(D,F) \in \mathbb{R}^2 : D \in (-1,0),\ F \in (0,1)\Big\}.$$

We take transverse sections $\Sigma_1$ and $\Sigma_2$ parametrized by $s \mapsto (1-s,0)$ and $s \mapsto (-1/s,0)$ with $s>0$, respectively, and define $T(s;\nu)$ to be the time that spends the solution of $X_\nu$ with initial condition at $(1-s,0) \in \Sigma_1$ to arrive at $\Sigma_2$. Thanks to the symmetry of $X_\nu$ with respect to $\{y=0\}$, it turns out that the period of the periodic orbit passing through $(1-s,0) \in \Sigma_1$ is precisely $2T(s;\nu)$. Consequently the emergence/disappearance of critical periodic orbits from $\Pi_\nu$ corresponds to zeros of $T'(s;\nu)$ bifurcating from $s=0$, more concretely to the number $\mathcal{Z}_0(T'(\,\cdot\,;\nu),\nu_\star)$ as introduced in Definition 2.3. A key point to study these bifurcations is that $T(s;\nu)$ is the Dulac time associated to the passage through a hyperbolic saddle, which is at infinity (see Fig. 3 again). Therefore we can apply [30, Theorem A] to obtain the asymptotic expansion of $T(s;\nu)$ at $s=0$ and use then [31, Theorem A] to compute its first coefficients $T_{ij}(\nu)$. Next result gathers all this information, where $\Gamma(\,\cdot\,)$ denotes the gamma function.

**Proposition 3.2.** *Let $T(s;\nu)$ be the Dulac time of the passage from $\Sigma_1$ to $\Sigma_2$ of the saddle at infinity of the vector field $X_\nu$ in (2) for $\nu \in V$. Then the coefficients $T_{00}$, $T_{01}$, $T_{10}$ and $T_{20}$ in its asymptotic expansion at $s = 0$ are meromorphic functions on $V$ that can be written as*

$$T_{00}(\nu) = \frac{\pi}{2\sqrt{F(D+1)}}, \qquad T_{01}(\nu) = \rho_1(\nu)\frac{\Gamma\left(-\frac{\lambda}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right)},$$

$$T_{10}(\nu) = \rho_2(\nu)(2D+1)\frac{\Gamma(1-\frac{1}{2\lambda})}{\Gamma(\frac{3}{2}-\frac{1}{2\lambda})}, \qquad T_{20}(\nu) = \frac{\sqrt{\pi}}{\sqrt{2F}}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{\lambda}\right)}{\Gamma\left(1-\frac{1}{\lambda}\right)} + \rho_3(\nu)(2D+1),$$

*where $\lambda(\nu) = \frac{F}{1-F}$ is the hyperbolicity ratio of the saddle,*

$$\rho_1(\nu) = \frac{\sqrt{\pi}}{2(1-F)}\left(\frac{F}{D+1}\right)^{\frac{\lambda+1}{2}}\left(\frac{D}{F-1}\right)^{\frac{\lambda}{2}} \text{ and } \rho_2(\nu) = \frac{\sqrt{\pi}}{2\sqrt{F(D+1)^3}},$$

*and $\rho_3$ is an analytic function on $V \cap \left\{\frac{2}{3} < F < 1\right\}$. In addition the following holds:*

(*a*) *If $\nu_0 \in V \cap \left\{\frac{2}{3} < F < 1\right\}$ then, for all $\upsilon > 0$ small enough,*

$$T(s;\nu) = T_{00}(\nu) + T_{10}(\nu)s + T_{20}(\nu)s^2 + \mathcal{F}^{\infty}_{L_0-\upsilon}(\nu_0)$$

*with $L_0 = \min\left(3, \lambda(\nu_0)\right)$. Moreover $T_{10}(-\frac{1}{2}, F) = 0$ and $T_{20}(-\frac{1}{2}, F) > 0$ for all $F \in (\frac{2}{3}, 1)$.*

(*b*) *If $\nu_0 \in V \cap \left\{\frac{1}{2} < F < \frac{2}{3}\right\}$ then, for all $\upsilon > 0$ small enough,*

$$T(s;\nu) = T_{00}(\nu) + T_{10}(\nu)s + T_{01}(\nu)s^{\lambda} + \mathcal{F}^{\infty}_{2-\upsilon}(\nu_0).$$

*Furthermore $T_{10}(-\frac{1}{2}, F) = 0$ and $T_{01}(-\frac{1}{2}, F) > 0$ for all $F \in (\frac{1}{2}, \frac{2}{3})$.*

(*c*) *If $\nu_0 \in V \cap \left\{F = \frac{2}{3}\right\}$ then, for all $\upsilon > 0$ small enough,*

$$T(s;\nu) = T_{00}(\nu) + T_{10}(\nu)s + T^2_{201}(\nu)s^2\omega_{2-\lambda}(s) + T^2_{200}(\nu)s^2 + \mathcal{F}^{\infty}_{3-\upsilon}(\nu_0),$$

*where $T^2_{200}$ and $T^2_{201}$ are analytic functions in a neighborhood of $V \cap \{F = \frac{2}{3}\}$. Moreover $T_{10}(-\frac{1}{2}, \frac{2}{3}) = 0$ and $T^2_{201}(-\frac{1}{2}, \frac{2}{3}) \neq 0$.*

(*d*) *If $\nu_0 \in V \cap \left\{F = \frac{1}{2}\right\}$ then, for all $\upsilon > 0$ small enough,*

$$T(s;\nu) = T_{00}(\nu) + T^1_{101}(\nu)s\omega_{1-\lambda}(s) + T^1_{100}(\nu)s + \mathcal{F}^{\infty}_{2-\upsilon}(\nu_0),$$

*where*

$$T^1_{101}(\nu) = -\rho_4(\nu)(F-1/2)^2 \text{ and } T^1_{100}(\nu) = \rho_5(\nu)(D+1/2) + \rho_6(\nu)(F-1/2)$$

*for some analytic positive functions $\rho_i$ in a neighborhood of $V \cap \{F = \frac{1}{2}\}$ with $\rho_5(-\frac{1}{2}, \frac{1}{2}) = \rho_6(-\frac{1}{2}, \frac{1}{2})$.*

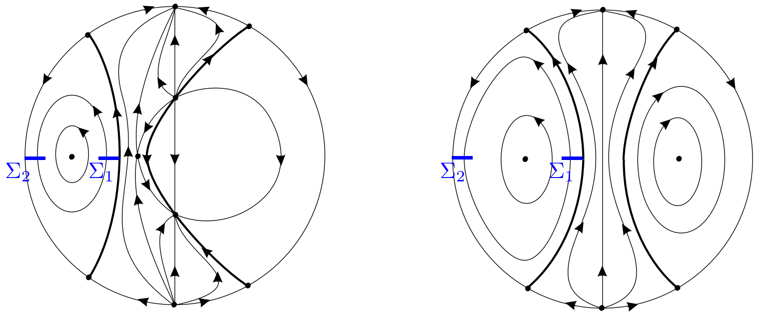


Fig. 4. Phase portrait in the Poincaré disc of $X_\nu$ for $\nu \in W$ with $D < -1$ (left) and $D > -1$ (right), where the center at $(0, 0)$ is shifted to the left, the vertical invariant line is $\{x = 1\}$ and the hyperbola $\{\frac{1}{2}y^2 - q(x) = 0\}$ appears in boldface type.

As we already explained, the proof of this result is postponed to Appendix A. The monomial order in each one of these asymptotic expansions is with respect to the strict partial order $\prec_{\nu_0}$ given in [29]. Let us recall in its regard that we write $f \prec_{\nu_0} g$ in case that

$$\lim_{(s,\nu)\to(0,\nu_0)} \frac{g(s;\lambda)}{f(s;\lambda)} = 0.$$

For the monomials under consideration this order is preserved after derivation with respect to $s$, and so it is the good flatness properties of the remainder. Thus, as it occurs with the Taylor's series of an analytic function, an upper bound for the number of zeros of $T'(s;\nu)$ that can bifurcate from $s = 0$ follows by identifying the first non-vanishing coefficient in the asymptotic expansion. For the proof and a precise statement of this result, which essentially follows by using the well-known derivation-division algorithm, the reader is referred to [29, Theorem C].

### 3.2. *Study of* $\{F = 2, D \in (-2, 0)\}$ *and* $\{D = \mathcal{G}(F) : F \in (1, 3/2)\}$

Fig. 4 shows the phase portrait in the Poincaré disc of the vector field $X_\nu$ in (2) for $\nu$ inside

$$W := \{(D, F) \in \mathbb{R}^2 : F + D > 0,\ D < 0 \text{ and } F > 1\}.$$

In this case the outer boundary of the period annulus of the center at $(0, 0)$ is contained in the union of the line at infinity and an invariant hyperbola $\mathcal{C} := \{\frac{1}{2}y^2 - q(x) = 0\}$, where $q(x) = ax^2 + bx + c$ with

$$a := \frac{D}{2(1-F)},\ b := \frac{D-F+1}{(1-F)(1-2F)} \text{ and } c := \frac{F-D-1}{2F(1-F)(1-2F)}.$$

One can verify that if $\nu \in W$ then $q$ has two distinct real zeros, that we shall denote by $p_1$ and $p_2$ taking $p_1 < p_2$. That being said, we place two transverse sections $\Sigma_1$ and $\Sigma_2$ parametrized by $s \mapsto (p_1 - s, 0)$ and $s \mapsto (-1/s, 0)$ with $s > 0$, respectively, and define $T(s;\nu)$ to be the time that takes to the solution of $X_\nu$ with initial condition at $(p_1 - s, 0) \in \Sigma_1$ to arrive at $\Sigma_2$. Then $T(s;\nu)$ is the Dulac time associated to the passage through a hyperbolic saddle at infinity, so that we can apply the results in [30,31] to obtain its asymptotic expansion at $s = 0$. This is important for the proof of Theorem A because, exactly as in the previous case, the symmetry of $X_\nu$ with respect to $\{y=0\}$ implies that the period of the periodic orbit passing through $(1-s,0)\in\Sigma_1$ is $2T(s;\nu)$. With regard to our next result we remark that $\frac{1-p_2}{1-p_1}<1$ for all $\nu\in W$, which is relevant since the hypergeometric function ${}_2F_1(a,b;c;\,\cdot\,)$ is holomorphic on $\mathbb{C}\setminus[1,+\infty)$, see Appendix C. Let us also mention that $\mathrm{B}(\cdot,\cdot)$ is the beta function.

**Proposition 3.3.** *Let $T(s;\nu)$ be the Dulac time of the passage from $\Sigma_1$ to $\Sigma_2$ of the saddle at infinity of the vector field $X_\nu$ in (2) for $\nu\in W$. Then the coefficients $T_{00}$, $T_{01}$, $T_{10}$ and $T_{20}$ in its asymptotic expansion at $s=0$ are meromorphic functions on $W$ that can be written as*

$$T_{00}(\nu)=\frac{\sqrt{2}}{\sqrt{a}(1-p_1)}\,{}_2F_1\big(1,\tfrac{1}{2};\tfrac{3}{2};\tfrac{1-p_2}{1-p_1}\big),$$
$$T_{01}(\nu)=\rho_1(\nu)\mathrm{B}\big(-\lambda,\tfrac{1}{2}\big),$$
$$T_{10}(\nu)=\rho_2(\nu)\mathrm{B}\big(1-\tfrac{1}{\lambda},-\tfrac{1}{2}\big){}_2F_1\big(-1-\tfrac{1}{\lambda},-\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{\lambda};\tfrac{1-p_2}{1-p_1}\big)$$

*and*

$$T_{20}(\nu)=\rho_3(\nu)\mathrm{B}\big(1-\tfrac{2}{\lambda},-\tfrac{3}{2}\big){}_2F_1\big(-\tfrac{2}{\lambda}-3,-\tfrac{3}{2};-\tfrac{1}{2}-\tfrac{2}{\lambda};\tfrac{1-p_2}{1-p_1}\big)+\rho_4(\nu)T_{10}(\nu),$$

*where $\lambda(\nu)=\frac{1}{2(F-1)}$ is the hyperbolicity ratio of the saddle and, for $i=1,2,3,4$, $\rho_i$ is an analytic positive function on $W$. In addition the following holds:*

(a) *If $\nu_0\in W\cap\left\{1<F<\frac{5}{4}\right\}$ then, for all $\upsilon>0$ small enough,*

$$T(s;\nu)=T_{00}(\nu)+T_{10}(\nu)s+T_{20}(\nu)s^2+\mathcal{F}^{\infty}_{L_0-\upsilon}(\nu_0)$$

*with $L_0=\min(3,\lambda(\nu_0))$. Moreover $T_{20}(\nu)\neq 0$ for all $\nu\in W\cap\left\{1<F<\frac{5}{4}\right\}$ such that $T_{10}(\nu)=0$.*

(b) *If $\nu_0\in W\cap\left\{\frac{5}{4}<F<\frac{3}{2}\right\}$ then, for all $\upsilon>0$ small enough,*

$$T(s;\nu)=T_{00}(\nu)+T_{10}(\nu)s+T_{01}(\nu)s^{\lambda}+T_{20}(\nu)s^2+\mathcal{F}^{\infty}_{L_0-\upsilon}(\nu_0)$$

*with $L_0=\lambda(\nu_0)+1$, and there exists a unique $\nu_\star\in W\cap\left\{\frac{5}{4}<F<\frac{3}{2}\right\}$ such that $T_{10}(\nu_\star)=0$ and $T_{01}(\nu_\star)=0$. Furthermore $T_{20}(\nu_\star)<0$, the gradients of $T_{01}$ and $T_{10}$ at $\nu_\star$ are linearly independent, and $\nu_\star=(D_\star,\frac{4}{3})$ with $D_\star=\mathcal{G}(\frac{4}{3})\approx -1.128$.*

(c) *If $\nu_0\in W\cap\left\{F=\frac{5}{4}\right\}$ then, for all $\upsilon>0$ small enough,*

$$T(s;\nu)=T_{00}(\nu)+T_{10}(\nu)s+T^2_{201}(\nu)s^2\omega_{2-\lambda}(s)+T^2_{200}(\nu)s^2+\mathcal{F}^{\infty}_{3-\upsilon}(\nu_0),$$

*where $T^2_{200}$ and $T^2_{201}$ are analytic functions in a neighborhood of $W\cap\{F=\frac{5}{4}\}$. Moreover $T_{10}(D,\frac{5}{4})=0$ if and only if $D=-1$, and $T^2_{201}(-1,\frac{5}{4})\neq 0$.*

(*d*) *If* $\nu_0 \in W \cap \{F = 2\}$ *then, for all* $\upsilon > 0$ *small enough,*

$$T(s;\nu) = T_{00}(\nu) + T_{01}(\nu)s^{\lambda} + T_{101}^{\frac{1}{2}}(\nu)s\omega_{1-2\lambda}(s) + T_{100}^{\frac{1}{2}}(\nu)s + \mathcal{F}^{\infty}_{3/2-\upsilon}(\nu_0),$$

*where* $T_{100}^{\frac{1}{2}}$ *and* $T_{101}^{\frac{1}{2}}$ *are analytic functions in a neighborhood of* $W \cap \{F = 2\}$. *Moreover* $T_{01}(D, 2) = 0$ *for all* $D \in (-2, 0)$, $T_{101}^{\frac{1}{2}}(D, 2) = 0$ *if and only if* $D = -\frac{1}{2}$, *and the gradients of* $T_{01}$ *and* $T_{101}^{\frac{1}{2}}$ *are linearly independent at* $(-\frac{1}{2}, 2)$.

The proof of this result is postponed to Appendix A.

**Remark 3.4.** The asymptotic expansions in Proposition 3.3 were already given in [23, Theorem 3.6] but only to second order. In that result it is given, among others, the expression of the coefficient $T_{10}(\nu)$ in terms of a definite improper integral. Furthermore, see [23, Proposition 3.11], it is proved by applying the Implicit Function Theorem that the set of those $\nu \in W_1 := W \cap \{F < 3/2\}$ such that $T_{10}(\nu) = 0$ is the graphic of an analytic function $D = \mathcal{G}(F)$. This is the function that appears in assertion (*b*) of Proposition 3.3. Thanks to the results in Appendix B we can now identify the improper integral as a hypergeometric function, so that we can write

$$\big\{\nu \in W_1 : D = \mathcal{G}(F)\big\} = \big\{\nu \in W_1 : {}_2F_1\big(-1 - \tfrac{1}{\lambda}, -\tfrac{1}{2}; \tfrac{1}{2} - \tfrac{1}{\lambda}; \tfrac{1-p_2}{1-p_1}\big) = 0\big\},$$

where $p_1$ and $p_2$ with $p_1 < p_2$ are the real roots of $q(x) = 0$ and $\lambda(\nu) = \frac{1}{2(F-1)}$. □

**Remark 3.5.** In the statement of Proposition 3.3 we refer to some positive functions $\rho_i \in \mathscr{C}^{\omega}(W)$. Let us mention that in the proof we show that

$$\rho_1(\nu) = \frac{1}{2\sqrt{2a}} \frac{(p_2 - p_1)^{\frac{1}{2(F-1)}}}{(F-1)(1-p_1)^{\frac{F}{F-1}}} \qquad \rho_2(\nu) = \frac{1}{2\sqrt{2a}} \frac{1}{(p_2 - p_1)(1-p_1)}$$

$$\rho_3(\nu) = \frac{3}{8\sqrt{2a}} \frac{1}{(p_2 - p_1)^2(1-p_1)} \qquad \rho_4(\nu) = \frac{p_1 - 1 + 2F(p_2 - p_1)}{(p_2 - p_1)(p_1 - 1)}$$

We do not use the explicit expressions in this paper but they may be relevant for future applications. □

### 3.3. *Study of* $\{F = 1, D \in (-1, 0)\}$

Our aim in this section is to study the period function of the center at the origin of $X_\nu$ for $\nu = (D, F)$ with $F \approx 1$ and $D \in (-1, 0)$. To this end we introduce transverse sections $\Sigma_1$ and $\Sigma_2$ parametrized, respectively, by means of

$$\sigma_1(s;\nu) := \begin{cases} (1-s, 0) & \text{if } F \leqslant 1, \\ (p_1 - s, 0) & \text{if } F > 1, \end{cases} \quad \text{and } \sigma_2(s;\nu) := (-1/s, 0),$$

where recall that $q(x) = a(x - p_1)(x - p_2)$ with $p_1 < p_2$ for $F > 1$. One can also check that $\lim_{F \to 1^+} p_1 = 1$. For each $\nu = (D, F) \in (-1, 0) \times (0, +\infty)$ we define $T(s;\nu)$ as the time that

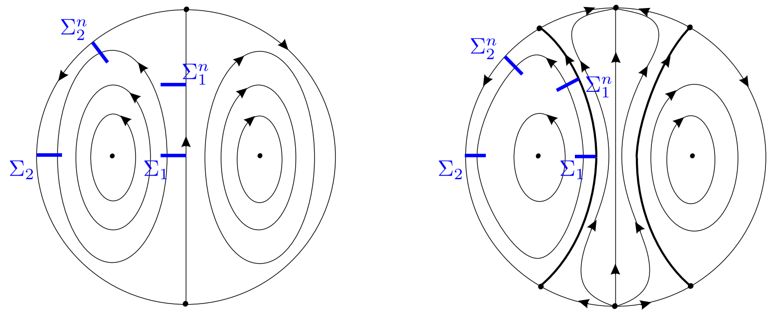


Fig. 5. Phase portrait of $X_\nu$ for $\nu = (D, F) \in (-1, 0) \times (0, +\infty)$ in the Poincaré disc with $F \leqslant 1$ (left) and $F > 1$ (right). In this case, contrary to the previous ones, the singularity at infinity for $F = 1$ is not a hyperbolic saddle but a saddle-node.

spends the solution of $X_\nu$ starting at $\sigma_1(s;\nu) \in \Sigma_1$ to arrive at $\Sigma_2$. A key feature of this Dulac time is that the singularity for $F = 1$ is not a hyperbolic saddle but a saddle-node. Our next result gives the asymptotic expansion of $T(s;\nu)$ at $s = 0$ for $F \approx 1$. We point out that this is relevant for the proof of Theorem A because the period of the periodic orbit of $X_\nu$ passing through $\sigma_1(s;\nu) \in \Sigma_1$ is precisely $2T(s;\nu)$ due to the symmetry of the vector field.

**Proposition 3.6.** *Let $T(s;\nu)$ be the Dulac time of the passage from $\Sigma_1$ to $\Sigma_2$ of the saddle-node unfolding at infinity $\{X_\nu\}$. Then there is an open neighborhood $\mathscr{U}$ of $(-1, 0) \times \{1\}$ such that*

$$T(s;\nu) = T_0(\nu) + T_1(\nu)s + T_2(\nu)s^2 + s^2 h(s;\nu),$$

*where $T_i \in \mathscr{C}^0(\mathscr{U})$ and, setting $\Theta = s\partial_s$, $\lim_{s\to 0^+} \Theta^k h(s;\nu) = 0$ uniformly on compact sets of $\mathscr{U}$ for $k = 0, 1, 2$. Moreover $T_1(D, 1) = 0$ if, and only if, $D = -\frac{1}{2}$. Finally $T_2(-1/2, 1) \neq 0$.*

**Proof.** To study the saddle-node bifurcation that occurs at infinity we work in the projective plane $\mathbb{RP}^2$ and perform the change of coordinates

$$(u, v) = p(x, y) := \left(\frac{1-x}{y}, \frac{1}{y}\right).$$

The meromorphic extension of $X_\nu$ in these coordinates is given by

$$\tilde{X}_\nu := p_* X_\nu = \frac{1}{v}\Big(uP(u, v;\nu)\partial_u + vQ(u, v;\nu)\partial_v\Big)$$

with

$$P(u, v;\nu) = 1 - F - Du^2 + (2D+1)uv - (D+1)v^2$$

and

$$Q(u, v;\nu) = -F - Du^2 + (2D+1)uv - (D+1)v^2.$$

Our first goal is to show that we can bring locally the saddle-node unfolding to a convenient normal form in order that we can apply the tools developed in [25] to study the asymptotic expansion of its Dulac map and Dulac time. With this aim, some long but easy computations show that the local analytic change of coordinates given by

$$(z,w)=\Psi(u,v):=\left(\frac{u}{\sqrt{g(u,v)}},\frac{v}{\sqrt{g(u,v)}}\right),$$

where $g(u,v):=\frac{(2D+1)}{(2F-1)D}uv-\frac{(D+1)}{2FD}v^2-\frac{1}{2D}$, brings the vector field $\tilde{X}_\nu$ to

$$\bar{X}_\nu:=\frac{1}{w\bar{U}(z,w;\nu)}\big(z(z^2-2(F-1))\partial_z-w(2F-z^2)\partial_w\big),$$

with $\bar{U}(z,w;\nu):=\left(\frac{(2D+1)}{2(2F-1)}zw-\frac{(D+1)}{4F}w^2-\frac{D}{2}\right)^{-\frac{1}{2}}$. A technical assumption in order to apply the results from [25] is that for each $\nu$ the Taylor's series of $(z,w)\mapsto\bar{U}(z,w;\nu)$ at $(0,0)$ is absolutely convergent for all $(z,w)\in[-1,1]^2$. This is not fulfilled unless we perform a rescaling which is only well defined provided that $\nu$ varies inside a compact subset of $(-1,0)\times(0,+\infty)$ and this forces us to work locally. For this reason, as a first step in the proof, we will show a local version of the statement. More concretely, that for each $\nu_\star=(D_\star,1)$ with $D_\star\in(-1,0)$ there exists an open ball $B_{\nu_\star}=\{\nu\in\mathbb{R}^2:\|\nu-\nu_\star\|<\delta\}$ such that

$$T(s;\nu)=T_0^{\nu_\star}(\nu)+T_1^{\nu_\star}(\nu)s+T_2^{\nu_\star}(\nu)s^2+s^2h^{\nu_\star}(s;\nu),$$

with $T_i^{\nu_\star}$ continuous functions on $B_{\nu_\star}$ and $\lim_{s\to0^+}\Theta^k h^{\nu_\star}(s;\nu)=0$ uniformly on $B_{\nu_\star}$ for $k=0,1,2$. To begin with we take $\delta>0$ small enough so that the closure of $B_{\nu_\star}$ is inside $(-1,0)\times(\frac{1}{2},+\infty)$ and define

$$r:=\inf\left\{\left|\frac{(D+1)}{DF}w^2-\frac{2(2D+1)}{D(2F-1)}zw\right|^{-1/2}:|z|\leqslant1,|w|\leqslant1\text{ and }\|\nu-\nu_\star\|\leqslant\delta\right\},\qquad(6)$$

which is strictly positive. The pull-back of $\bar{X}_\nu$ by the rescaling $\rho(z,w):=(rz,rw)$ can now be written as in [25, Eq. 13] because one can easily verify that

$$\rho^*\bar{X}_\nu=\frac{1}{wU(z,w;\nu)}\left(z(z^2-\varepsilon)\partial_z-w(2F/r^2-z^2)\partial_w\right)\text{ with }\varepsilon:=2(F-1)/r^2$$

and where the Taylor's series of

$$U(z,w;\nu):=\frac{\bar{U}(rz,rw;\nu)}{r}=\frac{-D}{2r}\left(1+r^2\left(\frac{(D+1)}{2FD}w^2-\frac{(2D+1)}{D(2F-1)}zw\right)\right)^{-\frac{1}{2}}$$

at $(z,w)=(0,0)$ is absolutely convergent for all $(z,w)\in[-1,1]^2$ and $\nu\in B_{\nu_\star}$ since, on account of (6),

$$r^2\left|\frac{(D+1)}{2FD}w^2-\frac{(2D+1)}{D(2F-1)}zw\right|\leqslant\frac{1}{2}\text{ for all }(z,w,\nu)\in[-1,1]^2\times B_{\nu_\star}.$$

In these new rescaled coordinates, that we still denote by $(z,w)$ for simplicity, the period annulus is inside the quadrant $\{w\geqslant 0, z\geqslant\vartheta_\varepsilon\}$ where

$$\vartheta_\varepsilon:=\begin{cases}\sqrt{\varepsilon} & \text{if }\varepsilon\geqslant 0,\\ 0 & \text{if }\varepsilon<0.\end{cases}$$

Setting $\Psi_\star:=\rho^{-1}\circ\Psi$, we take two auxiliary transverse sections $\Sigma_1^n:=\Psi_\star^{-1}(\{w=1\})$ and $\Sigma_2^n:=\Psi_\star^{-1}(\{z=1\})$ parameterized by $\sigma_1^n(s;\nu):=\Psi_\star^{-1}(s+\vartheta_\varepsilon,1)$ and $\sigma_2^n(s;\nu):=\Psi_\star^{-1}(1,s)$, respectively (see Fig. 5). We define $\mathcal{T}(s;\nu)$ and $\mathscr{D}(s;\nu)$ to be the Dulac time and Dulac map of $\tilde{X}_\nu$ from $\Sigma_1^n$ to $\Sigma_2^n$, respectively. We remark that, by construction, $\mathcal{T}(s;\nu)$ is the time that the solution of $\rho^*\bar{X}_\nu$ starting at the point $(s+\vartheta_\varepsilon,1)$ spends to arrive at $\{z=1\}$ and that the intersection point is precisely $(z,w)=(1,\mathscr{D}(s;\nu))$. In this regard, since $w=\mathscr{D}(z;\nu)$ is a trajectory of the vector field $z(z^2-\varepsilon)\partial_z+w(2F/r^2-z^2)\partial_w$, see [25, p. 6417] for details, the application of $(b)$ in Corollary A of [25] with $\{\mu=2,\ell=k=2,\lambda=2F/r^2\}$ shows that

$$\mathscr{D}(s;\nu)=s^2\mathcal{I}(B_{\nu_\star}),\tag{7}$$

by shrinking $\delta>0$ if necessary. Here, and in what follows, $\mathcal{I}(B_{\nu_\star})$ stands for some function $h(s;\nu)$ verifying that $\lim_{s\to 0^+}\Theta^k h(s;\nu)=0$ uniformly on $\nu\in B_{\nu_\star}$ for $k=0,1,2$. Furthermore, by applying Corollary B in the same paper with $\{\mu=2,\ell=k=2\}$ and shrinking $\delta>0$ again we can assert that

$$\mathcal{T}(s;\nu)=b_0(\nu)+b_1(\nu)s+b_2(\nu)s^2+s^2\mathcal{I}(B_{\nu_\star})\tag{8}$$

with $b_i\in\mathscr{C}^0(B_{\nu_\star})$ for $i=0,1,2$. Working in the original $(x,y)$ coordinates, we consider next the transition times $T_1(\,\cdot\,;\nu)$ and $T_2(\,\cdot\,;\nu)$ of $X_\nu$ from $\Sigma_1$ to $p^{-1}(\Sigma_1^n)$ and from $p^{-1}(\Sigma_2^n)$ to $\Sigma_2$, respectively. We define moreover $R(\,\cdot\,;\nu)$ to be the transition map from $\Sigma_1$ to $p^{-1}(\Sigma_1^n)$. Accordingly

$$T(s)=T_1(s)+\big(\mathcal{T}\circ R\big)(s)+\big(T_2\circ\mathscr{D}\circ R\big)(s),\tag{9}$$

where we omit the dependence on $\nu$ for the sake of shortness. By [22, Lemma 3.2], we have that $T_2(s;\nu)$ an analytic function at $\{0\}\times B_{\nu_\star}$ with $T_2(0;\nu)=0$. Observe at this point that, setting

$$\xi_\nu:=\begin{cases}0 & \text{if }F\leqslant 1,\\ 1-p_1 & \text{if }F>1,\end{cases}$$

we can write the parametrization of $\Sigma_1$ as $\sigma_1(s;\nu)=(1-\xi_\nu-s,0)$. We claim that there exist two functions $f(\hat{s};\nu)$ and $g(\hat{s};\nu)$, analytic at $\{0\}\times B_{\nu_\star}$, such that

$$T_1(s;\nu)=f(s+\xi_\nu;\nu)\text{ and }R(s;\nu)=g(s+\xi_\nu;\nu)-\vartheta_\varepsilon.\tag{10}$$

To show this let us consider two additional transverse sections $\hat{\Sigma}_1$ and $\hat{\Sigma}_1^n$ parameterized respectively by $\hat{\sigma}_1(\hat{s};\nu):=(1-\hat{s},0)$ and $\hat{\sigma}_1^n(\hat{s};\nu):=(\Psi_\star\circ p)^{-1}(\hat{s},1)$, which clearly are analytic at $\{0\} \times B_{\nu_\star}$. Moreover it is clear that they are related with $\Sigma_1$ and $\Sigma_1^n$ through $\sigma_1(s;\nu) = \hat{\sigma}_1(s+\xi_\nu;\nu)$ and $\sigma_1^n(s;\nu) = \hat{\sigma}_1^n(s+\vartheta_\varepsilon;\nu)$. That being said, the claim follows by noting that if we choose $f(\hat{s};\nu)$ and $g(\hat{s};\nu)$ to be, respectively, the transition time and transition map of $X_\nu$ from $\hat{\Sigma}_1$ to $\hat{\Sigma}_1^n$, which are clearly analytic at $\{0\} \times B_{\nu_\star}$, then the equalities in (10) hold. Note moreover that $g(\xi_\nu;\nu) = \vartheta_\varepsilon$ since $R(0;\nu) = 0$. On account of the claim, by considering the second order Taylor's development of $f(\hat{s};\nu)$ and $g(\hat{s};\nu)$ at $\hat{s} = \xi_\nu$, respectively, we get

$$T_1(s;\nu) = a_0(\nu) + a_1(\nu)s + a_2(\nu)s^2 + s^2\mathcal{I}(B_{\nu_\star}) \text{ and } R(s;\nu) = c_1(\nu)s + c_2(\nu)s^2 + s^2\mathcal{I}(B_{\nu_\star})$$

with $a_i, c_i \in \mathscr{C}^0(B_{\nu_\star})$ and where we also use that $\nu \mapsto \xi_\nu$ is a continuous function. The combination of the second expression above with (7) and (8) yields

$$\big(\mathscr{D} \circ R\big)(s) = s^2\mathcal{I}(B_{\nu_\star}) \text{ and } \big(\mathcal{T} \circ R\big)(s) = \hat{b}_0(\nu) + \hat{b}_1(\nu)s + \hat{b}_2(\nu)s^2 + s^2\mathcal{I}(B_{\nu_\star}),$$

respectively, with $\hat{b}_i \in \mathscr{C}^0(B_{\nu_\star})$. Summing up, since $\big(T_2 \circ \mathscr{D} \circ R\big)(s) = s^2\mathcal{I}(B_{\nu_\star})$ due to $T_2(0;\nu) = 0$, from (9) we can assert that

$$T(s;\nu) = T_0^{\nu_\star}(\nu) + T_1^{\nu_\star}(\nu)s + T_2^{\nu_\star}(\nu)s^2 + s^2 h^{\nu_\star}(s;\nu) \tag{11}$$

for some functions $T_i^{\nu_\star}$ that are continuous on $B_{\nu_\star}$ and some $h^{\nu_\star} \in \mathcal{I}(B_{\nu_\star})$. This concludes the proof of the local version of the statement, in which we remark that the coefficients $T_i^{\nu_\star}(\nu)$ and the remainder $s^2 h^{\nu_\star}(s;\nu)$ depend by construction on $\nu_\star$. Our next step will be to globalize them and to this end we define

$$\mathscr{U} := \bigcup_{\nu_\star \in (-1,0)\times\{1\}} B_{\nu_\star}$$

which is clearly an open neighborhood of $(-1,0)\times\{1\}$. Consider now any $\nu_1, \nu_2 \in (-1,0)\times\{1\}$ such that $B_{\nu_1} \cap B_{\nu_2} \neq 0$. Then, from (11), we get that

$$T_0^{\nu_1}(\nu) - T_0^{\nu_2}(\nu) + \big(T_1^{\nu_1}(\nu) - T_1^{\nu_2}(\nu)\big)s + \big(T_2^{\nu_1}(\nu) - T_2^{\nu_2}(\nu)\big)s^2 + s^2\big(h^{\nu_1}(s;\nu) - h^{\nu_2}(s;\nu)\big) = 0$$

for all $s > 0$ small enough and $\nu \in B_{\nu_1} \cap B_{\nu_2}$. Since $h^{\nu_1} - h^{\nu_2} \in \mathcal{I}(B_{\nu_1} \cap B_{\nu_2})$, taking the limit $s \to 0^+$ on both sides of the above equality we deduce that $T_0^{\nu_1} = T_0^{\nu_2}$ on $B_{\nu_1} \cap B_{\nu_2}$. Similarly, but taking the first and second derivatives with respect to $s$, respectively, we get that $T_1^{\nu_1} = T_1^{\nu_2}$ and $T_2^{\nu_1} = T_2^{\nu_2}$ on $B_{\nu_1} \cap B_{\nu_2}$. Hence, for $i = 0, 1, 2$, the local functions $T_i^{\nu_\star} \in \mathscr{C}^0(B_{\nu_\star})$ for $\nu_\star \in (-1,0)\times\{1\}$ glue together into a well defined continuous function $T_i$ on $\mathscr{U}$. Exactly the same argument shows that the local functions $h^{\nu_\star} \in \mathcal{I}(B_{\nu_\star})$ for $\nu_\star \in (-1,0)\times\{1\}$ glue together into a well defined function $h(s;\nu)$ satisfying that $\lim_{s\to 0^+} \Theta^k h(s;\nu) = 0$ uniformly on compact sets of $\mathscr{U}$ for $k = 0, 1, 2$. To show this last assertion it suffices to take a finite subcover $B_{\nu_1} \cup \ldots \cup B_{\nu_n}$ of the given compact subset of $\mathscr{U}$ and use that $h|_{B_{\nu_i}} \in \mathcal{I}(B_{\nu_i})$ for $i = 1, 2, \ldots, n$.

So far we have proved the first assertion in the statement. Let us turn to the proof of the second one. To this end the key point is that for those $\nu_0 \in \mathscr{U} \cap \{F < 1\}$ we can also apply (*a*) in Proposition 3.2 to obtain that

$$T(s;\nu) = T_{00}(\nu) + T_{10}(\nu)s + T_{20}(\nu)s^2 + \mathcal{F}^\infty_{3-\upsilon}(\nu_0), \tag{12}$$

where, setting $\lambda(\nu)=\frac{F}{1-F}$,

$$T_{10}(\nu)=\frac{\sqrt{\pi}(2D+1)}{2\sqrt{F(1+D)^3}}\frac{\Gamma\left(1-\frac{1}{2\lambda}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2\lambda}\right)} \text{ and } T_{20}(-\tfrac{1}{2},F)=\frac{\sqrt{\pi}}{\sqrt{2F}}\frac{\Gamma(\frac{1}{2}-\frac{1}{\lambda})}{\Gamma(1-\frac{1}{\lambda})}.$$

Hence, since $T_i\in\mathscr{C}^0(\mathscr{U})$, from (11) and (12) we can assert that

$$T_1(D,1)=\lim_{F\to 1^-}T_{10}(D,F)=\frac{2D+1}{(1+D)^{3/2}},$$

where we also use that $\lim_{F\to 1^-}\frac{\Gamma\left(1-\frac{1}{2\lambda}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2\lambda}\right)}=\frac{\Gamma(1)}{\Gamma(\frac{3}{2})}=\frac{2}{\sqrt{\pi}}$. Consequently, as desired, $T_1(D,1)=0$ if and only if $D=-\frac{1}{2}$. The same argument shows that

$$T_2(-1/2,1)=\lim_{F\to 1^-}T_{20}(-1/2,F)=\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1\right)}\neq 0,$$

and this completes the proof of the result. ■

## 4. Distinguished cases

This section is devoted to study three specific parameters. Recall that among the quadratic centers there are four nonlinear isochrones, see (3). Chicone and Jacobs show in [6, Theorem 3.1] that the criticality of the period function at the inner boundary (i.e., the center) of $\mathscr{P}$ is exactly 1 for each one of the nonlinear isochrones. In this section we prove that for two of them, namely $\nu=(-\frac{1}{2},2)$ and $\nu=(-\frac{1}{2},\frac{1}{2})$, the criticality at the outer boundary (i.e., the polycycle) is also 1, see Propositions 4.2 and 4.3, respectively. In the same vein it is also well-known that the criticality at the inner boundary of any quadratic center is at most two, see [6, Theorem 3.2]. This maximum criticality is achieved in three parameter values, the so-called Loud points, which following the notation in [6] are given by $\nu=L_i$ with

$$L_1:=\left(-\tfrac{3}{2},\tfrac{5}{2}\right),\ L_2:=\left(\tfrac{-11+\sqrt{105}}{20},\tfrac{15-\sqrt{105}}{20}\right) \text{ and } L_3:=\left(\tfrac{-11-\sqrt{105}}{20},\tfrac{15+\sqrt{105}}{20}\right). \tag{13}$$

As we already explained in the introduction, we conjecture that the criticality at the outer boundary of any quadratic center is at most two, and that there are only three parameter values where this maximum criticality is attained. In this paper we identify and prove the validity of the conjecture for two of these parameters. We investigate one of them in this section, see Proposition 4.4. The other one was already studied in [24] and we postpone its treatment until the proof of Theorem A.

The following is a sort of division theorem within the class of flat functions that will be used to study the criticality at the outer boundary for the above-mentioned isochrones. In its statement, and in what follows, we use the notation $0_n=(0,0,\ldots,0)\in\mathbb{R}^n$ for the sake of shortness.

**Lemma 4.1.** *Let us fix $K\in\mathbb{N}\cup\{\infty\}$, $L\geqslant 0$ and $n\in\mathbb{N}$. If $f(s;\mu_1,\ldots,\mu_n)\in\mathcal{F}_L^K(0_n)$ verifies*

$$f(s;\mu_1,\ldots,\mu_{k-1},0,\ldots,0)\equiv 0, \text{ for some } k\in\{1,2,\ldots,n\},$$

*then there exist* $f_k, \dots, f_n \in \mathcal{F}_L^{K-1}(0_n)$ *such that* $f = \sum_{i=k}^n \mu_i f_i$.

**Proof.** We proceed by induction on $n \in \mathbb{N}$. For the base case $n = 1$ we take $f(s; \mu_1) \in \mathcal{F}_L^K(0_1)$ with $f(s; 0) \equiv 0$ and define $f_1(s; \mu_1) := \int_0^1 \partial_2 f(s; \mu_1 t)dt$, so that $f = \mu_1 f_1$. To show that $f_1 \in \mathcal{F}_L^{K-1}(0_1)$ we use that, by hypothesis (see Definition A.2), for every $\nu = (\nu_0, \nu_1) \in \mathbb{Z}^2_{\geq 0}$ with $|\nu| = \nu_0 + \nu_1 \leqslant K - 1$ there exist a neighborhood $V \subset \mathbb{R}$ of 0 and $C, s_0 > 0$ such that $|\partial_s^{\nu_0}\partial_{\mu_1}^{\nu_1+1} f(s; \mu_1)| \leqslant C s^{L-\nu_0}$ for every $\mu_1 \in V$ and $s \in (0, s_0)$. On account of this and applying the Dominated Convergence Theorem [37, Theorem 11.30],

$$|\partial^\nu f_1(s; \mu_1)| \leqslant \int_0^1 \left|\partial^\nu(\partial_2 f(s; \mu_1 t))\right| dt \leqslant \int_0^1 |\partial_s^{\nu_0}\partial_2^{\nu_1+1} f(s; \mu_1 t)| t^{\nu_1} dt \leqslant \frac{C}{\nu_1 + 1} s^{L-\nu_0}$$

for every $\mu_1 \in V$ and $s \in (0, s_0)$. Hence $f_1 \in \mathcal{F}_L^{K-1}(0_1)$. To prove the inductive step we suppose that $n > 1$ and consider $f(s; \mu_1, \dots, \mu_n) \in \mathcal{F}_L^K(0_n)$ verifying that $f(s; \mu_1, \dots, \mu_{k-1}, 0, \dots, 0) \equiv 0$ for some $k \in \{1, 2, \dots, n\}$. It is clear that we can write

$$f(s; \mu_1, \dots, \mu_{n-1}, \mu_n) - f(s; \mu_1, \dots, \mu_{n-1}, 0) = \mu_n f_n(s; \mu_1, \dots, \mu_n) \tag{14}$$

with

$$f_n(s; \mu_1, \dots, \mu_n) := \int_0^1 \partial_{n+1} f(s; \mu_1, \dots, \mu_{n-1}, \mu_n t)dt.$$

Similarly as for the base case, taking $f \in \mathcal{F}_L^K(0_n)$ into account, one can easily show that $f_n \in \mathcal{F}_L^{K-1}(0_n)$. Since $f(s; \mu_1, \dots, \mu_{n-1}, 0)|_{\mu_k = \dots = \mu_{n-1} = 0} \equiv 0$, by the inductive hypothesis there exist

$$f_k(s; \mu_1, \dots, \mu_{n-1}), \dots, f_{n-1}(s; \mu_1, \dots, \mu_{n-1}) \in \mathcal{F}_L^{K-1}(0_{n-1})$$

such that

$$f(s; \mu_1, \dots, \mu_{n-1}, 0) = \sum_{i=k}^{n-1} \mu_i f_i(s; \mu_1, \dots, \mu_{n-1}).$$

Due to $\mathcal{F}_L^{K-1}(0_{n-1}) \subset \mathcal{F}_L^{K-1}(0_n)$, see Definition A.2, the combination of this identity with (14) shows that $f = \sum_{i=k}^n \mu_i f_i$ with $f_k, \dots, f_n \in \mathcal{F}_L^{K-1}(0_n)$ as desired. This shows the inductive step and concludes the proof of the result. ■

We state next our first result about the bifurcation of critical periodic orbits from the outer boundary of an isochronous center. With regard to its proof let us advance that, after a convenient division in the space of coefficients, we proceed as in the proofs of Bautin [4, §3] and Chicone and Jacobs [6, Theorem 2.2] for the analogous results about limit cycles and critical periods, respectively, bifurcating from the center. Here we tackle the bifurcation from the polycycle, which is more challenging because, contrary to the center, the period function cannot be analytically extended there. To overcome this difficulty it is crucial the fact that the flatness of the remainder in the asymptotic expansion is preserved after the derivation with respect to the parameters.

**Proposition 4.2.** *If* $\nu_0 = (-\frac{1}{2}, 2)$ *then* $\operatorname{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$.

**Proof.** We show first the upper bound $\operatorname{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \leqslant 1$, which constitutes the difficult part of the proof. To this end, following the notation introduced in Section 3.2, we define $P(s;\nu)$ to be the period of the periodic orbit of $X_\nu$ passing through the point $(p_1 - s, 0)$. Thanks to the reversibility of $X_\nu$ with respect to $\{y = 0\}$ it turns out that $P(s;\nu) = 2T(s;\nu)$ where $T(\cdot\,;\nu)$ is the Dulac time that we consider in Proposition 3.3. Thus, by applying $(d)$ in that result and setting $\lambda(\nu) = \frac{1}{2(F-1)}$, we can assert that

$$T(s;\nu) = T_{00}(\nu) + T_{01}(\nu)s^{\lambda} + T_{101}^{\frac{1}{2}}(\nu)s\omega_{1-2\lambda}(s) + T_{100}^{\frac{1}{2}}(\nu)s + r_1(s;\nu),$$

where $r_1 \in \mathcal{F}^{\infty}_{3/2-\upsilon}(\nu_0)$ for all $\upsilon > 0$ small enough, the coefficients are analytic in a neighborhood of $\nu_0 = (-\frac{1}{2}, 2)$ and, moreover, the gradients of $T_{01}$ and $T_{101}^{\frac{1}{2}}$ are linearly independent at $\nu_0$. Since one can verify that $\partial_s s\omega_\alpha(s) = (1-\alpha)\omega_\alpha(s) - 1$, the derivation of the above equality yields

$$s^{1-\lambda}T'(s;\nu) = \lambda T_{01}(\nu) + 2\lambda T_{101}^{\frac{1}{2}}(\nu)s^{1-\lambda}\omega_{1-2\lambda}(s) + \big(T_{100}^{\frac{1}{2}}(\nu) - T_{101}^{\frac{1}{2}}(\nu)\big)s^{1-\lambda} + r_2(s;\nu)$$

where, by using Lemmas A.3 and A.4 in [29], the remainder $r_2 := s^{1-\lambda}\partial_s r_1$ belongs to $\mathcal{F}^{\infty}_{1-\upsilon}(\nu_0)$ because $\partial_s r_1 \in \mathcal{F}^{\infty}_{1/2-\upsilon}(\nu_0)$ and, on the other hand, $s^{1-\lambda} \in \mathcal{F}^{\infty}_{1/2-\upsilon}(\nu_0)$ due to $\lambda(\nu_0) = 1/2$. Note furthermore that $\hat{\nu} = \Psi(\nu) := \big(\lambda(\nu)T_{01}(\nu), 2\lambda(\nu)T_{101}^{\frac{1}{2}}(\nu)\big)$ is local analytic change of coordinates at $\nu_0 = (-\frac{1}{2}, 2)$ such that $\Psi(\nu_0) = (0,0)$. We can thus write

$$\mathscr{R}_1(s;\hat{\nu}) := s^{1-\lambda(\nu)}T'(s;\nu)\Big|_{\nu=\Psi^{-1}(\hat{\nu})} = \hat{\nu}_1 + \hat{\nu}_2 s^{1-\hat{\lambda}}\omega_{1-2\hat{\lambda}}(s) + a(\hat{\nu})s^{1-\hat{\lambda}} + h(s;\hat{\nu}), \tag{15}$$

where we set $\hat{\lambda}(\hat{\nu}) := \lambda(\Psi^{-1}(\hat{\nu}))$ for shortness and define

$$a(\hat{\nu}) := \big(T_{100}^{\frac{1}{2}} - T_{101}^{\frac{1}{2}}\big)(\Psi^{-1}(\hat{\nu})) \text{ and } h(s;\hat{\nu}) := r_2(s;\Psi^{-1}(\hat{\nu})) \in \mathcal{F}^{\infty}_{1-\upsilon}(0_2).$$

Recall at this point that the center at the origin of $X_{\nu_0}$ is isochronous, so that $T'(s;\nu_0) \equiv 0$. Consequently, due to $\Psi(\nu_0) = (0,0)$,

$$a(0,0) = 0 \text{ and } h(s;0,0) \equiv 0.$$

By the Weierstrass Division Theorem (see for instance [11, Theorem 1.8]), the first equality implies that $a(\hat{\nu}) = \hat{\nu}_1 a_1(\hat{\nu}) + \hat{\nu}_2 a_2(\hat{\nu})$ with $a_1$ and $a_2$ analytic functions at $(0,0)$. On the other hand, by Lemma 4.1, $h(s;\hat{\nu}) = \hat{\nu}_1 h_1(s;\hat{\nu}) + \hat{\nu}_2 h_2(s;\hat{\nu})$ with $h_i \in \mathcal{F}^{\infty}_{1-\nu}(0_2)$. Therefore, from (15),

$$\mathscr{R}_1(s;\hat{\nu}) = \hat{\nu}_1\Big(1 + a_1(\hat{\nu})s^{1-\hat{\lambda}} + h_1(s;\hat{\nu})\Big) + \hat{\nu}_2\Big(s^{1-\hat{\lambda}}\omega_{1-2\hat{\lambda}}(s) + a_2(\hat{\nu})s^{1-\hat{\lambda}} + h_2(s;\hat{\nu})\Big)$$

Since $h_i \in \mathcal{F}^{\infty}_{1-\upsilon}(0_2)$, $h_i(s;\hat{\nu})$ tends to zero uniformly for $\hat{\nu} \approx (0,0)$ as $s \to 0^+$ (see Definition A.2). Due to $\lambda(\nu_0) = 1/2$, this is also the case of $s^{1-\hat{\lambda}}$ and $s^{1-\hat{\lambda}}\omega_{1-2\hat{\lambda}}(s)$ by (*c*) of Lemma A.4 in [29]. Hence there exists a neighborhood $U$ of $(0,0)$ such that $\lim_{s\to 0^+}(1 + a_1(\hat{\nu})s^{1-\hat{\lambda}} + h_1(s;\hat{\nu})) = 1$ uniformly on $U$. Accordingly, the function

$$\mathscr{R}_2(s;\hat{\nu}) := \frac{\mathscr{R}_1(s;\hat{\nu})}{1 + a_1(\hat{\nu})s^{1-\hat{\lambda}} + h_1(s;\hat{\nu})} = \hat{\nu}_1 + \hat{\nu}_2\ell(s;\hat{\nu}) \tag{16}$$

with

$$\ell(s;\nu) := \frac{s^{1-\hat{\lambda}}\omega_{1-2\hat{\lambda}}(s) + a_2(\hat{\nu})s^{1-\hat{\lambda}} + h_2(s;\hat{\nu})}{1 + a_1(\hat{\nu})s^{1-\hat{\lambda}} + h_1(s;\hat{\nu})}$$

belongs to the class $\mathscr{C}^{\infty}_{s>0}(U)$, see Definition A.1.

We claim that, by shrinking $U$, there exists $s_0 > 0$ such that $\mathscr{R}_2(s;\hat{\nu})$ has at most one zero on $(0,s_0)$, counted with multiplicities, for all $\hat{\nu} = (\hat{\nu}_1,\hat{\nu}_2) \in U \setminus \{(0,0)\}$. Indeed, to show this note first that if $\hat{\nu}_2 = 0$ then $\mathscr{R}_2(s;\hat{\nu}) = \hat{\nu}_1 \neq 0$, so that there is nothing to be proved in this case. Let us study consequently the case $\hat{\nu}_2 \neq 0$. To this end we observe that $\mathscr{R}_2'(s;\hat{\nu}) = \hat{\nu}_2\ell'(s;\hat{\nu})$ where, using a more compact notation,

$$\begin{aligned}
\ell'(s;\hat{\nu}) &= \partial_s\left(\frac{s^{1-\hat{\lambda}}\omega_{1-2\hat{\lambda}} + a_2 s^{1-\hat{\lambda}} + \mathcal{F}^{\infty}_{1-\upsilon}}{1 + a_1 s^{1-\hat{\lambda}} + \mathcal{F}^{\infty}_{1-\upsilon}}\right)\\
&= \frac{\omega_{1-2\hat{\lambda}}}{s^{\hat{\lambda}}}\left(\frac{\hat{\lambda} + \frac{a_2 - a_2\hat{\lambda} - 1}{\omega_{1-2\hat{\lambda}}} + \mathcal{F}^{\infty}_{\frac{1}{2}-\upsilon'}}{1 + a_1 s^{1-\hat{\lambda}} + \mathcal{F}^{\infty}_{1-\upsilon}} - \frac{\Big(s^{1-\hat{\lambda}} + a_2\frac{s^{1-\hat{\lambda}}}{\omega_{1-2\hat{\lambda}}} + \mathcal{F}^{\infty}_{1-\upsilon'}\Big)\Big((1-\hat{\lambda})a_1 + \mathcal{F}^{\infty}_{\frac{1}{2}-\upsilon'}\Big)}{(1 + a_1 s^{1-\hat{\lambda}} + \mathcal{F}^{\infty}_{1-\upsilon})^2}\right).
\end{aligned}$$

Here we use the identity $\partial_s s^b\omega_\alpha(s) = s^{b-1}((b-\alpha)\omega_\alpha(s) - 1)$ and that, by Lemmas A.3 and A.4 in [29], we have $1/\omega_{1-2\hat{\lambda}} \in \mathcal{F}^{\infty}_{-\upsilon}(0_2)$ and $s^{-\hat{\lambda}} \in \mathcal{F}^{\infty}_{-1/2-\upsilon}(0_2)$ for all $\upsilon > 0$ small enough due to $\hat{\lambda}(0,0) = \frac{1}{2}$ and, moreover, that the inclusion $\mathcal{F}^{\infty}_{L}\mathcal{F}^{\infty}_{L'} \subset \mathcal{F}^{\infty}_{L+L'}$ holds. We also remark that, by (*a*) of Lemma A.4 in [29],

$$\lim_{s\to 0^+}\frac{1}{\omega_{1-2\hat{\lambda}}(s)} = \frac{|1-2\hat{\lambda}| - (1-2\hat{\lambda})}{2} \text{ uniformly for } \hat{\nu} \approx (0,0).$$

On account of this, from the above expression of $\ell'$ we obtain that

$$\lim_{s\to 0^+}\frac{s^{\hat{\lambda}}\ell'(s;\hat{\nu})}{\omega_{1-2\hat{\lambda}}(s)} = b(\hat{\nu}) \text{ uniformly for } \hat{\nu} \approx (0,0),$$

where $b(\hat{\nu}) := \hat{\lambda} + \frac{1}{2}(a_2 - a_2\hat{\lambda} - 1)(|1-2\hat{\lambda}| - (1-2\hat{\lambda}))$. Since $\hat{\lambda}(0,0) = 1/2$, it is clear that $b(\hat{\nu})$ is a non-vanishing continuous function in a neighborhood of $(0,0)$. Accordingly, due to $\mathscr{R}_2'(s;\hat{\nu}) = \hat{\nu}_2\ell'(s;\hat{\nu})$, we can assert that

$$\lim_{s\to 0^+}\frac{s^{\hat\lambda}\mathscr{R}_2'(s;\hat\nu)}{\omega_{1-2\hat\lambda}(s)}=\hat\nu_2 b(\hat\nu) \text{ uniformly for } \hat\nu\approx(0,0).$$

Since $\omega_\alpha(s)$ only vanishes at $s=1$, by shrinking $U$ if necessary, we can assert the existence of some $s_0\in(0,1)$ such that $\mathscr{R}_2'(s;\hat\nu)\neq 0$ for all $s\in(0,s_0)$ and $\hat\nu=(\hat\nu_1,\hat\nu_2)\in U$ with $\hat\nu_2\neq 0$. Therefore, by Rolle's Theorem, $\mathscr{R}_2(s;\hat\nu)$ has at most one zero on $(0,s_0)$ counted with multiplicities. This shows the validity of the claim for the case $\nu_2\neq 0$.

Recall finally that the period function $P(s;\nu)$ is twice the Dulac time $T(s;\nu)$. Thus, taking the claim into account, from (15) and (16) it turns out that $V:=\Psi^{-1}(U)$ is an open neighborhood of $\nu_0=(-\frac{1}{2},2)$ verifying that $P'(s;\nu)$ has at most one isolated zero on $(0,s_0)$, counted with multiplicities, for all $\nu\in V$. (To be more precise, the claim applies for the punctured neighborhood $V\setminus\{\nu_0\}$ and, on the other hand, $P'(s;\nu_0)\equiv 0$, so that it has not any isolated zero.) Hence, see Definition 2.3, $\mathcal{Z}_0(P'(\cdot;\nu),\nu_0)\leqslant 1$. Therefore the upper bound $\mathrm{Crit}\big((\Pi_{\nu_0},X_{\nu_0}),X_\nu\big)\leqslant 1$ follows from assertion $(2a)$ in Lemma 2.4 since, using the notation in that result, $P(s;\nu)=\hat P(\sigma(s;\nu);\nu)$ with $\sigma(s;\nu)=(p_1-s,0)$ for $s\in[0,\delta)$. Thus it only remains to show that this upper bound is achieved. To this end we recall that, by [23, Theorem A], $\nu_0=(-\frac{1}{2},2)$ is a local bifurcation value of the period function at the outer boundary, see Definition 2.10. Then, since the period annulus of the centers under consideration varies continuously, see Remark 2.6, by applying $(a)$ in Lemma 2.16 we get that $\mathrm{Crit}\big((\Pi_{\nu_1},X_{\nu_1}),X_\nu\big)\geqslant 1$. This completes the proof of the result. ■

The following is our second result about the criticality of the quadratic isochrones.

**Proposition 4.3.** *If* $\nu_0=(-\frac{1}{2},\frac{1}{2})$ *then* $\mathrm{Crit}\big((\Pi_{\nu_0},X_{\nu_0}),X_\nu\big)=1$.

**Proof.** We prove $\mathrm{Crit}\big((\Pi_{\nu_0},X_{\nu_0}),X_\nu\big)\leqslant 1$ first, which is the most complicated part of the proof. To this end, for each $s\in(0,1)$ we denote by $P(s;\nu)$ the period of the periodic orbit of $X_\nu$ passing thought the point $(1-s,0)\in\mathbb{R}^2$, see Fig. 3. Then, on account of the reversibility of the vector field with respect to $\{y=0\}$, it follows that $P(s;\nu)=2T(s;\nu)$, where $T(\cdot;\nu)$ is the Dulac time introduced before Proposition 3.2. Thanks to that result we have thus the asymptotic expansion of $P(s;\nu)$ near the polycycle, which corresponds to $s=0$. On the other hand, it is well known that the period function can be analytically extended to the center (which corresponds to $s=1$ with this parametrization) because it is non-degenerated. The coefficients of the Taylor's series of $P'(s;\nu)$ at $s=1$ belong to the polynomial ring $\mathbb{R}[D,F]$. Chicone and Jacobs show (see Lemma 3.1 and Theorem 3.9 in [6]) that these coefficients are in the ideal generated by

$$p_2(D,F)=10D^2+10DF-D+4F^2-5F+1$$

and

$$\begin{aligned}p_4(D,F)=&\,1540D^4+4040D^3F+1180D^3+4692D^2F^2+1992D^2F+453D^2\\&+2768DF^3+228DF^2+318DF-2D+784F^4-616F^3-63F^2-154F+49\end{aligned}$$

over the local ring $\mathbb{R}\{D,F\}_{\nu_i}$ of convergent power series at $\nu_i$ localized at any of the four quadratic isochrones $\nu_0:=(-\frac{1}{2},\frac{1}{2})$, $\nu_1:=(0,1)$ $\nu_2:=(0,\frac{1}{4})$ and $\nu_3=(-\frac{1}{2},2)$. With regard to the first one, we claim that the ideal $\mathfrak{B}:=(p_2,p_4)$ is equal to $\big(D+F,(2F-1)^2\big)$ over the local ring $\mathbb{R}\{D,F\}_{\nu_0}$. Indeed, to prove this we use that

$$\begin{pmatrix} p_2 \\ p_4 \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} (2F-1)^2 \\ D+F \end{pmatrix} \tag{17}$$

with $q_{11}=1$, $q_{12}=10D-1$, $q_{21}=52F^2+44F+49$ and

$$q_{22}=576F^3+(2192D-584)F^2+(2500D^2+812D-135)F+1540D^3+1180D^2+453D-2.$$

(The idea to obtain this is that the zero of $p_{2i}|_{D=-F}$ at $F=1/2$ has multiplicity two.) From (17) we get that $p_{2i}\in\big(D+F,(2F-1)^2\big)$ over the polynomial ring $\mathbb{R}[D,F]$. Conversely, since one can verify that the determinant $q_{11}q_{22}-q_{21}q_{12}$ is different from zero at $\nu_0=(-\frac{1}{2},\frac{1}{2})$, by inverting the matrix in (17) it follows that $(2F-1)^2\in\mathfrak{B}$ and $D+F\in\mathfrak{B}$ over the local ring $\mathbb{R}\{D,F\}_{\nu_0}$. This proves the validity of the claim. Consequently, thanks to the result of Chicone and Jacobs mentioned above, we have the following equality between ideals over the local ring $\mathbb{R}\{D,F\}_{\nu_0}$:

$$\mathfrak{B}=\big(D+F,(2F-1)^2\big)=\big(P^{(i)}(1;\nu),i\in\mathbb{N}\big).$$

Now the crucial point is that the ideal $\big(P^{(i)}(s_0;\nu),i\in\mathbb{N}\big)$ does not depend on the point $s_0\in(0,1]$. Indeed, this follows verbatim the argument that R. Roussarie gives in [35, pp. 76–78] or [36, §4.3.1] to justify the same property about the ideal of the displacement map, the so-called Bautin ideal. Here we also use that, such as the displacement map, the period function $P(s;\nu)$ extends analytically to the non-degenerate center (i.e., $s=1$). Accordingly,

$$\mathfrak{B}=\big(D+F,(2F-1)^2\big)=\big(P^{(i)}(s_0;\nu),i\in\mathbb{N}\big)\text{ for all } s_0\in(0,1]. \tag{18}$$

We turn next to the study of the period function near the polycycle (i.e., $s=0$). In this regard by applying $(d)$ in Proposition 3.2 we can assert that, for all $\upsilon>0$ small enough,

$$P(s;\nu)=2T_{00}(\nu)+2T_{101}^1(\nu)s\omega_{1-\lambda}(s)+2T_{100}^1(\nu)s+\mathcal{F}_{2-\upsilon}^{\infty}(\nu_0),$$

where $\lambda(\nu)=\frac{F}{1-F}$ and

$$T_{101}^1(\nu)=-\rho_4(\nu)(F-1/2)^2 \text{ and } T_{100}^1(\nu)=\rho_5(\nu)(D+1/2)+\rho_6(\nu)(F-1/2) \tag{19}$$

for some analytic positive functions $\rho_4$, $\rho_5$ and $\rho_6$ in a neighborhood of $\nu_0=(-\frac{1}{2},\frac{1}{2})$. Consequently, on account of the identity $\partial_s s\omega_\alpha(s)=(1-\alpha)\omega_\alpha(s)-1$ and assertion $(f)$ of Lemma A.3 in [29],

$$P'(s;\nu)=2\lambda T_{101}^1(\nu)\omega_{1-\lambda}(s)+2\big(T_{100}^1-T_{101}^1\big)(\nu)+\mathcal{F}_{1-\upsilon}^{\infty}(\nu_0).$$

Furthermore, from (19) it follows that

$$\hat{\nu}=\Psi(\nu):=\Big((F-1/2)\sqrt{2\lambda\rho_4(\nu)},2\big(T_{100}^1-T_{101}^1\big)(\nu)\Big)$$

is an analytic local change of coordinates in a neighborhood of $\nu=\nu_0$ because one can verify that its Jacobian at $\nu_0=(-\frac{1}{2},\frac{1}{2})$ is equal to $-2\rho_5(\nu_0)\sqrt{2\rho_4(\nu_0)}\neq 0$. Setting $\hat{\nu}=(\hat{\nu}_1,\hat{\nu}_2)$, observe that then

$$P'(s;\Psi^{-1}(\hat{\nu})) = -\hat{\nu}_1^2\omega_{1-\hat{\lambda}}(s) + \hat{\nu}_2 + f(s;\hat{\nu}), \tag{20}$$

where $f \in \mathcal{F}^{\infty}_{1-\upsilon}(0_2)$ and we denote $\hat{\lambda} := \lambda(\Psi^{-1}(\hat{\nu}))$ for shortness.

We claim that $\mathfrak{B} = (\hat{\nu}_1^2, \hat{\nu}_2)$ over the local ring $\mathbb{R}\{D, F\}_{\nu_0}$. To show this we note that

$$\hat{\nu}_2\big|_{D=-F} = (2F-1)\big(\rho_6 - \rho_5\big)(-F, F) + 2(F-1/2)^2\rho_4(-F, F).$$

Since $\rho_5(\nu_0) = \rho_6(\nu_0)$ by $(d)$ in Proposition 3.2, it follows that $\big(\rho_6 - \rho_5\big)(-F, F) = (F - 1/2)r_1(F)$ for some analytic function $r_1$ at $F = 1/2$. Consequently $\hat{\nu}_2|_{D=-F} = (F-1/2)^2 r_2(F)$ with $r_2$ analytic at $F = 1/2$. Taking this into account, the Weierstrass Division Theorem (see [11, Theorem 1.8]) shows that

$$\hat{\nu}_2 = (D+F)q(\hat{\nu}) + (F-1/2)^2 r_2(F)$$

for some analytic function $q$ at $\hat{\nu} = (0,0)$ which, from (19), verifies $q(0,0) = 2\rho_5(\nu_0) \neq 0$. Hence we can write

$$\begin{pmatrix} \hat{\nu}_1^2 \\ \hat{\nu}_2 \end{pmatrix} = \begin{pmatrix} 0 & 2\lambda\rho_4(\nu) \\ q(\Psi(\nu)) & r_2(F) \end{pmatrix} \begin{pmatrix} D+F \\ (F-1/2)^2 \end{pmatrix},$$

where the matrix has an analytic inverse at $\nu = \nu_0$. Taking (18) into account this shows that $\mathfrak{B} = (\hat{\nu}_1^2, \hat{\nu}_2)$ over the local ring $\mathbb{R}\{D, F\}_{\nu_0}$, as desired.

Recall at this point that the center of $X_{\nu_0}$ is isochronous. Hence $P'(s;\nu_0) \equiv 0$. Thus, taking $\Psi(\nu_0) = (0,0)$ into account, from (20) we get that $f(s;0,0) \equiv 0$. Having this in mind we write the remainder in (20) as

$$f(s;\hat{\nu}) = f_1(s;\hat{\nu}) + f_2(s;\hat{\nu}_1)$$

with $f_1(s;\hat{\nu}) := f(s;\hat{\nu}_1,\hat{\nu}_2) - f(s;\hat{\nu}_1,0)$ and $f_2(s;\hat{\nu}_1) := f(s;\hat{\nu}_1,0)$. Since $f_1(s;\hat{\nu}_1,0) \equiv 0$, the application of Lemma 4.1 shows the existence of $g_1 \in \mathcal{F}^{\infty}_{1-\upsilon}(0_2)$ such that $f_1(s;\hat{\nu}) = \hat{\nu}_2 g_1(s;\hat{\nu})$. Due to $f_2(s;0) \equiv 0$ and again by Lemma 4.1, $f_2(s;\hat{\nu}_1) = \hat{\nu}_1 g_2(s;\hat{\nu}_1)$ with $g_2 \in \mathcal{F}^{\infty}_{1-\upsilon}(0_2)$. We also have $g_2(s;0) \equiv 0$ because, otherwise, it would exist $s_0 > 0$ such that $g_2(s_0;\hat{\nu}_1) \neq 0$ for all $\hat{\nu}_1 \approx 0$. In this case, taking the claim into account together with (18) and (20),

$$P'(s_0;\Psi^{-1}(\hat{\nu})) = -\hat{\nu}_1^2\omega_{1-\hat{\lambda}}(s_0) + \hat{\nu}_2 + \hat{\nu}_2 g_1(s_0;\hat{\nu}) + \hat{\nu}_1 g_2(s_0;\hat{\nu}_1) \in \mathfrak{B} = (\hat{\nu}_1^2, \hat{\nu}_2).$$

From here, since each $g_i(s_0;\hat{\nu})$ is analytic at $\hat{\nu} = (0,0)$ and $g_2(s_0;\hat{\nu}_1) \neq 0$ for $\hat{\nu}_1 \approx 0$, we would get that $\hat{\nu}_1 \in (\hat{\nu}_1^2, \hat{\nu}_2)$ over the local ring $\mathbb{R}\{D, F\}_{\nu_0}$, which is clearly a contradiction. Concerning the analyticity of $g_i(s_0;\hat{\nu})$, let us remark that it follows by applying the Weierstrass Division Theorem thanks to the analyticity of $f(s_0;\hat{\nu})$ at $\hat{\nu} = (0,0)$, which in its turn follows from (20) noting that:

- $P'(s_0;\nu)$ is analytic at $\nu = \nu_0$ because $\{X_\nu\}_{\nu\in\mathbb{R}^2}$ is an analytic family of the vector fields and hence, by Lemma 2.1, $(s,\nu) \mapsto P(s;\nu) = \hat{P}((1-s,0);\nu)$ is analytic on $(0,1)\times\mathbb{R}^2$,
- the change of coordinates $\hat{\nu} = \Psi(\nu)$ is analytic at $\nu = \nu_0$, and

- $\omega_\alpha(s_0)$ is analytic at $\alpha=0$ because $\omega_\alpha(s_0)=F(\alpha\ln s_0)\alpha$ with $F(x)=\frac{e^{-x}-1}{x}$.

Hence $g_2(s;0)\equiv 0$ and, by Lemma 4.1 once again, $f_2(s;\hat\nu_1)=\hat\nu_1^2 g_3(s;\hat\nu_1)$ with $g_3\in\mathcal{F}^\infty_{1-\upsilon}(0_2)$. Summing-up all this information about the remainder, from (20) we get that

$$P'(s;\Psi^{-1}(\hat\nu))=-\hat\nu_1^2\big(\omega_{1-\hat\lambda}(s)+\mathcal{F}^\infty_{1-\upsilon}(0_2)\big)+\hat\nu_2\big(1+\mathcal{F}^\infty_{1-\upsilon}(0_2)\big).$$

We are now in position to complete the proof by showing that there exist $s_0>0$ and an open neighborhood $U$ of $\hat\nu=(0,0)$ such that

$$G(s;\hat\nu):=\frac{P'(s;\Psi^{-1}(\hat\nu))}{\omega_{1-\hat\lambda}(s)+\mathcal{F}^\infty_{1-\upsilon}(0_2)}=-\hat\nu_1^2+\hat\nu_2\frac{1+\mathcal{F}^\infty_{1-\upsilon}(0_2)}{\omega_{1-\hat\lambda}(s)+\mathcal{F}^\infty_{1-\upsilon}(0_2)}$$

has at most one zero on $(0,s_0)$, counted with multiplicities, for all $\hat\nu=(\hat\nu_1,\hat\nu_2)\in U\setminus\{(0,0)\}$. This is clear in case that $\hat\nu_2=0$. To tackle the case $\hat\nu_2\neq 0$ we compute the derivative with respect to $s$ to obtain that

$$\begin{aligned}
G'(s;\hat\nu)&=\hat\nu_2\partial_s\left(\frac{1+\mathcal{F}^\infty_{1-\upsilon}}{\omega_{1-\hat\lambda}(s)+\mathcal{F}^\infty_{1-\upsilon}}\right)=\hat\nu_2\partial_s\left(\frac{1+\mathcal{F}^\infty_{1-\upsilon}}{\omega_{1-\hat\lambda}(s)(1+\mathcal{F}^\infty_{1-2\upsilon})}\right)\\
&=\hat\nu_2\partial_s\left(\frac{1+\mathcal{F}^\infty_{1-2\upsilon}}{\omega_{1-\hat\lambda}(s)}\right)=\frac{\hat\nu_2}{s^{2-\hat\lambda}\omega^2_{1-\hat\lambda}(s)}(1+\mathcal{F}^\infty_{1-2\upsilon})+\frac{\hat\nu_2}{\omega_{1-\hat\lambda}(s)}\mathcal{F}^\infty_{-2\upsilon}\\
&=\frac{\hat\nu_2}{s^{2-\hat\lambda}\omega^2_{1-\hat\lambda}(s)}\big(1+\mathcal{F}^\infty_{1-2\upsilon}+s^{2-\hat\lambda}\omega_{1-\hat\lambda}(s)\mathcal{F}^\infty_{-2\upsilon}\big)=\frac{\hat\nu_2}{s^{2-\hat\lambda}\omega^2_{1-\hat\lambda}(s)}\big(1+\mathcal{F}^\infty_{1-3\upsilon}\big).
\end{aligned}$$

Here, in the second equality we apply first assertion $(c)$ of Lemma A.4 in [29] to get that $1/\omega_{1-\hat\lambda}(s)\in\mathcal{F}^\infty_{-\upsilon}$ for all $\upsilon>0$ small enough, due to $\hat\lambda(0,0)=1$, and use next that $\mathcal{F}^\infty_{-\upsilon}\mathcal{F}^\infty_{1-\upsilon}\subset\mathcal{F}^\infty_{1-2\upsilon}$ from $(g)$ of Lemma A.3 in [29]. In the third equality, on account of $\frac{1}{1+s}-1\in\mathcal{F}^\infty_1$ and by $(h)$ of Lemma A.3 in [29], we use first the inclusion $\frac{1}{1+\mathcal{F}^\infty_{1-2\upsilon}}\subset 1+\mathcal{F}^\infty_{1-2\upsilon}$. Then, by using $(d)$ and $(g)$ of Lemma A.3 in [29], we expand the numerator to get that $(1+\mathcal{F}^\infty_{1-\upsilon})(1+\mathcal{F}^\infty_{1-2\upsilon})\subset 1+\mathcal{F}^\infty_{1-2\upsilon}$. Next, in the fourth equality we use that $\partial_s\omega_\alpha(s)=s^{-\alpha-1}$ and assertion $(f)$ of Lemma A.3 in [29] to deduce that $\partial_s\mathcal{F}^\infty_{1-2\upsilon}\subset\mathcal{F}^\infty_{-2\upsilon}$. Finally in the last equality we apply $(c)$ of Lemma A.4 in [29] to get that $s^{2-\hat\lambda}\omega_{1-\hat\lambda}(s)\in\mathcal{F}^\infty_{1-\upsilon}$ and we use again that $\mathcal{F}^\infty_{1-\upsilon}\mathcal{F}^\infty_{-2\upsilon}\subset\mathcal{F}^\infty_{1-3\upsilon}$. On account of Definition A.2 we can assert the existence of some $s_0\in(0,1)$ and a neighborhood $U$ of $(0,0)$ such that $G'(s;\hat\nu)\neq 0$ for all $s\in(0,s_0)$ and $\hat\nu\in U$ with $\hat\nu_2\neq 0$. Consequently $P'(s;\Psi^{-1}(\hat\nu))$ has at most one isolated zero on $(0,s_0)$, counted with multiplicities, for all $\hat\nu\in U\setminus\{(0,0)\}$. Thus, on account of Definition 2.3 and the fact that $\Psi(\nu_0)=(0,0)$, we get $\mathcal{Z}_0(P'(\cdot\,;\nu),\nu_0)\leqslant 1$. Finally the upper bound $\mathrm{Crit}\big((\Pi_{\nu_0},X_{\nu_0}),X_\nu\big)\leqslant 1$ follows from assertion $(2a)$ in Lemma 2.4 since, using the notation in that result, $P(s;\nu)=\hat P(\sigma(s;\nu);\nu)$ with $\sigma(s;\nu)=(1-s,0)$ for $s\in[0,\delta)$. Therefore it only remains to show that this upper bound is attained. To this end we recall that, by [23, Theorem A], $\nu_0=(-\frac12,\frac12)$ is a local bifurcation value of the period function at the outer boundary, see Definition 2.10. Then, since the period annulus of the centers under consideration varies continuously, see Remark 2.6, by applying $(a)$ in Lemma 2.16 we get that $\mathrm{Crit}\big((\Pi_{\nu_1},X_{\nu_1}),X_\nu\big)\geqslant 1$. This finishes the proof of the result. ■

As we explain at the beginning of this section, the maximum criticality of the period function at the inner boundary is 2 and it is achieved at the three Loud points $\nu = L_i$, see (13). We refer the interested reader to the paper of Chicone and Jacobs [6] for a proof of this result. In a joint paper with P. Mardešić, see [23, Theorem 4.3], we prove that at each $\nu = L_i$ there exists a germ of analytic curve that consists of local bifurcation values of the period function at the interior, see Definition 2.10. Since the period function extends analytically to the center, this follows readily by applying the Weierstrass Preparation Theorem. In our next result we identify a parameter $\nu = \nu_\star$ for which the criticality at the outer boundary is 2. Furthermore we prove that at $\nu = \nu_\star$ there exists a $\mathscr{C}^1$ germ of curve of local bifurcation values of the period function at the interior. Hence, roughly speaking, this parameter is the mirror image at the outer boundary of one of the Loud points, see Fig. 6 and Remark 4.5. In the statement, following the notation introduced at the beginning of Section 3.2, for each $s \in (0, p_1)$ and $\nu \approx \nu_\star$ we denote by $P(s;\nu)$ the period of the periodic orbit of $X_\nu$ passing through the point $(p_1 - s, 0) \in \mathbb{R}^2$. We also remark that $T_{10}$ and $T_{01}$ are the coefficients given in Proposition 3.3, which vanish at $\nu_\star = (\mathcal{G}(4/3), 4/3)$.

**Proposition 4.4.** *Let us consider $\nu_\star = (\mathcal{G}(4/3), 4/3)$. Then the following holds:*

($a$) $\mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big) = 2$.
($b$) *There exist an open neighborhood $U$ of $\nu_\star$ and $s_0 > 0$ such that the set*

$$\Delta := \{\nu \in U;\ \text{there exists } s \in (0, s_0) \text{ such that } P'(s;\nu) = P''(s;\nu) = 0\}$$

*satisfies the following conditions:*

- ($b1$) *Each $\nu \in \Delta$ is a local bifurcation value of the period function at the interior,*
- ($b2$) *there are $\varepsilon > 0$ and a $\mathscr{C}^1$ injective curve $\delta : (-\varepsilon, \varepsilon) \to U$ with $\delta(0) = \nu_\star$, $\delta((0,\varepsilon)) = \Delta$ and such that $\delta'(0) \neq (0,0)$ is tangent to $\{\nu \in U;\ T_{10}(\nu) = 0\}$,*
- ($b3$) *for each $\nu \in \Delta$ there exists a unique $s_\nu \in (0, s_0)$ such that $P'(s_\nu;\nu) = P''(s_\nu;\nu) = 0$ and, moreover, $\lim_{\nu \to \nu_\star} s_\nu = 0^+$,*
- ($b4$) *$\Delta \subset \{\nu \in U;\ T_{10}(\nu) < 0 \text{ and } T_{01}(\nu) > 0\}$, and*
- ($b5$) *for any $\nu_0 \in \Delta$ and any neighborhood $V$ of $\nu_0$ there exist $\bar{\nu} \in V$ and different $s_1, s_2 \in (0, s_0)$ such that $P'(s_1;\bar{\nu}) = P'(s_2;\bar{\nu}) = 0$.*

**Proof.** We observe first of all that $\sigma(s;\nu) := (p_1 - s, 0)$ is a parametrization of the outer boundary of the period annulus verifying the hypothesis in Lemma 2.4. This will enable us to relate $\mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big)$ with $\mathcal{Z}_0(P'(\cdot;\nu), \nu_\star)$. That being said, thanks to the reversibility of the vector field with respect to $\{y = 0\}$, we note that $P(s;\nu) = 2T(s;\nu)$, where $T(\cdot;\nu)$ is the Dulac time considered in Proposition 3.3. From point ($b$) in that result we can assert that, for all $\upsilon > 0$ small enough,

$$P(s;\nu) = 2T_{00}(\nu) + 2T_{10}(\nu)s + 2T_{01}(\nu)s^\lambda + 2T_{20}(\nu)s^2 + \mathcal{F}^\infty_{5/2-\upsilon}(\nu_\star),$$

where $\lambda(\nu) = \frac{1}{2(F-1)}$, $T_{10}(\nu_\star) = T_{01}(\nu_\star) = 0$, $T_{20}(\nu_\star) < 0$ and the gradients $\nabla T_{10}(\nu_\star)$ and $\nabla T_{01}(\nu_\star)$ are linearly independent. Due to $T_{20}(\nu_\star) \neq 0$, by applying [30, Theorem C] we get that $\mathcal{Z}_0(P'(\cdot;\nu), \nu_\star) \leqslant 2$. (For readers convenience, let us explain that [30, Theorem C] is a general result addressed to the Dulac time which, by using the well-known derivation-division algorithm, gives a bound for $\mathcal{Z}_0(T'(\cdot;\nu), \nu_0)$ in terms of the position of the first non-vanishing coefficient in the asymptotic expansion of $T(s;\nu)$ at $s=0$.) Consequently, by assertion $(2a)$ in Lemma 2.4, $\operatorname{Crit}\big((\Pi_{\nu_\star},X_{\nu_\star}),X_\nu\big)\leqslant 2$. In addition, since

$$F_1(s;\nu):=P'(s;\nu)=2T_{10}(\nu)+2\lambda T_{01}(\nu)s^{\lambda-1}+4T_{20}(\nu)s+\mathcal{F}^\infty_{3/2-\upsilon}(\nu_\star) \tag{21}$$

and the gradients $\nabla T_{10}(\nu_\star)$ and $\nabla T_{01}(\nu_\star)$ are linearly independent, by [30, Proposition 4.2] it turns out that $\mathcal{Z}_0(P'(\cdot;\nu),\nu_\star)\geqslant 2$. As a matter of fact, from the proof of that result, this lower bound is achieved by means of two different sequences of zeros of $P'(\cdot;\nu)$ and, therefore, by assertion $(2b)$ in Lemma 2.4, $\operatorname{Crit}\big((\Pi_{\nu_\star},X_{\nu_\star}),X_\nu\big)\geqslant 2$. Accordingly $\operatorname{Crit}\big((\Pi_{\nu_\star},X_{\nu_\star}),X_\nu\big)=2$ and this proves $(a)$.

Let us turn next to the proof of the assertions in $(b)$. For this purpose, from (21) and by applying Lemmas A.3 and A.4 in [29], we get

$$F_2(s;\nu):=s^{2-\lambda}P''(s;\nu)=2\lambda(\lambda-1)T_{01}(\nu)+4T_{20}(\nu)s^{2-\lambda}+\mathcal{F}^\infty_{1-\upsilon}(\nu_\star). \tag{22}$$

Setting $U_\varepsilon:=\{\nu\in\mathbb{R}^2:\|\nu-\nu_\star\|<\varepsilon\}$, the map $F:=(F_1,F_2)$ is well-defined for $(s,\nu)\in(0,\varepsilon)\times U_\varepsilon$ taking $\varepsilon>0$ small enough. Since $T_{10}(\nu_\star)=T_{01}(\nu_\star)=0$, $T_{20}(\nu_\star)\neq 0$ and the gradients $\nabla T_{10}(\nu_\star)$ and $\nabla T_{01}(\nu_\star)$ are linearly independent, we can assume by reducing $\varepsilon>0$ if necessary that $\hat\nu=\Psi(\nu)$, defined by means of

$$\Psi(\nu):=\left(\frac{T_{10}(\nu)}{2T_{20}(\nu)},\frac{\lambda(\nu)T_{01}(\nu)}{2T_{20}(\nu)}\right), \tag{23}$$

is an analytic change of coordinates from $U_\varepsilon$ to the neighborhood $\hat U_{\hat\varepsilon}:=(-\hat\varepsilon,\hat\varepsilon)^2$ of $(0,0)=\Psi(\nu_\star)$. Recall that our aim is to study the solutions of the system of equations $\{P'=0,P''=0\}$ which, on account of (21) and (22), is equivalent to $\{F_1=0,F_2=0\}$. In order to study the latter we first lift $\Psi$ to an analytic change of variables $\Phi$ given by

$$(\hat s,\hat\nu)=\Phi(s,\nu):=(s^{2-\lambda(\nu)},\Psi(\nu)),$$

which (diminishing $\varepsilon$ and $\hat\varepsilon$ if necessary) is defined from $\mathscr{U}_\varepsilon:=(0,\varepsilon)\times U_\varepsilon$ to $\hat{\mathscr{U}}_{\hat\varepsilon}:=(0,\hat\varepsilon)\times\hat U_{\hat\varepsilon}$, and then we consider the map $\hat F:\hat{\mathscr{U}}_{\hat\varepsilon}\to\mathbb{R}^2$ defined by $\hat F(\hat s,\hat\nu)=(\hat F_1(\hat s,\hat\nu),\hat F_2(\hat s,\hat\nu))$ with

$$\hat F_1(\hat s,\hat\nu):=\frac{F_1(\Phi^{-1}(\hat s,\hat\nu))}{4T_{20}(\Psi^{-1}(\hat\nu))}\text{ and }\hat F_2(\hat s,\hat\nu):=\frac{F_2(\Phi^{-1}(\hat s,\hat\nu))}{4T_{20}(\Psi^{-1}(\hat\nu))}.$$

By assertions $(h)$ and $(c)$ of Lemmas A.3 and A.4 in [29], respectively, it follows that

$$\hat F_1(\hat s,\hat\nu)=\hat\nu_1+\hat\nu_2\hat s^{\frac{\hat\lambda-1}{2-\hat\lambda}}+f_1(\hat s;\hat\nu)\text{ and }\hat F_2(\hat s,\hat\nu)=(\hat\lambda-1)\hat\nu_2+\hat s+f_2(\hat s;\hat\nu),$$

where $f_1,f_2\in\mathcal{F}^\infty_{2-\upsilon}(0_2)$ for some $\upsilon>0$ small enough and we set $\hat\lambda(\hat\nu):=\lambda(\Psi^{-1}(\hat\nu))$ for shortness. Here we also use that $\mathcal{F}^\infty_{3-\upsilon}(0_2)\subset\mathcal{F}^\infty_{2-\upsilon}(0_2)$ and $s=\hat s^{1/(2-\hat\lambda)}\in\mathcal{F}^\infty_{2-\upsilon}(0_2)$ due to $\lambda(\nu_\star)=3/2$. Observe on the other hand that, via the diffeomorphism $\Phi$, the system $\{P'(s;\nu)=0,P''(s;\nu)=0\}$ on $\mathscr{U}_\varepsilon$ is equivalent to the system $\{\hat F_1(\hat s,\hat\nu)=0,\hat F_2(\hat s,\hat\nu)=0\}$ on $\hat{\mathscr{U}}_{\hat\varepsilon}$. With regard to the latter note that, by [29, Lemma A.1], the remainders $f_1$ and $f_2$ extend to $\mathscr{C}^1$ functions in a neighborhood of $(0,0,0)$ satisfying that $\nabla f_1(0,0,0)=\nabla f_2(0,0,0)=(0,0,0)$. Observe in particular that $\hat{F}_2(\hat{s},\hat{\nu})$ extends to a $\mathscr{C}^1$ function in a neighborhood of $(0,0,0)$. Hence, taking $\hat{\lambda}(0,0)=3/2$ into account, by the Implicit Function Theorem there exists a $\mathscr{C}^1$ function $h(\hat{s},\hat{\nu}_1)$ in a neighborhood of $(0,0)$ such that, by shrinking $\hat{\varepsilon}>0$ if necessary,

$$\hat{F}_2(\hat{s},\hat{\nu})=0 \text{ with } (\hat{s},\hat{\nu})\in\hat{\mathscr{U}}_{\hat{\varepsilon}} \Leftrightarrow \hat{\nu}_2=h(\hat{s},\hat{\nu}_1).$$

Furthermore $h$ satisfies $h(0,\hat{\nu}_1)\equiv 0$ and $\nabla h(0,0)=(-2,0)$. Our next task is to substitute $\hat{\nu}_2=h(\hat{s},\hat{\nu}_1)$ in $\hat{F}_1(\hat{s},\hat{\nu}_1,\hat{\nu}_2)=0$ and analyze the resulting equation. To this end we extend $\hat{F}_1(\hat{s},\hat{\nu}_1,\hat{\nu}_2)|_{\hat{\nu}_2=h(\hat{s},\hat{\nu}_1)}$ on a neighborhood of $(\hat{s},\hat{\nu}_1)=(0,0)$ by means of

$$(\hat{s},\hat{\nu}_1)\mapsto \hat{\nu}_1+h(\hat{s},\hat{\nu}_1)|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}+\hat{f}_1(\hat{s},\hat{\nu}_1),$$

where $e(\hat{s},\hat{\nu}_1):=\frac{\hat{\lambda}-1}{2-\hat{\lambda}}\big|_{\hat{\lambda}=\hat{\lambda}(\hat{\nu}_1,h(\hat{s},\hat{\nu}_1))}$ and $\hat{f}_1(\hat{s},\hat{\nu}_1)=f_1(\hat{s};\hat{\nu}_1,h(\hat{s},\hat{\nu}_1))$ are clearly $\mathscr{C}^1$ in a neighborhood of $(0,0)$. We claim that the function $g(\hat{s},\hat{\nu}_1):=h(\hat{s},\hat{\nu}_1)|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}$ is $\mathscr{C}^1$ in a neighborhood of $(0,0)$ as well and that its gradient vanishes at $(0,0)$. To show this notice first that $g(0,\hat{\nu}_1)=0$ and, consequently, $\partial_{\hat{\nu}_1}g(0,\hat{\nu}_1)=0$. Moreover, using that $h(0,\hat{\nu}_1)=0$, we get

$$\partial_{\hat{s}}g(0,\hat{\nu}_1)=\lim_{\hat{s}\to 0}\frac{h(\hat{s},\hat{\nu}_1)|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}}{\hat{s}}=\lim_{\hat{s}\to 0}\frac{h(\hat{s},\hat{\nu}_1)-h(0,\hat{\nu}_1)}{\hat{s}}\lim_{\hat{s}\to 0}|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}=\partial_{\hat{s}}h(0,\hat{\nu}_1)\cdot 0=0$$

because $h$ is $\mathscr{C}^1$ and $e(0,0)=1$ implies $e(0,\hat{\nu}_1)>0$ for $\hat{\nu}_1\approx 0$. Similarly, if $\hat{s}\neq 0$ then

$$\begin{aligned}\partial_{\hat{s}}g(\hat{s},\hat{\nu}_1)&=(\partial_{\hat{s}}h(\hat{s},\hat{\nu}_1))|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}+h(\hat{s},\hat{\nu}_1)|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}\left(\log|\hat{s}|\partial_{\hat{s}}e(\hat{s},\hat{\nu}_1)+\frac{e(\hat{s},\hat{\nu}_1)}{\hat{s}}\right)\\&=(\partial_{\hat{s}}h(\hat{s},\hat{\nu}_1))|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}+\frac{h(\hat{s},\hat{\nu}_1)-h(0,\hat{\nu}_1)}{\hat{s}}|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}\Big(\hat{s}\log|\hat{s}|\partial_{\hat{s}}e(\hat{s},\hat{\nu}_1)+e(\hat{s},\hat{\nu}_1)\Big)\end{aligned}$$

and

$$\partial_{\hat{\nu}_1}g(\hat{s},\hat{\nu}_1)=(\partial_{\hat{\nu}_1}h(\hat{s},\hat{\nu}_1))|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}+h(\hat{s},\hat{\nu}_1)|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}\log|\hat{s}|\partial_{\hat{\nu}_1}e(\hat{s},\hat{\nu}_1)$$

tend to zero as $\hat{s}\to 0$ uniformly on $\hat{\nu}_1\approx 0$. This clearly implies that $g$ is $\mathscr{C}^1$ in a neighborhood of $(0,0)$ and $\nabla g(0,0)=(0,0)$, so that the claim is true. Thus, by applying the Implicit Function Theorem to the "extended equation"

$$\hat{\nu}_1+h(\hat{s},\hat{\nu}_1)|\hat{s}|^{e(\hat{s},\hat{\nu}_1)}+\hat{f}_1(\hat{s},\hat{\nu}_1)=0$$

and reducing $\hat{\varepsilon}>0$ once again, we obtain a $\mathscr{C}^1$ function $\ell(\hat{s})$ on $(-\hat{\varepsilon},\hat{\varepsilon})$ such that $\hat{F}_1(\hat{s},\hat{\nu}_1,h(\hat{s},\hat{\nu}_1))=0$ with $(\hat{s},\hat{\nu}_1)\in(0,\hat{\varepsilon})\times(-\hat{\varepsilon},\hat{\varepsilon})$ if, and only if, $\hat{\nu}_1=\ell(\hat{s})$. Moreover $\ell(0)=\ell'(0)=0$. Accordingly, after shrinking $\hat{\varepsilon}>0$ once again if necessary, we can assert that

$$\hat{F}_1(\hat{s},\hat{\nu})=\hat{F}_2(\hat{s},\hat{\nu})=0 \text{ with } (\hat{s},\hat{\nu})\in\hat{\mathscr{U}}_{\hat{\varepsilon}} \Leftrightarrow \hat{\nu}=(\hat{\nu}_1,\hat{\nu}_2)=\big(\ell(\hat{s}),h(\hat{s},\ell(\hat{s})\big).$$

At this point, since we reduced the original $\hat{\varepsilon}>0$, we also diminish $\varepsilon>0$ so that $\Phi(s,\nu)=(s^{2-\lambda(\nu)},\Psi(\nu))$ is still a diffeomorphism from $\mathscr{U}_\varepsilon$ into $\hat{\mathscr{U}}_{\hat{\varepsilon}}$. Then, from (21) and (22), the follow-

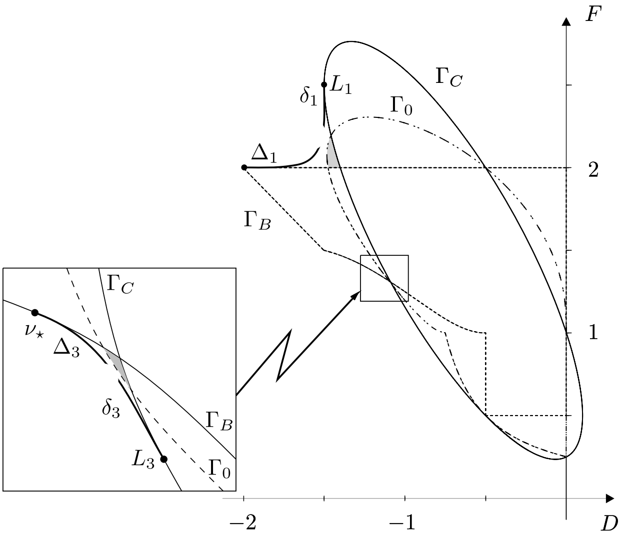


Fig. 6. Arrangement of the three types of local bifurcation curve (inner boundary, interior and outer boundary) near the parameters $\nu = L_3$ and $\nu = \nu_\star$, see Remark 4.5, and $\nu = L_1$ and $\nu = (2, 2)$, see Remark 5.1.

ing assertions are equivalent:

(1) $\nu_0 \in \Delta := \{\nu \in U_\varepsilon;$ there exists $s \in (0, \varepsilon)$ such that $P'(s; \nu) = P''(s; \nu) = 0\}$,
(2) $\Psi(\nu_0) = \big(\ell(\hat{s}_0), h(\hat{s}_0, \ell(\hat{s}_0))\big)$ for some $\hat{s}_0 \in (0, \hat{\varepsilon})$,
(3) $\nu_0 = \delta(t_0)$ for some $t_0 \in (0, \hat{\varepsilon})$, where $\delta(t) := \Psi^{-1}\big(\ell(t), h(t, \ell(t))\big)$.

It is clear from these equivalences that $\delta\colon (-\hat{\varepsilon}, \hat{\varepsilon}) \longrightarrow U_\varepsilon \subset \mathbb{R}^2$ is a $\mathscr{C}^1$ parametrized curve with $\delta(0) = \nu_\star$ satisfying that $\delta\big((0, \hat{\varepsilon})\big) = \Delta$. One can easily verify, taking $\ell'(0) = 0$ and $\partial_1 h(0, 0) \neq 0$ into account, together with the definition of $\Psi$ in (23), that $\delta'(0)$ is a non-zero vector tangent to $\{\nu \in U_\varepsilon;\ T_{10}(\nu) = 0\}$. In particular, on account of $\delta'(0) \neq (0, 0)$ and by reducing $\hat{\varepsilon} > 0$, we have that $\delta$ is one-to-one. This proves the assertion $(b2)$ in the statement. Due to $\mathcal{Z}_0(P'(\,\cdot\,; \nu), \nu_\star) \leqslant 2$, and after shrinking $\varepsilon > 0$ if necessary, note also that the zeros of $P'(\,\cdot\,; \nu)$ on $(0, \varepsilon)$ can have at most multiplicity two. Therefore, since the interior of $\Delta = \delta\big((0, \hat{\varepsilon})\big)$ is empty (as a subset of $\mathbb{R}^2$), by applying Lemma 2.15 we can assert that each $\nu_0 \in \Delta$ is a local bifurcation value of the period function at the interior, which shows the validity of $(b1)$ in the statement. With regard to the assertions in $(b3)$, we note that the uniqueness of $s_\nu$ and $\lim_{\nu \to \nu_\star} s_\nu = 0^+$ follow from the point (2) above using that $\hat{s} \mapsto h(\hat{s}, \ell(\hat{s}))$ is invertible at $\hat{s} = 0$ and that, by definition, $\hat{s} = s^{2-\lambda(\nu)}$. On the other hand, since $P''(s_\nu; \nu) = 0$ for all $\nu \in \Delta$, from (22) we get that $T_{01}(\nu) T_{20}(\nu) < 0$ for all $\nu \in \Delta$. Here we also use that $\lim_{\nu \to \nu_\star} s_\nu = 0^+$ to take advantage of the properties of the remainder and the fact that $\lambda(\nu_\star) = 3/2$. By arguing similarly, on account of $P'(s_\nu; \nu) = 0$ for all $\nu \in \Delta$, from (21) it follows that $T_{01}(\nu) T_{10}(\nu) < 0$ for all $\nu \in \Delta$. Taking this into account the assertion in $(b4)$ is a consequence of $T_{20}(\nu_\star) < 0$, see $(b)$ in Proposition 3.3. Finally, in order to prove $(b5)$, let us consider $\nu \in \Delta$ and note that from (21) we obtain

$$\lim_{\nu \to \nu_\star} \big(\partial_{\nu_1} P'(s; \nu), \partial_{\nu_2} P'(s; \nu)\big)\big|_{s = s_\nu} = 2 \nabla T_{10}(\nu_\star),$$

where we use that the flatness of the remainder $\mathcal{F}^{\infty}_{3/2-\upsilon}(\nu_\star)$ is preserved after derivation with respect to parameters, see Definition A.2. Similarly, in this case from (22) and using also $P''(s_\nu;\nu)\equiv 0$, we get

$$\lim_{\nu\to\nu_\star} s^{2-\lambda(\nu)}\big(\partial_{\nu_1}P''(s;\nu),\partial_{\nu_2}P''(s;\nu)\big)\Big|_{s=s_\nu}=\frac{3}{2}\nabla T_{01}(\nu_\star).$$

Thus, since the vectors $\nabla T_{10}(\nu_\star)$ and $\nabla T_{01}(\nu_\star)$ are linearly independent, so they are $\nabla P'(s;\nu)|_{s=s_\nu}$ and $\nabla P''(s;\nu)|_{s=s_\nu}$ for all $\nu\in U_\varepsilon$ (after shrinking $\varepsilon>0$ if necessary). That being said, we fix any $\nu_0\in\Delta$ and compute the second order Taylor's expansion of $P'(s;\nu)$ at $s=s_{\nu_0}$,

$$P'(s;\nu)=P'(s_{\nu_0};\nu)+P''(s_{\nu_0};\nu)(s-s_{\nu_0})+\mathrm{o}(s-s_{\nu_0}).$$

Then, due to $P'(s;\nu_0)\not\equiv 0$, $P'(s_{\nu_0};\nu_0)=P''(s_{\nu_0};\nu_0)=0$ and the fact that the gradients $\nabla P'(s_{\nu_0};\nu)$ and $\nabla P''(s_{\nu_0};\nu)$ are linearly independent at $\nu=\nu_0$, the application of [30, Proposition 4.2] shows that for each open neighborhood $V$ of $\nu_0$ there exist $\bar\nu\in V$ and two $s_1,s_2\in(0,\varepsilon)$ such that $P'(s_1;\bar\nu)=P'(s_2;\bar\nu)=0$. This proves the validity of the assertion in $(b5)$ and completes the proof of the result. ■

**Remark 4.5.** Let us finish this section contextualizing the results in Proposition 4.4. In Fig. 6 we display the ellipse $\Gamma_C$ that consists of local bifurcation values of the period function at the inner boundary (i.e., the center) of $\mathscr{P}$. It corresponds, see [6, Lemma 3.1], to the vanishing of the first period constant

$$p_2(\nu)=10D^2+10DF-D+4F^2-5F+1.$$

Moreover the curve $\Gamma_B$ consists of local bifurcation values at the outer boundary (i.e., the polycycle) of $\mathscr{P}$, see [23, Theorem A]. It is made of the arc $\{D=\mathcal{G}(F):F\in(1,\frac{3}{2})\}$ joining $(-\frac{3}{2},\frac{3}{2})$ and $(-\frac{1}{2},1)$ together with several straight segments. According to Proposition 4.4, the germ of curve $\Delta$ at $\nu=\nu_\star$ (that we depict as $\Delta_3$ in Fig. 6 for consistency) is inside the set of local bifurcation values of the period function at the interior of $\mathscr{P}$. Inside this set there is also the germ of a curve $\delta_3$ at $\nu=L_3$ by [23, Theorem 4.3]. At this moment we do not have any analytical tool to fully characterize this set of interior bifurcations. We conjecture that $\Delta_3$ and $\delta_3$ connect with each other to delimit a region of parameters for which the corresponding center has exactly two critical periodic orbits. With regard to this conjecture it is proved in [23, Theorem 5.2] that the center of any parameter inside one of the two light gray regions in Fig. 6 has *at least* two critical periodic orbits, cf. Fig. 7. The boundary of these regions is inside $\Gamma_C$, $\Gamma_B$ and $\Gamma_0$. For completeness let us explain that the curve $\Gamma_0$ consists of those parameters such that the period function tends to $2\pi$ as the periodic orbits tend to the outer boundary. □

## 5. Proof of Theorem A

**Proof of Theorem A.** The statement covers all the parameters $\nu_0\in\mathbb{R}^2$ outside the vertical segments $\ell_0:=\{D=-1,\,F\in[0,1]\}\cup\{D=0,\,F\in[0,\frac{1}{4})\}$. For simplicity in the exposition, instead of proving the five assertions in the statement separately, we split $\mathbb{R}^2\setminus\ell_0$ depending on the result and tool applied to study the corresponding criticality. For reader's convenience we enumerate the different cases that we obtain in this way.

1. Let us consider first of all the set $\ell_1 := \mathbb{R}^2 \setminus (\Gamma_B \cup \Gamma_U)$, where recall (see Fig. 1) that $\Gamma_U$ is the union of the dotted straight lines, whatever its color is, and $\Gamma_B$ is the Jordan curve in boldface type. Then, by [23, Theorem A], we know that any $\nu_0 \in \ell_1$ is a local regular value of the period function at the outer boundary, see $(c)$ in Definition 2.10. On account of this, by $(b)$ in Lemma 2.16 we get that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$. Here we also use that the period annuli of the Loud's centers vary continuously, see Remark 2.6, and that the outer boundary of $\mathscr{P}_\nu$ for $\nu \notin \Gamma_B \cup \Gamma_U$ is a hyperbolic polycycle, see for instance [23, §3.1].
2. The criticality at $\ell_2 := \{D = -\frac{1}{2}, F \in (\frac{1}{2}, 1)\} \cup \{F = \frac{1}{2}, D \in (-\frac{1}{2}, 0)\}$ and $\ell_3 := \{F = \frac{1}{2}, D \in (-1, -\frac{1}{2})\}$ follows from the results in Section 3.1. In this case $\sigma(s;\nu) = (1-s, 0)$ is a parametrization of the outer boundary of the period annulus verifying the hypothesis $(a)$, $(b)$ and $(c)$ in Lemma 2.4. Moreover denoting by $P(s;\nu)$ the period of the periodic orbit of $X_\nu$ passing through $\sigma(s;\nu)$, we have that $P(s;\nu) = 2T(s;\nu)$, where $T$ is the Dulac map considered in Proposition 3.2. By applying this result we know that the first non-vanishing coefficient in the asymptotic expansion of $P(s;\nu)$ at $s=0$ is the third one for all $\nu \in \ell_2$. Therefore [30, Theorem C] implies that $\mathcal{Z}_0(P'(\cdot\,;\nu), \nu_0) \leqslant 1$ for all $\nu_0 \in \ell_2$. On account of this, by assertion $(2a)$ in Lemma 2.4 it follows that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \leqslant 1$ for all $\nu_0 \in \ell_2$. On the other hand, due to $\ell_2 \subset \Gamma_B$, we know by [23, Theorem A] that these parameters are local bifurcation values of the period function at the outer boundary. Thus, since the period annuli of the Loud's centers vary continuously (see Remark 2.6), by applying $(a)$ in Lemma 2.16 we get that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \geqslant 1$ for all $\nu_0 \in \ell_2$. Hence $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$ for all $\nu_0 \in \ell_2$.

   We turn now to the criticality in the segment $\ell_3$. So let us fix any $\nu_0 = (D_0, \frac{1}{2})$ with $D_0 \in (-1, -\frac{1}{2})$ and note that then, by $(d)$ in Proposition 3.2,

$$T'(s;\nu) = -\rho_4(\nu)(F-1/2)^2\big(\lambda\omega_{1-\lambda}(s) - 1\big) + \rho_5(\nu)(D+1/2) + \rho_6(\nu)(F-1/2) + \mathscr{R}(s;\nu),$$

   where $\mathscr{R} \in \mathcal{F}^\infty_{1-\upsilon}(\nu_0)$ for all $\upsilon > 0$ small enough. To obtain the derivative of the Dulac time, we use that $\partial_s(s\omega_\alpha(s)) = (1-\alpha)\omega_\alpha(s) - 1$ and that, by $(f)$ in Lemma A.3 in [29], $\partial_s \mathcal{F}^\infty_{2-\upsilon}(\nu_0) \subset \mathcal{F}^\infty_{1-\upsilon}(\nu_0)$. From this equality, since $\lambda\omega_{1-\lambda}(s) - 1$ tends to $+\infty$ as $(s,\nu) \to (0,\nu_0)$ due to $\lambda(\nu_0) = 1$ (see Definition 3.1), $\mathscr{R}(s;\nu)$ tends to 0 as $(s,\nu) \to (0,\nu_0)$, $\rho_i(\nu_0) > 0$ and $D_0 + \frac{1}{2} < 0$, we can assert the existence of an open neighborhood $V$ of $\nu_0$ and $\varepsilon > 0$ such that $P'(s;\nu) = 2T'(s;\nu) < 0$ for all $\nu \in V$ and $s \in (0,\varepsilon)$. Consequently $\mathcal{Z}_0(P'(\cdot\,;\nu), \nu_0) = 0$ and so, by applying $(2c)$ in Lemma 2.4, we conclude that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$.
3. We turn next to study the horizontal segments $\ell_4 := \big\{F = 2, D \in (-2, 0) \setminus \{-\frac{1}{2}\}\big\}$ and the curve $\ell_5 := \big\{D = \mathcal{G}(F) : F \in (1, \frac{3}{2})\big\}$. Here we set $\nu_\star := (\mathcal{G}(4/3), 4/3)$ because this parameter yields to a distinguished case.

   We begin by noting (see the first paragraph in Section 3.2) that $\sigma(s;\nu) = (p_1 - s, 0)$ is a parametrization of the outer boundary of the period annulus verifying the assumptions in Lemma 2.4 and that if we denote the period of the periodic orbit of $X_\nu$ passing through $\sigma(s;\nu)$ by $P(s;\nu)$ then $P(s;\nu) = 2T(s;\nu)$, where $T$ is the Dulac map considered in Proposition 3.3. Thus, by applying first that result and then [30, Theorem C] we obtain that $\mathcal{Z}_0(P'(\cdot\,;\nu), \nu_0) \leqslant 1$ for all $\nu_0 \in \ell_4 \cup \ell_5 \setminus \{\nu_\star\}$. Moreover [23, Theorem A] shows that these parameters are local bifurcation values of the period function at the outer boundary because $\ell_4 \cup \ell_5 \subset \Gamma_B$. Since the period annuli of the Loud's centers vary continuously (see Remark 2.6), by applying $(a)$ in Lemma 2.16 we have $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \geqslant 1$ for all $\nu_0 \in \ell_4 \cup \ell_5 \setminus \{\nu_\star\}$. Therefore $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$ for all $\nu_0 \in \ell_4 \cup \ell_5 \setminus \{\nu_\star\}$.

On the other hand we have that $\mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big) = 2$ by assertion $(a)$ in Proposition 4.4. Finally the fact that there is a curve of local bifurcation values of the period function at the interior of $\mathscr{P}$ arriving at $\nu = \nu_\star$ tangent to $\Gamma_B$ follows from assertion $(b)$ in the same result.

4. Next we analyze the parameters in the segment $\ell_6 := \{F = 1, D \in (-1, 0)\}$, that corresponds to a case in which there is a saddle-node singularity at the outer boundary of the period annulus. This is treated in Section 3.3, where we introduce the map

$$\sigma(s;\nu) := \begin{cases} (1-s, 0) & \text{if } F \leqslant 1, \\ (p_1 - s, 0) & \text{if } F > 1, \end{cases}$$

that provides a parametrization of the outer boundary of the period annulus verifying the assumptions in Lemma 2.4. In addition if we denote by $P(s;\nu)$ the period of the periodic orbit of $X_\nu$ passing through $\sigma(s;\nu)$ then $P(s;\nu) = 2T(s;\nu)$, where $T$ is the Dulac map considered in Proposition 3.6. From that result we get the existence of an open neighborhood $\mathscr{U}$ of $\ell_6 = (-1, 0) \times \{1\}$ such that

$$P'(s;\nu) = 2T_1(\nu) + 4T_2(\nu)s + s\hat{h}(s;\nu)$$

with $T_1, T_2 \in \mathscr{C}^0(\mathscr{U})$ and where $\hat{h}(s;\nu)$ and $s\partial_s\hat{h}(s;\nu)$ tend to zero as $s \to 0^+$ uniformly on compact subsets of $\mathscr{U}$. We know moreover that $T_1(\nu) = 0$ if, and only if, $\nu = \nu_\star := (-\frac{1}{2}, 1)$ and that $T_2(\nu_\star) \neq 0$.

If we take any $\nu_0 \in \ell_6 \setminus \{\nu_\star\}$ then, thanks to the good properties of the remainder, we get that $\lim_{(s,\nu)\to(0,\nu_0)} P'(s;\nu) = 2T_1(\nu_0) \neq 0$ and this easily implies $\mathcal{Z}_0(P'(\,\cdot\,;\nu), \nu_0) = 0$. Hence, by applying assertion $(2c)$ in Lemma 2.4, $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$ for all $\nu_0 \in \ell_6 \setminus \{\nu_\star\}$.

In order to study the criticality of $\nu_\star$ we observe that $\lim_{(s,\nu)\to(0,\nu_\star)} P''(s;\nu) = 4T_2(\nu_\star) \neq 0$ and, consequently, $\mathcal{Z}_0(P'(\,\cdot\,;\nu), \nu_0) \leqslant 1$ by Rolle's Theorem. Therefore, by assertion $(2a)$ in Lemma 2.4, $\mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big) \leqslant 1$. On the other hand, the application of [23, Theorem A] together with $(a)$ in Lemma 2.16 shows that $\mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big) \geqslant 1$ due to $\nu_\star \in \Gamma_B$. Hence $\mathrm{Crit}\big((\Pi_{\nu_\star}, X_{\nu_\star}), X_\nu\big) = 1$.

5. We proceed with the study of the segment $\ell_7 := \{F = 0, D \in (-1, 0)\}$ which, as in the previous case, corresponds to period annuli having a saddle-node singularity at the outer boundary. In order to compute the criticality of any $\nu_0 \in \ell_7$ we apply the results obtained in [27]. In that paper it is proved that for each $\nu_0 \in \ell_7$ there exist $\delta > 0$, an open neighborhood $V$ of $\nu_0$ and a continuous function $\sigma : [0, \delta) \times V \longrightarrow \mathbb{RP}^2$ verifying the hypothesis in Lemma 2.4. Moreover, denoting the period of the periodic orbit of $X_\nu$ passing through $\sigma(s;\nu)$ by $P(s;\nu)$, the proof of [27, Theorem B] shows that $P'(s;\nu)$ tends to $-\infty$ as $(s,\nu) \to (0, \nu_0)$. Consequently $\mathcal{Z}_0(P'(\,\cdot\,;\nu), \nu_0) = 0$ and hence, by applying $(c)$ in Lemma 2.4, we get that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$.

6. We analyze next the parameters inside the segment $\ell_8 := \big\{D = 0, F \in [\frac{1}{4}, 2]\big\}$. So let us fix any $\nu_0 = (0, F_0)$ with $F_0 \in [\frac{1}{4}, 2]$. By [23, Theorem A] we can assert that if $F_0 \in [\frac{1}{2}, 2]$ then $\nu_0$ is a local bifurcation value of the period function at the outer boundary. On the other hand, if $F_0 \in [\frac{1}{4}, \frac{1}{2}]$ then we can conclude the same by applying [28, Theorem B]. Hence, since the period annuli of the Loud's centers vary continuously, see Remark 2.6, assertion $(a)$ in Lemma 2.16 shows that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \geqslant 1$ for all $\nu_0 \in \ell_8$.

7. From the results in [19,39] it follows that $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$ for any parameter $\nu_0$ inside the set $\ell_9 := \{D = 0, F \notin [0, 2]\} \cup \{D = -1, F < 0\} \cup \{D + F = 0, F < 0\}$. Indeed,

in those papers the authors determine a region $M$ in the parameter plane for which the corresponding centers have a globally monotonic period function. Taking this into account the assertion follows easily from the fact that $\ell_9$ is contained in the interior of $M$, see Definition 1.1.

8. We consider now the half-line $\ell_{10} := \{D + F = 0, F > 1\}$, so let us take a parameter $\nu_0 = (-F_0, F_0)$ with $F_0 > 1$. In this case the assertions with regard to its criticality follow from the results in [24]. It is proved there that there exists a function $\xi = \xi(\nu)$ in a neighborhood $U$ of $\nu_0$ such that $\sigma(s;\nu) = \big(0, \xi(\nu)(1-s)\big)$ is a $\mathscr{C}^0$ map on $[0,\delta) \times U$ verifying the hypothesis $(a)$, $(b)$ and $(c)$ in Lemma 2.4. Moreover if we denote by $P(s;\nu)$ the period of the periodic orbit of $X_\nu$ passing through $\sigma(s;\nu)$ then [24, Theorem B] shows that
   - $\mathcal{Z}_0(P'(\cdot\,;\nu),\nu_0) = 0$ if $F_0 \notin [3/2, 2]$,
   - $\mathcal{Z}_0(P'(\cdot\,;\nu),\nu_0) = 1$ if $F_0 \in [3/2, 2)$ and
   - $\mathcal{Z}_0(P'(\cdot\,;\nu),\nu_0) = 2$ if $F_0 = 2$.

   In the first case $\text{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$ by $(2c)$ in Lemma 2.4, whereas in the second case the combination of $(2a)$ and $(2b)$ implies $\text{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 1$. In the third case, by applying $(2a)$ we get $\text{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \leqslant 2$. To show that this upper bound is attained we also apply $(2b)$ in Lemma 2.4 but to this end we must check the assumption that for each open neighborhood $V$ of $\nu_0 = (-2, 2)$ and $\delta > 0$ there exist distinct $s_1, s_2 \in (0,\delta)$ and $\hat{\nu} \in V$ such that $P'(s_i;\hat{\nu}) = 0$ for $i = 1, 2$. To verify this we note first, see [24, §4], that we can write

$$P'(s;\nu) = \delta_1(\nu) f_1(s;\nu) + \delta_2(\nu) f_2(s;\nu) + f_3(s;\nu)$$

   where the coefficients $\delta_1$ and $\delta_2$ are independent at $\nu_0$ in the sense of [30, Definition 4.1] and, for $i = 1, 2$, $\lim_{s\to 0^+} \frac{f_{i+1}(s;\nu)}{f_i(s;\nu)} = 0$. On account of this and $P'(s;\nu_0) \not\equiv 0$, the fact that the mentioned assumption is verified follows from the proof of [30, Proposition 4.2]. Related with this let us also mention, see again [24, §4], that the ordered set $(f_1, f_2, f_3)$ is an extended complete Chebyshev system on $(0,\varepsilon)$ for $\varepsilon > 0$ sufficiently small (see [13] for a definition).

   On the other hand, by assertion $(a)$ in [24, Theorem C], there exist a neighborhood $U$ of $\nu_0 = (-2, 2)$, $s_0 > 0$ and a injective $\mathscr{C}^0$ curve $\rho\colon (-\varepsilon,\varepsilon) \longrightarrow U$ satisfying $\rho(0) = (-2, 2)$ and

$$\rho((0,\varepsilon)) = \Delta := \{\nu \in U;\ \text{there exists } s \in (0, s_0) \text{ such that } P'(s;\nu) = P''(s;\nu) = 0\}.$$

   Furthermore, assertion $(b)$ in that result shows that the curve $\Delta$ has an exponentially flat contact with the straight line $\{F = 2\}$ at $\nu_0 = (-2, 2)$, see Fig. 6 where we denote $\Delta$ by $\Delta_1$ for consistency. Since the interior of $\Delta$ is clearly empty and the Chebyshev property explained above prevents the zeros of $P'(\cdot\,;\nu)$ to have multiplicity greater than 2, the application of Lemma 2.15 shows that $\Delta$ consists of local bifurcation values of the period function at the interior.

9. Finally the fact that the criticality at the outer boundary of the isochrones $\nu_1 := (-\frac{1}{2}, 2)$ and $\nu_2 := (-\frac{1}{2}, \frac{1}{2})$ is 1 follows from Propositions 4.2 and 4.3, respectively.

Since $\mathbb{R}^2 \setminus \ell_0 = \big(\cup_{i=1}^{10} \ell_i\big) \cup \{\nu_1\} \cup \{\nu_2\}$, this concludes the proof of the result. ■

**Remark 5.1.** By assertion $(e)$ in Theorem A there exists a germ of curve $\Delta_1$ at $\nu = (-2, 2)$, see Fig. 6, which is inside the set of local bifurcation values of the period function at the interior

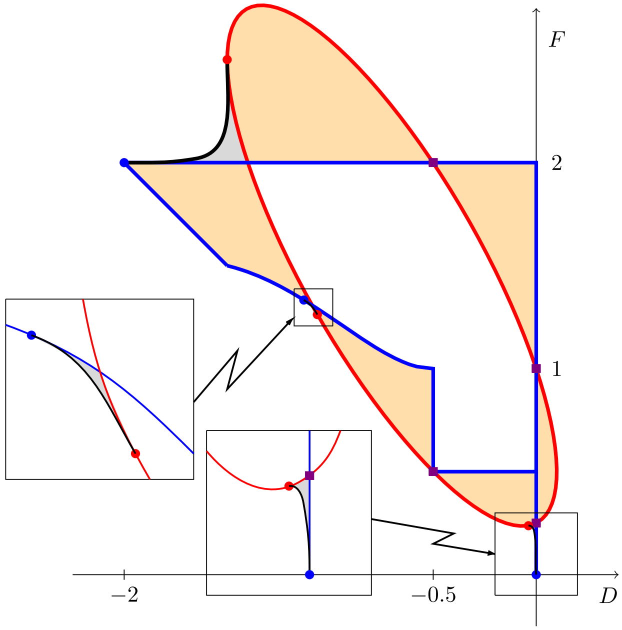


Fig. 7. In red, blue and black, respectively, the bifurcation curves at the inner boundary, the outer boundary and the interior. Rendered with a red (respectively, blue) circle, the three parameters with criticality 2 at the inner (respectively, outer) boundary. Depicted with a violet square, the four isochrones. We conjecture (see Remark 5.2 for details) that the parameters in gray, orange and uncolored, respectively, correspond to centers with *exactly* 2, 1 and 0 critical periodic orbits.

of $\mathscr{P}$. Inside this set there is also the germ of a curve $\delta_1$ at $\nu = L_1$ by [23, Theorem 4.3]. Similarly as we explain in Remark 4.5 for $\Delta_3$ and $\delta_3$, we conjecture that $\Delta_1$ and $\delta_1$ connect each other to delimit a region of parameters for which the period function has exactly two critical periodic orbits. In this regard [23, Theorem 5.2] shows that the center of any parameter inside the light gray sector has *at least* two critical periodic orbits. We know now, see point 8 in the proof of Theorem A, that $\nu = (-2, 2)$ is at the boundary of this region with exactly two critical periodic orbits. The numerical visualization of this fact is a challenging problem because $\Delta_1$ has a exponential flat contact with $\{F = 2\}$ at $\nu = (-2, 2)$. □

**Remark 5.2.** Following the observations made in Remarks 4.5 and 5.1 with regard to the local bifurcation curves displayed in Fig. 6, we conclude this section with the *conjecture* about the behavior of the period function in its whole domain. This is done in Fig. 7, which in turn is an update of the conjecture stated in our earliest paper on the issue, see [23, Figure 3]. Of course, as it is widely accepted, we conjecture that the center at the origin of (2) has at most two critical periodic orbits for any $\nu = (D, F) \in \mathbb{R}^2$. More specifically, see Fig. 7, we conjecture that the parameters inside the gray, orange and uncolored regions, respectively, have *exactly* 2, 1 and 0 critical periodic orbits. The main difference with respect to our original conjecture is about the size of the gray region in the middle, which we know now to be much smaller and not arriving to the diagonal $F = -D$. This conjecture has been partially proved. Indeed, besides the two subsets inside the gray region with *at least* 2 critical periodic orbits explained in Remark 5.1, we can quote [23, Theorem 5.1], which shows that all the parameters in the orange region have *at least* one critical periodic orbit. Furthermore the authors in [19,39] determine an unbounded and rather large subset in the parameter plane for which the corresponding center has no critical periodic orbits. □

## Appendix A. Coefficient formulas

### A.1. Previous results about the Dulac time

This appendix is entirely devoted to the proof of Propositions 3.2 and 3.3 in Section 3. For the parameter values under consideration in both results, and thanks to the symmetry of the vector field $X_\nu$ in (2), it turns out that the period function is twice the Dulac time associated to the passage through a hyperbolic saddle at infinity. The asymptotic expansion of this type of passage is the subject of our recent papers [29–31] and in order to prove the results in Section 3 we strongly rely on the tools developed there. For this reason we first summarize for reader's convenience the definitions and results from those papers that are indispensable here. We recap the results in three theorems. In short, Theorem A.3 will provide us with the monomial scale needed in each asymptotic expansion, which only depends on the hyperbolicity ratio of the saddle, whereas Theorem A.4 will give the explicit expression of their coefficients in terms of a sort of Mellin transform that is introduced in Theorem A.5.

In order to facilitate the application of the above-mentioned results we particularize them to fit in the context needed to prove Propositions 3.2 and 3.3. Thus, following the notation that we use in [30], let us consider the parameter $\hat{\mu} := (\lambda, \mu) \in \hat{W} := (0, +\infty) \times W$, where $W$ is an open set of $\mathbb{R}^N$, and the family of vector fields $\{X_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ with

$$X_{\hat{\mu}}(x_1, x_2) := \frac{1}{x_2}\Big(x_1 P_1(x_1, x_2; \hat{\mu})\partial_{x_1} + x_2 P_2(x_1, x_2; \hat{\mu})\partial_{x_2}\Big), \tag{24}$$

where

- $P_1$ and $P_2$ belong to $\mathscr{C}^\omega(\mathscr{U} \times \hat{W})$ for some open set $\mathscr{U}$ of $\mathbb{R}^2$ containing the origin,
- $P_1(x_1, 0; \hat{\mu}) > 0$ and $P_2(0, x_2; \hat{\mu}) < 0$ for all $(x_1, 0), (0, x_2) \in \mathscr{U}$ and $\hat{\mu} \in \hat{W}$,
- $\lambda = -\frac{P_2(0,0;\hat{\mu})}{P_1(0,0;\hat{\mu})}$.

Moreover, for $i = 1, 2$, let $\sigma_i : (-\varepsilon, \varepsilon) \times \hat{W} \longrightarrow \Sigma_i$ be a $\mathscr{C}^\omega$ transverse section to $X_{\hat{\mu}}$ at $x_i = 0$ defined by

$$\sigma_i(s; \hat{\mu}) = \big(\sigma_{i1}(s; \hat{\mu}), \sigma_{i2}(s; \hat{\mu})\big)$$

such that $\sigma_1(0, \hat{\mu}) \in \{(0, x_2); x_2 > 0\}$ and $\sigma_2(0, \hat{\mu}) \in \{(x_1, 0); x_1 > 0\}$ for all $\hat{\mu} \in \hat{W}$. Then Theorem A.3 is concerned with the time $T(s; \hat{\mu})$ that spends the solution of $X_{\hat{\mu}}$ passing through the point $\sigma_1(s; \hat{\mu}) \in \Sigma_1$ to arrive at $\Sigma_2$, see Fig. 8. More concretely it shows that $T(s; \hat{\mu})$ has an asymptotic expansion at $s = 0$ with the remainder having good flatness properties with respect to the parameters. We specify these properties in the following two definitions.

**Definition A.1.** Given an open subset $U \subset \hat{W} \subset \mathbb{R}^{N+1}$, we say that a function $\psi(s; \hat{\mu})$ belongs to the class $\mathscr{C}^\infty_{s>0}(U)$ if there exists an open neighborhood $\Omega$ of

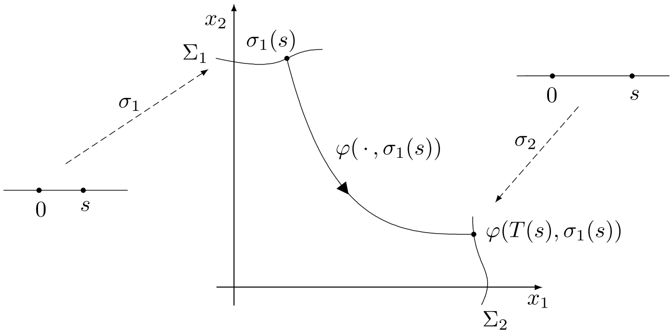


Fig. 8. Definition of $T(\cdot\,;\hat\mu)$, where $\varphi(t,p;\hat\mu)$ is the solution of $X_{\hat\mu}$ passing through the point $p\in\mathscr{U}$ at time $t=0$.

$$\{(s,\hat\mu)\in\mathbb{R}^{N+2};\, s=0,\, \hat\mu\in U\}=\{0\}\times U$$

in $\mathbb{R}^{N+2}$ such that $(s,\hat\mu)\mapsto\psi(s;\hat\mu)$ is $\mathscr{C}^\infty$ on $\Omega\cap\big((0,+\infty)\times U\big)$. □

**Definition A.2.** Consider an open subset $U\subset\hat W\subset\mathbb{R}^{N+1}$. Given any $L\in\mathbb{R}$ and $\hat\mu_0\in U$, we say that a function $\psi(s;\hat\mu)\in\mathscr{C}^\infty_{s>0}(U)$ is *$L$-flat with respect to $s$ at $\hat\mu_0$* if for each $\nu=(\nu_0,\ldots,\nu_{N+1})\in\mathbb{Z}^{N+2}_{\geq 0}$ there exist a neighborhood $V$ of $\hat\mu_0$ and $C,s_0>0$ such that

$$\left|\frac{\partial^{|\nu|}\psi(s;\hat\mu)}{\partial s^{\nu_0}\partial\hat\mu_1^{\nu_1}\cdots\partial\hat\mu_{N+1}^{\nu_{N+1}}}\right|\leqslant Cs^{L-\nu_0}\text{ for all } s\in(0,s_0)\text{ and }\hat\mu\in V,$$

where $|\nu|=\nu_0+\cdots+\nu_{N+1}$. In this case we write $\psi\in\mathcal{F}^\infty_L(\hat\mu_0)$. If $W$ is a (not necessarily open) subset of $U$ then we define $\mathcal{F}^\infty_L(W):=\bigcap_{\hat\mu_0\in W}\mathcal{F}^\infty_L(\hat\mu_0)$. □

Next result merges the statements of Theorem 1.6, Theorem 4.3 and Corollary B in [31]. Following the notation in that paper, we particularize them to the case $(n_1,n_2)=(0,1)$ for simplicity. Moreover, for the sake of shortness, we only include those items that will be used in the present paper.

**Theorem A.3.** *Let $T(s;\hat\mu)$ be the Dulac time of the hyperbolic saddle (24) from $\Sigma_1$ and $\Sigma_2$. Then, setting $D_{00}=\emptyset$, $D_{10}=\frac{1}{\mathbb{N}}$, $D_{01}=\mathbb{N}$, $D_{20}=\frac{2}{\mathbb{N}}$ and $D_{02}=\frac{\mathbb{N}}{2}$, for each $(i,j)\in\{(0,0),(1,0),(0,1),(2,0),(0,2)\}$ there exists a meromorphic function $T_{ij}(\hat\mu)$ on $\hat W=(0,+\infty)\times W$, having poles only along $D_{ij}\times W$, such that the following assertions hold:*

(1) *If $\lambda_0\in(1,2)$ then $T(s;\hat\mu)=T_{00}(\hat\mu)+T_{10}(\hat\mu)s+T_{01}(\hat\mu)s^\lambda+T_{20}(\hat\mu)s^2+\mathcal{F}^\infty_L(\{\lambda_0\}\times W)$ for any $L\in\big[2,\lambda_0+1\big)$.*
(2) *If $\lambda_0>2$ then $T(s;\hat\mu)=T_{00}(\hat\mu)+T_{10}(\hat\mu)s+T_{20}(\hat\mu)s^2+\mathcal{F}^\infty_L(\{\lambda_0\}\times W)$ for any $L\in\big[2,\min(3,\lambda_0)\big)$.*
(3) *If $\lambda_0=\frac{1}{2}$ then $T(s;\hat\mu)=T_{00}(\hat\mu)+T_{01}(\hat\mu)s^\lambda+s\boldsymbol{T}^{\lambda_0}_{10}(\omega;\hat\mu)+\mathcal{F}^\infty_L(\{\lambda_0\}\times W)$ for any $L\in[1,\frac{3}{2})$, where $\omega=\omega(s;\alpha)$, $\alpha=1-2\lambda$ and $\boldsymbol{T}^{\lambda_0}_{10}(w;\hat\mu)\in\mathscr{C}^\infty(\hat U)[w]$ for some open neighborhood $\hat U$ of $\{\lambda_0\}\times W$. Moreover*

$$\boldsymbol{T}_{10}^{\lambda_0}(\omega;\hat{\mu}) = T_{10}(\hat{\mu}) + T_{02}(\hat{\mu})(1+\alpha\omega) \text{ for } \lambda \neq \lambda_0.$$

(4) *If* $\lambda_0 = 1$ *then* $T(s;\hat{\mu}) = T_{00}(\hat{\mu}) + s\boldsymbol{T}_{10}^{\lambda_0}(\omega;\hat{\mu}) + \mathcal{F}_L^{\infty}(\{\lambda_0\}\times W)$ *for any* $L \in (1,2)$*, where* $\omega = \omega(s;\alpha)$*,* $\alpha = 1-\lambda$ *and* $\boldsymbol{T}_{10}^{\lambda_0}(w;\hat{\mu}) \in \mathscr{C}^{\infty}(\hat{U})[w]$ *for some open neighborhood* $\hat{U}$ *of* $\{\lambda_0\}\times W$*. Moreover*

$$\boldsymbol{T}_{10}^{\lambda_0}(\omega;\hat{\mu}) = T_{10}(\hat{\mu}) + T_{01}(\hat{\mu})(1+\alpha\omega) \text{ for } \lambda \neq \lambda_0.$$

(5) *If* $\lambda_0 = 2$ *then* $T(s;\hat{\mu}) = T_{00}(\hat{\mu}) + T_{10}(\hat{\mu})s + s^2\boldsymbol{T}_{20}^{\lambda_0}(\omega;\hat{\mu}) + \mathcal{F}_L^{\infty}(\{\lambda_0\}\times W)$ *for any* $L \in \left[2,3\right)$*, where* $\omega = \omega(s;\alpha)$*,* $\alpha = 2-\lambda$ *and* $\boldsymbol{T}_{20}^{\lambda_0}(w;\hat{\mu}) \in \mathscr{C}^{\infty}(\hat{U})[w]$ *for some open neighborhood* $\hat{U}$ *of* $\{\lambda_0\}\times W$*. Moreover*

$$\boldsymbol{T}_{20}^{\lambda_0}(\omega;\hat{\mu}) = T_{20}(\hat{\mu}) + T_{01}(\hat{\mu})\,(1+\alpha\omega) \text{ for } \lambda \neq \lambda_0.$$

We focus next on the expression of the coefficients $T_{ij}$ and the result that we state below in this regard follows from assertion $(c)$ in [31, Theorem A] particularized to $(n_1,n_2)=(0,1)$. In its statement we use the following functions:

$$\begin{aligned}
&L_1(u) := \exp\int_0^u \left(\frac{P_1(0,z)}{P_2(0,z)} + \frac{1}{\lambda}\right)\frac{dz}{z} && L_2(u) := \exp\int_0^u \left(\frac{P_2(z,0)}{P_1(z,0)} + \lambda\right)\frac{dz}{z}\\
&A_1(u) := \frac{1}{P_2(0,u)} && A_2(u) := \frac{L_2(u)}{P_1(u,0)}\\
&M_1(u) := L_1(u)\partial_1\left(\frac{P_1}{P_2}\right)(0,u) && B_1(u) := L_1(u)\partial_1 P_2^{-1}(0,u)\\
&C_1(u) := L_1^2(u)\partial_1^2 P_2^{-1}(0,u) + 2L_1(u)\hat{M}_1(1/\lambda,u)\partial_1 P_2^{-1}(0,u)
\end{aligned} \tag{25}$$

Here, given $\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\geq 0}$ and a real valued function $f(x)$ that is $\mathscr{C}^{\infty}$ in an open interval containing $x=0$, $\hat{f}(\alpha,x)$ is a sort of incomplete Mellin transform (see Theorem A.5 below). Moreover, for the sake of shortness, in the following statement we use the compact notation $\sigma_{ijk}$ for the $k$th derivative at $s=0$ of the $j$th component of $\sigma_i(s;\hat{\mu})$, i.e.,

$$\sigma_{ijk}(\hat{\mu}) := \partial_s^k \sigma_{ij}(0;\hat{\mu}).$$

Also with regard to the statement, note that $D_{ij}$ refers to the discrete sets in Theorem A.3.

**Theorem A.4.** *For each* $(i,j) \in \{(0,0),(1,0),(0,1),(2,0)\}$*, the following expression of* $T_{ij}(\hat{\mu})$ *is valid provided that* $\lambda \notin D_{ij}$*:*

$$T_{00}(\hat{\mu}) = -\,\sigma_{120}\hat{A}_1(-1,\sigma_{120}),$$

$$T_{01}(\hat{\mu}) = \frac{\sigma_{120}\sigma_{111}^{\lambda}}{\sigma_{210}^{\lambda}L_1^{\lambda}(\sigma_{120})}\hat{A}_2(\lambda,\sigma_{210}),$$

$$T_{10}(\hat{\mu}) = -\frac{\sigma_{121}}{P_2(0,\sigma_{120})} - \frac{\sigma_{120}\sigma_{111}}{L_1(\sigma_{120})}\hat{B}_1\big(1/\lambda - 1, \sigma_{120}\big),$$

*and*

$$T_{20}(\hat{\mu}) = -\frac{\sigma_{120}\sigma_{122}}{2\sigma_{120}P_2(0,\sigma_{120})} - \frac{1}{2}\sigma_{121}^2\partial_2 P_2^{-1}(0,\sigma_{120}) - \sigma_{121}\sigma_{111}\partial_1 P_2^{-1}(0,\sigma_{120})$$
$$- \frac{\sigma_{120}\sigma_{111}^2}{2L_1^2(\sigma_{120})}\hat{C}_1(2/\lambda - 1, \sigma_{120}) - S_1\frac{\sigma_{120}\sigma_{111}}{L_1(\sigma_{120})}\hat{B}_1(1/\lambda - 1, \sigma_{120}),$$

*where*

$$S_1 = \frac{\sigma_{112}}{2\sigma_{111}} - \frac{\sigma_{121}}{\sigma_{120}}\left(\frac{P_1}{P_2}\right)(0,\sigma_{120}) - \frac{\sigma_{111}}{L_1(\sigma_{120})}\hat{M}_1(1/\lambda, \sigma_{120}).$$

As we already explained, the following result (that merges Theorem B.1 and Corollary B.3 in [31]) is the third ingredient needed in the proof of Propositions 3.2 and 3.3.

**Theorem A.5.** *Consider an open interval $I$ of $\mathbb{R}$ containing $x = 0$ and an open subset $U$ of $\mathbb{R}^N$.*

(*a*) *Given $f(x;\nu) \in \mathscr{C}^\infty(I \times U)$, there exists a unique $\hat{f}(\alpha, x;\nu) \in \mathscr{C}^\infty((\mathbb{R}\setminus\mathbb{Z}_{\geq 0}) \times I \times U)$ such that*

$$x\partial_x\hat{f}(\alpha,x;\nu) - \alpha\hat{f}(\alpha,x;\nu) = f(x;\nu).$$

(*b*) *If $x \in I\setminus\{0\}$ then $\partial_x(\hat{f}(\alpha,x;\nu)|x|^{-\alpha}) = f(x;\nu)\frac{|x|^{-\alpha}}{x}$ and, taking any $k \in \mathbb{Z}_{\geq 0}$ with $k > \alpha$,*

$$\hat{f}(\alpha,x;\nu) = \sum_{i=0}^{k-1}\frac{\partial_x^i f(0;\nu)}{i!(i-\alpha)}x^i + |x|^\alpha\int_0^x \Big(f(s;\nu) - T_0^{k-1}f(s;\nu)\Big)|s|^{-\alpha}\frac{ds}{s},$$

*where $T_0^k f(x;\nu) = \sum_{i=0}^k \frac{1}{i!}\partial_x^i f(0;\nu)x^i$ is the $k$-th degree Taylor polynomial of $f(x;\nu)$ at $x = 0$.*

(*c*) *If $f(x;\nu)$ is analytic on $I \times U$ then $\hat{f}(\alpha,x;\nu)$ is analytic on $(\mathbb{R}\setminus\mathbb{Z}_{\geq 0}) \times I \times U$. Finally, for each $(\alpha_0,x_0,\nu_0) \in \mathbb{Z}_{\geq 0} \times I \times U$ the function $(\alpha,x,\nu) \mapsto (\alpha_0 - \alpha)\hat{f}(\alpha,x;\nu)$ extends analytically to $(\alpha_0,x_0,\nu_0)$.*

(*d*) *If $f(x;\nu) = x^n g(x;\nu)$ with $g \in \mathscr{C}^\infty(I \times U)$ and $n \in \mathbb{N}$ then $\hat{f}(\alpha,x;\nu) = x^n\hat{g}(\alpha - n, x;\nu)$.*

The following simple observation will be useful in order to study the coefficients of the asymptotic expansions that we shall deal with.

**Remark A.6.** If $\sum_{i=1}^m a_i x^{\lambda_i} + \psi(x) = 0$ for all $x \in (0,\varepsilon)$, where $\lambda_i \in \mathbb{R}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, $a_1, a_2, \ldots, a_m \in \mathbb{R}$ and $\psi(x) = \mathrm{o}(x^{\lambda_m})$ then $a_1 = a_2 = \cdots = a_m = 0$. □

We are now in position to begin the proof of the two first results in Section 3.

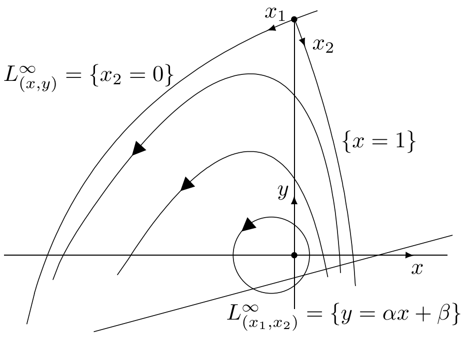


Fig. 9. Projective coordinate change in the proof of Proposition 3.2.

## A.2. *Proof of Proposition 3.2*

**Proof of Proposition 3.2.** We follow the approach in [22, §5] to take advantage of the general setting developed in [30]. To this end we will work on an extended parameter space $\bar\nu \in \bar V$ that we specify as follows. Firstly, introducing an auxiliary parameter $\eta \approx 0$, we consider two local transverse sections $\Sigma_1^\eta$ and $\Sigma_2^\eta$ parametrized respectively by $s \mapsto (1-s, \eta)$ and $s \mapsto (-\frac{1}{s}, \frac{\eta}{s})$, for $s > 0$, cf. Fig. 3. Secondly, taking any $\alpha, \beta \in \mathbb{R}$ such the straight line $y = \alpha x + \beta$ does not intersect any solution of $X_\nu$ while traveling from $\Sigma_1^\eta$ to $\Sigma_2^\eta$. One can readily see that a sufficient condition for this to hold is that

$$\alpha + \beta < \eta \text{ and } \eta > -\alpha.$$

Then, setting $\bar\nu := (D, F, \alpha, \beta, \eta)$, we will work on the extended parameter space

$$\bar V := \left\{ \bar\nu \in \mathbb{R}^5 : D \in (-1, 0), F \in (0, 1), \alpha + \beta < \eta, -\alpha < \eta \right\}.$$

Taking this into account, we consider the projective change of coordinates, see Fig. 9,

$$(x_1, x_2) = \left( \frac{1-x}{y - \alpha x - \beta}, \frac{1}{y - \alpha x - \beta} \right).$$

One can verify that in these coordinates the parametrizations of $\Sigma_1^\eta$ and $\Sigma_2^\eta$ become

$$\sigma_1(s) := \left( \frac{s}{\eta + \alpha s - \alpha - \beta}, \frac{1}{\eta + \alpha s - \alpha - \beta} \right) \text{ and } \sigma_2(s) := \left( \frac{1+s}{\alpha + \eta - \beta s}, \frac{s}{\alpha + \eta - \beta s} \right), \tag{26}$$

respectively, whereas the vector field (2) is brought to

$$\bar X_{\bar\nu} := \frac{1}{x_2} \big( x_1 P_1(x_1, x_2; \bar\nu) \partial_{x_1} + x_2 P_2(x_1, x_2; \bar\nu) \partial_{x_2} \big) \tag{27}$$

with

$$P_1(x_1,x_2;\bar\nu)=(1-\alpha x_1)^2+(\alpha+\beta)x_2-x_2^2+(1-\alpha^2-\alpha\beta)x_1x_2$$
$$-F(1+\alpha(x_2-x_1)+\beta x_2)^2-D(x_1-x_2)^2$$

and

$$P_2(x_1,x_2;\bar\nu)=\alpha^2x_1^2-\alpha x_1-x_2^2+(1-\alpha^2-\alpha\beta)x_1x_2$$
$$-F(1+\alpha(x_2-x_1)+\beta x_2)^2-D(x_1-x_2)^2.$$

The reason why we introduce the auxiliary parameters $\alpha$, $\beta$ and $\eta$ is because the computations are much easier taking the projective change of coordinates that sends $y=0$ to infinity (i.e., with $\alpha=\beta=0$), which is not compatible with the placement of the original transverse sections (i.e., with $\eta=0$). Since the parameters $\bar\nu$ with $\alpha=\beta=\eta=0$ are in the boundary of the admissible set $\bar V$, we will work in the interior and then make a limit argument. By introducing these auxiliary parameters we end up in a setting where the assumptions to apply the results in Section A.1 are fulfilled. Observe in particular that $P_1$ and $P_2$ are analytic on $\mathbb{R}^2\times\bar V$. Following the notation introduced there, note that the hyperbolicity ratio of the saddle

$$\lambda=-\frac{P_2(0,0)}{P_1(0,0)}=\frac{F}{1-F}$$

depends on $\mu=\bar\nu$. We denote the Dulac time of $\bar X_{\bar\nu}$ from $\Sigma_1^\eta$ to $\Sigma_2^\eta$ by $\bar T(s;\bar\nu)$. Note that, by construction, it does not depend on $\alpha$ and $\beta$ as long as $\alpha+\beta<\eta$ and $\eta>-\alpha$ holds. Moreover, and this is the key point for our purposes, the Dulac time $T(s;\nu)$ in the statement is precisely $\bar T(s;\bar\nu)$ for $\eta=0$.

Let us fix any $\bar\nu_0=(D_0,F_0,\alpha_0,\beta_0,\eta_0)\in\bar V$ with $F_0\in[\frac12,1)$. Observe that $\lambda(\bar\nu_0)>2$ if $F_0\in(\frac23,1)$, $\lambda(\bar\nu_0)\in(1,2)$ if $F_0\in(\frac12,\frac23)$, $\lambda(\bar\nu_0)=2$ if $F_0=\frac23$ and $\lambda(\bar\nu_0)=1$ if $F_0=\frac12$. Consequently, by applying (2), (1), (5) and (4) in Theorem A.3, respectively, we get that

($a'$) $\bar T(s;\bar\nu)=\bar T_{00}(\bar\nu)+\bar T_{10}(\bar\nu)s+\bar T_{20}(\bar\nu)s^2+\mathcal{F}^\infty_{L_0-\upsilon}(\bar\nu_0)$ if $F_0\in(\frac23,1)$, where $L_0=\min(3,\lambda(\bar\nu_0))$,

($b'$) $\bar T(s;\bar\nu)=\bar T_{00}(\bar\nu)+\bar T_{10}(\bar\nu)s+\bar T_{01}(\bar\nu)s^\lambda+\mathcal{F}^\infty_{2-\upsilon}(\bar\nu_0)$ if $F_0\in(\frac12,\frac23)$,

($c'$) $\bar T(s;\bar\nu)=\bar T_{00}(\bar\nu)+\bar T_{10}(\bar\nu)s+\bar T^2_{201}(\bar\nu)s^2\omega_{2-\lambda}(s)+\bar T^2_{200}(\bar\nu)s^2+\mathcal{F}^\infty_{3-\upsilon}(\bar\nu_0)$ if $F_0=\frac32$, where $\bar T^2_{201}(\bar\nu)$ and $\bar T^2_{200}(\bar\nu)$ are smooth in a neighborhood of $\{\bar\nu\in\bar V:\lambda(\bar\nu)=2\}$ and, moreover,

$$\bar T^2_{201}(\bar\nu)=(2-\lambda)\bar T_{01}(\bar\nu)\text{ and }\bar T^2_{200}(\bar\nu)=\bar T_{20}(\bar\nu)+\bar T_{01}(\bar\nu)\text{ for }\lambda(\bar\nu)\neq2,$$

($d'$) $\bar T(s;\bar\nu)=\bar T_{00}(\bar\nu)+\bar T^1_{101}(\bar\nu)s\omega_{1-\lambda}(s)+\bar T^1_{100}(\bar\nu)s+\mathcal{F}^\infty_{2-\upsilon}(\bar\nu_0)$ if $F_0=\frac12$, where $\bar T^1_{101}(\bar\nu)$ and $\bar T^1_{100}(\bar\nu)$ are smooth in a neighborhood of $\{\bar\nu\in\bar V:\lambda(\bar\nu)=1\}$ and, moreover,

$$\bar T^1_{101}(\bar\nu)=(1-\lambda)\bar T_{01}(\bar\nu)\text{ and }\bar T^1_{100}(\bar\nu)=\bar T_{10}(\bar\nu)+\bar T_{01}(\bar\nu)\text{ for }\lambda(\bar\nu)\neq1.$$

Here $\upsilon$ is a small enough positive number depending on $\bar\nu_0$. Furthermore the coefficients $\bar T_{ij}(\bar\nu)$ are meromorphic functions on $\bar V$ with poles only at those $F\in(0,1)$ such that $\lambda(\bar\nu)=\frac{F}{1-F}\in D_{ij}$, where $D_{00}=\emptyset$, $D_{10}=\frac{1}{\mathbb{N}}$, $D_{01}=\mathbb{N}$ and $D_{20}=\frac{2}{\mathbb{N}}$.

We claim that the coefficients $\bar{T}_{ij}(\bar{\nu})$ do not depend on $\alpha$ and $\beta$. Indeed, to see this recall that the Dulac time $\bar{T}(s;\bar{\nu})$ does not depend on $\alpha$ and $\beta$ provided that $\alpha+\beta<\eta$ and $\eta>-\alpha$, which is verified for $\bar{\nu}\in\bar{V}$. Hence $\partial_\alpha\bar{T}(s;\bar{\nu})\equiv 0$ and $\partial_\beta\bar{T}(s;\bar{\nu})\equiv 0$. Thus from $(b')$ we get that, for each fixed $\bar{\nu}_\star\in\bar{V}\cap\{\frac{1}{2}<F<\frac{2}{3}\}$,

$$\partial_\alpha\bar{T}_{00}(\bar{\nu}_\star)+\partial_\alpha\bar{T}_{10}(\bar{\nu}_\star)s+\partial_\alpha\bar{T}_{01}(\bar{\nu}_\star)s^{\lambda(\bar{\nu}_\star)}+\mathrm{o}(s^{2-2\upsilon})=0 \text{ for all } s>0,$$

where we use that the flatness order of the remainder is preserved when derived with respect to parameters (see Definition A.2). Then, since $1<\lambda<2$ for $F\in(\frac{1}{2},\frac{2}{3})$ and we can choose $\upsilon>0$ arbitrary small (depending on $\bar{\nu}_\star$), by taking Remark A.6 into account we can assert that

$$\partial_\alpha\bar{T}_{00}(\bar{\nu}_\star)=0,\ \partial_\alpha\bar{T}_{10}(\bar{\nu}_\star)=0 \text{ and } \partial_\alpha\bar{T}_{01}(\bar{\nu}_\star)=0.$$

Since $\bar{V}\cap\{\frac{1}{2}<F<\frac{2}{3}\}$ is open and the coefficients are meromorphic on $\bar{V}$, by Lemma B.1 it follows that $\partial_\alpha\bar{T}_{00}$, $\partial_\alpha\bar{T}_{10}$ and $\partial_\alpha\bar{T}_{01}$ are identically zero. The claim for $\partial_\beta$ and the other coefficients follows verbatim.

On account of the claim and the fact that $\bar{T}(s;\bar{\nu})|_{\eta=0}=T(s;\nu)$, the assertions in $(a)$–$(d)$ with regard to the asymptotic expansion of $T(s;\nu)$ follow from $(a')$–$(d')$, respectively, by setting

$$T_{ij}(\nu):=\bar{T}_{ij}(\bar{\nu})\big|_{\eta=0}.$$

Next we proceed with the computation of the expression of each coefficient. To this end observe that

$$T_{ij}(\nu)=\bar{T}_{ij}(D,F,\alpha,\beta,\eta)\big|_{\eta=0}=\lim_{\eta\to 0^+}\bar{T}_{ij}(D,F,\alpha,\beta,\eta)=\lim_{\eta\to 0^+}\bar{T}_{ij}(D,F,0,0,\eta), \tag{28}$$

where the second follows by the continuity of $\bar{\nu}\mapsto\bar{T}_{ij}(\bar{\nu})$ on $\bar{V}$ and the last one on account of the previous claim. In view of this the plan is to compute $\bar{T}_{ij}(D,F,0,0,\eta)$ with $\eta>0$ by applying Theorem A.4 and then make $\eta\to 0$. With this aim it is first necessary to obtain the functions in (25). In doing so, and setting

$$\delta_1=\frac{1}{2F},\ \kappa_1=\frac{D+1}{F},\ \delta_2=\frac{1}{2(1-F)} \text{ and } \kappa_2=\frac{D}{F-1},$$

for shortness, one can verify that

$$L_i(u)=h(u;\delta_i,\kappa_i) \text{ for } i=1,2, \text{ where } h(u;\delta,\kappa):=(1+\kappa u^2)^\delta.$$

We stress that $L_1$ and $L_2$ are defined in (25) in terms of the functions $P_1(x_1,x_2;\bar{\nu})$ and $P_1(x_1,x_2;\bar{\nu})$ given in (27) and that, as we already explained, we take $\alpha=\beta=0$ here and in what follows. Similarly, setting

$$\gamma_1=-\frac{2D+1}{F^2}$$

for the sake of shortness again, some computations show that $M_1(u)=\gamma_1 u h(u;\delta_1-2,\kappa_1)$,

$$A_1(u) = -2\delta_1 h(u; -1, \kappa_1),\ B_1(u) = M_1(u) \text{ and } C_1(u)|_{D=-\frac{1}{2}} = -4\delta_1^2 h(u; 2\delta_1 - 2, \delta_1). \quad (29)$$

Moreover $A_2(u) = 2\delta_2 h(u; \delta_2 - 1, \kappa_2)$. With regard to the parametrization of the transverse sections in the expression of the coefficients, see (26), using the compact notation $\sigma_{ijk}(\bar{\nu}) := \partial_s^k \sigma_{ij}(0; \bar{\nu})$ we get that

$$\sigma_{112} = \sigma_{121} = \sigma_{122} = 0 \text{ and } \sigma_{111} = \sigma_{120} = \sigma_{210} = \sigma_{221} = 1/\eta. \quad (30)$$

We are now in position to apply Theorem A.4 to obtain the coefficients $\bar{T}_{ij}(\bar{\nu})$. (We omit the computation leading to $T_{00}(\nu)$ because it is given in [22, Proposition 5.2].) In doing so we obtain that

$$\bar{T}_{01}(\nu, 0, 0, \eta) = \frac{\sigma_{120}\sigma_{111}^{\lambda}}{\sigma_{210}^{\lambda} L_1^{\lambda}(\sigma_{120})} \hat{A}_2(\lambda, \sigma_{210}) = 2\delta_2 \frac{\eta^{\lambda}}{(\eta^2 + \kappa_1)^{\delta_2}} \hat{h}(\lambda, 1/\eta; \delta_2 - 1, \kappa_2).$$

Therefore

$$\begin{aligned} T_{01}(\nu) &= \lim_{\eta \to 0^+} \bar{T}_{01}(\nu, 0, 0, \eta) = \frac{2\delta_2}{\kappa_1^{\delta_2}} \frac{\kappa_2^{\frac{\lambda}{2}}}{2} \mathrm{B}\left(-\frac{\lambda}{2}, -\delta_2 + 1 + \frac{\lambda}{2}\right) \\ &= \frac{\delta_2}{\sqrt{\kappa_2}} \left(\frac{\kappa_2}{\kappa_1}\right)^{\delta_2} \mathrm{B}\left(-\frac{\lambda}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2(1-F)} \left(\frac{F}{D+1}\right)^{\frac{\lambda+1}{2}} \left(\frac{D}{F-1}\right)^{\frac{\lambda}{2}} \frac{\Gamma(-\frac{\lambda}{2})}{\Gamma(\frac{1-\lambda}{2})}, \end{aligned}$$

where the first equality follows from (28), the second one by $(a)$ in Proposition B.3 (provided that $\lambda \notin \mathbb{N}$), and the last one by using (46) and that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Since $\rho_1(\nu) = \frac{\sqrt{\pi}}{2(1-F)}\big(\frac{F}{D+1}\big)^{\frac{\lambda+1}{2}}\big(\frac{D}{F-1}\big)^{\frac{\lambda}{2}}$ is an analytic positive function on $V$, this proves the equality in the statement because we know that $T_{01}(\nu)$ is meromorphic on $V$ with poles only at those $F \in (0, 1)$ such that $\lambda(\nu) \in \mathbb{N}$.

Let us turn next to the computation of $T_{10}$. With this aim we note that

$$\begin{aligned} \bar{T}_{10}(\nu, 0, 0, \eta) &= -\sigma_{120}\left(\frac{\sigma_{121}}{\sigma_{120} P_2(0, \sigma_{120})} + \frac{\sigma_{111}}{L_1(\sigma_{120})} \hat{B}_1\big(1/\lambda - 1, \sigma_{120}\big)\right) \\ &= -\frac{\eta^{-2}}{L_1(\eta^{-1})} \hat{B}_1(1/\lambda - 1, \eta^{-1}) = -\gamma_1 \frac{\eta^{-3+2\delta_1}}{(\eta^2 + \kappa_1)^{\delta_1}} \hat{h}(1/\lambda - 2, \eta^{-1}; \delta_1 - 2, \kappa_1). \end{aligned}$$

Here the first equality follows by Theorem A.4, the second one from (30) and the last one by applying $(d)$ in Theorem A.5 to the function $B_1(u) = \gamma_1 u h(u; \delta_1 - 2, \kappa_1)$, see (29). Since $-3 + 2\delta_1 = \frac{1}{\lambda} - 2$, by applying $(a)$ in Proposition B.3 we get

$$T_{10}(\nu) = \lim_{\eta \to 0^+} T_{10}(\nu, 0, 0, \eta) = -\frac{\gamma_1 \kappa_1^{\frac{1}{2\lambda} - 1}}{2\kappa_1^{\delta_1}} \mathrm{B}\left(1 - \frac{1}{2\lambda}, \frac{1}{2}\right) = \frac{\sqrt{\pi}(2D+1)}{2\sqrt{F(1+D)^3}} \frac{\Gamma(1 - \frac{1}{2\lambda})}{\Gamma(\frac{3}{2} - \frac{1}{2\lambda})}$$

and, due to $\rho_2(\nu) = \frac{\sqrt{\pi}}{2\sqrt{F(1+D)^3}}$, this proves the validity of the expression for $T_{10}$ given in the statement. Let us finally compute the coefficient $T_{20}$. In this case, on account of (30), by Theorem A.4 we get

$$\bar{T}_{20}(\nu,0,0,\eta) = -\eta^{-1}\left(\frac{\eta^{-2}}{2L_1^2(\eta^{-1})}\hat{C}_1(2/\lambda-1,\eta^{-1}) + \frac{\eta^{-1}S_1}{L_1(\eta^{-1})}\hat{B}_1(1/\lambda-1,\eta^{-1})\right).$$

Thus, since $\gamma_1 = 0$ for $D = -\frac{1}{2}$, from (29) it turns out that

$$\bar{T}_{20}(-1/2,F,0,0,\eta) = \frac{2\delta_1^2\eta^{-3+4\delta_1}}{(\eta^2+\kappa_1)^{2\delta_1}}\hat{h}(2/\lambda-1,\eta^{-1};2\delta_1-2,\delta_1)\Big|_{D=-\frac{1}{2}}$$

Hence, due to $-3+4\delta_1 = \frac{2}{\lambda}-1$, by applying $(a)$ in Proposition B.3 once again,

$$\begin{aligned}T_{20}(-1/2,F) &= \lim_{\eta\to 0^+}\bar{T}_{20}(-1/2,F,0,0,\eta) = \frac{\kappa_1^{\frac{1}{\lambda}-\frac{1}{2}}}{4F^2\kappa_1^{2\delta_1}}\Big|_{D=-\frac{1}{2}}\mathrm{B}\left(\frac{1}{2}-\frac{1}{\lambda},\frac{1}{2}\right)\\ &= \frac{\sqrt{\pi}}{\sqrt{2F}}\frac{\Gamma(\frac{1}{2}-\frac{1}{\lambda})}{\Gamma(1-\frac{1}{\lambda})}.\end{aligned}$$

Since $T_{20}(\nu)$ is a meromorphic function having poles only at those $\nu_0 \in V$ such that $\lambda(\nu_0) \in D_{20} = \frac{2}{\mathbb{N}}$ and, on the other hand, $\lambda(\nu) > 2$ for all $\nu \in V \cap \{\frac{2}{3} < F < 1\}$, by applying the Weierstrass Division Theorem (see for instance [11, Theorem 1.8]), we can assert the existence of an analytic function $\rho_3$ on $V \cap \{\frac{2}{3} < F < 1\}$ such that

$$T_{20}(\nu) = \frac{\sqrt{\pi}}{\sqrt{2F}}\frac{\Gamma(\frac{1}{2}-\frac{1}{\lambda})}{\Gamma(1-\frac{1}{\lambda})} + \rho_3(\nu)(2D+1).$$

It only remains to prove the assertions with regard to the properties of the coefficients in the respective asymptotic expansions. Being the ones in $(a)$ and $(b)$ an easy consequence of well-known properties of the gamma function (see for instance [1]), we proceed with the other two:

$(c)$ Let us take any $\nu_0 \in V \cap \{F = \frac{2}{3}\}$ and note that, from $(c')$, the functions $T_{201}^2(\nu) := \bar{T}_{201}^2(\bar{\nu})\big|_{\eta=0}$ and $T_{200}^2(\nu) := \bar{T}_{200}^2(\bar{\nu})\big|_{\eta=0}$ are smooth in a neighborhood of $\{\nu \in V : \lambda(\nu) = 2\} = V \cap \{F = \frac{2}{3}\}$ and

$$T_{201}^2(\nu) = \big(2-\lambda(\nu)\big)T_{01}(\nu) \text{ and } T_{200}^2(\nu) = T_{20}(\nu) + T_{01}(\nu) \text{ for } \lambda(\nu) \neq 2.$$

Recall that $T_{01}(\nu)$ and $T_{20}(\nu)$ are meromorphic with a pole at those $\nu$ such that $\lambda(\nu) = 2 \in D_{01} \cap D_{20}$. What is more, by Propositions 3.2 and 3.6 in [31], respectively, we know that in both cases the pole is simple. Consequently by the Weierstrass Division Theorem (or, more directly, by [31, Lemma 2.8]) it follows that $T_{201}^2(\nu)$ and $T_{200}^2(\nu)$ are analytic in a neighborhood of $V \cap \{F = \frac{2}{3}\}$. On the other hand, from the already proved part of the statement, if $\nu_\star = (-\frac{1}{2}, \frac{2}{3})$ then $T_{10}(\nu_\star) = 0$ and

$$T_{201}^2(\nu_\star) = \lim_{\nu\to\nu_\star}(2-\lambda)T_{01}(\nu) = \frac{\rho_1(\nu_\star)}{\Gamma(-\frac{1}{2})}\lim_{\nu\to\nu_\star}\Gamma(-\lambda/2)(2-\lambda) = -2\frac{\rho_1(\nu_\star)}{\Gamma(-\frac{1}{2})} \neq 0$$

because $\lim_{x\to -1}(x+1)\Gamma(x) = -1$ (see [1, §6] for instance).

(*d*) Consider finally any $\nu_0 \in V \cap \{F = \frac{1}{2}\}$. Then, from assertion $(d')$, $T^1_{101}(\nu) := \bar{T}^1_{101}(\bar{\nu})\big|_{\eta=0}$ and $T^1_{100}(\nu) := \bar{T}^1_{100}(\bar{\nu})\big|_{\eta=0}$ are smooth functions in a neighborhood of $\{\nu \in V : \lambda(\nu) = 1\} = V \cap \{F = \frac{1}{2}\}$ and, in addition,

$$T^1_{101}(\nu) = \big(1 - \lambda(\nu)\big)T_{01}(\nu) \text{ and } T^1_{100}(\nu) = T_{10}(\nu) + T_{01}(\nu) \text{ for } \lambda(\nu) \neq 1.$$

Since $T_{10}(\nu)$ and $T_{01}(\nu)$ are meromorphic with a pole at those $\nu$ such that $\lambda(\nu) = 1 \in D_{10} \cap D_{01}$, the above equality shows exactly as before that $T^1_{101}(\nu)$ and $T^1_{100}(\nu)$ are analytic in a neighborhood of $V \cap \{F = \frac{1}{2}\}$. Moreover, from the expression for $T_{01}$ in the statement that we already proved and using that $1 - \lambda = 2\frac{F-1/2}{F-1}$, we can write

$$T^1_{101}(\nu) = (1-\lambda)T_{01}(\nu) = -\rho_4(\nu)(F - 1/2)^2 \text{ with } \rho_4(\nu) := \frac{\rho_1(\nu)}{(F-1)^2}\frac{4\Gamma(-\frac{\lambda}{2})}{\Gamma(\frac{1-\lambda}{2})(\lambda - 1)}.$$

The function $\rho_4$ is analytic at $\nu_0 = (D_0, \frac{1}{2})$ because $\lambda(\nu_0) = 1$ and $\Gamma(z)$ has simple pole at $z = 0$. In addition, since $\lim_{z\to 0} z\Gamma(z) = 1$ and $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, we get that $\rho_4(\nu_0) = 16\sqrt{\pi}\rho_1(\nu_0) > 0$. From the expressions in the statement as well, we get

$$T^1_{100}(\nu) = \rho_5(\nu)(D + 1/2) + \rho_6(\nu)(F - 1/2)$$

with $\rho_5(\nu) := 2\rho_2(\nu)\frac{\Gamma(1-\frac{1}{2\lambda})}{\Gamma(\frac{3}{2}-\frac{1}{2\lambda})}$ and $\rho_6(\nu) := \frac{1-F}{2}\rho_4(\nu)$, that are analytic and positive at $\nu_0 = (D_0, \frac{1}{2})$ due to $\lambda(\nu_0) = 1$. Finally a computation shows that $\rho_5(-\frac{1}{2}, \frac{1}{2}) = \rho_6(-\frac{1}{2}, \frac{1}{2})$.

This concludes the proof of the result. ■

### A.3. *Proof of Proposition 3.3*

**Proof of Proposition 3.3.** We will adapt the arguments in [23, §3.2.1] to take advantage of the general setting developed in [31]. To this end, as we did in the proof of the previous result, we will work in an extended parameter space $\bar{W}$ to be specified. In this case the computations are a little bit more involved because we also need to straighten the separatrices of the saddle, see Fig. 4. With this aim in view we first take $\varepsilon \in \mathbb{R}$ and consider the local change of coordinates given by

$$(x_1, x_2) = \phi_\varepsilon(x, y) := \left(\frac{q(x) - \frac{1}{2}y^2}{a(p_2 - x + \varepsilon y)^2}, \frac{p_2 - p_1}{p_2 - x + \varepsilon y}\right),$$

where recall that $q(x) = a(x - p_1)(x - p_2)$ with $a = \frac{D}{2(1-F)} > 0$ and $p_1, p_2 \in \mathbb{R}$, $p_1 < p_2$, for all $\nu \in W$. In what follows, for the sake of shortness we set

$$\kappa_1 := p_2 - p_1 \text{ and } \kappa_2 := 1/\sqrt{2a}.$$

One can check that the Jacobian determinant of $\phi_\varepsilon$ vanishes at $(x, y)$ if and only if $y - \varepsilon q'(x) = 0$, where

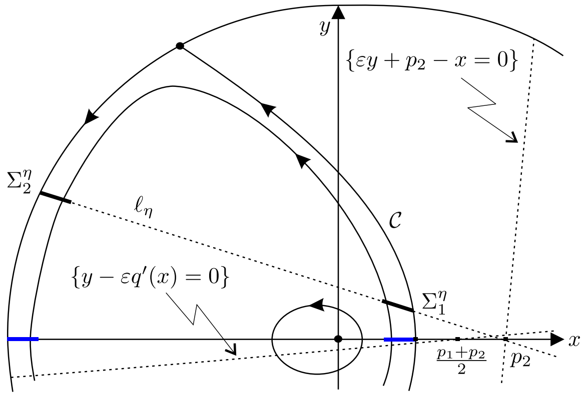


Fig. 10. Auxiliary transverse sections in the proof of Proposition 3.3.

$$q'(x) = 2ax - a(p_1 + p_2)x,$$

and that this straight line is mapped by $\phi_\varepsilon$ to

$$\mathscr{D}_\varepsilon(x_1, x_2) := 2a(1 - x_1 - x_2) + a^2\varepsilon^2(4x_1 + x_2^2) = 0.$$

We claim that $\phi_\varepsilon$ is an analytic map from

$$\Omega_\varepsilon := \{(x, y) \in \mathbb{R}^2 : p_2 - x + \varepsilon y > 0,\ y - \varepsilon q'(x) > 0\}$$

to $\mathscr{U}_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0,\ \mathscr{D}_\varepsilon(x_1, x_2) > 0\}$ with a well defined analytic inverse given by

$$\psi_\varepsilon(x_1, x_2) := \left(\frac{\kappa_1(\varepsilon\mathscr{D}_\varepsilon(x_1, x_2)^{\frac{1}{2}} - 1) + (p_2 - \varepsilon^2 a(p_1 + p_2))x_2}{(1 - 2a\varepsilon^2)x_2},\right.$$
$$\left.\frac{\kappa_1(\mathscr{D}_\varepsilon(x_1, x_2)^{\frac{1}{2}} + a\varepsilon(x_2 - 2))}{(1 - 2a\varepsilon^2)x_2}\right).$$

Indeed, the claim follows by checking that $\phi_\varepsilon \circ \psi_\varepsilon = \mathrm{Id}$ on $\{\mathscr{D}_\varepsilon(x_1, x_2) > 0, x_2 \neq 0\}$ and that $\psi_\varepsilon \circ \phi_\varepsilon = \mathrm{Id}$ on $\{\frac{y-\varepsilon q'(x)}{p_2-x+\varepsilon y} > 0\}$. To show the second identity we use that $(\mathscr{D}_\varepsilon \circ \phi_\varepsilon)(x, y) = \left(\frac{y-\varepsilon q'(x)}{p_2-x+\varepsilon y}\right)^2$. Consequently $(x_1, x_2) = \phi_\varepsilon(x, y)$ is an analytic global change of variables from $\Omega_\varepsilon$ to $\mathscr{U}_\varepsilon$ for all $\varepsilon$. In what follows we require $|\varepsilon| < \frac{1}{\sqrt{2a}}$ in order that the straight line $\{\varepsilon y + p_2 - x = 0\}$ does not intersect the left branch of the hyperbola $\mathcal{C} = \{\frac{1}{2}y^2 - q(x) = 0\}$, see Fig. 10.

Next we introduce a second auxiliary parameter $\eta \in \mathbb{R}$ and consider two additional transverse sections $\Sigma_1^\eta$ and $\Sigma_2^\eta$ laying on the straight line $\ell_\eta := \{y = \eta(p_2 - x)\}$, see Fig. 10. Observe that $\ell_\eta$ intersects the right branch of the hyperbola $\mathcal{C}$ at $(p_2, 0)$ for all $\eta$. We require $|\eta| < \sqrt{2a}$ additionally in order that $\ell_\eta$ intersects the hyperbola at a point $(x_\eta, y_\eta)$ in the left branch. Then we parametrize $\Sigma_1^\eta$ and $\Sigma_2^\eta$, respectively, by

$$s \mapsto \big(x_\eta - s, \eta(p_2 - x_\eta + s)\big) \text{ and } s \mapsto \big(-1/s, \eta(p_2 + 1/s)\big), \tag{31}$$

for $s > 0$ small enough. We also require $\varepsilon(\eta^2 + 2a) + 2\eta > 0$ so that $(x_\eta, y_\eta) \in \Omega_\varepsilon$. Summing up, the admissible conditions

$$|\varepsilon| < \frac{1}{\sqrt{2a}},\ |\eta| < \sqrt{2a} \text{ and } \varepsilon(\eta^2 + 2a) + 2\eta > 0$$

guarantee that any solution of $X_\nu$ going from $\Sigma_1^\eta$ to $\Sigma_2^\eta$ is inside the domain $\Omega_\varepsilon$ of the coordinate change $(x_1, x_2) = \phi_\varepsilon(x, y)$. Thus, setting $\bar{\nu} := (\nu, \varepsilon, \eta)$, we will work on the extended parameter space

$$\bar{W} := \left\{\bar{\nu} \in \mathbb{R}^4 : F + D > 0, D < 0, F > 1, |\varepsilon| < \tfrac{1}{\sqrt{2a}},\ |\eta| < \sqrt{2a} \text{ and } \varepsilon(\eta^2 + 2a) + 2\eta > 0\right\}.$$

Clearly the sets $\{(\nu, \varepsilon, 0) : \nu \in W \text{ and } \varepsilon \in (0, \frac{1}{\sqrt{2a}})\}$ and $\{(\nu, 0, \eta) : \nu \in W \text{ and } \eta \in (0, \sqrt{2a})\}$ are inside $\bar{W}$, that will be crucial in the forthcoming steps.

At this point we define $\sigma_1(\,\cdot\,; \bar{\nu})$ and $\sigma_2(\,\cdot\,; \bar{\nu})$ to be, respectively, the composition with $\phi_\varepsilon$ of the parametrization of $\Sigma_1^\eta$ and $\Sigma_2^\eta$ given in (31). In its regard one can check that

$$\sigma_1(s; \bar{\nu})|_{\varepsilon=0} = \left(\frac{s(1-\kappa_2^2\eta^2)^2}{\kappa_1 + s(1-\kappa_2^2\eta^2)}, \frac{\kappa_1(1-\kappa_2^2\eta^2)}{\kappa_1 + s(1-\kappa_2^2\eta^2)}\right) \tag{32}$$

and

$$\sigma_2(s; \bar{\nu})|_{\varepsilon=0} = \left(\frac{1+p_1 s}{1+p_2 s} - \kappa_2^2\eta^2, \frac{\kappa_1 s}{1+p_2 s}\right). \tag{33}$$

One can also verify that the coordinate change $(x_1, x_2) = \phi_\varepsilon(x, y)$ brings the vector field $X_\nu$ in (2) to

$$\bar{X}_{\bar{\nu}}(x_1, x_2) = \frac{1}{x_2}\big(x_1 P_1(x_1, x_2; \bar{\nu})\partial_{x_1} + x_2 P_2(x_1, x_2; \bar{\nu})\partial_{x_2}\big),$$

where $P_1$ and $P_2$ analytic functions on $\{(x_1, x_2, \bar{\nu}) \in \mathbb{R}^2 \times \bar{W} : \mathscr{D}_\varepsilon(x_1, x_2) > 0\}$. The hyperbolicity ratio of the saddle at the origin is

$$\lambda = -\frac{P_2(0,0)}{P_1(0,0)} = \frac{1}{2(F-1)}.$$

Moreover $P_1|_{\varepsilon=0} = R\bar{P}_1$ and $P_2|_{\varepsilon=0} = R\bar{P}_2$ where $R(x_1, x_2) = \frac{1}{\kappa_2}\sqrt{1 - x_1 - x_2}$,

$$\bar{P}_1(x_1, x_2) = 2\kappa_1(F-1) + 2(p_2 - 1)x_2 \text{ and } \bar{P}_2(x_1, x_2) = -\kappa_1 + (p_2 - 1)x_2. \tag{34}$$

(It will be clear in a moment the reason why it suffices to restrict to $\varepsilon = 0$.) For each $\bar{\nu} \in \bar{W}$, we define $\bar{T}(s; \bar{\nu})$ to be the Dulac time of $\bar{X}_{\bar{\nu}}$ between the transverse sections $\phi_\varepsilon(\Sigma_1^\eta)$ and $\phi_\varepsilon(\Sigma_2^\eta)$ parametrized by $\sigma_1(\,\cdot\,; \bar{\nu})$ and $\sigma_2(\,\cdot\,; \bar{\nu})$, respectively. We point out that, by construction, $\bar{T}(s; \bar{\nu})$ does not depend on $\varepsilon$ and that, furthermore, $\bar{T}(s; \bar{\nu})\big|_{\eta=0} = T(s; \nu)$.

Next we will apply Theorem A.3 to obtain the asymptotic expansion of $\bar{T}(s;\bar{\nu})$ at $s=0$. Note to this end that, by construction, given any $\bar{\nu}_0 \in \bar{W}$ there exists a relatively compact neighborhood $\mathcal{V}_0$ of

$$\big\{(x_1,0): x_1 \in [0,\bar{\sigma}_{21}(0;\bar{\nu}_0)]\big\} \cup \big\{(0,x_2): x_2 \in [0,\bar{\sigma}_{12}(0;\bar{\nu}_0)]\big\}$$

in $\mathbb{R}^2$ and a neighborhood $\bar{W}_0$ of $\bar{\nu}_0$ in $\bar{W}$ such that $\phi_\varepsilon\big(\Sigma_1^\eta \cup \Sigma_2^\eta\big) \subset \mathcal{V}_0$ for all $\bar{\nu} \in \bar{W}_0$ and

$$\mathcal{V}_0 \times \bar{W}_0 \subset \{(x_1,x_2,\bar{\nu}) \in \mathbb{R}^2 \times \bar{W} : \mathscr{D}_\varepsilon(x_1,x_2) > 0\}.$$

Here we use (see also Fig. 10) that $\phi_\varepsilon$ maps the straight line $\{y-\varepsilon q'(x)=0\}$ to $\{\mathscr{D}_\varepsilon(x_1,x_2)=0\}$. The above inclusion guarantees that $P_1$ and $P_2$ are analytic on $\mathcal{V}_0 \times \bar{W}_0$, so that we can apply Theorem A.3 to study the Dulac time of $\bar{X}_{\bar{\nu}}$ for $\bar{\nu} \approx \bar{\nu}_0$. Accordingly, with this aim, let us fix any $\bar{\nu}_0 = (D_0,F_0,\varepsilon_0,\eta_0) \in \bar{W}$ with $F_0 \in (1,\frac{3}{2}) \cup \{2\}$. Observe that $\lambda(\bar{\nu}_0) > 2$ if $F_0 \in (1,\frac{5}{4})$, $\lambda(\bar{\nu}_0) \in (1,2)$ if $F_0 \in (\frac{5}{4},\frac{3}{2})$, $\lambda(\bar{\nu}_0)=2$ if $F_0=\frac{5}{4}$ and $\lambda(\bar{\nu}_0)=\frac{1}{2}$ if $F_0=2$. Then, by applying (2), (1), (5) and (3) in Theorem A.3, respectively, we can assert that

$(a')$ $\bar{T}(s;\bar{\nu}) = \bar{T}_{00}(\bar{\nu}) + \bar{T}_{10}(\bar{\nu})s + \bar{T}_{20}(\bar{\nu})s^2 + \mathcal{F}^\infty_{L_0-\upsilon}(\bar{\nu}_0)$ if $F_0 \in (1,\frac{5}{4})$, where $L_0 = \min(3, \lambda(\bar{\nu}_0))$,

$(b')$ $\bar{T}(s;\bar{\nu}) = \bar{T}_{00}(\bar{\nu}) + \bar{T}_{10}(\bar{\nu})s + \bar{T}_{01}(\bar{\nu})s^\lambda + \bar{T}_{20}(\bar{\nu})s^2 + \mathcal{F}^\infty_{L_0-\upsilon}(\bar{\nu}_0)$ if $F_0 \in (\frac{5}{4},\frac{3}{2})$, where $L_0 = \lambda(\bar{\nu}_0)+1$,

$(c')$ $\bar{T}(s;\bar{\nu}) = \bar{T}_{00}(\bar{\nu}) + \bar{T}_{10}(\bar{\nu})s + \bar{T}^2_{201}(\bar{\nu})s^2\omega_{2-\lambda}(s) + \bar{T}^2_{200}(\bar{\nu})s^2 + \mathcal{F}^\infty_{3-\upsilon}(\bar{\nu}_0)$ if $F_0=\frac{5}{4}$, where $\bar{T}^2_{201}(\bar{\nu})$ and $\bar{T}^2_{200}(\bar{\nu})$ are smooth in a neighborhood of $\{\bar{\nu} \in \bar{W} : \lambda(\bar{\nu})=2\}$ and, moreover,

$$\bar{T}^2_{201}(\bar{\nu}) = (2-\lambda)\bar{T}_{01}(\bar{\nu}) \text{ and } \bar{T}^2_{200}(\bar{\nu}) = \bar{T}_{20}(\bar{\nu}) + \bar{T}_{01}(\bar{\nu}) \text{ for } \lambda(\bar{\nu}) \neq 2,$$

$(d')$ $\bar{T}(s;\bar{\nu}) = \bar{T}_{00}(\bar{\nu}) + \bar{T}_{01}(\bar{\nu})s^\lambda + \bar{T}^{\frac{1}{2}}_{101}(\bar{\nu})s\omega_{1-2\lambda}(s) + \bar{T}^{\frac{1}{2}}_{100}(\bar{\nu})s + \mathcal{F}^\infty_{3/2-\upsilon}(\bar{\nu}_0)$ if $F_0=2$, where $\bar{T}^1_{101}(\bar{\nu})$ and $\bar{T}^1_{100}(\bar{\nu})$ are smooth in a neighborhood of $\{\bar{\nu} \in \bar{W} : \lambda(\bar{\nu})=1/2\}$ and, moreover,

$$\bar{T}^{\frac{1}{2}}_{101}(\bar{\nu}) = (1-2\lambda)\bar{T}_{02}(\bar{\nu}) \text{ and } \bar{T}^{\frac{1}{2}}_{100}(\bar{\nu}) = \bar{T}_{10}(\bar{\nu}) + \bar{T}_{02}(\bar{\nu}) \text{ for } \lambda(\bar{\nu}) \neq 1/2.$$

Here $\upsilon$ is a small enough positive number depending on $\bar{\nu}_0$. Furthermore by applying locally Theorem A.3 we know that the coefficients $\bar{T}_{ij}(\bar{\nu})$ are meromorphic functions on $\bar{W}$ with poles only at those $F>1$ such that $\lambda(\bar{\nu}) = \frac{1}{2(F-1)} \in D_{ij}$, where $D_{00}=\emptyset$, $D_{10}=\frac{1}{\mathbb{N}}$, $D_{01}=\mathbb{N}$, $D_{20}=\frac{2}{\mathbb{N}}$ and $D_{02}=\frac{\mathbb{N}}{2}$.

We claim that the coefficients $\bar{T}_{ij}(\bar{\nu})$ do not depend on $\varepsilon$. To prove this observes that the Dulac time $\bar{T}(s;\bar{\nu})$ does not depend on $\varepsilon$ as long as $\bar{\nu} \in \bar{W}$. Accordingly $\partial_\varepsilon \bar{T}(s;\bar{\nu}) \equiv 0$. Thus from $(b')$ we get that, for each fixed $\bar{\nu}_\star \in \bar{W} \cap \{\frac{5}{4} < F < \frac{3}{2}\}$,

$$\partial_\varepsilon \bar{T}_{00}(\bar{\nu}_\star) + \partial_\varepsilon \bar{T}_{10}(\bar{\nu}_\star)s + \partial_\varepsilon \bar{T}_{01}(\bar{\nu}_\star)s^{\lambda(\bar{\nu}_\star)} + \partial_\varepsilon \bar{T}_{20}(\bar{\nu}_\star)s^2 + \mathrm{o}(s^{L_0-2\upsilon}) = 0 \text{ for all } s>0,$$

where $L_\star = \lambda(\bar{\nu}_\star)+1$ and we use that the flatness order of the remainder is preserved when derived with respect to parameters (see Definition A.2). Then, since $1<\lambda<2$ for $F \in (\frac{5}{4},\frac{3}{2})$ and we can choose $\upsilon>0$ arbitrary small (depending on $\bar{\nu}_\star$), the application of Remark A.6 shows that

$$\partial_\varepsilon \bar{T}_{00}(\bar{\nu}_\star)=0,\ \partial_\varepsilon \bar{T}_{10}(\bar{\nu}_\star)=0,\ \partial_\varepsilon \bar{T}_{01}(\bar{\nu}_\star)=0 \text{ and } \partial_\varepsilon \bar{T}_{20}(\bar{\nu}_\star)=0.$$

Since $\bar{W}\cap\{\frac{5}{4}<F<\frac{3}{2}\}$ is open and the coefficients are meromorphic on $\bar{W}$, by Lemma B.1 it follows that $\partial_\varepsilon \bar{T}_{00}$, $\partial_\varepsilon \bar{T}_{10}$, $\partial_\varepsilon \bar{T}_{01}$ and $\partial_\varepsilon \bar{T}_{20}$ are identically zero. The claim for $\partial_\varepsilon \bar{T}_{02}$ follows verbatim.

Thanks to the claim and the fact that $\bar{T}(s;\bar{\nu})\big|_{\eta=0}=T(s;\nu)$ by construction, the assertions in $(a)$–$(d)$ concerning the asymptotic expansion of $T(s;\nu)$ at $s=0$ follow from $(a')$–$(d')$, respectively, by setting

$$T_{ij}(\nu):=\bar{T}_{ij}(\bar{\nu})\big|_{\eta=0}.$$

We proceed next with the computation of these coefficients and for this purpose the idea is that if $\nu\in W$ and $\varepsilon>0$ then

$$T_{ij}(\nu)=\bar{T}_{ij}(\nu,\varepsilon,0)=\lim_{\eta\to0^+}\bar{T}_{ij}(\nu,\varepsilon,\eta)=\lim_{\eta\to0^+}\bar{T}_{ij}(\nu,0,\eta), \tag{35}$$

where in the second equality we use the continuity of $\bar{T}_{ij}(\bar{\nu})$ at any $\bar{\nu}_0\in\bar{W}$ with $\lambda(\bar{\nu}_0)=\frac{1}{2(F_0-1)}\notin D_{ij}$ and in the last one the fact that $\bar{T}_{ij}(\bar{\nu})$ does not depend on $\varepsilon$. Hence our first goal is to obtain $\bar{T}_{ij}(\bar{\nu})$ for $\varepsilon=0$ and to this end we shall apply Theorem A.4. (We point out that from now on all the computations are performed taking $\varepsilon=0$ and $\eta>0$.) In doing so, and setting

$$\kappa_0:=\frac{p_2-1}{p_2-p_1}$$

for shortness, from (25) we obtain that $L_1(u)=(1-\kappa_0u)^{2F}$, $L_2(u)\equiv1$, $M_1(u)\equiv0$ and

$$\begin{aligned} A_1(u)&=-\frac{\kappa_2}{\kappa_1}(1-\kappa_0u)^{-1}(1-u)^{-\frac12} & A_2(u)&=\frac{\kappa_2}{2\kappa_1(F-1)}(1-u)^{-\frac12}\\ B_1(u)&=-\frac{\kappa_2}{2\kappa_1}(1-\kappa_0u)^{2F-1}(1-u)^{-\frac32} & C_1(u)&=-\frac{3\kappa_2}{4\kappa_1}(1-\kappa_0u)^{4F-1}(1-u)^{-\frac52}. \end{aligned} \tag{36}$$

From (32) and (33), the necessary information with regard to the transverse sections is the following:

$$\sigma_{120}=\sigma_{210}=1-\kappa_2^2\eta^2,\ \ \sigma_{111}=-\sigma_{121}=\tfrac{(1-\kappa_2^2\eta^2)^2}{\kappa_1} \text{ and } \sigma_{122}=-\sigma_{112}=\tfrac{2(1-\kappa_2^2\eta^2)^3}{\kappa_1^2}. \tag{37}$$

Taking this into account we obtain that

$$\begin{aligned} T_{00}(\nu)&=\lim_{\eta\to0^+}\bar{T}_{00}(\nu,0,\eta)=-\lim_{\eta\to0^+}(1-\kappa_2^2\eta^2)\hat{A}_1(-1,1-\kappa_2^2\eta^2)\\ &=\tfrac{\kappa_2}{\kappa_1}\,\mathrm{B}(1,\tfrac12)\,{}_2F_1(1,1;\tfrac32;\kappa_0)=\tfrac{2\kappa_2}{\kappa_1(1-\kappa_0)}\,{}_2F_1(1,\tfrac12;\tfrac32;\tfrac{\kappa_0}{\kappa_0-1}) \end{aligned}$$

where in the second equality we use Theorem A.4, in the third one we apply $(b)$ in Proposition B.3 taking $\{\alpha=-1,\gamma=1,\delta=\frac12,x=\kappa_0\}$ and in the last one [1, §15.3]. Since $\frac{\kappa_0}{\kappa_0-1}=\frac{1-p_2}{1-p_1}$, this shows the validity of the expression for $T_{00}$ given in the statement. Similarly, from (36) and (37) again,

$$T_{01}(\nu) = \lim_{\eta\to 0^+} \bar{T}_{01}(\nu,0,\eta)$$
$$= \tfrac{\kappa_1^{-\lambda}}{(1-\kappa_0)^{2\lambda F}} \lim_{\eta\to 0^+} \hat{A}_2(\lambda, 1-k_2^2\eta^2) = \tfrac{\kappa_2(1-\kappa_0)^{-2\lambda F}}{2\kappa_1^{\lambda+1}(F-1)} \mathrm{B}(-\lambda, \tfrac{1}{2}).$$

Here the last equality follows by applying (*b*) in Proposition B.3 taking $\{\alpha=\lambda, \delta=\frac{1}{2}, x=0\}$, so that $\alpha=\lambda(\nu)=\frac{1}{2(F-1)}\notin\mathbb{Z}_{\geq 0}$, and noting that ${}_2F_1(a,b;c;0)=1$, see (47). Since $\rho_1(\nu) := \frac{\kappa_2(1-\kappa_0)^{-2\lambda F}}{2\kappa_1^{\lambda+1}(F-1)}$ is an analytic positive function on $W$, this proves the expression for $T_{01}$ given in the statement.

Let us study next the coefficient $T_{10}$. For this purpose we apply Theorem A.4, which on account of (37) shows that if $\eta>0$ and $\lambda(\nu)\notin D_{10}=\frac{1}{\mathbb{N}}$ then

$$\bar{T}_{10}(\nu,0,\eta) = \frac{1}{\kappa_1}\Bigg(\frac{-1+\kappa_2^2\eta^2}{\eta\,(\kappa_1+(p_2-1)(1-\kappa_2^2\eta^2))} - \frac{(1-\kappa_2^2\eta^2)^3}{(1-\kappa_0(1-\kappa_2^2\eta^2))^{2F}}\,\hat{B}_1\left(1/\lambda-1, 1-\kappa_2^2\eta^2\right)\Bigg). \tag{38}$$

Following the notation in Proposition B.3, see (36), we can write $B_1(1-\kappa_2^2\eta^2) = -\frac{\kappa_2}{2\kappa_1} g(y;\delta,\gamma;x)$ with $\{y=1-\kappa_2^2\eta^2, \delta=-\frac{1}{2}, \gamma=2F, x=\kappa_0\}$ but we cannot apply it to get the limit of $\hat{B}_1\left(1/\lambda-1, 1-\kappa_2^2\eta^2\right)$ as $\eta\to 0^+$ because the condition $\delta>0$ is not satisfied. As a matter of fact this is coherent because, since the first summand in (38) is divergent as $\eta\to 0^+$, it happens that $\lim_{\eta\to 0^+}\hat{B}_1\left(1/\lambda-1, 1-\kappa_2^2\eta^2\right)$ diverges as well. Hence the approach to compute this coefficient hast to be different. The idea is to take advantage of [23, Theorem 3.6], which shows that if $F\in(1,\frac{3}{2})$ then $T_{10}(\nu)=\rho_2(\nu)\bar{\rho}_2(\nu)$ where

$$\rho_2(\nu) := \frac{\kappa_2}{2\kappa_1(1-p_1)} \text{ and } \bar{\rho}_2(\nu) := -2 + \int_0^1 \left((1-s)^{-\frac{1}{\lambda}}(1-\bar{\kappa}s)^{1+\frac{1}{\lambda}} - 1\right) s^{-\frac{1}{2}}\frac{ds}{s}$$

with $\bar{\kappa} := \frac{1-p_2}{1-p_1}$. By applying assertion (*b*) in Theorem A.5 taking $f(s;\nu)=(1-s)^{-\frac{1}{\lambda}}(1-\bar{\kappa}s)^{1+\frac{1}{\lambda}}$, $k=1$ and $\alpha=\frac{1}{2}$ we can assert that $\bar{\rho}_2(\nu)=\lim_{s\to 1^-}\hat{f}(\frac{1}{2},s;\nu)$. Observe on the other hand that, following the notation in Proposition B.3, we can write $f(s;\nu)=g(y;\delta,\gamma;x)$ with $\{y=s, \delta=1-\frac{1}{\lambda}, \gamma=-1-\frac{1}{\lambda}, x=\bar{\kappa}\}$. Thus, since one can verify that $\delta=1-\frac{1}{\lambda}>0$ and $x=\bar{\kappa}<1$ for all $\nu\in W\cap\{1<F<\frac{3}{2}\}$, the application of assertion (*b*) in that result gives

$$\bar{\rho}_2(\nu) = \lim_{y\to 1^-}\hat{f}(\tfrac{1}{2},y;\nu) = \mathrm{B}\left(-\tfrac{1}{2}, 1-\tfrac{1}{\lambda}\right){}_2F_1\left(-1-\tfrac{1}{\lambda}, -\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{\lambda}; \bar{\kappa}\right).$$

Due to $\mathrm{B}(-\frac{1}{2}, 1-\frac{1}{\lambda}) = \mathrm{B}(1-\frac{1}{\lambda}, -\frac{1}{2})$, see (46), this proves the validity of the expression for $T_{10}(\nu)$ in the statement for all $\nu\in W\cap\{1<F<\frac{3}{2}\}$. Accordingly, since $T_{10}$ is meromorphic on $W$, this is also the case of $\bar{\rho}_2$ thanks to Lemma B.2, and $\rho_2$ is analytic on $W$, the application of the real version of Lemma B.1 implies the validity of the equality on $W$. Observe moreover that $\rho_2$ is positive on $W$.

We proceed with the computation of the coefficient $T_{20}$. In first instance, for the sake of convenience we shall work with $\nu \in W \cap \{1 < F < \frac{5}{4}\}$, so that $\lambda(\nu) > 2$. Due to $M_1 \equiv 0$, Theorem A.4 shows that if $\lambda(\nu) \notin D_{20} = \frac{2}{\mathbb{N}}$ then

$$\begin{aligned}\bar{T}_{20}(\nu,0,\eta) = &-\frac{\sigma_{120}\sigma_{122}}{2\sigma_{120}P_2(0,\sigma_{120})} - \frac{1}{2}\sigma_{121}^2\partial_2 P_2^{-1}(0,\sigma_{120}) - \sigma_{121}\sigma_{111}\partial_1 P_2^{-1}(0,\sigma_{120}) \\ &- \frac{\sigma_{120}\sigma_{111}^2}{2L_1^2(\sigma_{120})}\hat{C}_1(2/\lambda - 1,\sigma_{120}) \\ &- \left(\frac{\sigma_{112}}{2\sigma_{111}} - \frac{\sigma_{121}}{\sigma_{120}}\left(\frac{P_1}{P_2}\right)(0,\sigma_{120})\right)\frac{\sigma_{120}\sigma_{111}}{L_1(\sigma_{120})}\hat{B}_1(1/\lambda - 1,\sigma_{120}).\end{aligned} \tag{39}$$

Since $P_2|_{\varepsilon=0} = R\bar{P}_2$, $R(0,\sigma_{120}) = \frac{1}{\kappa_2}\sqrt{1-x_2}\big|_{x_2=1-\kappa_2^2\eta^2} = \eta$ and $\bar{P}_2(0,1) = p_1 - 1 \neq 0$, see (34) and (37), it follows that we can write

$$-\frac{\sigma_{120}\sigma_{122}}{2\sigma_{120}P_2(0,\sigma_{120})} - \frac{1}{2}\sigma_{121}^2\partial_2 P_2^{-1}(0,\sigma_{120}) - \sigma_{121}\sigma_{111}\partial_1 P_2^{-1}(0,\sigma_{120}) = \frac{\phi_1(\eta^2)}{\eta^3} \tag{40}$$

and

$$-\frac{\sigma_{112}}{2\sigma_{111}} + \frac{\sigma_{121}}{\sigma_{120}}\left(\frac{P_1}{P_2}\right)(0,\sigma_{120}) = \phi_2(\eta^2), \tag{41}$$

where here (and in what follows) $\phi_i(x)$ stands for an analytic function at $x = 0$. (In fact $\phi_i$ depends also on $\nu$ and this dependence is analytic on $W$. We omit this dependence for brevity when there is no risk of confusion.) Following this notation, from (38) and using that $\lim_{\eta\to 0^+} L_1(1-\kappa_2^2\eta^2) = (1-\kappa_0)^{2F} \neq 0$, we can assert that

$$\frac{\sigma_{120}\sigma_{111}}{L_1(\sigma_{120})}\hat{B}_1(1/\lambda - 1,\sigma_{120}) = \phi_3(\eta^2)\bar{T}_{10}(\nu,0,\eta) + \frac{\phi_4(\eta^2)}{\eta} \tag{42}$$

as long as $\lambda(\nu) \notin D_{10} = \frac{1}{\mathbb{N}}$. On the other hand, since $\lambda(\nu) > 2$ for all $\nu \in W \cap \{1 < F < \frac{5}{4}\}$, by applying assertion (*b*) in Theorem A.5 with $k = 0$ and $\alpha = \frac{2}{\lambda} - 1$, we get

$$\begin{aligned}\sigma_{120}^{1-\frac{2}{\lambda}}\hat{C}_1(2/\lambda - 1,\sigma_{120}) &= \int_0^{\sigma_{120}} C_1(u)u^{1-\frac{2}{\lambda}}\frac{du}{u} = \int_0^{1-\kappa_2^2\eta^2} C_1(u)u^{-\frac{2}{\lambda}}du \\ &= -\frac{3\kappa_2}{4\kappa_1}(1-\kappa_0)^{4F-1}\underbrace{\int_{\kappa_2^2\eta^2}^{1}(1-\bar{\kappa}x)^{4F-1}x^{-\frac{5}{2}}(1-x)^{-\frac{2}{\lambda}}dx}_{I},\end{aligned}$$

where in the second equality we perform the change of variable $u = 1 - x$ and use that $\frac{\kappa_0}{\kappa_0 - 1} = \frac{1-p_2}{1-p_1} = \bar{\kappa}$. Next we split the above integral as $I = I_1 + I_2$ with

$$I_1 := \int_{\bar{\kappa}_2^2\eta^2}^{1} \Big((1-\bar{\kappa}x)^{4F-1}(1-x)^{-\frac{2}{\lambda}} - (1+\beta x)\Big)x^{-\frac{3}{2}}\frac{dx}{x} \text{ and } I_2 := \int_{\bar{\kappa}_2^2\eta^2}^{1} (1+\beta x)x^{-\frac{3}{2}}\frac{dx}{x},$$

where we take $\beta := \frac{2}{\lambda} - (4F-1)\bar{\kappa}$ so that $I_1$ converges as $\eta \to 0$. Due to $I_2 = -\frac{2}{3} - 2\beta + \frac{2}{3}\frac{1}{(\kappa_2\eta)^3} + \frac{2\beta}{\kappa_2\eta}$, we can write $I = J + \frac{2}{3}\frac{1}{(\kappa_2\eta)^3} + \frac{2\beta}{\kappa_2\eta}$ with

$$J(\nu,\eta) := \int_{\kappa_2^2\eta^2}^{1} \Big((1-\bar{\kappa}x)^{4F-1}(1-x)^{-\frac{2}{\lambda}} - (1+\beta x)\Big)x^{-\frac{3}{2}}\frac{dx}{x} - \frac{2}{3} - 2\beta.$$

Consequently we obtain that

$$\hat{C}_1(2/\lambda - 1, \sigma_{120}) = -\frac{3\kappa_2}{4\kappa_1}\frac{(1-\kappa_0)^{4F-1}}{(1-\kappa_2^2\eta^2)^{1-\frac{2}{\lambda}}}\left(J(\nu,\eta) + \frac{2}{3(\kappa_2\eta)^3} + \frac{2\beta}{\kappa_2\eta}\right). \tag{43}$$

On the other hand, setting $g(x;\nu) := (1-\bar{\kappa}x)^{4F-1}(1-x)^{-\frac{2}{\lambda}}$ we get that

$$\lim_{\eta\to 0} J(\nu,\eta) = \lim_{x\to 1^-} \hat{g}(3/2, x;\nu) = \mathrm{B}\left(1-\frac{2}{\lambda}, -\frac{3}{2}\right) {}_2F_1\left(1-4F, -\frac{3}{2}; -\frac{1}{2}-\frac{2}{\lambda}; \bar{\kappa}\right) =: J_0(\nu).$$

Here the first equality follows by $(b)$ in Theorem A.5 taking $\{\alpha = \frac{3}{2}, k = 2\}$ and the second one by $(b)$ in Proposition B.3 taking $\{\alpha = \frac{3}{2}, \gamma = 1-4F, \delta = 1-\frac{2}{\lambda}, x = \bar{\kappa}\}$ and using that $\delta > 0$ thanks to $\lambda(\nu) > 2$ for all $\nu \in W \cap \{1 < F < \frac{5}{4}\}$. That being said, substituting (40), (41), (42) and (43) into (39) and gathering next the analytic functions at $\eta = 0$ we obtain that

$$\bar{T}_{20}(\nu,0,\eta) = \phi_5(\nu,\eta^2)J(\nu,\eta) + \phi_6(\nu,\eta^2)\bar{T}_{10}(\nu,0,\eta) + \phi_7(\nu,\eta^2)/\eta^3,$$

with $\phi_i(\nu,x)$ analytic at $x = 0$. (Here we specify again the dependence on $\nu$ for the sake of consistency in the exposition.) Consequently, from (35),

$$T_{20}(\nu) = \lim_{\eta\to 0^+} \bar{T}_{20}(\nu,0,\eta) = \phi_5(\nu,0)J_0(\nu) + \phi_6(\nu,0)T_{10}(\nu),$$

where we use that $\phi_7(\nu,\eta^2) = \mathrm{O}(\eta^4)$ because the limit must be finite. One can easily check that

$$\rho_3(\nu) := \phi_5(\nu,0) = \frac{3\kappa_2}{8\kappa_1^3(1-\kappa_0)} \text{ and } \rho_4(\nu) := \phi_6(\nu,0) = \frac{p_1 - 1 + 2F\kappa_1}{\kappa_1(p_1-1)}.$$

This proves the validity of the expression for $T_{20}(\nu)$ for all $\nu \in W \cap \{1 < F < \frac{5}{4}\}$. Similarly as before, by applying Lemma B.1 this equality extends to $W$ since $\rho_3$ and $\rho_4$ are analytic on $W$

and, on the other hand, $J_0$ is meromorphic on $W$ by Lemma B.2 and so are $\bar{T}_{10}$ and $\bar{T}_{20}$. Finally an easy computation shows that $\rho_3$ and $\rho_4$ are positive on $W$.

So far we have proved the validity of the expression of the coefficients $T_{ij}(\nu)$ that we give in the first part of the statement. Moreover, since $T(s;\nu) = \bar{T}(s;\bar{\nu})|_{\eta=0}$ and $T_{ij}(\nu) = \bar{T}_{ij}(\bar{\nu})|_{\eta=0}$, the assertions with regard to the asymptotic expansion of the Dulac map at $s=0$ in $(a)$–$(d)$ follow, respectively, from $(a')$–$(d')$. The only remaining point concerns the behavior of the coefficients in each one of these cases. This is our final task, that we carry out case by case:

$(a)$ Let us consider any $\nu_0 = (D_0, F_0) \in W \cap \{1 < F < \frac{5}{4}\}$ such that $T_{10}(\nu_0) = 0$. We claim that then $D_0 \in (-1, -\frac{1}{2})$. Indeed, to prove this we first use Proposition 3.11 in [23], which shows that the set $\{\nu \in W : T_{10}(\nu) = 0 \text{ with } 1 < F < \frac{3}{2}\}$ is the graphic of an analytic function $D = \mathcal{G}(F)$ verifying $-F < \mathcal{G}(F) < -\frac{1}{2}$ and $\lim_{F\to 1^+}\mathcal{G}(F) = -\frac{1}{2}$. Therefore it is clear that the claim will follow once we prove that $T_{10}(-1, F) \neq 0$ for all $F \in (1, \frac{5}{4})$. In order to show this we note that $p_2|_{D=-1} = 1$ and, consequently,

$$T_{10}(-1,F) = \rho_2(\nu)\mathrm{B}\big(1-\tfrac{1}{\lambda}, -\tfrac{1}{2}\big)\, {}_2F_1\big(-1-\tfrac{1}{\lambda}, -\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{\lambda}; \tfrac{1-p_2}{1-p_1}\big)\Big|_{D=-1} \neq 0$$

because ${}_2F_1(a,b;c;0) = 1$ by definition and, on the other hand, $\lambda(\nu) = \frac{1}{2(1-F)} > 2$ for all $F \in (1, \frac{5}{4})$ and one can check that $\mathrm{B}\big(1-\frac{1}{\lambda}, -\frac{1}{2}\big) = \frac{\Gamma(1-\frac{1}{\lambda})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{\lambda})} \neq 0$ for all $\lambda > 2$. This proves the claim.

Recall at this point that

$$T_{20}(\nu) = \rho_3(\nu)\mathrm{B}\big(1-\tfrac{2}{\lambda}, -\tfrac{3}{2}\big)\, {}_2F_1\big(-\tfrac{2}{\lambda}-3, -\tfrac{3}{2}; -\tfrac{1}{2}-\tfrac{2}{\lambda}; \tfrac{1-p_2}{1-p_1}\big) + \rho_4(\nu)T_{10}(\nu). \tag{44}$$

Accordingly, since $T_{10}(\nu_0) = 0$ with $\nu_0 \in Q := W \cap \{1 < F < \frac{5}{4}\} \cap \{-1 < D < -\frac{1}{2}\}$ and $\rho_2$ and $\rho_3$ are positive functions, in order to prove that $T_{20}(\nu_0) \neq 0$ it suffices to show that the linear combination

$$\begin{aligned}&\mathrm{B}\big(1-\tfrac{2}{\lambda}, -\tfrac{3}{2}\big)\, {}_2F_1\big(-\tfrac{2}{\lambda}-3, -\tfrac{3}{2}; -\tfrac{1}{2}-\tfrac{2}{\lambda}; \tfrac{1-p_2}{1-p_1}\big)\\ &\quad - 4\mathrm{B}\big(1-\tfrac{1}{\lambda}, -\tfrac{1}{2}\big)\, {}_2F_1\big(-1-\tfrac{1}{\lambda}, -\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{\lambda}; \tfrac{1-p_2}{1-p_1}\big)\end{aligned}$$

does not vanish on $Q$. Since one can easily verify that $\frac{2}{\lambda} \in (0,1)$ and $\frac{1-p_2}{1-p_1} \in (-1, 0)$ for all $\nu \in Q$, this follows directly by applying Proposition C.1 with $\{a = \frac{2}{\lambda}, b = -\frac{1-p_2}{1-p_1}\}$.

$(b)$ We already proved that $T_{01}(\nu) = \rho_1(\nu)\mathrm{B}\big(-\lambda, \frac{1}{2}\big)$ with $\rho_1$ an analytic positive function on $W$. The function $\mathrm{B}(-\lambda, \frac{1}{2}) = \frac{\Gamma(-\lambda)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\lambda)}$ vanishes only when $\frac{1}{2} - \lambda(\nu) \in \mathbb{Z}_{\leq 0}$ and, for $\nu \in W \cap \{\frac{5}{4} < F < \frac{3}{2}\}$, this occurs if and only if $\lambda(\nu) = \frac{3}{2}$, i.e., $F = \frac{4}{3}$. Recall moreover that, by Proposition 3.11 in [23], the set $\{\nu \in W : T_{10}(\nu) = 0 \text{ with } 1 < F < \frac{3}{2}\}$ is the graphic of an analytic function $D = \mathcal{G}(F)$ on $(1, \frac{3}{2})$. Consequently there exists a unique $\nu_\star = (D_\star, \frac{4}{3})$ inside $W \cap \{\frac{5}{4} < F < \frac{3}{2}\}$ such that $T_{10}(\nu_\star) = 0$ and one can prove that $D_\star \in (-1.15, -1.10)$. The gradients of $T_{10}$ and $T_{01}$ are linearly independent at $\nu_\star$ because $\partial_D T_{10}(\nu) \neq 0$ for all $\nu \in W \cap \{1 < F < \frac{3}{2}\}$ by $(a)$ of Lemma 3.13 in [23] and, on the other hand, $\partial_D T_{01}(\nu_\star) =$

$\mathrm{B}\big(-\lambda,\frac{1}{2}\big)|_{\nu=\nu_\star}\partial_D\rho_1(\nu_\star)=0$ and $\partial_F T_{10}(\nu_\star)\neq 0$ since the gamma function has simple poles at $\mathbb{Z}_{\leq 0}$ with non-zero residue. Finally the fact that $T_{20}(\nu_\star)<0$ follows noting that, from (44) and $\lambda(\nu_\star)=\frac{3}{2}$, we get

$$T_{20}(\nu_\star)=\rho_3(\nu_\star)\,\mathrm{B}\big(-\tfrac{1}{3},-\tfrac{3}{2}\big){}_2F_1\big(-\tfrac{13}{3},-\tfrac{3}{2};-\tfrac{11}{6};\tfrac{1-p_2}{1-p_1}\big)\Big|_{\nu=\nu_\star},$$

which is negative because one can easily check that $D\mapsto {}_2F_1\big(-\frac{13}{3},-\frac{3}{2};-\frac{11}{6};\frac{1-p_2}{1-p_1}\big)|_{F=\frac{4}{3}}$ is positive on $(-1.15,-1.10)$ and $\mathrm{B}\big(-\frac{1}{3},-\frac{3}{2}\big)\approx -2.6$.

$(c)$ Let us fix any $\nu_0\in W\cap\{F=\frac{5}{4}\}$, so that $\lambda(\nu_0)=2$. Then, from $(c')$, $T^2_{201}(\nu):=\bar{T}^2_{201}(\bar{\nu})|_{\eta=0}$ and $T^2_{200}(\nu):=\bar{T}^2_{200}(\bar{\nu})|_{\eta=0}$ are smooth functions in a neighborhood of $\{\nu\in W:\lambda(\nu)=2\}$ and, in addition,

$$T^2_{201}(\nu)=\big(2-\lambda(\nu)\big)T_{01}(\nu)\text{ and }T^2_{200}(\nu)=T_{20}(\nu)+T_{01}(\nu)\text{ for }\lambda(\nu)\neq 2.$$

The functions $T_{01}(\nu)$ and $T_{20}(\nu)$ are meromorphic with a pole at those $\nu$ such that $\lambda(\nu)=2\in D_{01}\cap D_{20}$, and the pole is simple in both cases by Propositions 3.2 and 3.6 in [31], respectively. Therefore by the Weierstrass Division Theorem (or, more directly, by [31, Lemma 2.8]) it follows that $T^2_{201}(\nu)$ and $T^2_{200}(\nu)$ are analytic in a neighborhood of $W\cap\{F=\frac{5}{4}\}$. Furthermore, by [23, Lemma 3.13] once again, $T_{10}(D,\frac{5}{4})=0$ if and only if $D=-1$. Finally, since $\lambda(-1,\frac{5}{4})=2$, $T_{01}(\nu)=\rho_1(\nu)\mathrm{B}(-\lambda,\frac{1}{2})$ and $\mathrm{B}(-\lambda,\frac{1}{2})=\frac{\Gamma(-\lambda)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\lambda)}$, we have that

$$T^2_{201}(-1,5/4)=\lim_{\nu\to(-1,\frac{5}{4})}\big(2-\lambda(\nu)\big)T_{01}(\nu)=\rho_1(-1,5/4)\frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{3}{2})}\lim_{\lambda\to 2}(2-\lambda)\Gamma(-\lambda)\neq 0,$$

because the gamma function has simple poles with non-zero residues at $\mathbb{Z}_{\leq 0}$.

$(d)$ Consider finally any $\nu_0\in W\cap\{F=2\}$, so that $\lambda(\nu_0)=\frac{1}{2}$. Since $T_{01}(\nu)=\rho_1(\nu)\mathrm{B}\big(-\lambda,\frac{1}{2}\big)$ with $\rho_1\neq 0$, $\lambda(\nu)=\frac{1}{2(F-1)}$ and $\mathrm{B}\big(-\lambda,\frac{1}{2}\big)=\frac{\Gamma(-\lambda)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\lambda)}$, there exists an analytic non-vanishing function $\ell_1$ in a neighborhood of $W\cap\{F=2\}$ such that $T_{01}(\nu)=(F-2)\ell_1(\nu)$. On the other hand, from $(d')$ and arguing as in the previous case, the functions $T^{\frac{1}{2}}_{101}(\nu):=\bar{T}^{\frac{1}{2}}_{101}(\bar{\nu})|_{\eta=0}$ and $T^{\frac{1}{2}}_{100}(\nu):=\bar{T}^{\frac{1}{2}}_{100}(\bar{\nu})|_{\eta=0}$ are analytic in a neighborhood of $\{\nu\in W:\lambda(\nu)=\frac{1}{2}\}$ and

$$T^{\frac{1}{2}}_{101}(\nu)=(1-2\lambda)T_{02}(\nu)\text{ and }T^{\frac{1}{2}}_{100}(\nu)=T_{10}(\nu)+T_{02}(\nu)\text{ for }\lambda(\nu)\neq 1/2.$$

In particular we have that the sum of residues of $T_{10}$ and $T_{02}$ along $\{\nu\in W:\lambda(\nu)=\frac{1}{2}\}=W\cap\{F=2\}$ is equal to zero, which saves us from computing the explicit value of $T_{02}$. Indeed, since $\lambda(\nu_0)=\frac{1}{2}$ and $\mathrm{B}(1-\frac{1}{\lambda},-\frac{1}{2})=\frac{\Gamma(1-\frac{1}{\lambda})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{\lambda})}$, we obtain that

$$T^{\frac{1}{2}}_{101}(\nu_0)=\lim_{\nu\to\nu_0}(1-2\lambda(\nu))T_{02}(\nu)=-\lim_{\nu\to\nu_0}(1-2\lambda(\nu))T_{10}(\nu)$$

$$= -\rho_2(\nu_0)\,{}_2F_1\big(-3,-\tfrac{1}{2};-\tfrac{3}{2};\tfrac{1-p_2}{1-p_1}\big|_{\nu=\nu_0}\big)\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{3}{2})}\lim_{\lambda\to\frac{1}{2}}(1-2\lambda)\Gamma(1-\tfrac{1}{\lambda})$$
$$= \tfrac{3}{4}\rho_2(\nu_0)\,{}_2F_1\big(-3,-\tfrac{1}{2};-\tfrac{3}{2};\tfrac{1-p_2}{1-p_1}\big|_{\nu=\nu_0}\big),$$

where in the second equality we use the expression of $T_{10}$ already proved and in the last one that $\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{3}{2})} = -\frac{3}{2}$ and $\lim_{x\to-1}(x+1)\Gamma(x) = -1$, see for instance [1, §15]. From the same reference we get that ${}_2F_1\big(-3,-\frac{1}{2};-\frac{3}{2};x\big) = (x+1)(x-1)^2$. Moreover one can verify that $D \mapsto \frac{1-p_2}{1-p_1}\big|_{F=2}$ maps diffeomorphically $(-2,0)$ to $(-\infty,1)$ and that it is equal to $-1$ at $D=-\frac{1}{2}$. Accordingly we can assert that $T_{101}^{\frac{1}{2}}(D,2) = (D+\frac{1}{2})\ell_2(D)$ where $\ell_2$ is a non-vanishing analytic function on $(-2,0)$ and, consequently, $\partial_D T_{101}^{\frac{1}{2}}(-\frac{1}{2},2) \neq 0$. Since $\partial_D T_{01}(-\frac{1}{2},2) = 0$ and $\partial_F T_{01}(-\frac{1}{2},2) = \ell_1(-\frac{1}{2},2) \neq 0$, this proves that the gradients of $T_{01}$ and $T_{101}^{\frac{1}{2}}$ are linearly independent at $\nu = (-\frac{1}{2},2)$ as desired.

This finishes the proof of the result. ■

## Appendix B. Beta and hypergeometric functions

In this appendix we are concerned with the integral representation of the Beta and hypergeometric functions (see [2]). The Beta integral is defined for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$ by

$$\mathrm{B}(z,w) := \int_0^1 t^{z-1}(1-t)^{w-1}dt = \int_0^{+\infty} t^{z-1}(1+t)^{-z-w}dt. \tag{45}$$

This function can be analytically extended for $z, w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ thanks to the identity

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{46}$$

where $\Gamma$ is the gamma function. Recall in this regard that $1/\Gamma(z)$ is an entire function with simple zeros at $z \in \mathbb{Z}_{\leq 0}$. On the other hand, if we consider $a, b, c \in \mathbb{C}$ with $c \notin \mathbb{Z}_{\leq 0}$ and $z$ inside the complex open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then Gauss hypergeometric function is defined by the power series

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{47}$$

where for a given $x \in \mathbb{C}$ we use the Pochhammer symbol $(x)_n := x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$.

In this section by a meromorphic function of several complex variables we mean a function that locally writes as a quotient of two holomorphic functions. Recall that a function $f : \Omega \longrightarrow \mathbb{C}$, where $\Omega$ is a connected open set of $\mathbb{C}^n$, is holomorphic if for each $z_0 \in \Omega$ there exists an open polydisc $D_r(z_0)$ such that $f$ can be written as an absolutely and uniformly convergent power series at $z_0$, i.e., $f(z) = \sum_\alpha a_\alpha (z - z_0)^\alpha$ for all $z \in D_r(z_0)$. On account of this one can readily obtain the following result about uniqueness of meromorphic continuation.

**Lemma B.1.** *Consider two functions $\phi$ and $\varphi$ that are meromorphic on a connected open set $\Omega \subset \mathbb{C}^n$. If there exists an open subset $V$ of $\Omega$ such that $\phi|_V = \varphi|_V$ then $\phi = \varphi$.*

Let us remark that the previous result is also true in the real setting, i.e., for functions in $\mathbb{R}^n$ that locally write as a quotient of real analytic functions. The following result is well-known but since we did not find its statement in its fullness we give it here for the sake of completeness.

**Lemma B.2.** *The function $(a,b,c,z) \mapsto \frac{{}_2F_1(a,b;c;z)}{\Gamma(c)}$ extends holomorphically to $\mathbb{C}^3 \times (\mathbb{C} \setminus [1,+\infty))$.*

**Proof.** Following [2, p. 65], we show first that the function extends holomorphically to $\mathbb{C}^3 \times \mathbb{D}$. To prove this claim we write

$$\frac{{}_2F_1(a,b;c;z)}{\Gamma(c)} = \sum_{n=0}^{\infty} f_n(a,b,c,z) \text{ with } f_n(a,b,c,z) := \frac{\Gamma(a+n)\Gamma(b+n)z^n}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(1+n)}.$$

Stirling's asymptotic formula $\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}$ as $\operatorname{Re}(z) \to +\infty$ (see [2, Theorem 1.4.1]) shows that

$$\left| \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} \right| \sim n^{\operatorname{Re}(a+b+c-1)} \text{ as } n \to +\infty.$$

Fix any compact set $K \subset \mathbb{C}^3 \times \mathbb{D}$ and suppose that $\operatorname{Re}(a+b+c-1) \leqslant m \in \mathbb{N}$ and $|z| \leqslant r < 1$ for all $(a,b,c,z) \in K$. Then, on account of the above asymptotic estimate and the fact that $1/\Gamma(z)$ is an entire function, there exists $C > 0$ such that $|f_n(a,b,c,z)| \leqslant C n^m r^n$ for all $(a,b,c,z) \in K$. By applying the Weierstrass M-test this proves that the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on compact sets of $\mathbb{C}^3 \times \mathbb{D}$. So the claim follows because the uniform limit of holomorphic functions is holomorphic (see [16, Proposition 2]).

Finally the result follows by Pfaff and Kummer's formulas (see [1, §15] or [2, Theorem 2.3.2]) relating the values of ${}_2F_1(a,b;c;\cdot)$ at $z$, $\frac{z}{z-1}$ and $\frac{1}{z}$, which enable to extend holomorphically to $\mathbb{C}^3 \times (\mathbb{C} \setminus [1,+\infty))$ the map $(a,b,c,z) \mapsto \frac{{}_2F_1(a,b;c;z)}{\Gamma(c)}$. (These formulas are usually proved under some restrictions on the parameters $a$ $b$ and $c$ but they are always satisfied thanks to the claim and Lemma B.1.) This concludes the proof of the result. ■

It is worth to mention that by Hartogs's theorem (see [14, §1.2]), a function of several complex variables is holomorphic if, and only if, it is holomorphic (in the classical one-variable sense) in each variable separately. The main concern in this section is Euler's integral representation of ${}_2F_1$, see for instance [2, Theorem 2.2.1], that is given by

$$ {}_2F_1(a,b;c;z) = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt \tag{48}$$

provided that $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $z \in \mathbb{C} \setminus [1, +\infty)$. Our goal is to use this formula to compute $\hat{f}(\alpha, x)$ (see Theorem A.5) for some specific functions $f(x)$. Next result is addressed to this problem.

**Proposition B.3.** *The following holds:*

(*a*) *Consider* $h(y;\delta;\kappa) = (1+\kappa y^2)^\delta$ *with* $\kappa > 0$. *Then, for any* $\delta \in \mathbb{R}$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ *such that* $2\delta < \alpha$,

$$\lim_{y\to+\infty} y^{-\alpha}\hat{h}(\alpha, y; \delta; \kappa) = \frac{\kappa^{\frac{\alpha}{2}}}{2}\mathrm{B}\left(-\frac{\alpha}{2}, -\delta + \frac{\alpha}{2}\right).$$

(*b*) *Consider* $g(y;\delta,\gamma;x) = (1-y)^{\delta-1}(1-xy)^{-\gamma}$ *with* $y \in (0,1)$ *and* $x < 1$. *Then, for any* $\delta > 0$, $\gamma \in \mathbb{R}$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$,

$$\lim_{y\to1^-} \hat{g}(\alpha, y; \delta, \gamma; x) = \mathrm{B}(-\alpha, \delta)\,{}_2F_1(\gamma, -\alpha; \delta-\alpha; x).$$

**Proof.** In order to prove (*a*) we define $\Omega := \{(\alpha, \delta, \kappa) \in \mathbb{R}^3 : 2\delta < \alpha \text{ and } \kappa > 0\}$, which is connected. Note then that we must show the validity of the identity on $\Omega \cap \{\alpha \notin \mathbb{Z}_{\geq 0}\}$. We will show first the identity on an open set of $V \subset \Omega$ and then extend it by using the real version of Lemma B.1. With this aim observe that if we work on $V := \Omega \cap \{\alpha < 0\}$ then the application of assertion (*b*) of Theorem A.5 with $k = 0$ yields

$$\lim_{y\to+\infty} y^{-\alpha}\hat{h}(\alpha, y; \delta, \kappa) = \int_0^{+\infty} (1+\kappa u^2)^\delta u^{-\alpha-1} du = \frac{\kappa^{\frac{\alpha}{2}}}{2}\int_0^{+\infty} (1+v)^\delta v^{-\frac{\alpha}{2}-1} dv$$
$$= \frac{\kappa^{\frac{\alpha}{2}}}{2}\mathrm{B}\left(-\frac{\alpha}{2}, -\delta + \frac{\alpha}{2}\right),$$

where in the second equality we perform the change of variable $v = \kappa u^2$ and in the third one we use (45). Note at this point that the function of the right hand side is meromorphic on $\Omega$ because $1/\Gamma$ is entire and $\mathrm{B}(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$. We claim that the function on the left hand side is also meromorphic on $\Omega$. To show this we work first on $\Omega \cap \{\alpha \notin \mathbb{Z}_{\geq 0}\}$ because in doing so we can apply assertion (*b*) of Theorem A.5 with any $k > \alpha$ to obtain

$$y^{-\alpha}\hat{h}(\alpha, y) = \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{i!(i-\alpha)} y^{i-\alpha} + \int_0^y \big(h(u) - T_0^{k-1}h(u)\big)\frac{du}{u^{\alpha+1}}$$
$$= \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{i!(i-\alpha)} y^{i-\alpha} + \int_0^1 \big(h(u) - T_0^{k-1}h(u)\big)\frac{du}{u^{\alpha+1}} + \int_1^y h(u)\frac{du}{u^{\alpha+1}}$$
$$- \int_1^y T_0^{k-1}h(u)\frac{du}{u^{\alpha+1}}$$

$$= \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{i!(i-\alpha)} + \int_0^1 \left(h(u) - T_0^{k-1}h(u)\right)\frac{du}{u^{\alpha+1}} + \int_1^y h(u)\frac{du}{u^{\alpha+1}}$$

$$= \hat{h}(\alpha, 1; \delta, \kappa) + \int_1^y (1+\kappa u^2)^{\delta} u^{-\alpha-1} du.$$

Here we denote $\partial_y^i h(y;\delta,\kappa) = h^{(i)}(y)$ for shortness. Consequently, since $2\delta < \alpha$,

$$\lim_{y\to+\infty} y^{-\alpha}\hat{h}(\alpha, y; \delta, \kappa) = \hat{h}(\alpha, 1; \delta, \kappa) + \int_1^{+\infty} (1+\kappa u^2)^{\delta} u^{-\alpha-1} du$$

$$= \hat{h}(\alpha, 1; \delta, \kappa) + \kappa^{\delta}\int_0^1 (1+\kappa^{-1}v^2)^{\delta} v^{\alpha-2\delta-1} dv$$

$$= \hat{h}(\alpha, 1; \delta, \kappa) + \kappa^{\delta}\hat{h}(2\delta-\alpha, 1; \delta, \kappa^{-1}),$$

where in the second equality we make the change of variable $v = 1/u$ and in the last one we apply $(b)$ in Theorem A.5 with $k = 0$. By $(c)$ in Theorem A.5 the second summand is analytic on $\Omega$, whereas the first one is meromorphic on $\Omega$. This shows the validity of the claim and so the result follows by applying the real version of Lemma B.1.

In order to prove $(b)$ we fix $\delta$, $\gamma$ and $x$ and apply $(b)$ in Theorem A.5 to the function $y \mapsto g(y;\delta,\gamma;x)$. Then, taking any $k \in \mathbb{N}$ with $k > \alpha$ and setting $\partial_y^i g(y;\delta,\gamma;x) = g^{(i)}(y)$ for shortness, we get

$$\lim_{y\to 1^-} \hat{g}(\alpha, y) = \sum_{r=0}^{k-1} \frac{g^{(r)}(0)}{r!(r-\alpha)} + \int_0^1 \left(g(u) - \sum_{r=0}^{k-1}\frac{g^{(r)}(0)}{r!}u^r\right)\frac{du}{u^{\alpha+1}}$$

$$= 2^{\alpha}\left(\sum_{r=0}^{k-1} \frac{g^{(r)}(0)}{r!(r-\alpha)2^r} + 2^{-\alpha}\int_0^{\frac{1}{2}} \left(g(u) - \sum_{r=0}^{k-1}\frac{g^{(r)}(0)}{r!}u^r\right)\frac{du}{u^{\alpha+1}}\right)$$

$$+ \int_{\frac{1}{2}}^1 (1-u)^{\delta-1}(1-xu)^{-\gamma}u^{-\alpha-1}du - \sum_{r=0}^{k-1}\frac{g^{(r)}(0)}{r!}\int_{\frac{1}{2}}^1 u^{r-\alpha}\frac{du}{u}$$

$$+ \sum_{r=0}^{k-1}\frac{g^{(r)}(0)}{r!(r-\alpha)}(1-2^{\alpha-r})$$

$$= 2^{\alpha}\hat{g}\,(\alpha, 1/2; \delta, \gamma; x) + \int_0^{\frac{1}{2}} s^{\delta-1}(1-x(1-s))^{-\gamma}(1-s)^{-\alpha-1}ds$$

$$= 2^{\alpha}\hat{g}\,(\alpha, 1/2; \delta, \gamma; x) + 2^{-\delta}(1-x)^{-\gamma}\hat{g}\left(-\delta, \frac{1}{2}; -\alpha, \gamma; \frac{x}{x-1}\right)$$

where in the last equality we use $(b)$ in Theorem A.5 with $k = 0$ and also take $\delta > 0$ into account. Thus, by applying $(c)$ in Theorem A.5 to each summand in the last expression, this shows that the function

$$(\gamma, \alpha, \delta, x) \mapsto \lim_{y\to 1^-} \hat{g}(\alpha, y; \delta, \gamma; x)$$

is meromorphic on the open connected set $\hat{\Omega} := \mathbb{R}^2 \times (0, +\infty) \times (-\infty, 1)$. Note also that if we consider parameter values in $\hat{V} := \hat{\Omega} \cap \{\alpha < 0\} \cap \{\delta > 0\}$ then

$$\lim_{y\to 1^-} \hat{g}(\alpha, y; \delta, \gamma; x) = \int_0^1 (1-u)^{\delta-1}(1-xu)^{-\gamma}u^{-\alpha-1}du = \mathrm{B}(-\alpha, \delta)\,{}_2F_1(\gamma, -\alpha; \delta-\alpha; x),$$

where in the first equality we apply $(b)$ in Theorem A.5 with $k = 0$ and in the second one we use Euler's integral representation (48). We have just proved that the left hand side expression is a meromorphic function on $\hat{\Omega}$. Furthermore, by applying Lemma B.2 and taking $\mathrm{B}(-\alpha, \delta) = \frac{\Gamma(-\alpha)\Gamma(\delta)}{\Gamma(\delta-\alpha)}$ into account, we can assert that the right hand side is also a meromorphic function on $\hat{\Omega}$. In view of this the identity in $(b)$ for the parameters under consideration follows by applying the real version of Lemma B.1. ■

It is worth to point out that the application of $(b)$ in Proposition B.3 provides integral representations of the hypergeometric function in a range of parameters not covered by Euler's formula (48). Indeed, by applying also $(b)$ in Theorem A.5 with any $k > \alpha$ we get

$$\mathrm{B}(-\alpha, \delta)\,{}_2F_1(\gamma, -\alpha; \delta-\alpha; x) = \sum_{r=0}^{k-1} \frac{g^{(r)}(0)}{r!(r-\alpha)} + \int_0^1 t^{-\alpha-1}\left((1-t)^{\delta-1}(1-xt)^{-\gamma} - \sum_{r=0}^{k-1} \frac{g^{(r)}(0)}{r!}t^r\right)dt,$$

which holds for any $\delta > 0$, $\gamma \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$, where $g(t) = (1-t)^{\delta-1}(1-xt)^{-\gamma}$. We stress that (48) gives an integral representation for ${}_2F_1(\gamma, -\alpha; \delta-\alpha; x)$ only in case that $\delta > 0$ and $\alpha < 0$.

## Appendix C. A technical result for the proof of Proposition 3.3

**Proposition C.1.** *The function*

$$\Phi(a, b) = \mathrm{B}(1-a, -\tfrac{3}{2})\,{}_2F_1(-3-a, -\tfrac{3}{2}; -\tfrac{1}{2}-a; -b) - 4\mathrm{B}(1-\tfrac{a}{2}, -\tfrac{1}{2})\,{}_2F_1(-1-\tfrac{a}{2}, -\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{a}{2}; -b)$$

*is strictly positive for all $a, b \in (0, 1)$.*

**Proof.** In what follows given a smooth function of several variables $\psi(u)$ with $u=(u_1,u_2,\dots,u_n)\in\mathbb{R}^n$, for each fixed $i\in\{1,2,\dots,n\}$ and $m\in\mathbb{N}$ we denote by $T^m_{u_i=0}\psi(u)$ the $m$-th order Taylor polynomial of $u_i\mapsto\psi(u)$ at $u_i=0$, i.e., $T^m_{u_i=0}\psi(u)=\sum_{k=0}^m\frac{1}{k!}\left.\frac{\partial^k\psi}{\partial u_i^k}\right|_{u_i=0}u_i^k$. Setting $P(a,b):=T^3_{b=0}\Phi(a,b)$ we claim that the following inequalities hold:

(*i*) $\Phi(a,b)\geqslant P(a,b)$ for all $a,b\in(0,1)$, and
(*ii*) $P(a,b)>0$ for all $a,b\in(0,1)$.

It is clear that the result will follow once we prove this. For this purpose our first task will be to express the function $\Phi$ in terms of a definite integral and to this end we define

$$\Phi_1(a,b):=\mathrm{B}(-\tfrac{3}{2},1-a)\,{}_2F_1(-3-a,-\tfrac{3}{2};-\tfrac{1}{2}-a;-b)$$

and

$$\Phi_2(a,b):=\mathrm{B}(-\tfrac{1}{2},1-\tfrac{a}{2})\,{}_2F_1(-1-\tfrac{a}{2},-\tfrac{1}{2};\tfrac{1}{2}-\tfrac{a}{2};-b),$$

so that, taking $\mathrm{B}(x,y)=\mathrm{B}(y,x)$ into account, $\Phi=\Phi_1-4\Phi_2$. Then by applying (*b*) in Proposition B.3 with $\{\alpha=\frac{3}{2},\delta=1-a,\gamma=-3-a,x=-b\}$ we can assert that $\Phi_1(a,b)=\lim_{y\to1^-}\hat g_1(\frac{3}{2},y;a,b)$ where $g_1(y;a,b):=(1-y)^{-a}(1+by)^{3+a}$. Next we apply (*b*) in Theorem A.5 taking $\{f=g_1,\alpha=\frac{3}{2},k=2,x=1\}$ to obtain that

$$\begin{aligned}\Phi_1(a,b)=\lim_{y\to1^-}\hat g_1(3/2,y;a,b)&=2\kappa_1+\int_0^1\big(g_1(s)-r_1(s)\big)s^{-\frac{5}{2}}ds\\&=\int_0^1\big(g_1(s)-r_1(s)+\kappa_1s^2\big)s^{-\frac{5}{2}}ds,\end{aligned}$$

where $\kappa_1=-\frac{1}{3}-\bar\kappa$ and $r_1(s;a,b)=1+\bar\kappa s$ with $\bar\kappa:=(3+a)b+a$. Similarly by (*b*) in Proposition B.3 with $\{\alpha=\frac{1}{2},\delta=1-\frac{a}{2},\gamma=-1-\frac{a}{2},x=-b\}$ we obtain that $\Phi_2(a,b)=\lim_{y\to1^-}\hat g_2(\frac{1}{2},y;a,b)$ where $g_2(y;a,b):=(1-y)^{-\frac{a}{2}}(1+by)^{1+\frac{a}{2}}$. Next we apply (*b*) in Theorem A.5 with $\{f=g_2,\alpha=\frac{1}{2},k=2,x=1\}$ to get that

$$\begin{aligned}\Phi_2(a,b)=\lim_{y\to1^-}\hat g_2(1/2,y;a,b)&=2\kappa_2+\int_0^1\big(g_2(s)-r_2(s)\big)s^{-\frac{3}{2}}ds\\&=\int_0^1\big(sg_2(s)-sr_2(s)+\kappa_2s^2\big)s^{-\frac{5}{2}}ds,\end{aligned}$$

where $\kappa_2=-1+\bar\kappa$ and $r_2(s;a,b)=1+\bar\kappa s$. Accordingly an easy computation yields

$$\Phi(a,b)=\Phi_1(a,b)-4\Phi_2(a,b)=\int_0^1 h(s;a,b)s^{-\frac{5}{2}}ds,$$

where

$$h(s;a,b):=(1+bs)^{3+a}(1-s)^{-a}-4s(1+bs)^{1+\frac{a}{2}}(1-s)^{-\frac{a}{2}}-1+(4-\bar{\kappa})s+(11/3-\bar{\kappa})s^2$$

and consequently

$$P(a,b)=T^3_{b=0}\Phi(a,b)=\int_0^1 h_0(s;a,b)s^{-\frac{5}{2}}ds \text{ where } h_0:=T^3_{b=0}h.$$

On account of this definition and the fact that if $\frac{\partial\varphi}{\partial u_i}\equiv 0$ then $T^m_{u_i=0}(\varphi\psi)=\varphi T^m_{u_i=0}(\psi)$, we get

$$\Phi(a,b)-P(a,b)=\int_0^1 \big(h(s;a,b)-h_0(s;a,b)\big)s^{-\frac{5}{2}}ds=\int_0^1 \frac{g(s;a,b)-g_0(s;a,b)}{(1-s)^a s^{\frac{5}{2}}}ds,$$

where

$$g(s;a,b):=(1+bs)^{3+a}-4s(1-s)^{\frac{a}{2}}(1+bs)^{1+\frac{a}{2}} \text{ and } g_0:=T^3_{b=0}g.$$

Therefore the assertion in $(i)$ will follow once we prove that $g(x;a,b)\geqslant g_0(x;a,b)$ for all $a,b,x\in(0,1)$. As a first step to this aim let us prove that, setting

$$\ell(a,y):=(1+y)^{3+a}-4(1+y)^{1+\frac{a}{2}} \text{ and } \ell_0:=T^3_{y=0}\ell,$$

then

$$g(x;a,b)-g_0(x;a,b)\geqslant \ell(a,bx)-\ell_0(a,bx) \text{ for all } a,b,x\in(0,1).$$

In order to show this we note that

$$\begin{aligned} g(x;a,b)-g_0(x;a,b)&=(1+bx)^{3+a}-T^3_{b=0}(1+bx)^{3+a}\\ &\quad-4x(1-x)^{\frac{a}{2}}\left((1+bx)^{1+\frac{a}{2}}-T^3_{b=0}(1+bx)^{1+\frac{a}{2}}\right)\\ &\geqslant(1+bx)^{3+a}-T^3_{b=0}(1+bx)^{3+a}\\ &\quad-4\left((1+bx)^{1+\frac{a}{2}}-T^3_{b=0}(1+bx)^{1+\frac{a}{2}}\right) \end{aligned}$$

because $x(1-x)^{\frac{a}{2}}\leqslant 1$ and, thanks to the remainder's formula in Taylor's Theorem, one can easily verify that $(1+y)^\eta-T^m_{y=0}(1+y)^\eta\geqslant 0$ for any $m$ odd and $\eta\in(1,2)$. It is clear then that a sufficient condition for the inequality in $(i)$ to hold is that $\ell(a,y)-\ell_0(a,y)\geqslant 0$ for all

$a, y \in (0,1)$. In order to show that this is indeed true we note that, by the remainder's formula in Taylor's Theorem again,

$$\ell(a,y) - \ell_0(a,y) = \frac{\partial_y^4 \ell(a,y_0)}{4!} y^4 \text{ for some } y_0 \in (0,y),$$

and consequently it suffices to verify that

$$\partial_y^4 \ell(a,y) = (a+3)(a+2)(a+1)a(1+y)^{a-1} - \tfrac{1}{4}(a+2)a(a-1)(a-4)(1+y)^{\frac{a}{2}-3} \geqslant 0$$

for all $a, y \in (0,1)$. This inequality is equivalent to $(1+y)^{2+\frac{a}{2}} \geqslant \frac{1}{4}\frac{(a-2)(a-4)}{(a+3)(a+1)}$, which is obviously true due to $(1+y)^{2+\frac{a}{2}} \geqslant 1 \geqslant \frac{1}{4}\frac{(a-2)(a-4)}{(a+3)(a+1)}$ for all $a, y > 0$. This proves the first assertion of the claim.

We now turn to the proof of the inequality in $(ii)$. On account of the definition of the Gauss hypergeometric function, see (47), together with the definition of the function $\Phi(a,b)$ given in the statement it easily follows that

$$P(a,b) = T^3_{b=0}\Phi(a,b) = \rho_0(a) - \rho_1(a)b + \rho_2(a)b^2 - \rho_3(a)b^3$$

with

$$\rho_n(a) := \frac{(-3-a)_n(-\frac{3}{2})_n}{(-\frac{1}{2}-a)_n n!}\mathrm{B}\left(1-a,-\frac{3}{2}\right) - 4\frac{(-1-\frac{a}{2})_n(-\frac{1}{2})_n}{(\frac{1}{2}-\frac{a}{2})_n n!}\mathrm{B}\left(1-\frac{a}{2},-\frac{1}{2}\right). \quad (49)$$

Observe that $P(a,b)$ is a polynomial in $b$ for each fixed $a$. In order to prove the inequality in $(ii)$ we consider $P(a,\cdot)$ as a polynomial family depending on a parameter $a \in (0,1)$. In doing so it is clear that the following three conditions imply that the number of zeros of $b \mapsto P(a,b)$ on the interval $(0,1)$, counted with multiplicities, is the same for all $a \in (0,1)$:

$(a)$ $P(a,0) > 0$ for all $a \in (0,1)$,
$(b)$ $P(a,1) > 0$ for all $a \in (0,1)$ and
$(c)$ $\mathrm{Discrim}_b P(a,b) \neq 0$ for all $a \in (0,1)$.

Since one can readily show that, for instance, $P(\frac{1}{2},b) > 0$ for all $b \in (0,1)$, it is clear that $(ii)$ will follow once we prove that these three conditions are true. This constitutes our next task. In order to prove the inequality in $(a)$ we first note that, from (46),

$$\begin{aligned} P(a,0) = \rho_0(a) &= \mathrm{B}\left(1-a,-\frac{3}{2}\right) - 4\mathrm{B}\left(1-\frac{a}{2},-\frac{1}{2}\right) \\ &= 8\sqrt{\pi}\left(\frac{\Gamma(1-\frac{a}{2})}{\Gamma(\frac{1}{2}-\frac{a}{2})} - \frac{a+\frac{1}{2}}{6}\frac{\Gamma(1-a)}{\Gamma(\frac{1}{2}-a)}\right), \end{aligned}$$

where we use that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$, see [1, §6.1]. Taking this into account, the fact that $P(a,0) > 0$ for all $a \in [\frac{1}{2},1)$ is clear because $\Gamma(z)$ is negative for $z \in (-1,0)$ and positive for $z > 0$ and $\lim_{z\to 0} \frac{1}{\Gamma(z)} = 0$, see [1, §6.1] again. To show that this is also true for $a \in (0, \frac{1}{2})$ we use that then

$$\frac{\Gamma(1-\frac{a}{2})}{\Gamma(\frac{1}{2}-\frac{a}{2})} > \frac{\Gamma(1-a)}{\Gamma(\frac{1}{2}-a)} > \frac{a+\frac{1}{2}}{6}\frac{\Gamma(1-a)}{\Gamma(\frac{1}{2}-a)}.$$

The second inequality above is obvious whereas the first one follows noting that $z \mapsto \frac{\Gamma(1-z)}{\Gamma(\frac{1}{2}-z)}$ is positive and decreasing on $(0, \frac{1}{2})$. In its turn this is true due to $\left(\log \frac{\Gamma(1-z)}{\Gamma(\frac{1}{2}-z)}\right)' = \Psi(\frac{1}{2}-z) - \Psi(1-z) < 0$ for all $z \in (0, \frac{1}{2})$ since thedigamma function

$$\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^1 \frac{1-x^{z-1}}{1-x}dx \tag{50}$$

is a well defined monotonous increasing function for $z > 0$, see [1, §6.3]. Here $\gamma \approx 0.577$ is the Euler-Mascheroni constant. This proves the validity of the inequality in $(a)$.

Let us turn next to the proof of the assertion with regard to $P(a, 1)$. To this end, for the sake of convenience, we introduce the function

$$F(a) := \frac{3}{16}\frac{((a-1)(a-3)(a-5))^2}{(2a-3)(4a^2-1)}\frac{\mathrm{B}\left(1-a,-\frac{3}{2}\right)}{\mathrm{B}\left(1-\frac{a}{2},-\frac{1}{2}\right)} = \frac{\Gamma^2\left(\frac{7}{2}-\frac{a}{2}\right)}{2^a\sqrt{\pi}\,\Gamma\left(\frac{5}{2}-a\right)}, \tag{51}$$

where the identity follows using the so called duplication formula for $\Gamma$, see [1, §6.1.18]. This function will enable us to write $P(a, 1) = \rho_0(a) - \rho_1(a) + \rho_2(a) - \rho_3(a)$ in a more convenient form taking advantage of the fact that each $\rho_n(a)$ is linear in $\mathrm{B}\left(1-a,-\frac{3}{2}\right)$ and $\mathrm{B}\left(1-\frac{a}{2},-\frac{1}{2}\right)$, see (49). In doing so, some easy computations using a symbolic manipulator (see [21] for instance) show that

$$P(a,1) = \frac{40a(a+2)(3a-5)}{((a-1)(a-3)(a-5))^2}\mathrm{B}\left(1-\frac{a}{2},-\frac{1}{2}\right)\Big(F(a)-g(a)\Big),$$

where

$$g(a) := \frac{(23a-94)(a-1)(a-3)(a-4)(a-5)}{160(3a-5)(a+2)}.$$

Thus, since $\mathrm{B}\left(1-\frac{a}{2},-\frac{1}{2}\right) = -\frac{2\sqrt{\pi}\,\Gamma(1-\frac{a}{2})}{\Gamma(\frac{1}{2}-\frac{a}{2})}$ is negative for all $a \in (0, 1)$, the assertion in $(b)$ will follow once we prove that $F(a) > g(a)$ for all $a \in (0, 1)$. As an intermediate step to this end we claim that if $a \in (0, 1)$ then $F(a) > 0$, $F'(a) < 0$ and $F''(a) > 0$. The first inequality is clear from (51) because $\Gamma(z) > 0$ for all $z > 0$. The second inequality is also easy because some computations show that

$$\frac{F'(a)}{F(a)} = -\big(h(a)+\ln 2\big) \text{ with } h(a) := \Psi(7/2-a/2) - \Psi(5/2-a)$$

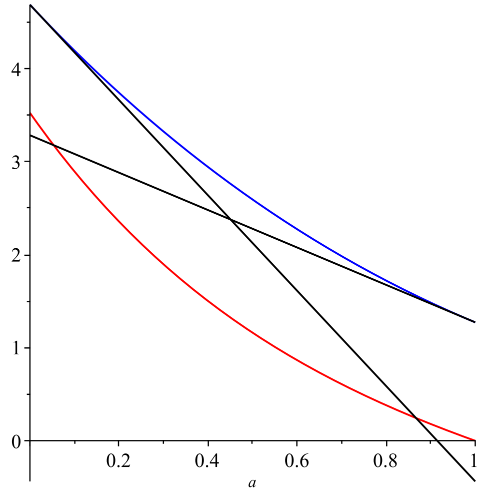


Fig. 11. The graphs of the transcendental function $F(a)$ in blue, its tangent lines $\ell_0(a)$ and $\ell_1(a)$ at $a=0$ and $a=1$, respectively, in black and the rational function $g(a)$ in red.

and, on the other hand, $h(a)>0$ for all $a\in(0,1)$ due to $\Psi'(z)>0$ for $z>0$. Finally, in order to show the third inequality we first note that

$$\frac{F''(a)}{F(a)}=\big(h(a)+\log 2\big)^2-h'(a).$$

Furthermore, due to $x^{\frac{3}{2}-a}>x^{\frac{5}{2}-\frac{a}{2}}>2^{-n}x^{\frac{5}{2}-\frac{a}{2}}$ for all $x\in(0,1)$ and $a>0$, from (50) it turns out that

$$\begin{aligned}h^{(n)}(a)&=(-1/2)^n\Psi^{(n)}(7/2-a/2)-(-1)^n\Psi^{(n)}(5/2-a)\\&=\int_0^1\Big(x^{\frac{3}{2}-a}-2^{-n}x^{\frac{5}{2}-\frac{a}{2}}\Big)\frac{(-\log x)^n}{1-x}dx>0\text{ for all }n\in\mathbb{Z}_{\geq 0}.\end{aligned}$$

Hence $h^{(n)}$ is increasing on $(0,+\infty)$ for all $n\in\mathbb{Z}_{\geq 0}$ and, consequently, $h^{(n)}(1)>h^{(n)}(a)>h^{(n)}(0)$ for all $a\in(0,1)$. Thus if $a\in(0,1)$ then

$$\frac{F''(a)}{F(a)}>\big(h(0)+\log 2\big)^2-h'(1)=\left(\frac{2}{5}+\log 2\right)^2+\frac{27}{8}-\frac{5\pi^2}{12}\approx 0.46.$$

Accordingly, $F''(a)>0$ for all $a\in(0,1)$ and this concludes the proof of the claim. We proceed now with the proof of $(b)$, which let us recall that it will follow once we prove that $F(a)>g(a)$ for all $a\in(0,1)$. With this aim in view we take the tangent lines to the graph of $F$ at $a=0$ and $a=1$, that are given by

$$\ell_0(a)=\tfrac{75}{16}-(\tfrac{15}{8}+\tfrac{75}{16}\log 2)a\text{ and }\ell_1(a)=\tfrac{4}{\pi}+\tfrac{2}{\pi}(1-6\log 2)(a-1),$$

respectively, see Fig. 11. Since $F$ is convex, in order to show that $F(a) > g(a)$ for all $a \in (0, 1)$, it suffices to verify that $\max\{\ell_0(a), \ell_1(a)\} > g(a)$ for all $a \in (0, 1)$. To see this we consider the unique solution of $\ell_0(a) = \ell_1(a)$, which one can check that it is given by

$$a = \hat{a} := \frac{75\pi - 32 - 192\log 2}{(75\pi - 192)\log 2 + 30\pi + 32} \approx 0.45.$$

One can also verify that, for $i = 0, 1$, $\ell_i(a) - g(a) = \frac{p_i(a)}{160\pi^i(2+a)(5-3a)}$ with

$$\begin{aligned} p_0(a) = {} & 23a^5 - 393a^4 + (3479 + 2250\log 2)\,a^3 + (-9957 + 750\log 2)\,a^2 \\ & + (7688 - 7500\log 2)\,a + 1860 \end{aligned}$$

and

$$\begin{aligned} p_1(a) = {} & 23\pi a^5 - 393\pi a^4 + (-960 + 2579\pi + 5760\log 2)\,a^3 \\ & - (8007\pi + 1280 + 3840\log 2)\,a^2 \\ & + (11438\pi + 2880 - 21120\log 2)\,a + 3200 - 5640\pi + 19200\log 2. \end{aligned}$$

By applying Sturm's Theorem we can assert that $p_0$ is positive on $(0, 0.46)$ and that $p_1$ is positive on $(0.44, 1)$, which imply that $\max\{\ell_0(a), \ell_1(a)\} > g(a)$ for all $a \in (0, 1)$ as desired. This proves $(b)$.

Our last task is to prove the assertion in $(c)$. To this end we use a symbolic manipulator in order to show that

$$\mathrm{Discrim}_b P(a, b) = \frac{-2(a+2)\mathrm{B}(1 - \frac{a}{2}, -\frac{1}{2})^4}{3((a-1)(a-3)(a-5))^8}\mathscr{R}\big(a, F(a)\big),$$

where

$$\begin{aligned} \mathscr{R}(a, t) = {} & -16384(2a-1)(8a^6 + 36a^5 - 126a^4 - 413a^3 + 429a^2 + 576a \\ & - 512)(a+3)^2(2a-3)^2 t^4 + 3072(a-1)(a-3)(a-5)(2a-3)(a+3)\big(48a^8 \\ & + 252a^7 - 1904a^6 - 2305a^5 + 11568a^4 - 2566a^3 - 14160a^2 - 2784a + 11520\big)t^3 \\ & - 24a(a+2)\big(768a^9 - 7808a^8 + 3616a^7 + 135520a^6 - 221032a^5 - 557976a^4 \\ & + 823685a^3 + 1082256a^2 - 894960a - 915840)(a-5)^2(a-1)^2(a-3)^2 t^2 \\ & - 4(a-4)(a+2)(320a^8 - 1400a^7 + 1830a^6 - 5491a^5 + 4678a^4 + 32889a^3 \\ & - 4482a^2 - 47520a - 64800\big)(a-1)^3(a-3)^3(a-5)^4 t + 15a(a-2)(a \\ & - 4)(a+2)(5a^4 - 15a^3 - 5a^2 + 27a + 36)(a-1)^4(a-3)^5(a-5)^6. \end{aligned}$$

Let us mention that in order to ease this computation and introduce $F$ we use that the coefficients of $P(a, b) = \rho_0(a) - \rho_1(a)b + \rho_2(a)b^2 - \rho_3(a)b^3$, see (49), are linear in $\mathrm{B}\left(1 - a, -\frac{3}{2}\right)$ and $\mathrm{B}\left(1 - \frac{a}{2}, -\frac{1}{2}\right)$ and that, on the other hand, the discriminant of a third degree polynomial is a homogeneous polynomial of degree 4 in its coefficients. On account of the above expression it is clear that $(c)$ will follow once we prove that $\mathscr{R}(a, F(a)) \neq 0$ for all $a \in (0,1)$. To this end we note that $F(0) = \frac{75}{16}$ and $F(1) = \frac{4}{\pi}$. Therefore, see Fig. 11, the graph $t = F(a)$ for $a \in (0,1)$ verifies $\max\{\ell_0(a), \frac{4}{\pi}\} < F(a) < \frac{75}{16}$ because we previously proved that $F' < 0$ and $F'' > 0$ on the interval $(0,1)$. Accordingly it suffices to show that $\mathscr{R}(a,t) \neq 0$ for all $(a,t)$ inside the trapezium given by $\max\{\ell_0(a), \frac{4}{\pi}\} < t < \frac{75}{16}$ and $a \in (0,1)$. We will prove this taking $t \in (\frac{4}{\pi}, \frac{75}{16})$ as a fixed parameter and showing that the polynomial $a \mapsto \mathscr{R}(a,t)$ has not any root on $(\ell_0^{-1}(t), 1)$, where $\ell_0^{-1}(t) = \frac{75-16t}{30-75\log 2}$. To this effect we show first that $\mathscr{R}\left(\frac{75-16t}{30-75\log 2}, t\right)$, $\mathscr{R}(1,t)$ and $\mathrm{Discrim}_a \mathscr{R}(a,t)$ do not vanish for all $t \in (\frac{4}{\pi}, \frac{75}{16})$. This implies that the number of roots of $\mathscr{R}(a,t) = 0$ on $(\ell_0^{-1}(t), 1)$ does not change for $t \in (\frac{4}{\pi}, \frac{75}{16})$. Taking this into account the desired result follows by checking, for instance, that this number is zero for $t = 2 \in (\frac{4}{\pi}, \frac{75}{16})$. All these assertions can be checked systematically by applying Sturm's Theorem because only one variable polynomials are involved. This shows the validity of $(c)$ and so the inequality in $(ii)$ is true. This concludes the proof of the result. ∎

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