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An elementary proof of the existence of the Leontief inverse

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Abstract

We provide a new and simpler proof for the existence and non-negativity of the Leontief inverse that does not rely on advanced mathematics. Instead, we start from an economic condition related to the equilibrium in quantities between demand and supply and proceed using only elementary linear algebra to establish the nature of the inverse.

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1 Preliminaries

Consider the system of equations:

$$\mathbf{A} \cdot x + f = x \tag{1}$$

where $\mathbf{A}$ is a non-negative $n \times n$ square matrix, and x and f are n-dimensional column vectors. Vector f is a datum whereas vector x represents the solution (if it exists) of the system of equations. As it stands, this system is just a mathematical construct with no underlying interpretation. It may or may not have a solution and this depends on the nature of $\mathbf{A}$ and the values in f.

Assume now that matrix $\mathbf{A} = (a_{ij})$ embodies the technology of an interindustry economy with n goods, fixed technical coefficients and constant returns to scale. In this case, we interpret a_{ij} as the physical quantity of good i needed as input to yield one unit of good j as output. Because of the monotonicity properties of technologies, these coefficients will be positive or zero. The full matrix $\mathbf{A}$ will be non-negative but we rule out the extreme case $\mathbf{A}=0$ since this would entail production with free lunch in terms of inputs, which is only possible in the mythical land of Cockaigne (Koopmans 1951; Debreu 1959). In turn, vector f represents exogenous final demand and vector x the endogenous output of the economy. Equation (1) shows now the balance, or supply-demand equilibrium, between total produced output x and its allocation to satisfy final demand f and intermediate demand $\mathbf{A} \cdot x$. The latter is the demand made by productive units to other productive units and that is required to generate the production x that allows fulfilling the bill of goods in f. Under the economic interpretation of (1), vectors f and x need now to be non-negative to be economically meaningful. We can rewrite (1) as:

$$x - \mathbf{A} \cdot x = (\mathbf{I} - \mathbf{A}) \cdot x = f \tag{2}$$

If matrix $(\mathbf{I} - \mathbf{A})$ happens to be invertible then we could solve (2) as:

$$x = (\mathbf{I} - \mathbf{A})^{-1} \cdot f \tag{3}$$

Nothing would guarantee, however, that the solution x in (3) would be non-negative. It would be so if both f and $(\mathbf{I} - \mathbf{A})^{-1}$ were non-negative. Since final demand f will always be non-negative, it all relies on whether or not $(\mathbf{I} - \mathbf{A})^{-1}$ is a non-negative matrix as well. This is therefore the key issue in input-output economics, as we well know. The matrix $(\mathbf{I} - \mathbf{A})^{-1}$ plays a key role in the solution and we commonly refer to it as Leontief's inverse.

The mathematical conditions that ensure that inverse matrix $(\mathbf{I}-\mathbf{A})^{-1}$ exists and is non-negative have been extensively explored in the literature in well-known foundational contributions. Debreu and Herstein (1953) used fixed point theorems to show the property (in fact, under more general terms than needed in (3)) and how it relates to the maximal eigenvalue of matrix $\mathbf{A}$. Nikaido (1968, 1972) and Takayama (1985) also show the link between the maximal eigenvalue of **A** and the non-negative invertibility of $(\mathbf{I}-\mathbf{A})$. In a different vein, Hawkins and Simon (1948) relate the desired property to the positive sign of all leading principal minors of matrix $(\mathbf{I}-\mathbf{A})$. This mathematical condition, however, does not have an easy and straightforward economic interpretation (see Jeong 1982, for a discussion). Both the eigenvalue property and the Hawkins-Simon conditions turn out to be mathematically equivalent to the non-negative invertibility of $(\mathbf{I}-\mathbf{A})$. In turn, the Brauer-Solow conditions in Solow (1952) provide sufficient conditions for the invertibility property based on lower and upper bounds on the maximal eigenvalue of **A**. For computational issues the Brauer-Solow conditions turns out to be quite handy. Miller and Blair (2009) cleverly simplify the Brauer-Solow condition from eigenvalues to the norm of any non-negative matrix **A** constructed form the monetary values in empirical input-output tables. Dietzenbacher (2005) also uses the Brauer-Solow condition with physical input-output data to show the existence of the needed Leontief inverse.

In here, we will look at the situation from a different and simpler perspective. We will use an economic definition as starting point and will prove that the economic definition is all we need to establish the desired properties of the Leontief inverse. The usual advanced mathematical theorems mentioned above continue, of course, to be valid[1]. Since learning costs are important, the question is whether we can reach the same type of conclusion using tools that are more elementary and more affordable to acquire. Gale (1960), for instance, links the properties of the Leontief inverse to a productivity condition of the matrix **A** using an algebraic proof that does not need any of the mentioned topological properties. Simpler results, when they exist, are both useful and needed because they contribute to consolidating the bases of analysis among a broader base of practitioners and we achieve this with substantially lower learning costs. We share this approach and so we provide here a simple justification for the existence of the Leontief inverse that is based on economic definitions rather than on mathematical conditions that are not always easy to follow and to interpret.

[1] In a more general setting, however, the equivalence may fail. See Sancho (2019) for a counterexample in a trade model.

2 The main result

We now return to equations (1) and (2) and observe that the specific solution *x*, given that **A** is a description of the economy's technology that we assume is invariant, will depend on the values in *f*. The question is if for any non-negative value of *f*, the system of equations in (1), or equivalently in (2), will always have a non-negative output solution *x*.

Definition: The technology represented by a non-negative, non-zero, $n \times n$ square matrix **A** represents a ***viable*** economy if for any non-negative vector of final demand $f \geqq 0$ there exists a non-negative output vector $x \geqq 0$ such that $(\mathbf{I}-\mathbf{A}) \cdot x = f$.

In words, no matter what the (non-negative) levels of final demand *f* may be, a viable economy will always be able to produce output *x* that is also non-negative and allocates in balance between all sources of demand, final and intermediate.

Proposition: An economy **A** is ***viable*** if and only if the matrix $(\mathbf{I}-\mathbf{A})$ is invertible and the inverse $(\mathbf{I}-\mathbf{A})^{-1}$ is non-negative.

Proof: Firstly, necessity. It is obvious that if the inverse of $(\mathbf{I}-\mathbf{A})$ exists and is non-negative then from equation (3) we see that the technology always yields a viable economy. Secondly, sufficiency. If the economy is viable then for any $f \geqq 0$ the system of equations (2) has a solution. Therefore the rank[2] of matrix $(\mathbf{I}-\mathbf{A})$ and the rank of the matrix $(\mathbf{I}-\mathbf{A})$ enlarged adjoining the column vector *f* will coincide: $rank(\mathbf{I}-\mathbf{A}) = rank(\mathbf{I}-\mathbf{A} \mid f)$. Let us take the canonical basis in R^n and let e^i represent the *i-th* vector in the said basis. Since $e^i \geqq 0$ we also conclude that the rank of $(\mathbf{I}-\mathbf{A})$ and the rank of the matrix enlarged with vector e^i in column format $(\mathbf{I}-\mathbf{A} \mid e^i)$ will be equal for any *i*. From here:

$$rank(\mathbf{I}-\mathbf{A}) = rank(\mathbf{I}-\mathbf{A} \mid e^1, e^2, \ldots, e^n) = n$$

since the *n* columns of the canonical basis are, by definition of a basis, linearly independent. Therefore, $(\mathbf{I}-\mathbf{A})$ has full rank and is invertible. Finally, to prove that the inverse $(\mathbf{I}-\mathbf{A})^{-1}$ must be non-negative, let us assume that it is not. In this case there would exist at least a negative element in $(\mathbf{I}-\mathbf{A})^{-1}$, say $m_{ij} < 0$. Take now the non-negative vector e^j in the canonical basis. Since $(\mathbf{I}-\mathbf{A})^{-1}$ exists we find that the vector *x* from:

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[2] The *rank* of a rectangular matrix is the maximal number of linearly independent rows or columns. The canonical basis in R^n is the set of *n* vectors $e^1=(1, 0, \ldots, 0)$, $e^2 = (0, 1, 0, \ldots, 0), \ldots, e^n=(0, 0, \ldots, 1)$.

$$x = (\mathbf{I} - \mathbf{A})^{-1} \cdot e^j$$

yields $x^i = m_{ij} < 0$. This violates the viability assumption and gives rise to a contradiction. In conclusion, the inverse of $(\mathbf{I} - \mathbf{A})$ will exist and will be non-negative.

3 Concluding remarks

We have developed a different and hopefully simpler proof for the existence of a non-negative Leontief inverse. This short contribution has two points of interest. On a conceptual level, it directly links the mathematical properties of the inverse to an economic condition of the type "supply equals demand" instead that on topological properties of the technology matrix. In doing so we also verify the equivalence between the economic and the mathematical conditions. In a similar vein, but far more technically complex, Uzawa (1962) and Debreu (1982) pointed out the remarkable fact that the existence of a competitive equilibrium—an economic condition—also implies the validity of the fixed-point theorems of Brouwer and Kakutani. We learn in graduate school that the fixed-point theorems imply the existence of an equilibrium but what they amazingly state—and is not really well known—is that the converse is also true. The economic condition also implies the technical one and thus there is in fact a profound equivalence property underlying the existence theorems.

On a methodological level, undergraduate level elementary linear algebra is all we need. This provides the benefit of making more accessible to input-output analysts this key property of the Leontief inverse with quite a smaller learning cost. Besides complementing the current pool of knowledge, it is also reassuring—from the perspective of applied economics—that any viable economy of the input-output type will give rise to a well-behaved Leontief inverse.

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