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ACTIONS ON PRODUCTS OF CAT(−1) SPACES

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ABSTRACT. We show that for X a proper CAT(−1) space there is a maximal open subset of the horofunction compactification of $X \times X$, with respect to the maximum metric, that compactifies the diagonal action of an infinite quasi-convex group of the isometries of X . We also consider the product action of two quasi-convex representations of an infinite hyperbolic group on the product of two different proper CAT(−1) spaces.

1. INTRODUCTION

The action of a discrete group of isometries Γ on the ideal boundary of a proper CAT(−1) space X has a dynamical decomposition $\partial_\infty X = \Omega_\Gamma \sqcup \Lambda_\Gamma$, where Λ_Γ is the limit set and Ω_Γ is the domain of discontinuity [6]. In addition, if Γ is quasi-convex, then the action on $X \cup \Omega_\Gamma$ is also properly discontinuous and cocompact, so Ω_Γ compactifies the action of Γ on X [18].

This dynamical decomposition of the visual boundary may not hold for CAT(0) spaces: there may be no uniqueness of maximal discontinuity domains, or there may be no discontinuity domain at all, even if the limit set is proper. We mention the work of Papasoglu and Swenson [17], or Kapovich, Leeb, and Porti [12], in the context of symmetric spaces. In this paper we consider the case of the product of two proper CAT(−1) spaces, which is a CAT(0) space, see the work on products by Geninska [10] and by Link [14].

As a motivating example, consider a cocompact fuchsian group $\Gamma < \text{Isom}(\mathbb{H}^2)$ acting diagonally on $\mathbb{H}^2 \times \mathbb{H}^2$. The ideal boundary of \mathbb{H}^2 is $\partial_\infty \mathbb{H}^2 \cong S^1$, so the visual boundary of the product is the spherical join of two circles, $\partial_\infty(\mathbb{H}^2 \times \mathbb{H}^2) \cong S^1 \times S^1 \times [0, \pi/2]/\sim$, where \sim is the relation that collapses each subset $\{*\} \times S^1 \times \{0\}$ or $S^1 \times \{*\} \times \{\pi/2\}$ to a point. The diagonal action on the ideal boundary preserves the sets $S^1 \times S^1 \times \{\theta\}$ for each $\theta \in [0, \pi/2]$, so finding a domain of discontinuity amounts to find a domain of discontinuity for the diagonal action on $S^1 \times S^1$. This is not possible because the action on $S^1 \times S^1$ has a dense orbit [15, Thm. 3.6.1], hence it has empty domain of discontinuity (even if the limit set is $S^1 \times S^1 \times \{\pi/4\}$). Notice that the visual compactification of $\mathbb{H}^2 \times \mathbb{H}^2$ is the horofunction compactification with respect to the product metric, or ℓ^2 metric. Instead, here we work with the ℓ^∞ or maximum metric, which happens to be better suited for those compactifications.

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For X_1 and X_2 two proper $\text{CAT}(-1)$ spaces, we denote the ideal or Gromov boundary of $X_1 \times X_2$ with respect to the max metric by $\partial_\infty^{\max}(X_1 \times X_2)$. In Proposition 3.13 we show that $\partial_\infty^{\max}(X_1 \times X_2)$ is homeomorphic to the join of the boundaries of each factor. In particular, the ideal boundaries for both metrics ℓ^2 and ℓ^∞ are homeomorphic, but their compactifications are not equivalent, since the identity does not extend continuously to the compactifications. The max compactification is adapted to diagonal actions, as it allows to find an ideal subset where the diagonal action is properly discontinuous and which compactifies the action. The main theorem of this paper is:

Theorem 1.1. *Let X be a proper $\text{CAT}(-1)$ space and Γ an infinite quasi-convex group of isometries of X . There exists an open set $\Omega_\Gamma^{\max} \subset \partial_\infty^{\max}(X \times X)$ such that:*

- (a) *The diagonal action of Γ on $X \times X \cup \Omega_\Gamma^{\max}$ is properly discontinuous and cocompact.*
- (b) *Ω_Γ^{\max} is the largest open subset of $\partial_\infty^{\max}(X \times X)$ where the diagonal action is properly discontinuous.*

When Γ acts cocompactly on X , the theorem has been proved in [9]. To prove Theorem 1.1 we show that the nearest point projection from $X \times X$ to the diagonal extends continuously to a map on $\partial_\infty^{\max}(X \times X)$ with image in the visual compactification of the diagonal.

The ideal boundary $\partial_\infty^{\max}(X_1 \times X_2)$ decomposes in two parts defined in Section 3, regular and singular. The regular part $\partial_\infty^{\max}(X_1 \times X_2)_{\text{reg}}$ consist of points that correspond to the maximum of two Busemann functions, one on each factor, and it is homeomorphic to $\partial_\infty X_1 \times \partial_\infty X_2 \times \mathbb{R}$ (Proposition 3.10). The singular part $\partial_\infty^{\max}(X_1 \times X_2)_{\text{sing}}$ consists of points that are Busemann functions in one of the factors and it is homeomorphic to the disjoint union $\partial_\infty X_1 \sqcup \partial_\infty X_2$ (Proposition 3.9).

In a $\text{CAT}(0)$ space the limit set Λ_Γ is the set of accumulation points in the ideal boundary of an orbit and it is independent of the choice of the orbit. In our setting, since the max metric is not $\text{CAT}(0)$, the set of accumulation points of an orbit depends on the orbit, so we consider the *large limit set*, consisting of accumulation points of any orbit. For a diagonal action it turns out that the large limit set is contained in the regular part of the boundary and that Ω_Γ^{\max} is the complement of the closure of the large limit set. In the particular case in which Γ is a cocompact group, the set Ω_Γ^{\max} is naturally homeomorphic to the set of parameterized geodesics in one factor, as shown in [9].

This max metric is a Finsler metric. Bordifications through Finsler metrics of symmetric spaces have been used by Kapovich and Leeb [11] to obtain a characterization of Anosov representations. In a product of $\text{CAT}(-1)$ spaces, this corresponds to the ℓ^1 metric.

The max compactification is very convenient for diagonal actions, but it would be interesting to see in what other situations it is useful. For Γ an infinite hyperbolic group, we consider ρ_1 and ρ_2 two quasi-convex representations in the respective group of isometries of $\text{CAT}(-1)$ spaces X_1 and X_2 , and their product action $\rho_1 \times \rho_2$ on $X_1 \times X_2$: an element γ in Γ

maps $(x_1, x_2) \in X_1 \times X_2$ to $(\rho_1(\gamma)x_1, \rho_2(\gamma)x_2)$. Since the max metric on $X_1 \times X_2$ is not CAT(0), the accumulation set of an orbit may depend on the orbit. The union of all possible accumulation sets is called the *large limit set* and it is denoted by $\Lambda_{\rho_1 \times \rho_2} \subset \partial_\infty^{\max}(X_1 \times X_2)$. In analogy to the diagonal case, it is reasonable to ask under what conditions the large limit set $\Lambda_{\rho_1 \times \rho_2}$ of the product action also remains inside the regular part of the boundary.

Proposition 1.2. *Let X_1 and X_2 be proper CAT(−1) spaces, Γ an infinite hyperbolic group, and $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two quasi-convex representations. The following are equivalent:*

- (a) $\Lambda_{\rho_1 \times \rho_2} \subset \partial_\infty^{\max}(X_1 \times X_2)_{\text{reg}}$.
- (b) *There exists $C > 0$ depending on $o \in X_1$ and $o' \in X_2$ such that*
 $|d_1(\rho_1(\gamma)o, \rho_1(\gamma')o) - d_2(\rho_2(\gamma)o', \rho_2(\gamma')o')| < C,$ *for all $\gamma \in \Gamma$.*
- (c) *The length spectrum is the same: $\tau(\rho_1(\gamma)) = \tau(\rho_2(\gamma))$ for all $\gamma \in \Gamma$, where $\tau(\rho_i(\gamma))$ denotes the translation length of $\rho_i(\gamma)$.*

When item (b) holds we say that ρ_1 and ρ_2 are *coarsely equivalent*. If both representations ρ_1 and ρ_2 are coarsely equivalent and cocompact, then the spaces X_1 and X_2 are *almost-isometric*. This means that there exists an *almost-isometry* between the spaces, which is a quasi-isometry with multiplicative constant one. This almost-isometry allows to construct a coarse equivariant map between the regular parts of the ideal boundaries of $X_1 \times X_1$ and $X_1 \times X_2$, so that the open set in $\partial_\infty^{\max}(X_1 \times X_1)$ of Theorem 1.1 is mapped to an open set $\Omega_\Gamma^{\max} \subset \partial_\infty^{\max}(X_1 \times X_2)$ with good properties:

Theorem 1.3. *Let X_1 and X_2 be proper CAT(−1) spaces, Γ a hyperbolic group, and $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two cocompact discrete representations. If ρ_1 and ρ_2 are coarsely equivalent, then there exists an open subset $\Omega_\Gamma^{\max} \subset \partial_\infty^{\max}(X_1 \times X_2)$ such that the product action of Γ on $X_1 \times X_2 \cup \Omega_\Gamma^{\max}$ is properly discontinuous and cocompact.*

2. PRELIMINARIES

A metric space is said to be *proper* if all its closed balls are compact, and *geodesic* if any two points can be joined by a geodesic segment.

A CAT(−1) space X is a geodesic metric space where triangles are thinner than comparison triangles in the hyperbolic plane. Similarly, a CAT(0) space satisfies the same condition placing the comparison triangles in the Euclidean plane. In particular, CAT(−1) spaces are also CAT(0) spaces. A reference for these spaces is for instance [4].

Two rays $c(t)$ and $c'(t)$ in a metric space are said to be *asymptotic* if there exists $C < \infty$ such that $d(c(t), c'(t)) \leq C$ for any $t \geq 0$. The *visual boundary* $\partial_\infty X$ of a metric space X is the set of equivalent classes of asymptotic rays. In a proper CAT(0) space $\bar{X} = X \cup \partial_\infty X$ can be given a topology (the cone topology, see [4]) such that both \bar{X} and $\partial_\infty X$ are compact. The space \bar{X} is denoted the *visual compactification*.

A discrete group action on a topological space X is *properly discontinuous* if every compact subset intersects finitely many of its translates. For isometric actions on proper metric spaces, this is equivalent to the fact that every point has an open neighborhood which intersects only finitely many

of its translates. The action is *cocompact* if there exists a compact subset $K \subset X$ whose translates cover X . For Γ a discrete group of isometries of a proper CAT(0) space, the *limit set* Λ_Γ is defined as the set of accumulation points of an orbit in $\partial_\infty X$ and it is independent of the orbit. For a CAT(-1) space X , the complement of Λ_Γ in $\partial_\infty X$ is the *domain of discontinuity* Ω_Γ and Γ acts properly discontinuously on Ω_Γ [6].

A subset $S \subset X$ is *quasi-convex* if an ε -neighborhood of $S \cap X$ contains its quasi-convex hull (the union of segments between points in S), for some $\varepsilon > 0$. A group Γ of the isometries of a CAT(-1) space X is *quasi-convex* if it acts properly discontinuously on X and any orbit is quasi-convex.

A map between metric spaces $f: X \rightarrow Y$ is a *quasi-isometric embedding* if there are constants $A \geq 1$ and $C \geq 0$ satisfying that for all $x_1, x_2 \in X$:

$$\frac{1}{A}d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + C.$$

The map $f: X \rightarrow Y$ is a *quasi-isometry* if it is a quasi-isometric embedding that is *coarsely onto*, namely if for each $y \in Y$ there exists $x \in X$ with:

$$d_Y(f(x), y) \leq C,$$

for some C . An *almost-isometry* is a quasi-isometry with multiplicative constant $A = 1$.

A quasi-convex group of isometries of a proper CAT(-1) space X is hyperbolic and finitely generated. Moreover, the orbit map:

$$\Gamma \rightarrow X$$

$$\gamma \mapsto \gamma o$$

is a quasi-isometric embedding for any $o \in X$ and it extends to an equivariant homeomorphism (which is also Lipschitz and quasi-conformal) from $\partial_\infty \Gamma$ to its limit set Λ_Γ [3]. The action of a quasi-convex group on $X \cup \Omega_\Gamma$ is properly discontinuous [6] and cocompact [18].

3. THE MAX COMPACTIFICATION

Let (X_1, d_1) and (X_2, d_2) be two proper CAT(-1) spaces, consider the product space $X_1 \times X_2$ equipped with the max metric d_{\max} , or ℓ^∞ metric:

$$d_{\max}((x, y), (x', y')) = \max\{d_1(x, x'), d_2(y, y')\}$$

for any $(x, y), (x', y') \in X_1 \times X_2$. The metrics d_{\max} and the product metric

$$d_{\ell^2}((x, y), (x', y')) = \sqrt{d_1(x, x')^2 + d_2(y, y')^2}$$

are comparable so they induce the same topology in $X_1 \times X_2$. For X_1 and X_2 proper geodesic spaces, $(X_1 \times X_2, d_{\max})$ is also a proper geodesic space [16, Prop.2.6.6]. In this section we compute its horofunction compactification.

For a proper metric space X , let $C_*(X)$ denote space of continuous functions on X up to additive constants, equipped with the topology of uniform convergence on compact subsets. The *Gromov* or *horofunction compactification* of X is the closure in $C_*(X)$ of the image of the map

$$\begin{aligned} \iota: X &\rightarrow C_*(X) \\ x &\mapsto [d(x, \cdot)], \end{aligned}$$

see [2]. We denote the horofunction compactification of X by \overline{X} . The ideal boundary is the set $\overline{X} \setminus \iota(X)$ and it is denoted by $\partial_\infty X$. For a proper metric space both \overline{X} and $\partial_\infty X$ are compact and metrizable spaces.

Remark 3.1. *Fixing a base point $o \in X$, the sequence $[d(x_n, \cdot)]$ converges to a class of functions $[f] \in C_*(X)$ if, and only if, the sequence of corresponding normalized distance functions $d(x_n, \cdot) - d(x_n, o)$ converges to $f - f(o)$ uniformly on all balls $B(o, r)$. In addition, $C_*(X)$ is homeomorphic to the subspace of continuous functions on X satisfying $f(o) = 0$.*

Definition 3.2. *A horofunction h is a continuous function on X such that its class $[h]$ belongs to $\partial_\infty X$.*

Remark 3.3. *A class of horofunctions is called an ideal point and it is denoted it by ξ . The horofunction h in the class ξ satisfying $h(o) = 0$ is denoted by h_ξ^o .*

Notation 3.4. *When we say that a sequence $(x_n)_n$ converges to an ideal point ξ in the horofunction compactification, $x_n \rightarrow \xi$, we mean that for a base point $o \in X$ the corresponding sequence of normalized distance functions converges uniformly on compact subsets to the horofunction h_ξ^o .*

The level sets of a horofunction are called *horospheres* and the sublevel sets, *horoballs*. Notice that two horofunctions in the same equivalence class differ by a constant and share the same set of horospheres and horoballs. The horofunctions of a proper CAT(0) space are Busemann functions:

Definition 3.5 (See [1] or [2]). *A Busemann function in a metric space (X, d) is a function defined as:*

$$z \mapsto \lim_{t \rightarrow +\infty} d(c(t), z) - t$$

for some geodesic ray $c(t)$ in X .

In a proper CAT(0) space X , given a point $o \in X$ and an ideal point $\xi \in \partial_\infty X$ there is a unique ray $c(t)$ such that $c(0) = o$ and its associated Busemann function is in the class ξ . This Busemann function is denoted by

$$\beta_\xi^o(z) = \lim_{t \rightarrow +\infty} d(c(t), z) - t,$$

The horofunction compactification and the visual compactification of a proper CAT(0) space are equivalent [4, Cor. 8.20].

Lemma 3.6 (cf. [4]). *For a CAT(0) space X :*

- (i) *If $\sigma: [0, +\infty) \rightarrow X$ is a ray in the class $\xi \in \partial_\infty X$, then $\beta_\eta^o(\sigma(s))$ converges to $+\infty$ if $\eta \neq \xi$ and to $-\infty$ if $\eta = \xi$.*
- (ii) *For any $p, q \in X$, $\beta_\eta^p - \beta_\eta^q$ is a constant function.*

The Gromov or horofunction compactification of $(X_1 \times X_2, d_{\max})$ is denoted by $\overline{X_1 \times X_2}^{\max}$ and its ideal boundary by $\partial_\infty^{\max}(X_1 \times X_2)$. We choose a base point $O = (o, o')$ with $o \in X_1$ and $o' \in X_2$. As a representative of a class of normalized distance functions, we have the function:

$$\begin{aligned} d_{\max}^O((x, y), \cdot) &= d_{\max}((x, y), \cdot) - d_{\max}((x, y), (o, o')) \\ &= \max\{d_1(x, \cdot), d_2(y, \cdot)\} - \max\{d_1(x, o), d_2(y, o')\}. \end{aligned}$$

Then, by Remark 3.1, $[d_{\max}((x_n, y_n), \cdot)] \rightarrow \xi \in \partial_{\infty}^{\max}(X_1 \times X_2)$ if and only if $d_{\max}^O((x_n, y_n), \cdot) \rightarrow h_{\xi}^O$, where h_{ξ}^O is the horofunction in the class ξ that satisfies $h_{\xi}^O(O) = 0$.

Given a diverging sequence $(x_n, y_n) \subset (X_1 \times X_2, d_{\max})$, we distinguish two cases, up to a subsequence:

- (I) either $|d_1(x_n, o) - d_2(y_n, o')| \rightarrow \infty$,
- (II) or $|d_1(x_n, o) - d_2(y_n, o')|$ remains bounded.

Notice that if one of $d_1(x_n, o)$ or $d_2(y_n, o')$ is bounded, then we are in the first case, as we assume that (x_n, y_n) diverges.

Proposition 3.7. *Let (X_1, d_1) and (X_2, d_2) be proper CAT(-1) metric spaces. Let (x_n, y_n) be a diverging sequence in $(X_1 \times X_2, d_{\max})$.*

- (I) *If $|d_1(x_n, o) - d_2(y_n, o')| \rightarrow \infty$, then, up to subsequence and up to permuting X_1 and X_2 , there exists $\xi \in \partial_{\infty} X_1$ such that*

$$\lim_{n \rightarrow \infty} d_{\max}^O((x_n, y_n), (z, z')) = \beta_{\xi}^o(z).$$

- (II) *If $|d_1(x_n, o) - d_2(y_n, o')|$ remains bounded, then, up to a subsequence, there exist $\xi \in \partial_{\infty} X_1$ and $\xi' \in \partial_{\infty} X_2$ such that*

$$\lim_{n \rightarrow \infty} d_{\max}^O((x_n, y_n), (z, z')) = \max\{\beta_{\xi}^o(z), \beta_{\xi'}^{o'}(z') - C\},$$

for some constant $C \in \mathbb{R}$.

Proof. We prove case (II), the proof for case (I) being similar. For each n , denote $C_n = d_1(x_n, o) - d_2(y_n, o')$ and assume that $C_n \geq 0$. Then

$$d_{\max}((x_n, y_n), (o, o')) = d_1(x_n, o) = d_2(y_n, o') + C_n$$

and

$$\begin{aligned} d_{\max}^O((x_n, y_n), (z, z')) &= \\ d_{\max}((x_n, y_n), (x, y)) - d_{\max}((x_n, y_n), (o, o')) &= \\ \max\{d_1(x_n, x) - d_1(x_n, o), d_2(y_n, y) - d_2(y_n, o') - C_n\}. \end{aligned}$$

Both sequences x_n and y_n subconverge to an ideal point, and since C_n is bounded we are done. \square

In the remaining of the section, (X_1, d_1) and (X_2, d_2) denote proper CAT(-1) metric spaces, as in Proposition 3.7.

Definition 3.8. *We define the singular part of the ideal boundary as the subset of ideal points with a representative of the form $\beta_{\xi}^o(z)$ or $\beta_{\xi'}^{o'}(z')$, i.e. case (I) in Proposition 3.7. We denote it by $\partial_{\infty}^{\max}(X_1 \times X_2)_{\text{sing}}$.*

The regular part of the ideal boundary is its complement, namely the subset of ideal points with a representative of the form $\max\{\beta_{\xi}^o(z), \beta_{\xi'}^{o'}(z') + C\}$ with $C \in \mathbb{R}$, i.e. case (II). We denote it by $\partial_{\infty}^{\max}(X_1 \times X_2)_{\text{reg}}$.

Using that the set of Busemann functions in one factor is naturally identified to the boundary of this factor, we have:

Proposition 3.9. *There is a natural homeomorphism*

$$\varphi_{\text{sing}}: \partial_{\infty}^{\max}(X_1 \times X_2)_{\text{sing}} \longrightarrow \partial_{\infty} X_1 \sqcup \partial_{\infty} X_2$$

that consists in associating to a Busemann function that takes values only in the first (second) factor of $X_1 \times X_2$ the same Busemann function viewed as a point of the first (second) summand in $\partial_\infty X_1 \sqcup \partial_\infty X_2$.

For the regular part, notice that we can get rid of the additive constant in Proposition 3.7 by changing the base point. Thus regular points are the classes modulo constant of the functions $\max\{\beta_\xi^p(z), \beta_{\xi'}^{p'}(z')\}$ for all $p \in X_1$, $p' \in X_2$, $\xi \in \partial_\infty X_1$ and $\xi' \in \partial_\infty X_2$.

Proposition 3.10. *For each choice of base point $(o, o') \in X_1 \times X_2$ there is a natural homeomorphism*

$$(1) \quad \begin{aligned} \varphi_{\text{reg}}: \partial_\infty^{\text{max}}(X_1 \times X_2)_{\text{reg}} &\longrightarrow \partial_\infty X_1 \times \partial_\infty X_2 \times \mathbb{R} \\ \left[\max\{\beta_\xi^p(z), \beta_{\xi'}^{p'}(z')\} \right] &\mapsto (\xi, \xi', \beta_\xi^p(o) - \beta_{\xi'}^{p'}(o')). \end{aligned}$$

Remark 3.11. *If we fix $p = o$ and $p' = o'$, then homeomorphism (1) can be written as:*

$$(2) \quad \begin{aligned} \varphi_{\text{reg}}: \partial_\infty^{\text{max}}(X_1 \times X_2)_{\text{reg}} &\longrightarrow \partial_\infty X_1 \times \partial_\infty X_2 \times \mathbb{R} \\ \max\{\beta_\xi^o(z), \beta_{\xi'}^{o'}(z') - C\} &\mapsto (\xi, \xi', C) \end{aligned}$$

where $C \in \mathbb{R}$.

Proof of Prop. 3.10. Notice that $\lim_{t \rightarrow +\infty} \max\{\beta_\xi^p(c(t)), \beta_{\xi'}^{p'}(c'(t))\} = -\infty$ if and only if $c(+\infty) = \xi$ and $c'(+\infty) = \xi'$; otherwise this limit is $+\infty$, by Lemma 3.6. Thus ξ and ξ' in the construction of φ_{reg} are uniquely determined, and it follows easily that φ_{reg} is well defined and injective. In addition, surjectivity of φ_{reg} and continuity of $\varphi_{\text{reg}}^{-1}$ follow easily from construction and the properties of Busemann functions (Lemma 3.6).

To prove continuity of φ_{reg} , as ideal boundaries are metrizable, we use sequences. Let $\max\{\beta_{\xi_n}^o(z), \beta_{\xi'_n}^{o'}(z') - C_n\}$ be a sequence that converges to $\max\{\beta_\eta^o(z), \beta_{\eta'}^{o'}(z') - C\}$. The third coordinate of φ_{reg} in (1) is clearly continuous, hence $C_n \rightarrow C$. By compactness of $\partial_\infty X_i$, up to subsequence $\xi_n \rightarrow \xi_\infty$ and $\xi'_n \rightarrow \xi'_\infty$. By injectivity of φ_{reg} , $\xi_\infty = \eta$ and $\xi'_\infty = \eta'$ and we get continuity. \square

Remark 3.12. *Observe that a sequence (x_n, y_n) converges to (ξ, ξ', C) if, and only if, $x_n \rightarrow \xi$, $y_n \rightarrow \xi'$, and $d_1(x_n, o) - d_2(y_n, o') \rightarrow C$.*

Let $\text{Join}(\partial_\infty X_1, \partial_\infty X_2)$ denote the topological join of $\partial_\infty X_1$ and $\partial_\infty X_2$. Propositions 3.9 and 3.10 can be improved:

Proposition 3.13. *There is a natural homeomorphism*

$$\partial_\infty^{\text{max}}(X_1 \times X_2) \cong \text{Join}(\partial_\infty X_1, \partial_\infty X_2).$$

Proof. In view of Propositions 3.9 and 3.10 and Remark 3.11, we have to prove the following claim: for sequences (ξ_n) in $\partial_\infty X_1$, (ξ'_n) in $\partial_\infty X_2$, and (C_n) in \mathbb{R} we have $\xi_n \rightarrow \xi$, $\xi'_n \rightarrow \xi'$, and $C_n \rightarrow +\infty$ as $n \rightarrow \infty$ if and only if the function

$$(z, z') \mapsto \max\{\beta_{\xi_n}^o(z), \beta_{\xi'_n}^{o'}(z') - C_n\}$$

converges to $(z, z') \mapsto \beta_\xi^o(z)$ uniformly on compact subsets of $X_1 \times X_2$. Notice that we do not require convergence on (ξ'_n) . We also need the symmetric claim when $C_n \rightarrow -\infty$, and after replacing $\max\{\beta_{\xi_n}^o(z), \beta_{\xi'_n}^{o'}(z') - C_n\}$ by

$\max\{\beta_{\xi_n}^o(z) + C_n, \beta_{\xi_n'}^{o'}(z')\}$ (a different function in the same equivalence class) but the proof is symmetric.

To prove the claim assume first that $C_n \rightarrow +\infty$. Take as compact set the ball centered at (o, o') : $B(o, R) \times B(o', R)$. Since Busemann functions are 1-Lipschitz and as we chose normalizations so that $\beta_{\xi_n}^o(o) = \beta_{\xi_n'}^{o'}(o') = 0$, for $C_n \geq 2R$ we have $\max\{\beta_{\xi_n}^o(z), \beta_{\xi_n'}^{o'}(z') - C_n\} = \beta_{\xi_n}^o(z)$ for $(z, z') \in B(o, R) \times B(o', R)$. Here uniform convergence of $\beta_{\xi_n}^o$ on compact subsets follows from the horosphere compactification of X_1 . Next assume $C_n \in [-R, R]$. Here $\max\{\beta_{\xi_n}^o(z), \beta_{\xi_n'}^{o'}(z') - C_n\}$ has a converging subsequence to $\max\{\beta_{\xi}^o(z), \beta_{\xi'}^{o'}(z') - C\}$, uniformly on compact subsets. Using that the Busemann functions have slope -1 in rays pointing to the ideal point, we see that the limit $\max\{\beta_{\xi}^o(z), \beta_{\xi'}^{o'}(z') - C\}$ cannot be expressed a Busemann function in a single factor, β_{ξ}^o or $\beta_{\xi'}^{o'}$. \square

4. DIAGONAL ACTIONS

Let Γ be an infinite quasi-convex group of isometries of a proper $\text{CAT}(-1)$ space X . In this section we consider the diagonal action of Γ on $X \times X$:

$$\begin{aligned} \Gamma \times X \times X &\rightarrow X \times X \\ (\gamma, x, y) &\mapsto (\gamma x, \gamma y). \end{aligned}$$

The diagonal action extends continuously to the ideal boundary of the max compactification. The following is straightforward:

Lemma 4.1. *The diagonal action on the points of $\partial_{\infty}^{\max}(X \times X)$ is given by:*

$$\begin{aligned} \gamma[\beta_{\xi}^o] &= [\beta_{\gamma\xi}^o] \\ \gamma[\max\{\beta_{\xi}^o, \beta_{\xi'}^{o'} - C\}] &= [\max\{\beta_{\gamma\xi}^o, \beta_{\gamma\xi'}^{o'} - C + \beta_{\xi'}^{o'}(\gamma^{-1}o) - \beta_{\xi}^o(\gamma^{-1}o)\}]. \end{aligned}$$

Remark 4.2. *Under the identification in Remark 3.11 the diagonal action maps a singular point ξ to $\gamma\xi$, and a regular point (ξ, ξ', C) , to $(\gamma\xi, \gamma\xi', C + \beta_{\xi}^o(\gamma^{-1}o) - \beta_{\xi'}^{o'}(\gamma^{-1}o))$.*

In this section we prove that there is an open subset $\Omega_{\Gamma}^{\max} \subset \partial_{\infty}^{\max}(X \times X)$ where the diagonal action of Γ is properly discontinuous and cocompact. In Subsection 4.1 we prove that the nearest point projection of $X \times X$ to the diagonal $\Delta \subset X \times X$ extends continuously to $\overline{X} \times \overline{X}^{\max}$. In Subsection 4.2 we use this projection to show that there exist a proper domain of discontinuity $\Omega_{\Gamma}^{\max} \subset \partial_{\infty}^{\max}(X \times X)$. Furthermore we see that the action on the whole $X \times X \cup \Omega_{\Gamma}^{\max}$ is properly discontinuous and cocompact, and that Ω_{Γ}^{\max} is the largest open set of the boundary that satisfies these conditions.

4.1. Extending the projection to the diagonal. The nearest point projection from $X \times X$ to the diagonal for the max distance is given by the midpoint:

$$(3) \quad \begin{aligned} \pi: X \times X &\rightarrow \Delta \\ (x, y) &\mapsto (m, m), \end{aligned}$$

where Δ is the diagonal in $X \times X$:

$$\Delta = \{(x, x) \mid x \in X\},$$

and m is the midpoint of the geodesic segment joining x and y . By construction, π is continuous and equivariant.

In this section we extend it continuously to a map

$$\tilde{\pi}: \overline{X \times X}^{\max} \rightarrow \overline{\Delta}^{\max} = \Delta \cup \Delta_{\infty},$$

where $\overline{\Delta}^{\max}$ is the closure of Δ in $\overline{X \times X}^{\max}$, and

$$\Delta_{\infty} = \{(\xi, \xi) \mid \xi \in \partial_{\infty} X\},$$

denotes the diagonal in $\partial_{\infty} X \times \partial_{\infty} X$. For this purpose, we consider the decomposition

$$\partial_{\infty}^{\max}(X \times X) = \partial_{\infty}^{\max}(X \times X)_{\text{sing}} \sqcup \varphi_{\text{reg}}^{-1}(\Delta_{\infty} \times \mathbb{R}) \sqcup \Omega^{\max}$$

where

$$\Omega^{\max} = \varphi_{\text{reg}}^{-1}((\partial_{\infty} X \times \partial_{\infty} X \setminus \Delta_{\infty}) \times \mathbb{R}) \subset \partial_{\infty}^{\max}(X \times X)_{\text{reg}},$$

and φ_{reg} is the homeomorphism in Proposition 3.10. In [9] the projection is extended continuously to a map

$$\Omega^{\max} \rightarrow \Delta.$$

Following [9], the extension uses that Ω^{\max} is naturally homeomorphic to the set G of parameterized geodesics in X (with the topology of uniform convergence on compact sets) through the map:

$$\begin{aligned} \varphi: G &\longrightarrow \Omega^{\max} \\ g &\mapsto \lim_{n \rightarrow \infty} (g(n), g(-n)) \end{aligned}$$

Via this identification, by [9] the projection extends continuously to

$$(4) \quad \begin{aligned} G &\rightarrow \Delta \\ g &\mapsto (g(0), g(0)) \end{aligned}$$

Thus it remains to extend it to $\varphi_{\text{reg}}(\Delta_{\infty} \times \mathbb{R})$ and to $\partial_{\infty}^{\max}(X \times X)_{\text{sing}}$.

Definition 4.3. *The extended projection*

$$\tilde{\pi}: \overline{X \times X}^{\max} \rightarrow \overline{\Delta}^{\max} \cong \overline{X}$$

is defined by (3) and (4) on $X \times X \cup \Omega^{\max}$.

On $\varphi_{\text{reg}}(\Delta_{\infty} \times \mathbb{R})$ it is the projection to Δ_{∞} , and on $\partial_{\infty}^{\max}(X \times X)_{\text{sing}} \cong \partial_{\infty} X \sqcup \partial_{\infty} X$ it is the identification $\partial_{\infty} X \cong \Delta_{\infty}$.

Remark 4.4. *We have an equivariant homeomorphism $\varphi' = \varphi_{\text{reg}} \circ \varphi$, given by:*

$$\begin{aligned} \varphi': G &\rightarrow ((\partial_{\infty} X \times \partial_{\infty} X) \setminus \Delta_{\infty}) \times \mathbb{R} \\ g &\mapsto (g(+\infty), g(-\infty), C_g), \end{aligned}$$

where:

$$C_g = \lim_{n \rightarrow \infty} d(g(n), o) - d(g(-n), o) = \beta_{g(-\infty)}^o(g(0)) - \beta_{g(+\infty)}^o(g(0)).$$

Therefore, a geodesic g corresponds to a point

$$(g(+\infty), g(-\infty), C_g) \in ((\partial_{\infty} X \times \partial_{\infty} X) \setminus \Delta_{\infty}) \times \mathbb{R}$$

which in its turn corresponds to the regular point:

$$[\max\{\beta_{g(+\infty)}^o, \beta_{g(-\infty)}^o - C_g\}] \in \Omega^{\max}.$$

To prove the continuity of $\tilde{\pi}$, we use of that the Gromov product extends continuously to the boundary of a proper CAT(-1) space [5, Proposition 3.4.2]. The *Gromov product* of two points $x, y \in X$ with respect to a base point $o \in X$ is defined as:

$$(x|y)_o = \frac{1}{2} [d(x, o) + d(y, o) - d(x, y)].$$

Given ξ and ξ' two points in the visual boundary of a proper CAT(-1), the Gromov product is defined as:

$$\lim_{i,j} (x_i|y_j)_o = (\xi|\xi')_o,$$

for any sequences $x_i \rightarrow \xi$, $y_j \rightarrow \xi'$.

Let g be a geodesic in X , the Gromov product of the ideal points $g(+\infty)$ and $g(-\infty)$ with respect to a base point o , can be written in terms of Busemann functions as:

$$(g(+\infty)|g(-\infty))_o = \frac{1}{2} [\beta_{g(+\infty)}^{g(0)}(o) + \beta_{g(-\infty)}^{g(0)}(o)].$$

The Gromov product for ideal points satisfies:

$$(\xi|\xi')_o = +\infty \text{ if and only if } \xi = \xi',$$

see [5]. Similarly two sequences x_i, y_j have the same limit iff:

$$(x_i|y_j)_o = +\infty.$$

Theorem 4.5. *The map $\tilde{\pi}: \overline{X \times X}^{\max} \rightarrow \overline{X}$ is continuous and equivariant.*

Proof. The equivariance follows from naturality. To prove the continuity, we have also shown in [9] that $\tilde{\pi}$ restricted to $X \times X \cup \Omega^{\max}$ is continuous, but it remains to be proved in $\partial_{\infty}^{\max}(X \times X) \setminus \Omega^{\max}$. We have to check two cases. (I) Firstly, we shall see that the image of a sequence of points (x_n, y_n) in $X \times X$ that converges to an ideal point, either in the singular part or in the diagonal of the regular part of the boundary, converges to the image of this ideal point. (II) Secondly, we shall check that the image of a sequence of ideal points that converges to an ideal point either in the diagonal of the regular part or in the singular part of the boundary, converges to the image of the ideal point. Along this proof, m_n denotes the midpoint of the segment joining x_n and y_n .

Case (I). Consider a sequence (x_n, y_n) in $X \times X$ converging to an ideal point. We distinguish two subcases: either (a) the limit of the sequence is a singular point, or (b) the limit is a point in the diagonal of the regular part.

Subcase (a). Suppose, up to permuting factors, that the sequence converges to a singular point in the boundary of the first factor: $(x_n, y_n) \rightarrow [\beta_{\xi}^o]$. Therefore, $x_n \rightarrow \xi$ and $d(x_n, o) - d(y_n, o) \rightarrow +\infty$. By the triangle inequality: $d(m_n, o) \geq d(m_n, y_n) - d(y_n, o)$, and using the definition of the Gromov product, we have:

$$(x_n|m_n)_o \geq \frac{1}{2} [d(x_n, o) - d(y_n, o)].$$

Henceforth $(x_n|m_n)_o \rightarrow +\infty$, and by the properties of the Gromov product, x_n and m_n have the same limit.

Subcase (b). Now suppose that the sequence converges to a diagonal point in the regular part of the boundary: $(x_n, y_n) \rightarrow [\max\{\beta_\xi^o, \beta_\xi^o + C\}]$. In this case $x_n \rightarrow \xi$, $y_n \rightarrow \xi$ and $d(x_n, o) - d(y_n, o) \rightarrow -C$.

Using the definition of the Gromov product again and reorganizing terms, we have:

$$(5) \quad 2(x_n|m_n)_o = (x_n|y_n)_o + d(m_n, o) + \frac{1}{2} [d(x_n, o) - d(y_n, o)].$$

Notice that $\frac{1}{2} [d(x_n, o) - d(y_n, o)]$ is uniformly bounded and that $(x_n|y_n)_o \rightarrow +\infty$, since both x_n and y_n converge to the same point. From (5) we deduce that $(x_n|m_n)_o \rightarrow +\infty$, which implies that $m_n \rightarrow \xi$.

Case (II). Next we deal with a sequence of regular ideal points of the form $[\max\{\beta_{\xi_n}^o, \beta_{\xi_n}^o + C_n\}]$ in $\partial_\infty^{\max}(X \times X)$ with limit either a regular point in the diagonal, subcase (a), or a singular point, subcase (b). From now on, g_n denotes the geodesic corresponding to a point $[\max\{\beta_{\xi_n}^o, \beta_{\xi_n}^o + C_n\}]$ under the identification $\Omega^{\max} \cong G$.

Subcase (a). Suppose that the sequence converges to a regular diagonal point: $[\max\{\beta_{\xi_n}^o, \beta_{\xi_n}^o + C_n\}] \rightarrow [\max\{\beta_\xi^o, \beta_\xi^o + C\}]$. In this case $\xi_n \rightarrow \xi$, $\xi_n' \rightarrow \xi$ and $C_n \rightarrow C$. For each n , we consider a sequence of points x_k in X such that $x_k \rightarrow \xi_n$. Using the fact that the Gromov product extends continuously to the boundary of a CAT(-1) space, and the definition of Busemann function, we write:

$$\begin{aligned} (g_n(0)|\xi_n)_o &= \lim_{k \rightarrow \infty} (g_n(0)|x_k)_o = \lim_{k \rightarrow \infty} \frac{1}{2} [d(g_n(0), o) + d(x_k, o) \\ &\quad - d(g_n(0), x_k)] = \frac{1}{2} [d(g_n(0), o) + \beta_{\xi_n}^{g_n(0)}(o)]. \end{aligned}$$

Similarly, taking a sequence y_k in X for each n , with $y_k \rightarrow \xi_n'$:

$$(g_n(0)|\xi_n')_o = \frac{1}{2} [d(g_n(0), o) + \beta_{\xi_n'}^{g_n(0)}(o)].$$

Adding both equalities above we obtain:

$$\begin{aligned} (6) \quad (g_n(0)|\xi_n)_o + (g_n(0)|\xi_n')_o &= d(g_n(0), o) + \frac{1}{2} \beta_{\xi_n}^{g_n(0)}(o) + \frac{1}{2} \beta_{\xi_n'}^{g_n(0)}(o) \\ &= d(g_n(0), o) + (\xi_n|\xi_n')_o. \end{aligned}$$

By compactness of \overline{X} , after passing to a subsequence we may assume that $g_n(0) \rightarrow \eta \in \overline{X}$. Then, since $(\xi_n|\xi_n')_o \rightarrow (\xi|\xi)_o = +\infty$, by (6) we have that $(\eta|\xi)_o = +\infty$. So $\xi = \eta$ and $g_n(0) \rightarrow \xi$.

Subcase (b). Next we suppose that the sequence converges to a singular point, a Busemann function in the second factor (up to permutation of factors): $[\max\{\beta_{\xi_n}^o, \beta_{\xi_n}^o + C_n\}] \rightarrow [\beta_{\xi'}^o]$ with $\xi_n \rightarrow \xi$, $\xi_n' \rightarrow \xi'$ and since $\beta_{\xi'}^o$ is a Busemann function in the second factor, $C_n \rightarrow +\infty$.

Similarly to the preceding case, for each n :

$$(g_n(0)|\xi_n')_o = \frac{1}{2} [d(g_n(0), o) - \beta_{\xi_n'}^o(g_n(0))],$$

$$(g_n(0)|\xi_n)_o = \frac{1}{2} [d(g_n(0), o) - \beta_{\xi_n}^o(g_n(0))],$$

and we can combine both equalities to get:

$$\begin{aligned} (g_n(0)|\xi'_n)_o &= \frac{1}{2}d(g_n(0), o) - \frac{1}{2}\beta_{\xi'_n}^o(g_n(0)) \\ &= (g_n(0)|\xi_n)_o + \frac{1}{2}\beta_{\xi_n}^o(g_n(0)) - \frac{1}{2}\beta_{\xi'_n}^o(g_n(0)) \\ &= (g_n(0)|\xi_n)_o + \frac{1}{2}C_n. \end{aligned}$$

Here we have used that $C_n = \beta_{\xi_n}^o(g_n(0)) - \beta_{\xi'_n}^o(g_n(0))$ by Remark 4.4. Now, since $(g_n(0)|\xi_n)_o \geq 0$ and $C_n \rightarrow +\infty$ we have that $(g_n(0)|\xi'_n)_o \rightarrow +\infty$ and $g_n(0) \rightarrow \xi'$. \square

4.2. The ideal domain Ω_Γ^{\max} . Let Γ be an infinite quasi-convex group of isometries of X . We denote by Λ_Γ its limit set, which is the set of accumulation points of any orbit in $\partial_\infty X$, and by $\Omega_\Gamma = \partial_\infty X \setminus \Lambda_\Gamma$ its domain of discontinuity. The action of Γ on $X \cup \Omega_\Gamma$ is properly discontinuous and cocompact [6, 18]. Next we show that the diagonal action of Γ on the inverse image under the projection $\tilde{\pi}$ of $X \cup \Omega_\Gamma$ is also properly discontinuous and cocompact:

Theorem 4.6. *Let X be a proper CAT(-1) space and let $\Gamma \subset \text{Isom}(X)$ be an infinite quasi-convex group. The diagonal action of Γ on $\tilde{\pi}^{-1}(X \cup \Omega_\Gamma)$ is properly discontinuous and cocompact.*

Proof. Besides being continuous and equivariant, $\tilde{\pi}: \overline{X} \times \overline{X}^{\max} \rightarrow \overline{X}$ is proper, since it is a continuous map from a compact to a Hausdorff space.

In [18] it is shown that for a Dirichlet domain $D \subset X$ its closure in $\overline{D} \subset \overline{X}$ is a compact set that satisfies:

- (i) $\overline{D} \subset X \cup \Omega_\Gamma$,
- (ii) $\bigcup_{\gamma \in \Gamma} \gamma \overline{D} = X \cup \Omega_\Gamma$, and
- (iii) for every compact $K \subset X \cup \Omega_\Gamma$, $|\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}| < \infty$.

Therefore $\tilde{\pi}^{-1}(\overline{D})$ satisfies the corresponding properties for the action on $\tilde{\pi}^{-1}(X \cup \Omega_\Gamma)$. This proves the theorem. \square

Now, let Ω_Γ^{\max} be the intersection of $\tilde{\pi}^{-1}(X \cup \Omega_\Gamma)$ with the ideal boundary of $\overline{X} \times \overline{X}^{\max}$:

$$\Omega_\Gamma^{\max} = \tilde{\pi}^{-1}(X \cup \Omega_\Gamma) \cap \partial_\infty^{\max}(X \times X),$$

and let Δ_Γ be the subset of the diagonal in $\partial_\infty X \times \partial_\infty X$ that corresponds to the limit points of the action of Γ on X :

$$\Delta_\Gamma = \{(\xi, \xi) \in \partial_\infty X \times \partial_\infty X \text{ with } \xi \in \Lambda_\Gamma\}.$$

Remark 4.7. *Via the homeomorphism in Proposition 3.13 that identifies $\partial_\infty^{\max}(X \times X)$ with $\text{Join}(\partial_\infty X, \partial_\infty X)$:*

$$\Omega_\Gamma^{\max} \cong ((\partial_\infty X \times \partial_\infty X) \setminus \Delta_\Gamma) \times \mathbb{R} \cup \Omega_\Gamma^1 \cup \Omega_\Gamma^2,$$

where Ω_Γ^i denotes Ω_Γ viewed in the factor $\partial_\infty X_i$, for each $i = 1, 2$. Observe that when Γ is a cocompact group Ω_Γ^{\max} is just the set Ω^{\max} of the previous subsection.

In Proposition 4.10 we shall show that Ω_Γ^{\max} is the largest open set of the boundary where the diagonal action is properly discontinuous. But first, let us study the limit set of this action on $\partial_\infty^{\max}(X \times X)$. Since $(X \times X, d_{\max})$ is not CAT(0), the accumulation set of each orbit depends on the orbit. We define the *large limit set* of the diagonal action as the union of all the accumulation sets of orbits on $\partial_\infty^{\max}(X \times X)$:

$$\Lambda = \bigcup_{(x,y) \in X \times X} \overline{\Gamma(x,y)} \cap \partial_\infty^{\max}(X \times X)$$

Lemma 4.8. *For Λ and Δ_Γ as above,*

$$\varphi_{\text{reg}}(\Lambda) = \Delta_\Gamma \times \mathbb{R}.$$

Proof. First, observe that the limit of any sequence $(\gamma_n x, \gamma_n y)$ that converges to the ideal boundary is contained in $\Delta_\Gamma \times \mathbb{R}$. Indeed, by the triangle inequality, $|d(\gamma_n x, o) - d(\gamma_n y, o)| \leq d(x, y)$, so the limit is a regular point. Furthermore, if $\gamma_n x \rightarrow \xi$ then $\gamma_n y \rightarrow \xi$ since $d(\gamma_n x, \gamma_n y) = d(x, y)$, i.e. the sequences $\gamma_n x$ and $\gamma_n y$ stay within a bounded distance. Therefore the limit point lies in $\Delta_\Gamma \times \mathbb{R}$.

Finally we show that any point (ξ, ξ, C) for $\xi \in \Lambda_\Gamma$ belongs to the limit set. Take any sequence γ_n such that $\gamma_n o \rightarrow \xi$. Let ξ' be an accumulation point for $\gamma_n^{-1} o$ and $x, y \in X$ satisfying $\beta_{\xi'}^o(y) - \beta_{\xi'}^o(x) = C$. Then it follows easily that (ξ, ξ, C) is the limit of the sequence $(\gamma_n x, \gamma_n y)$. \square

Remark 4.9. *The large limit set Λ is not closed but notice that the complement of Ω_Γ^{\max} is its closure $\overline{\Lambda}$.*

From this remark we easily deduce:

Proposition 4.10. *The set Ω_Γ^{\max} is the largest open subset of $\partial_\infty^{\max}(X \times X)$ such that the action of Γ on $X \times X \cup \Omega_\Gamma^{\max}$ is properly discontinuous.*

5. EXAMPLES

In this section we consider some examples for the max compactification of diagonal actions. The first one is the diagonal action of a cocompact group of isometries of a riemannian manifold. The second one is the action of convex cocompact Kleinian groups on $\mathbb{H}^n \times \mathbb{H}^n$. Finally we describe an example of the max compactification of a diagonal action on the product of two trees. This is also an example where the nearest point projection to the diagonal is not a fibration.

5.1. Compact riemannian manifolds. As in the previous section, let $\pi: X \times X \rightarrow \Delta$ denote the nearest point projection to the diagonal for the max distance. By (3), the fibre $\pi^{-1}(a, a)$ is the set of pairs $(x, y) \in X \times X$ such that a is the midpoint of the segment joining x and y . If $X = \mathbb{H}^2$, then the fibre $F_a = \pi^{-1}(a, a)$ is:

$$F_a = \{(x, s_a x) \mid x \in \mathbb{H}^2\}$$

where $s_a x$ is the symmetric point of x with respect to a . Then $F_a \cong \mathbb{H}^2$ for any a . The boundary at infinity of the fibre F_a is the set of parameterized geodesics with $g(0) = a$, so:

$$\partial_\infty^{\max} F_a = \{g \mid g(0) = a\} \cong (T_a \mathbb{H}^2)^1$$

For $S = \mathbb{H}^2/\Gamma$ a compact hyperbolic surface, the max compactification of $(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma$ is the fibration by closed disks of $S = \mathbb{H}^2/\Gamma$, so:

$$\overline{(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma}^{\max} \cong US,$$

where $US = \{(x, v) \in TS \mid |v| \leq 1\}$.

The same is true for a Cartan-Hadamard manifold X of dimension n and sectional curvature ≤ -1 : $\partial_\infty F_x$ is identified with the unit tangent sphere at x , $(T_x X)^1 \cong S^{n-1}$, and the fibre over each point of the diagonal is a closed disk. If $M = X/\Gamma$ is a compact manifold, then the compactification of $(X \times X)/\Gamma$ with respect to the max metric is homeomorphic to the fibration by closed disks of $M = X/\Gamma$:

$$\overline{(X \times X)/\Gamma}^{\max} \cong UM,$$

where $UM = \{(x, v) \in TM \mid |v| \leq 1\}$.

5.2. Convex cocompact Kleinian groups. Let $\Gamma < \text{Isom}^+(\mathbb{H}^n)$ be a discrete torsion free subgroup, that is convex cocompact. Assume that $M = \mathbb{H}^n/\Gamma$ is not compact, then it has finitely many ends and its compactification consists in adding a compact conformal $(n-1)$ manifold $N_i^{n-1} = \Omega_i/\Gamma_i$, where Ω_i is a connected component of the discontinuity domain $\Omega = \partial_\infty \mathbb{H}^n \setminus \Lambda_\Gamma$.

The compactification $\overline{(\mathbb{H}^n \times \mathbb{H}^n)/\Gamma}^{\max}$ is the union of the fibration by compact balls on M and a finite collection of products of the conformal ideal manifolds with intervals, $N_i^{n-1} \times \mathbb{R}$, where $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\} \cong [0, 1]$.

To understand how these products $N_i^{n-1} \times [0, 1]$ are attached, we consider a diverging geodesic ray in $r: [0, +\infty) \rightarrow M$, corresponding to a point in $\partial_\infty M$. For each $t \in [0, +\infty)$, the fibre $\pi^{-1}(r(t))$ is a compactified hyperbolic space $\overline{\mathbb{H}^n}$, we aim to understand how they fit with a compactified \mathbb{R} when $t \rightarrow \infty$. Assume that r is a ray in \mathbb{H}^n . Points in $\pi^{-1}(r(t))$ are of the form $(\exp_{r(t)}(v), \exp_{r(t)}(-v))$, for some $v \in T_{r(t)} \mathbb{H}^n$. To compare different fibers, let V be a parallel vector field along r that is unitary. Let $\theta \in [0, 2\pi)$ be the angle between $r'(t)$ and $V(t)$, which is constant by parallelism. Then every point of $\pi^{-1}(r(t))$ is written as

$$(\exp_{r(t)}(sV(t)), \exp_{r(t)}(-sV(t)))$$

for some V as above and $s \in \mathbb{R}_{\geq 0}$.

Proposition 5.1. *Given V as above and $s \in \mathbb{R}_{\geq 0}$,*

$$\lim_{t \rightarrow +\infty} (\exp_{r(t)}(sV(t)), \exp_{r(t)}(-sV(t))) = \max\{\beta_{r(+\infty)}^{r(0)}, \beta_{r(+\infty)}^{r(0)} - 2d\}$$

where $d \in \mathbb{R}$ is defined by the relation

$$(7) \quad \tanh d = \cos \theta \tanh s$$

and θ is the (constant) angle between r' and V .

The relation (7) means that d is the signed distance from $r(t)$ to the orthogonal projection of $\exp_{r(t)}(sV(t))$ to the ray r , see Figure 1. This proposition explains how the $\overline{\mathbb{H}^n}$ in the fibre are attached to a segment in the limit: the whole \mathbb{H}^n is projected orthogonally to the geodesic containing r , and, by continuous extension, $\overline{\mathbb{H}^n}$ is projected to $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$.

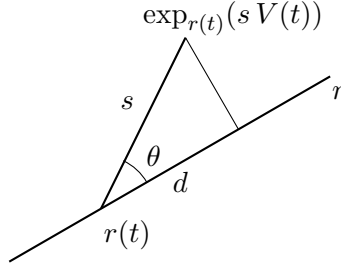


FIGURE 1. The triangle for Equality (7) in Proposition 5.1

Proof. As the distance from $\exp_{r(t)}(\pm s V(t))$ to $r(t)$ is s , we know that the limit is a maximum of Busemann functions centered at $r(+\infty)$. Therefore we only have to compute the limit

$$\lim_{t \rightarrow +\infty} d(\exp_{r(t)}(s V(t)), r(0)) - d(\exp_{r(t)}(-s V(t)), r(0)).$$

Set $d_{\pm}(t) = d(\exp_{r(t)}(\pm s V(t)), r(0))$. By the hyperbolic cosine formula:

$$\cosh d_+(t) = \cosh s \cosh t - \cos \theta \sinh s \sinh t,$$

$$\cosh d_-(t) = \cosh s \cosh t + \cos \theta \sinh s \sinh t.$$

Hence

$$\lim_{t \rightarrow +\infty} e^{d_+(t) - d_-(t)} = \lim_{t \rightarrow +\infty} \frac{\cosh d_+(t)}{\cosh d_-(t)} = \frac{\cosh s - \cos \theta \sinh s}{\cosh s + \cos \theta \sinh s}$$

By taking logarithms:

$$\lim_{t \rightarrow +\infty} d_+(t) - d_-(t) = \log \frac{1 - \cos \theta \tanh s}{1 + \cos \theta \tanh s} = \log \frac{1 - \tanh d}{1 + \tanh d} = -2d. \quad \square$$

5.3. Constant valence trees. Let T be the tree of valence 4 and edges of unit length, it is the Cayley graph of \mathbb{F}_2 , the free group on two generators. Let $\Delta \subset T \times T$ denote the diagonal. If we consider the nearest point projection to the diagonal, then the fibres over different points of Δ might be different. In fact, given $(a, a) \in \Delta$ there are three possible fibres, which are topologically not equivalent, depending on whether $a \in T$ is a vertex, a midpoint of an edge or a generic point in an edge.

Theorem 5.2. *For T the tree of valence four and edges of unit length, the fibres over Δ are of one of the following mutually exclusive types:*

- (1) *Generic fibres.* For $a \in T$ a generic point in an edge, the fibre over (a, a) is a tree of valence 4. The metric in the fibre depends on the distance from a to its nearest vertex in T , L with $0 < L < 1/2$. Along any path through (a, a) , the length of consecutive edges alternate between $2L$ and $L' = 1 - 2L$. The point (a, a) is the midpoint of an edge of length $2L$.
- (2) *Midpoint fibres.* For $a \in T$ a midpoint of an edge, the fibre over (a, a) is a tree of constant valence 10. All edges have length 1 and the point (a, a) is the midpoint of an edge. This is the limit of the previous case when $L \rightarrow 1/2$.

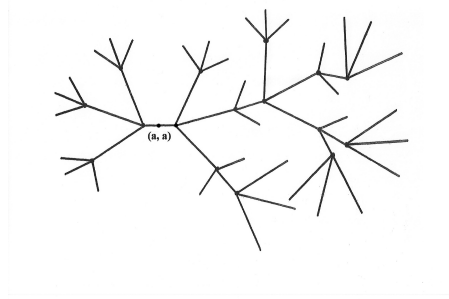


FIGURE 2. Case 1

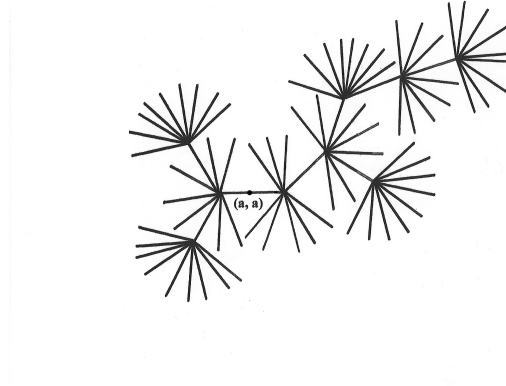


FIGURE 3. Case 2

- (3) *Vertex fibres.* For $a \in T$ a vertex, the fibre over (a, a) is a tree of constant valence 10, except in the point (a, a) , which is a valence 12 vertex. All edges have length 1. This is the limit of the generic case when $L \rightarrow 0$, taking into account that there are four fibres approaching to the base point, one for each edge in T incident to a .

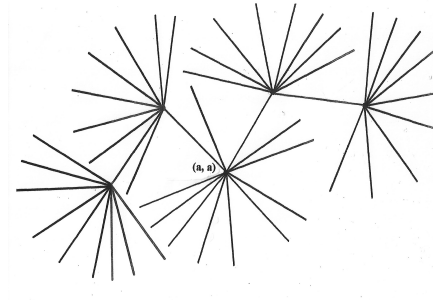


FIGURE 4. Case 3

Proof. The fibre of (a, a) is the set of pairs $(x, y) \in T \times T$ so that the midpoint of the segment \overline{xy} is a . To reach all such a pairs, we start from a and consider pairs of paths obtained by moving at speed one along T and pointing away from a . The first requirement is that we start in different directions, ie for

small times the points (x, y) of the pair are already different. When one of the points reach a vertex, we consider all possible continuations along different edges.

This construction provides a graph structure on the fibre: when x and y move along an edge this yields an edge of the fibre, when at least one of them reaches a vertex of T , this yields a vertex of the fibre. This fibre is in fact a tree, because we can orient each edge of the initial tree T so that it points in the direction opposite to a (when a lies in the interior of an edge, we split this edge along a). This yields an orientation of the edges of the fibre, so that each edge points away from (a, a) . In addition, at every vertex only edge points to this vertex, the other edges point away, hence it is a tree.

We describe the tree for the fibre of a vertex in T , case 3, the other two cases are a follow from similar arguments. Let a be a vertex and denote by (x, y) a point in the fibre over (a, a) . For each x in a edge incident to a , there are three possibilities for y , such that a is the midpoint of x and y , one from each of the three remaining edges. In total there are 4 edges incident to a , so there are $4 \cdot 3 = 12$ edges incident to (a, a) . Next we follow x and y along two edges, always satisfying $d(x, a) = d(y, a)$, until both x and y are two vertices v and v' . Then, for x there are three new possibilities, one for each new edge incident to v , and if we follow the path along one of the edges there are three possibilities for y , such that $d(x, a) = d(y, a)$, one for each new edge incident to the vertex v' . So there are $3 \cdot 3 + 1 = 10$ edges incident to (v, v') . This pattern repeats, given rise to a tree with valence 10 in all vertices except the base point. The distance between two consecutive vertices, for instance (a, a) and (v, v') , is $d_{\max}((a, a), (v, v')) = \max\{d(a, v), d(a, v')\} = d(a, v) = 1$, so the edges have length 1. \square

Remark 5.3. *Let T be again the tree of valence 4, also the Cayley graph of \mathbb{F}_2 . The projection $T \times T \rightarrow \Delta \cong T$ induces a map from $(T \times T)/\mathbb{F}_2$ to T/\mathbb{F}_2 , which is a wedge of two circles. The fibres are trees, as in Theorem 5.2, and the tree depends on the point on the wedge T/\mathbb{F}_2 : Case 3 for the vertex of the wedge, Case 2 for the midpoints of the edges, and Case 1 for the remaining (generic) points. The ideal boundary of each fibre in the max compactification is its boundary at infinity as a tree, which is a Cantor set in any case.*

The previous considerations of course apply to free groups of rank n and their Cayley graphs.

6. PRODUCT ACTIONS

Let (X_1, d_1) and (X_2, d_2) be two proper CAT(-1) spaces and let Γ be an infinite hyperbolic group with $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two *quasi-convex representations* (by ρ_i *quasi-convex* we mean that it has finite kernel and that $\rho_i(\Gamma)$ is a discrete quasiconvex group). In this section we study under what conditions the large limit set $\Lambda_{\rho_1 \times \rho_2}$ of the product action

$$\begin{aligned} \Gamma \times X_1 \times X_2 &\rightarrow X_1 \times X_2 \\ (\gamma, x, y) &\mapsto (\rho_1(\gamma)x, \rho_2(\gamma)y) \end{aligned}$$

lies inside the regular part of the boundary. We shall see that asking the large limit set to lie in the regular part of the boundary is in fact a very restrictive condition, which is related to the marked length spectrum conjecture. Indeed, in Subsection 6.1 we prove the following proposition:

Proposition 6.1. *Let X_1, X_2 be proper CAT(-1) spaces and $\overline{X_1 \times X_2}^{\max}$ the horofunction compactification with respect to d_{\max} . Let Γ be an infinite hyperbolic group and $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two quasi-convex representations. The following are equivalent:*

- (a) $\Lambda_{\rho_1 \times \rho_2} \subset \partial_{\infty}^{\max}(X_1 \times X_2)_{\text{reg}}$.
- (b) $\rho_1 \simeq_{C.E.} \rho_2$.
- (c) $\tau(\rho_1(\gamma)) = \tau(\rho_2(\gamma))$ for all $\gamma \in \Gamma$.

Here τ denotes the translation length of an isometry, see Definition 6.6 below, so condition (c) means that both representations have the same translation lengths. Condition (b) requires the following definition:

Definition 6.2. *The representations ρ_1, ρ_2 are said to be coarsely equivalent if there exists $C > 0$ such that:*

$$|d_1(\rho_1(\gamma)o, \rho_1(\gamma')o) - d_2(\rho_2(\gamma)o', \rho_2(\gamma')o')| \leq C$$

for some $o \in X_1$ and $o' \in X_2$, and for all $\gamma, \gamma' \in \Gamma$. When ρ_1 and ρ_2 are coarsely equivalent, we write $\rho_1 \simeq_{C.E.} \rho_2$.

If the representations are coarsely equivalent for some base-points $o \in X_1$ and $o' \in X_2$ then they are coarsely equivalent for any choice of base-points in X_1 and X_2 . We see later in Lemma 6.12 how the bound depends on the choice of base-points.

Definition 6.3. *A K -almost-isometry between two metric spaces is a map $f: X_1 \rightarrow X_2$ such that*

$$|d(x, y) - d(f(x), f(y))| \leq K, \quad \forall x, y \in X_1,$$

and X_2 lies in the K -neighborhood of $f(X_1)$.

If ρ_1 and ρ_2 are coarsely equivalent and cocompact, then the spaces X_1 and X_2 are equivariantly almost-isometric. In this case it is possible to find a subset Ω_{Γ}^{\max} of $\partial_{\infty}^{\max}(X_1 \times X_2)$ where the product action is properly discontinuous and cocompact. In Subsection 6.2 we use the existence of this almost-isometry between X_1 and X_2 and its extension to the ideal boundaries of the spaces to prove the following theorem:

Theorem 6.4. *Let X_1, X_2 be proper CAT(-1) spaces and $\overline{X_1 \times X_2}^{\max}$ the horofunction compactification with respect to d_{\max} . Let Γ be a hyperbolic group and $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two cocompact discrete representations with finite kernel. If ρ_1 and ρ_2 are coarsely equivalent, then there exists a subset $\Omega_{\Gamma}^{\max} \subset \partial_{\infty}^{\max}(X_1 \times X_2)$ where the product action of Γ is properly discontinuous and cocompact.*

6.1. Regular limit sets and coarsely equivalent representations. Let Γ be a group acting on a space X with two metrics d_1 and d_2 that are Γ -invariant.

Definition 6.5. *Two metrics d_1 and d_2 on a space X are coarsely equivalent if there exists a constant $C \geq 0$ such that for all $x, y \in X$,*

$$|d_1(x, y) - d_2(x, y)| \leq C.$$

Definition 6.6. *The metrics d_1, d_2 have the same marked length spectrum with respect to the action of Γ if $\tau_1(\gamma) = \tau_2(\gamma)$ for all $\gamma \in \Gamma$, where $\tau_i(\gamma)$ is the translation length of γ for d_i defined by*

$$\tau_i(\gamma) = \lim_{n \rightarrow \infty} \frac{d_i(x, \gamma^n(x))}{n}$$

for any $x \in X$.

The equivalence of both definitions for hyperbolic groups follows from results of Furman [8] and Krat [13].

Theorem 6.7. *(Furman, Krat) Let Γ be a hyperbolic group acting on itself with two left invariant metrics d_1, d_2 which are quasi-isometric to a word metric by the identity map. Then d_1 and d_2 are coarsely equivalent if and only if d_1 and d_2 have the same marked length spectrum.*

We are interested in a hyperbolic group acting on two proper CAT(-1) spaces $(X_1, d_1), (X_2, d_2)$ via quasi-convex representations ρ_1, ρ_2 into their respective groups of isometries. A hyperbolic group Γ acts on itself by left translations. Moreover, fixing $o_i \in X_i$, each representation ρ_i induces an orbit map from Γ to the target space X_i for $i = 1, 2$:

$$\begin{aligned} O_i: \Gamma &\rightarrow X_i \\ \gamma &\mapsto \rho_i(\gamma)o_i \end{aligned}$$

These orbit maps induce left invariant metrics d_{Γ_i} in Γ by:

$$d_{\Gamma_i}(\gamma, \gamma') = d_i(\rho_i(\gamma)o_i, \rho_i(\gamma')o_i)$$

so that for $i = 1, 2$ (Γ, d_{Γ_i}) are Γ invariant metric spaces. Moreover, since (X_i, d_i) are proper CAT(-1) spaces and the representations are quasi-convex, these metrics are quasi-isometric to a word metric by the identity map, see [3].

Remark 6.8. *The metrics d_{Γ_1} and d_{Γ_2} are coarsely equivalent if and only if $\rho_1 \simeq_{C.E.} \rho_2$. Indeed, in both cases the condition to be satisfied is that there exists a constant C such that*

$$|d_1(\rho_1(\gamma)o_1, \rho_1(\gamma')o_1) - d_2(\rho_2(\gamma)o_2, \rho_2(\gamma')o_2)| \leq C$$

for some $o_1 \in X_1, o_2 \in X_2$ and for all $\gamma, \gamma' \in \Gamma$.

Using Remark 6.8, Theorem 6.7 yields:

Proposition 6.9. *Let X_1, X_2 be proper CAT(-1) spaces. Let Γ be an infinite hyperbolic group and $\rho_1: \Gamma \rightarrow \text{Isom}(X_1), \rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two quasi-convex representations. Then:*

$$\rho_1 \simeq_{C.E.} \rho_2 \Leftrightarrow \tau(\rho_1(\gamma)) = \tau(\rho_2(\gamma))$$

for all $\gamma \in \Gamma$.

Next we prove the equivalence (a) \Leftrightarrow (b) in Proposition 6.1. Recall that the *large limit set* of the product action $\rho_1 \times \rho_2$ is the union of accumulation sets of all orbits on $\partial_\infty^{\max}(X_1 \times X_2)$:

$$\Lambda_{\rho_1 \times \rho_2} = \bigcup_{(x,y) \in X_1 \times X_2} \overline{\{(\rho_1(\gamma)x, \rho_2(\gamma)y) \mid \gamma \in \Gamma\}} \cap \partial_\infty^{\max}(X_1 \times X_2).$$

Recall also that the regular part of the ideal boundary can be identified with the product of the ideal boundaries of each factor and \mathbb{R} :

$$\partial_\infty(X_1 \times X_2)_{\text{reg}} \cong \partial X_1 \times \partial X_2 \times \mathbb{R}.$$

Fixing a base point $(o, o') \in X_1 \times X_2$, a sequence $(x_n, y_n) \subset X_1 \times X_2$ converges to a point (ξ, ξ', C) in the regular part if:

$$\begin{aligned} x_n &\rightarrow \xi \in \partial X_1, \\ y_n &\rightarrow \xi' \in \partial X_2, \text{ and} \\ d_1(x_n, o) - d_2(y_n, o') &\rightarrow C \in \mathbb{R}. \end{aligned}$$

Proposition 6.10. *If $\Lambda_{\rho_1 \times \rho_2} \subset \partial_\infty^{\max}(X_1 \times X_2)_{\text{reg}}$ then $\rho_1 \simeq_{C.E.} \rho_2$.*

Proof. Suppose that ρ_1 and ρ_2 are not coarsely equivalent. This means that there is a sequence γ_n in Γ such that $|d_1(o, \rho_1(\gamma_n)o) - d_2(o', \rho_2(\gamma_n)o')|$ is unbounded. By definition of singular point, this means that $(\rho_1(\gamma_n)o, \rho_2(\gamma_n)o')$ accumulates in the singular part, which is a contradiction. \square

For the implication (b) \Rightarrow (a) we need a couple of lemmas. The first one is a direct consequence of the triangle inequality:

Lemma 6.11. *Let x, y, z and t be four points in a metric space (X, d) . Then:*

$$|d(x, y) - d(z, t)| \leq d(x, z) + d(y, t).$$

Lemma 6.12. *If $\rho_1 \simeq_{C.E.} \rho_2$ then for any $x \in X_1, y \in X_2$:*

$$|d_1(\rho_1(\gamma)x, \rho_1(\gamma')x) - d_2(\rho_2(\gamma)y, \rho_2(\gamma')y)| \leq C + 2(d_1(x, o) + d_2(y, o'))$$

for all γ, γ' in Γ and for a C depending only on o and o' .

Proof. We add and subtract $d_1(\rho_1(\gamma)o, \rho_1(\gamma')o)$ and $d_2(\rho_2(\gamma)o', \rho_2(\gamma')o')$ and apply the triangle inequality:

$$\begin{aligned} &|d_1(\rho_1(\gamma)x, \rho_1(\gamma')x) - d_2(\rho_2(\gamma)y, \rho_2(\gamma')y)| \\ &\leq |d_1(\rho_1(\gamma)x, \rho_1(\gamma')x) - d_1(\rho_1(\gamma)o, \rho_1(\gamma')o)| \\ &\quad + |d_1(\rho_1(\gamma)o, \rho_1(\gamma')o) - d_2(\rho_2(\gamma)o', \rho_2(\gamma')o')| \\ &\quad + |d_2(\rho_2(\gamma)o', \rho_2(\gamma')o') - d_2(\rho_2(\gamma)y, \rho_2(\gamma')y)| \end{aligned}$$

Next we find a bound for each summand of the right-hand side. By Lemma 6.11:

$$\begin{aligned} &|d_1(\rho_1(\gamma)x, \rho_1(\gamma')x) - d_1(\rho_1(\gamma)o, \rho_1(\gamma')o)| \\ &\leq d_1(\rho_1(\gamma)x, \rho_1(\gamma)o) + d_1(\rho_1(\gamma')x, \rho_1(\gamma')o) = 2d_1(x, o) \end{aligned}$$

and

$$\begin{aligned} &|d_2(\rho_2(\gamma)o', \rho_2(\gamma')o') - d_2(\rho_2(\gamma)y, \rho_2(\gamma')y)| \\ &\leq d_2(\rho_2(\gamma)o', \rho_2(\gamma)y) + d_2(\rho_2(\gamma')o', \rho_2(\gamma')y) = 2d_2(y, o'). \end{aligned}$$

In addition, by assumption we have:

$$|d_1(\rho_1(\gamma)o, \rho_1(\gamma')o) - d_2(\rho_2(\gamma)o', \rho_2(\gamma')o')| \leq C,$$

so the result follows. \square

Remark 6.13. *Observe that Lemma 6.12 implies that the definition of coarse equivalence does not depend on the orbit.*

Proposition 6.14. *If $\rho_1 \simeq_{C.E.} \rho_2$ then $\Lambda_{\rho_1 \times \rho_2} \subset \partial_\infty^{\max}(X_1 \times X_2)_{\text{reg}}$.*

Proof. We want to show that sequences of the form $(\rho_1(\gamma_n)x, \rho_2(\gamma_n)y)$ accumulate in the regular part; equivalently $|d_1(\rho_1(\gamma_n)x, o) - d_2(\rho_2(\gamma_n)y, o')|$ is bounded, so every accumulation point of the sequence is in the regular part. Applying Lemma 6.12 with $x = o$, $y = o'$, $\gamma = \gamma_n$ and $\gamma' = \text{Id}$ we get that $|d_1(\rho_1(\gamma_n)o, o) - d_2(\rho_2(\gamma_n)o', o')| \leq C$. Then, it follows from the triangle inequality that $|d_1(\rho_1(\gamma_n)x, o) - d_2(\rho_2(\gamma_n)y, o')| \leq d_1(x, o) + d_2(y, o') + C$. \square

6.2. Compactification of product actions. In this section we consider $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ two discrete cocompact coarsely equivalent representations. We shall show that, as in the diagonal case, there exists an open subset $\Omega \subset \partial_\infty^{\max}(X_1 \times X_2)$ of the ideal boundary such that the product action on $X_1 \times X_2 \cup \Omega$ is properly discontinuous and cocompact.

Lemma 6.15. *If $\rho_1: \Gamma \rightarrow \text{Isom}(X_1)$, $\rho_2: \Gamma \rightarrow \text{Isom}(X_2)$ are coarsely equivalent cocompact representations, then there exists an equivariant almost-isometry $f: X_1 \rightarrow X_2$.*

Proof. Since the action is cocompact on both spaces X_1 and X_2 , each of these spaces is equivariantly almost-isometric to any orbit of Γ . The condition of coarse equivalence implies that the orbits of Γ in X_1 are equivariantly almost-isometric to the orbits of Γ in X_2 . \square

Remark 6.16. *The almost-isometry f is not unique.*

To find $\Omega \subset \partial_\infty^{\max}(X_1 \times X_2)$ such that Γ acts properly discontinuously and cocompactly on $X_1 \times X_2 \cup \Omega$, consider $f: X_1 \rightarrow X_2$ the almost-isometry of Lemma 6.15 and use the map

$$\text{Id} \times f: X_1 \times X_1 \rightarrow X_1 \times X_2$$

to translate the properties of the diagonal action $\rho_1 \times \rho_1$ on $X_1 \times X_1$ to the product action $\rho_1 \times \rho_2$ on $X_1 \times X_2$.

The almost-isometry f of Lemma 6.15 has an almost-inverse $f^{-1}: X_2 \rightarrow X_1$ such that:

$$d_1(f^{-1}(f(x_1)), x_1) \leq K \quad \text{and} \quad d_2(f(f^{-1}(x_2)), x_2) \leq K,$$

for all $x_1 \in X_1$ and $x_2 \in X_2$. Since quasi-isometries between CAT(−1) spaces extend to homeomorphisms of the boundaries, f extends to an equivariant homeomorphism:

$$f_\infty: \partial_\infty X_1 \rightarrow \partial_\infty X_2,$$

whose inverse is the extension of the almost-isometry f^{-1} .

Remark 6.17. *All the choices of almost-isometries $f: X_1 \rightarrow X_2$ extend to the same map $f_\infty: \partial_\infty X_1 \rightarrow \partial_\infty X_2$.*

For $i = 1, 2$, let

$$\begin{aligned} \varphi_i: (\partial_\infty^{\max}(X_1 \times X_i))_{\text{reg}} &\rightarrow \partial_\infty X_1 \times \partial_\infty X_i \times \mathbb{R} \\ z &\mapsto (\xi_i(z), \eta_i(z), h_i(z)) \end{aligned}$$

be the homeomorphism of Proposition 3.10. Choose $o \in X_1$ and $f(o) \in X_2$ as base points to compute h_1 and h_2 as in Proposition 3.10:

$$\begin{aligned} h_1([(x, x') \mapsto \max\{\beta(x), \beta'(x')\}]) &= \beta(o) - \beta'(o), \\ h_2([(x, x') \mapsto \max\{\beta(x), \beta''(x'')\}]) &= \beta(o) - \beta''(f(o)), \end{aligned}$$

where β and β' are Busemann functions on X_1 and β'' on X_2 .

The domain of discontinuity of the diagonal action is

$$\Omega_1 = \varphi_1^{-1}((\partial_\infty X_1 \times \partial_\infty X_1 \setminus \Delta_\infty) \times \mathbb{R}),$$

where Δ_∞ denotes the diagonal of $\partial_\infty X_1$. For the action on $X_1 \times X_2$ define

$$\Omega_2 = \varphi_2^{-1}((\partial_\infty X_1 \times \partial_\infty X_2 \setminus \Delta_{f_\infty}) \times \mathbb{R}),$$

where Δ_{f_∞} is graph of f_∞ :

$$\Delta_{f_\infty} = \{(\xi, \eta) \in \partial_\infty X_1 \times \partial_\infty X_2 \mid \eta = f_\infty(\xi)\}.$$

By Remark 4.9, $\Omega_1 = \partial_\infty^{\max}(X_1 \times X_1) \setminus \overline{\Lambda}_1$, where Λ_1 denotes the large limit set of the diagonal action. For Ω_2 we also have:

Remark 6.18. *Let Λ_2 denote the large limit set of the $(\rho_1 \times \rho_2)$ -action. Then $\Omega_2 = \partial_\infty^{\max}(X_1 \times X_2) \setminus \overline{\Lambda}_2$.*

As Remark 4.9, this remark follows from the fact that $\Lambda_2 \cong \Delta_{f_\infty} \times \mathbb{R}$ (the proof of this equality is similar to Lemma 4.8).

We next prove that Ω_2 is the set Ω_F^{\max} in the statement of Theorem 6.4. For this purpose we consider a map $F: \Omega_1 \rightarrow \Omega_2$ defined as follows. Every $z \in \Omega_1$ can be written as

$$z = \lim_{n \rightarrow +\infty} (g(-n), g(n)),$$

for a unique geodesic g in X_1 . This construction yields a homeomorphism between the set of bi-infinite geodesics in X_1 and Ω_1 . Next, if $\varphi_1(z) = (\xi(z), \eta(z), h_1(z))$, then define $F(z)$ by

$$\varphi_2(F(z)) = (\xi(z), f_\infty(\eta(z)), h_2(F(z))),$$

where

$$h_2(F(z)) = \limsup_{n \rightarrow +\infty} (d_1(g(n), o) - d_2(f(g(-n)), f(o))).$$

Thus, for any bi-infinite geodesic g in X_1 ,

$$F\left(\lim_{n \rightarrow +\infty} (g(-n), g(n))\right) = \lim_{k \rightarrow +\infty} (g(-n_k), f(g(n_k)))$$

for some diverging subsequence $(n_k)_k$. Notice that the map $F: \Omega_1 \rightarrow \Omega_2$ may be non continuous and depends on the choice of f .

Lemma 6.19. *Let K be the constant of almost isometry of f . Then:*

- (i) $|h_2(F(z)) - h_1(z)| \leq K, \forall z \in \Omega_1$.
- (ii) For $i = 1, 2$, if $z, z' \in \Omega_i$ satisfy $\xi_i(z) = \xi_i(z')$ and $\eta_i(z) = \eta_i(z')$, then, $\forall \gamma \in \Gamma$,

$$h_i(\gamma z) - h_i(\gamma z') = h_i(z) - h_i(z').$$

$$(iii) |h_2(F(\gamma z)) - h_2(\gamma F(z))| \leq 4K, \forall z \in \Omega_1, \forall \gamma \in \Gamma.$$

Proof. (i) Write $z \in \Omega_1$ as the limit $z = \lim_{n \rightarrow +\infty} (g(-n), g(n))$ for a (unique) geodesic g in Ω_1 . Then

$$\begin{aligned} h_1(z) &= \lim_{n \rightarrow +\infty} d(g(-n), o) - d(g(n), o), \\ h_2(F(z)) &= \limsup_{n \rightarrow +\infty} d(g(-n), o) - d(f(g(n)), f(o)). \end{aligned}$$

From these expressions we get

$$|h_2(F(z)) - h_1(z)| \leq \limsup_{n \rightarrow +\infty} |d(f(g(n)), f(o)) - d(g(n), o)| \leq K.$$

(ii) We prove it for $i = 2$, as the proof for $i = 1$ is analogous. By Lemma 4.1:

$$h_2(\gamma z) - h_2(z) = \beta_{\xi_2(z)}^o(\gamma^{-1}o) - \beta_{\eta_2(z)}^{f(o)}(\gamma^{-1}f(o)).$$

As we assume $\xi_2(z) = \xi_2(z')$ and $\eta_2(z) = \eta_2(z')$, assertion (ii) is proved.

(iii) We write:

$$\begin{aligned} h_2(F(\gamma z)) - h_2(\gamma F(z)) &= (h_2(F(\gamma z)) - h_1(\gamma z)) + (h_1(\gamma z) - h_1(z)) \\ &\quad + (h_1(z) - h_2(F(z))) + (h_2(F(z)) - h_2(\gamma F(z))) \\ &= (I) + (II) + (III) + (IV). \end{aligned}$$

The terms (I) and (III) are bounded in absolute value by K by item (i). By Lemma 4.1:

$$\begin{aligned} (II) &= h_1(\gamma z) - h_1(z) = \beta_{\xi_+}^o(\gamma^{-1}o) - \beta_{\xi_-}^o(\gamma^{-1}o), \\ (IV) &= h_2(F(z)) - h_2(\gamma F(z)) = -\beta_{\xi_+}^o(\gamma^{-1}o) + \beta_{f_\infty(\xi_-)}^{f(o)}(\gamma^{-1}f(o)). \end{aligned}$$

Hence

$$(8) \quad (II) + (IV) = \beta_{f_\infty(\xi_-)}^{f(o)}(f(\gamma^{-1}o)) - \beta_{\xi_-}^o(\gamma^{-1}o).$$

For $r: [0, +\infty) \rightarrow X_1$ the geodesic ray with $r(0) = o$ that converges to ξ_- :

$$(9) \quad \beta_{\xi_-}^o(\gamma^{-1}o) = \lim_{t \rightarrow +\infty} d_1(r(t), \gamma^{-1}o) - d_1(r(t), o).$$

On the other hand, $f \circ r: [0, +\infty) \rightarrow X_2$ is a quasi-geodesic that converges to $f_\infty(\xi_-)$. Since the visual compactification and the compactification by horofunctions are the same for a CAT(-1)-space, there is a diverging sequence $(t_k) \rightarrow +\infty$ such that

$$(10) \quad \beta_{f_\infty(\xi_-)}^{f(o)}(f(\gamma^{-1}o)) = \lim_{k \rightarrow +\infty} d_2(f(r(t_k)), f(\gamma^{-1}o)) - d_1(f(r(t_k)), f(o)).$$

Since f is a K -almost isometry, it follows from (8), (9) and (10) that $|(II) + (IV)| \leq 2K$. \square

Lemma 6.20. *Let $z \in \Omega_1$ and $y \in \Omega_2$ be such that $\xi_1(z) = \xi_2(y)$, $f_\infty(\eta_1(z)) = \eta_2(y)$, and $h_1(z) = h_2(y)$. Then*

$$|h_1(\gamma z) - h_2(\gamma y)| \leq 6K, \quad \forall \gamma \in \Gamma.$$

Proof. It is a consequence of the following three inequalities:

$$\begin{aligned} |h_1(\gamma z) - h_2(F(\gamma z))| &\leq K, \\ |h_2(F(\gamma z)) - h_2(\gamma F(z))| &\leq 4K, \\ |h_2(\gamma F(z)) - h_2(\gamma y)| &= |h_2(F(z)) - h_2(y)| = |h_2(F(z)) - h_1(z)| \leq K. \end{aligned}$$

Here we have used Lemma 6.19, item (i) for the first line, item (iii) for the second, and items (ii) and (i) for the last one. \square

Proposition 6.21. *The action of Γ on Ω_2 is properly discontinuous and cocompact.*

Proof. We prove proper discontinuity by showing that no two points in Ω_2 are dynamically related. Recall that two points x, y in a metric space Z are dynamically related by Γ if there exist a sequences $(z_n)_n$ in Z and $(\gamma_n)_n$ in Γ such that $z_n \rightarrow x$, $\gamma_n \rightarrow \infty$, and $\gamma_n z_n \rightarrow y$, see [7]. Proper discontinuity is equivalent to the property that any two points (possibly equal) are not dynamically related.

By contradiction, we assume y_∞ and y'_∞ in Ω_2 are dynamically related, and we shall show that two points in Ω_1 are dynamically related. Namely, assume that there exists a sequence $(y_n)_n$ in Ω_2 and a diverging sequence $(\gamma_n)_n$ in Γ such that $y_n \rightarrow y_\infty \in \Omega_2$ and $\gamma_n y_n \rightarrow y'_\infty \in \Omega_2$. For each $n \in \mathbb{N}$ let $z_n \in \Omega_1$ be such that $\xi_1(z_n) = \xi_2(y_n)$, $f_\infty(\eta_1(z_n)) = \eta_2(y_n)$, and $h_1(z_n) = h_2(y_n)$ (we have defined $\varphi_i = (\xi_i, \eta_i, h_i)$). Since ϕ_1 and ϕ_2 are homeomorphisms, $z_n \rightarrow z_\infty \in \Omega_1$. On the other hand, the coordinates $\xi_1(\gamma_n z_n)$ and $\eta_1(\gamma_n z_n)$ also converge and it remains to bound $|h_1(\gamma_n z_n)|$: by Lemma 6.20 $|h_1(\gamma_n z_n) - h_2(\gamma_n y_n)| \leq 6K$ and $h_2(\gamma_n y_n) \rightarrow h_2(y'_\infty)$.

Next we prove cocompactness. Let $(y_n)_n$ be a sequence in Ω_2 . For every $n \in \mathbb{N}$ we consider $z_n \in \Omega_1$ as above: $\xi_1(z_n) = \xi_2(y_n)$, $\eta_1(z_n) = f_\infty(\eta_2(y_n))$, and $h_1(z_n) = h_2(y_n)$. As the action is cocompact in Ω_1 , there exists a sequence γ_n in Γ such that $\gamma_n z_n$ converges, and all we need to prove is that $|h_2(\gamma_n y_n)|$ is bounded. This is a consequence of the inequality $|h_2(\gamma_n y_n) - h_1(\gamma_n z_n)| \leq 6K$ (by Lemma 6.20) and that $h_1(\gamma_n z_n)$ converges. \square

Now we consider the action on the whole $X_1 \times X_2 \cup \Omega_2$. We require the following lemma:

Lemma 6.22. *Let $(x_n, y_n)_n$ be a diverging sequence in $X_1 \times X_1$. The accumulation set of $(x_n, y_n)_n$ is contained in Ω_1 if and only if the accumulation set of $(x_n, f(y_n))_n$ is contained in Ω_2 .*

Proof. First assume that $(x_n, y_n)_n$ converges to a point in Ω_1 . Namely $x_n \rightarrow \xi \in \partial_\infty X_1$, $y_n \rightarrow \eta \neq \xi \in \partial_\infty X_1$ and $|d_1(x_n, o) - d_1(y_n, o)|$ is bounded. Thus, as $x_n \rightarrow \xi$ and $f(y_n) \rightarrow f_\infty(\eta) \neq f_\infty(\xi)$, the assertion follows from the estimate

$$\begin{aligned} |d_1(x_n, o) - d_2(f(y_n), f_n(o))| \\ \leq |d_1(x_n, o) - d_1(y_n, o)| + |d_1(y_n, o) - d_2(f(y_n), f_n(o))|, \end{aligned}$$

that is bounded because f is K -almost isometry.

For the converse, assuming that $|d_1(x_n, o) - d_2(y_n, f(o))|$ is bounded, we write:

$$\begin{aligned} & |d_1(x_n, o) - d_1(f^{-1}(y_n), o)| \\ & \leq |d_1(x_n, o) - d_2(y_n, f(o))| + |d_2(y_n, f(o)) - d_1(f^{-1}(y_n), o)|, \end{aligned}$$

that is bounded because:

$$\begin{aligned} & |d_2(y_n, f(o)) - d_1(f^{-1}(y_n), o)| \leq |d_2(y_n, f(o)) - d_1(f^{-1}(y_n), f^{-1}(f(o)))| \\ & \quad + |d_1(f^{-1}(y_n), f^{-1}(f(o))) - d_1(f^{-1}(y_n), o)| \\ & \leq |d_2(y_n, f(o)) - d_1(f^{-1}(y_n), f^{-1}(f(o)))| + d_1(f^{-1}(f(o)), o) \leq 2K. \quad \square \end{aligned}$$

Theorem 6.23. *The action of Γ on $X_1 \times X_2 \cup \Omega_2$ is properly discontinuous and cocompact.*

Proof. For proper discontinuity we will prove that no two points in $X_1 \times X_2 \cup \Omega_2$ are dynamically related, as in the proof of Proposition 6.21. Since the action is properly discontinuous on both $X_1 \times X_2$ and Ω_2 , it is enough to check that if (x_n, y_n) is a sequence in $X_1 \times X_2$ that converges to a point $z \in \Omega_2$, then there is no divergent sequence $(\gamma_n)_n \subset \Gamma$ such that $(\rho_1(\gamma_n)x_n, \rho_2(\gamma_n)y_n)$ accumulates in $X_1 \times X_2 \cup \Omega_2$. By contradiction, assume that such sequences exist. If $(\rho_1(\gamma_n)x_n, \rho_2(\gamma_n)y_n)$ converges to a point $(x, y) \in X_1 \times X_2$, then

$$d_{\max}((\rho_1(\gamma_n^{-1})(x), \rho_2(\gamma_n^{-1})(y)), (x_n, y_n))$$

is uniformly bounded and $(\rho_1(\gamma_n^{-1})(x), \rho_2(\gamma_n^{-1})(y))_n$ converges to the same point as $(x_n, y_n)_n$. Hence $z \in \Omega_2$ is the accumulation point of an orbit and we get a contradiction with Remark 6.18. Therefore, we assume that $(\rho_1(\gamma_n)x_n, \rho_2(\gamma_n)y_n)$ accumulates in Ω_2 . By Lemma 6.22, both sequences $(x_n, f^{-1}(y_n))$ and $(\rho_1(\gamma_n)x_n, \rho_1(\gamma_n)f^{-1}(y_n))$ accumulate in Ω_1 , which contradicts that Γ acts properly discontinuously on $X_1 \times X_1 \cup \Omega_1$.

To prove cocompactness and using Proposition 6.21, consider a sequence (x_n, y_n) in $X_1 \times X_2$. There exists a sequence γ_n of elements in Γ such that $(\rho_1(\gamma_n)(x_n), \rho_1(\gamma_n)(f^{-1}(y_n)))$ accumulates in $X_1 \times X_1 \cup \Omega_1$. Again by Lemma 6.22 $(\rho_1(\gamma_n)(x_n), \rho_2(\gamma_n)(y_n))$ accumulates in $X_1 \times X_2 \cup \Omega_2$. \square

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