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ON RECURSIVE CONSTRUCTIONS OF $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-LINEAR HADAMARD CODES

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Abstract. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes are subgroups of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$. A $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code is a Hadamard code, which is the Gray map image of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. In this paper, we generalize some known results for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes to $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. First, we give a recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$. It is known that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, this family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes. We show that, for example, for length 2^{11}, the constructed nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard, nor to any previously constructed $\mathbb{Z}_{2^s}$-Hadamard code, with $s \geq 2$. Finally, we also present other recursive constructions of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes having the same type, and we show that, after applying the Gray map, the codes obtained are equivalent to the previous ones.

1. **Introduction.** Let $\mathbb{Z}_{2^s}$ be the ring of integers modulo 2^s with $s \geq 1$. The set of n-tuples over $\mathbb{Z}_{2^s}$ is denoted by $\mathbb{Z}_{2^s}^n$. In this paper, the elements of $\mathbb{Z}_{2^s}^n$ are also called vectors. A code over $\mathbb{Z}_2$ of length n is a nonempty subset of $\mathbb{Z}_2^n$, and it is linear if it is a subspace of $\mathbb{Z}_2^n$. Similarly, a nonempty subset of $\mathbb{Z}_{2^s}^n$ is a $\mathbb{Z}_{2^s}$-additive code if it is a subgroup of $\mathbb{Z}_{2^s}^n$. A $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a subgroup of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$. Note that a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a linear code over $\mathbb{Z}_2$ when $\alpha_2 = \alpha_3 = 0$, a $\mathbb{Z}_4$-additive or $\mathbb{Z}_8$-additive code when $\alpha_1 = \alpha_3 = 0$ or $\alpha_1 = \alpha_2 = 0$, respectively, and a $\mathbb{Z}_2\mathbb{Z}_4$-additive code when $\alpha_3 = 0$. The order of a vector $u \in \mathbb{Z}_{2^s}^n$, denoted by $o(u)$, is the smallest positive integer m such that $mu = (0, \dots, 0)$. Also, the order of a vector $\mathbf{u} \in \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$, denoted by $o(\mathbf{u})$, is the smallest positive integer m such that $m\mathbf{u} = (0, \dots, 0 \mid 0, \dots, 0 \mid 0, \dots, 0)$.

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The Hamming weight of a vector $u \in \mathbb{Z}_2^n$, denoted by $\mathrm{wt}_H(u)$, is the number of nonzero coordinates of u. The Hamming distance of two vectors $u, v \in \mathbb{Z}_2^n$, denoted by $d_H(u,v)$, is the number of coordinates in which they differ. Note that $d_H(u,v) = \mathrm{wt}_H(u-v)$. The minimum distance of a code C over $\mathbb{Z}_2$ is $d(C) = \min\{d_H(u,v) : u, v \in C, u \neq v\}$.

In [20], a Gray map from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is defined as $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$ and $\phi(3) = (1,0)$. There exist different generalizations of this Gray map, which go from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$ [15, 12, 16, 21, 25]. The one given in [21] can be defined in terms of the elements of a Hadamard code [25], and Carlet's Gray map [15] is a particular case of the one given in [25] satisfying $\sum \lambda_i \phi(2^i) = \phi(\sum \lambda_i 2^i)$ [17]. In this paper, we focus on Carlet's Gray map [15], from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$, which is also a particular case of the one given in [34]. Specifically,

$$\phi_s(u) = (u_{s-1}, u_{s-1}, \dots, u_{s-1}) + (u_0, \dots, u_{s-2})Y_{s-1}, \tag{1}$$

where $u \in \mathbb{Z}_{2^s}$; $[u_0, u_1, \dots, u_{s-1}]_2$ is the binary expansion of u, that is, $u = \sum_{i=0}^{s-1} u_i 2^i$ with $u_i \in \{0,1\}$; and Y is a matrix of size $(s-1) \times 2^{s-1}$ whose columns are all the vectors in $\mathbb{Z}_2^{s-1}$. Without loss of generality, we assume that the columns of Y_{s-1} are ordered in ascending order by considering the elements of $\mathbb{Z}_2^{s-1}$ as the binary expansions of the elements of $\mathbb{Z}_{2^{s-1}}$. Note that ϕ_1 is the identity map, and

$$\begin{array}{rrcl} \phi_2 : & \mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_2^2 \\ & 0 & \mapsto & (0,0) \\ & 1 & \mapsto & (0,1) \\ & 2 & \mapsto & (1,1) \\ & 3 & \mapsto & (1,0) \end{array} \qquad \begin{array}{rrcl} \phi_3 : & \mathbb{Z}_8 & \longrightarrow & \mathbb{Z}_2^4 \\ & 0 & \mapsto & (0,0,0,0) \\ & 1 & \mapsto & (0,1,0,1) \\ & 2 & \mapsto & (0,0,1,1) \\ & 3 & \mapsto & (0,1,1,0) \\ & 4 & \mapsto & (1,1,1,1) \\ & 5 & \mapsto & (1,0,1,0) \\ & 6 & \mapsto & (1,1,0,0) \\ & 7 & \mapsto & (1,0,0,1). \end{array}$$

We define $\Phi_s : \mathbb{Z}_{2^s}^n \to \mathbb{Z}_2^{n2^{s-1}}$ as the component-wise extended map of ϕ_s. We can also define a Gray map Φ from $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ to $\mathbb{Z}_2^n$, where $n = \alpha_1 + 2\alpha_2 + 4\alpha_3$, as follows:

$$\Phi(u_1 \mid u_2 \mid u_3) = (u_1, \Phi_2(u_2), \Phi_3(u_3)),$$

for any $u_i \in \mathbb{Z}_{2^i}^{\alpha_i}$, where $1 \leq i \leq 3$.

Let $\mathcal{C} \subseteq \mathbb{Z}_{2^s}^n$ be a $\mathbb{Z}_{2^s}$-additive code of length n. We say that the Gray map image of $\mathcal{C}$, say $C = \Phi_s(\mathcal{C})$, is a $\mathbb{Z}_{2^s}$-linear code of length $n2^{s-1}$. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_{2^s}^n$, it is isomorphic to $\mathbb{Z}_{2^s}^{t_1} \times \mathbb{Z}_{2^{s-1}}^{t_2} \times \cdots \times \mathbb{Z}_2^{t_s}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi_s(\mathcal{C})$, is of type $(n; t_1, \dots, t_s)$. Note that $|\mathcal{C}| = 2^{st_1} 2^{(s-1)t_2} \cdots 2^{t_s}$. Similarly, if $\mathcal{C} \subseteq \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, we say that its Gray map image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length $\alpha_1 + 2\alpha_2 + 4\alpha_3$. Since $\mathcal{C}$ can be seen as a subgroup of $\mathbb{Z}_8^{\alpha_1+\alpha_2+\alpha_3}$, it is isomorphic to $\mathbb{Z}_8^{t_1} \times \mathbb{Z}_4^{t_2} \times \mathbb{Z}_2^{t_3}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$. We have that a $\mathbb{Z}_2\mathbb{Z}_4$-linear code $\mathcal{C}$ [10, 11] can can be seen as a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of type $(\alpha_1, \alpha_2, 0; 0, t_2, t_3)$. In this case, we also write that the type of $\mathcal{C}$ is directly $(\alpha_1, \alpha_2; t_2, t_3)$. Unlike linear codes over finite fields, linear codes over rings do not have a basis, but a generator matrix exists for these codes with a minimum number of rows. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, then $|\mathcal{C}| = 8^{t_1} 4^{t_2} 2^{t_3}$ and there exist a generator matrix with $t_1 + t_2 + t_3$ rows.

Two structural properties of codes over $\mathbb{Z}_2$ are the rank and dimension of the kernel. The rank of a code C over $\mathbb{Z}_2$ is simply the dimension of the linear span of C, say $\langle C\rangle$. The kernel of a code C over $\mathbb{Z}_2$ is defined as $\mathrm{K}(C) = \{\mathbf{x} \in \mathbb{Z}_2^n : \mathbf{x}+C = C\}$ [3]. If the all-zero vector belongs to C, then $\mathrm{K}(C)$ is a linear subcode of C. Note also that if C is linear, then $K(C) = C = \langle C\rangle$. We denote the rank of C as $\mathrm{rank}(C)$ and the dimension of the kernel as $\ker(C)$. These parameters can be used to distinguish between nonequivalent codes since equivalent ones have the same rank and dimension of the kernel.

A binary code with length n, $2n$ codewords, and minimum distance $n/2$ is called a Hadamard code. Hadamard codes can be constructed from Hadamard matrices [1, 27]. Note that linear Hadamard codes are first-order Reed-Muller codes, or equivalently, the dual of extended Hamming codes [27, Ch.13 §3]. The $\mathbb{Z}_{2^s}$-additive codes such that after the Gray map Φ_s give Hadamard codes are called $\mathbb{Z}_{2^s}$-additive Hadamard codes, and the corresponding images are called $\mathbb{Z}_{2^s}$-linear Hadamard codes. Similarly, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes such that after the Gray map Φ give Hadamard codes are called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes, and the corresponding images are called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes.

It is well-known that $\mathbb{Z}_4$-linear Hadamard codes (that is, $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 = 0$) and $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ can be classified by using either the rank or the dimension of the kernel [24, 29]. Moreover, in [26], it is shown that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$. Later, in [17, 6, 19, 4], a recursive construction for $\mathbb{Z}_{p^s}$-linear Hadamard codes, with p prime, is described, the linearity is established, and a partial classification by using the dimension of the kernel is obtained, giving the exact amount of nonequivalent such codes for some parameters. In [18], a complete classification of $\mathbb{Z}_8$-linear Hadamard codes by using the rank and dimension of the kernel is provided, giving the exact amount of nonequivalent such codes. For any $t \geq 2$, the full classification of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear Hadamard codes of length p^t, with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $p \geq 3$ prime, is given in [5, 7], by using just the dimension of the kernel.

The paper contributes to the study of codes over rings $\mathbb{Z}_{p^s}$, which were first studied by Blake [9] and Shankar [31] in 1975 and 1979, respectively. These codes have become more significant after the publication of [20]. It is also important to note that Hadamard codes are two weight codes, which have been widely studied in [32, 33]. On the other hand, the classification of nonlinear Hadamard codes is still an open problem. By giving an additive structure, as $\mathbb{Z}_{p^s}$-linear, $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear or $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes, to some of them, and showing whether they are equivalent or not among them, we are providing a partial classification for these codes.

From a more practical point of view, since Hadamard codes are optimal and have a high correction capability, they appear in different aspects related to the transmission of information, such as in digital communication with satellites [22], in CDMA phones to modulate the transmission of information and minimize interference with other transmissions [35] and, in general, in different OCDMA multiple access systems to allow access to multiple users asynchronously and simultaneously [23]. Other applications are found in cryptography [28] or in information hiding (steganography and watermarking) [36]. See [22] for more applications in other fields.

This paper is focused specifically on $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, generalizing some results given for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ in [29, 30] related to a recursive construction of such codes. These codes are also compared with the $\mathbb{Z}_4$-linear, $\mathbb{Z}_8$-linear, and in general $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$ considered in [17]. In general, the construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes allows to construct codes which are not equivalent to $\mathbb{Z}_{2^s}$-linear Hadamard codes, with $s \geq 2$. It is known that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, this family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadeamard codes does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes, or $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 4$. In Example 2.3, we show that all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17], with $s \geq 2$, of the same length 2^{11}. This paper is organized as follows. In Section 2, we describe a recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. We emphasise that, unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, this family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadeamard codes does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes. In Section 3, we present other recursive constructions and show that we obtain $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes equivalent to the previous ones. Finally, in Section 4, we give some conclusions and further research on this topic.

2. Recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes.

The description of generator matrices having a minimum number of rows for $\mathbb{Z}_4$-additive Hadamard, some families of $\mathbb{Z}_{2^s}$-additive Hadamard, and in general $\mathbb{Z}_{p^s}$-additive Hadamard codes, with $s \geq 2$ and p prime, are given in [24], [17], and [6], respectively. Similarly, generator matrices having a minimum number of rows for $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive Hadamard codes with $\alpha_1 \neq 0, \alpha_2 \neq 0$ and p prime, as long as a recursive construction of these matrices, are given in [29, 30] when $p = 2$ and in [5] when $p \geq 3$. In this section, we generalize these results for $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. Specifically, we define a recursive construction for the generator matrices of a family of these codes and establish that they generate $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes.

Let $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{7}$ be the vectors having the elements $0, 1, 2, \dots, 7$ repeated in each coordinate, respectively. If A is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, that is, a subgroup of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ for some integers $\alpha_1, \alpha_2, \alpha_3 \geq 0$, then we denote by A_1 the submatrix of A with the first α_1 columns over $\mathbb{Z}_2$, A_2 the submatrix with the next α_2 columns over $\mathbb{Z}_4$, and A_3 the submatrix with the last α_3 columns over $\mathbb{Z}_8$. We have that $A = (A_1 \mid A_2 \mid A_3)$, where the number of columns of A_i is α_i for $i \in \{1, 2, 3\}$.

Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Now, we construct recursively matrices A^{t_1,t_2,t_3} having t_1 rows of order 8, t_2 rows of order 4, and t_3 rows of order 2 as follows. First, we consider the following matrix:

$$A^{1,0,1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right). \tag{2}$$

Then, we apply the following constructions. If we have a matrix $A^{\ell-1,0,1} = (A_1 \mid A_2 \mid A_3)$, with $\ell \geq 2$, we may construct the matrix

$$A^{\ell,0,1} = \left(\begin{array}{cc|ccccc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & M_2 & A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right), \tag{3}$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{\ell-1}\}$ and $M_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0,2,4,6\}^{\ell-1}\}$. We perform Construction (3) until $\ell = t_1$. If we have a matrix $A^{t_1,\ell-1,1} = (A_1 \mid A_2 \mid A_3)$, with $t_1 \geq 1$ and $\ell \geq 1$, we may construct the matrix

$$A^{t_1,\ell,1} = \left(\begin{array}{cc|ccccc|cccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array} \right), \tag{4}$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1+\ell-1}\}$. We repeat Construction (4) until $\ell = t_2$. Finally, if we have a matrix $A^{t_1,t_2,\ell-1} = (A_1 \mid A_2 \mid A_3)$, with $t_1 \geq 1$, $t_2 \geq 0$, and $\ell \geq 2$, we may construct the matrix

$$A^{t_1,t_2,\ell} = \left(\begin{array}{cc|cc|cc} A_1 & A_1 & A_2 & A_2 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{4} \end{array} \right). \tag{5}$$

We repeat Construction (5) until $\ell = t_3$. Thus, in this way, we obtain A^{t_1,t_2,t_3}.

Summarizing, in order to achieve A^{t_1,t_2,t_3} from $A^{1,0,1}$, first we add $t_1 - 1$ rows of order 8 by applying Construction (3) $t_1 - 1$ times, starting from $A^{1,0,1}$ up to obtain $A^{t_1,0,1}$; then we add t_2 rows of order 4 by applying Construction (4) t_2 times, up to generate $A^{t_1,t_2,1}$; and, finally, we add $t_3 - 1$ rows of order 2 by applying Construction (5) $t_3 - 1$ times to achieve A^{t_1,t_2,t_3}. Note that the first row always has the row $(\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})$.

Example 2.1. By using the constructions described in (3), (4), and (5), we obtain the following matrices $A^{2,0,1}$, $A^{1,1,1}$ and $A^{1,1,2}$, respectively, starting from $A^{1,0,1}$ given in (2):

$$A^{2,0,1} = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & 1111 & 0246 & 11111111 \\ 00 & 11 & 11 & 0123 & 1111 & 01234567 \end{array} \right), \tag{6}$$

$$A^{1,1,1} = \left(\begin{array}{cc|cc|c} 11 & 11 & 22 & 2222 & 4444 \\ 01 & 01 & 02 & 1111 & 1111 \\ 00 & 11 & 11 & 0123 & 0246 \end{array} \right), \tag{7}$$

$$A^{1,1,2} = \left(\begin{array}{cc|cc|cc} 1111 & 1111 & 222222 & 222222 & 4444 & 4444 \\ 0101 & 0101 & 021111 & 021111 & 1111 & 1111 \\ 0011 & 0011 & 110123 & 110123 & 0246 & 0246 \\ 0000 & 1111 & 000000 & 222222 & 0000 & 4444 \end{array} \right).$$

In order to obtain $A^{2,1,1}$, we start with $A^{1,0,1}$, we apply Construction (3) to obtain $A^{2,0,1} = (A_1 \mid A_2 \mid A_3)$ given in (6), and then we apply (4) to obtain

$$A^{2,1,1} = \left(\begin{array}{cc|ccccc|cccc} & & 2222 & & & & & & & & \\ A_1 & A_1 & 0022 & A_2 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_3 \\ & & 0202 & & & & & & & & \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array} \right).$$

The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by A^{t_1,t_2,t_3} is denoted by $\mathcal{H}^{t_1,t_2,t_3}$, and the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code $\Phi(\mathcal{H}^{t_1,t_2,t_3})$ by H^{t_1,t_2,t_3}.

Lemma 2.1. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. Let $\mathcal{H}^{t_1,t_2,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ generated by $A^{t_1,t_2,1}$. Then, $2^{t_1+t_2} = \alpha_1$, $4^{t_1+t_2} = \alpha_1 + 2\alpha_2$ and $8^{t_1}4^{t_2} = \alpha_1 + 2\alpha_2 + 4\alpha_3$.*

Proof. First, we prove this lemma for the code $\mathcal{H}^{t_1,0,1}$ by induction on $t_1 \geq 1$. Note that the lemma holds for the code $\mathcal{H}^{1,0,1}$ of type $(2,1,1;1,0,1)$. Assume that the lemma holds for the code $\mathcal{H}^{t_1,0,1}$ of type $(\alpha_1,\alpha_2,\alpha_3;t_1,0,1)$, that is,

$$2^{t_1} = \alpha_1, 4^{t_1} = \alpha_1 + 2\alpha_2 \textit{ and } 8^{t_1} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{8}$$

By using Construction (3), the type of $\mathcal{H}^{t_1+1,0,1}$ is $(\alpha'_1,\alpha'_2,\alpha'_3;t_1+1,0,1)$, where

$$\alpha'_1 = 2\alpha_1, \alpha'_2 = 2^{t_1} + 4\alpha_2 \textit{ and } \alpha'_3 = 4^{t_1} + 8\alpha_3. \tag{9}$$

Thus, from (8) and (9), $2^{t_1+1} = 2\alpha_1 = \alpha'_1$, $4^{t_1+1} = 4\alpha_1 + 8\alpha_2 = 2\alpha_1 + 2\alpha_1 + 8\alpha_2 = \alpha'_1 + 2^{t_1+1} + 8\alpha_2 = \alpha'_1 + 2\alpha'_2$ and $8^{t_1+1} = 8\alpha_1 + 16\alpha_2 + 32\alpha_3 = 2\alpha_1 + (2\alpha_1 + 8\alpha_2) + (4\alpha_1 + 8\alpha_2 + 32\alpha_3) = 2\alpha_1 + (2^{t_1+1} + 8\alpha_2) + (4^{t_1+1} + 32\alpha_3) = \alpha'_1 + 2\alpha'_2 + 4\alpha'_3$. Therefore, the lemma holds for the code $\mathcal{H}^{t_1,0,1}$.

Next, we prove this lemma for the code $\mathcal{H}^{t_1,t_2,1}$ by induction on $t_2 \geq 0$. Assume that the lemma holds for the code $\mathcal{H}^{t_1,t_2,1}$ of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,1)$, that is,

$$2^{t_1+t_2} = \alpha_1, 4^{t_1+t_2} = \alpha_1 + 2\alpha_2, \textit{ and } 8^{t_1}4^{t_2} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{10}$$

By using Construction (4), the type of $\mathcal{H}^{t_1,t_2+1,1}$ is $(\alpha'_1,\alpha'_2,\alpha'_3;t_1,t_2+1,1)$, where

$$\alpha'_1 = 2\alpha_1, \alpha'_2 = 2^{t_1+t_2} + 4\alpha_2 \textit{ and } \alpha'_3 = 4\alpha_3. \tag{11}$$

Thus, from (10) and (11), $2^{t_1+(t_2+1)} = 2\alpha_1 = \alpha'_1$, $4^{t_1+(t_2+1)} = 4\alpha_1 + 8\alpha_2 = 2\alpha_1 + 2\alpha_1 + 8\alpha_2 = \alpha'_1 + 2^{t_1+t_2+1} + 8\alpha_2 = \alpha'_1 + 2\alpha'_2$ and $8^{t_1}4^{t_2+1} = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 = 2\alpha_1 + (2\alpha_1 + 8\alpha_2) + 16\alpha_3 = \alpha'_1 + (2^{t_1+t_2+1} + 8\alpha_2) + 4\alpha'_3 = \alpha'_1 + 2\alpha'_2 + 4\alpha'_3$. Therefore, the lemma holds for the code $\mathcal{H}^{t_1,t_2+1,1}$. This completes the proof. □

Proposition 2.2. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,t_3)$ generated by A^{t_1,t_2,t_3}. Then,*

$$\begin{aligned} \alpha_1 &= 2^{t_1+t_2+t_3-1}, \\ \alpha_1 + 2\alpha_2 &= 4^{t_1+t_2}2^{t_3-1}, \\ \alpha_1 + 2\alpha_2 + 4\alpha_3 &= 8^{t_1}4^{t_2}2^{t_3-1}. \end{aligned} \tag{12}$$

Proof. We prove this result for the code $\mathcal{H}^{t_1,t_2,t_3}$ by induction on $t_3 \geq 1$. By Lemma 2.1, the proposition holds for $t_3 = 1$, that is, for the code $\mathcal{H}^{t_1,t_2,1}$. Assume that it holds for the code $\mathcal{H}^{t_1,t_2,t_3}$ of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,t_3)$, that is, (12) holds. By using Construction (5), the type of $\mathcal{H}^{t_1,t_2,t_3+1}$ is $(\alpha'_1,\alpha'_2,\alpha'_3;t_1,t_2,t_3+1)$, where

$$\alpha'_1 = 2\alpha_1, \alpha'_2 = 2\alpha_2, \text{ and } \alpha'_3 = 2\alpha_3. \tag{13}$$

Thus, from (12) and (13), $2^{t_1+t_2+t_3} = 2\alpha_1 = \alpha'_1$, $4^{t_1+t_2}2^{t_3} = 2\alpha_1 + 4\alpha_2 = \alpha'_1 + 2\alpha'_2$ and $8^{t_1}4^{t_2}2^{t_3} = 2\alpha_1 + 4\alpha_2 + 8\alpha_3 = \alpha'_1 + 2\alpha'_2 + 4\alpha'_3$. Therefore, the proposition is true for the code $\mathcal{H}^{t_1,t_2,t_3+1}$. This completes the proof. □

Corollary 2.3. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,t_3)$ generated by A^{t_1,t_2,t_3}. Then,*

$$\begin{aligned} \alpha_1 &= 2^{t_1+t_2+t_3-1}, \\ \alpha_2 &= 4^{t_1+t_2}2^{t_3-2} - 2^{t_1+t_2+t_3-2}, \\ \alpha_3 &= 8^{t_1}4^{t_2-1}2^{t_3-1} - 4^{t_1+t_2-1}2^{t_3-1}. \end{aligned}$$

Remark 2.4. By Corollary 2.3, we have that the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes $\mathcal{H}^{t_1,t_2,t_3}$ of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,t_3)$ generated by A^{t_1,t_2,t_3}, so constructed recursively from (3), (4), and (5), satisfy that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$.

Remark 2.5. The construction of the generator matrices A^{t_1,t_2,t_3} is a generalization of the recursive construction of the generator matrices of the $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes of type $(\alpha_1, \alpha_2; t_2, t_3)$ with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, given in [30]. Note that if we do not consider the coordinates over $\mathbb{Z}_8$ in Constructions (3), (4), and (5), we have that (3) and (4) become

$$A^{\ell,1} = \left(\begin{array}{cc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \right), \tag{14}$$

where $A^{\ell-1,1} = (A_1 \mid A_2)$ and $M_1 = 2A_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{\ell-1}\}$ (up to a column permutation); and Construction (5) become

$$A^{t_2,\ell} = \left(\begin{array}{cc|cc} A_1 & A_1 & A_2 & A_2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \end{array} \right), \tag{15}$$

where $A^{t_2,\ell-1} = (A_1 \mid A_2)$. Then, starting from the following matrix:

$$A^{1,1} = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array} \right), \tag{16}$$

and applying (14) and (15) in the same way as above, we obtain the generator matrices A^{t_2,t_3} of the known $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes of type $(\alpha_1, \alpha_2; t_2, t_3)$ with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ [29, 30]. The $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by A^{t_2,t_3} is denoted by $\mathcal{H}^{t_2,t_3}$, and the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $\Phi(\mathcal{H}^{t_2,t_3})$ by H^{t_2,t_3}.

When we include all the elements of $\mathbb{Z}_{2^i}$, where $1 \leq i \leq 3$, as coordinates of a vector, we place them in increasing order. For a set $S \subseteq \mathbb{Z}_{2^i}$ and $\lambda \in \mathbb{Z}_{2^i}$, where $i \in \{1,2,3\}$, we define $\lambda S = \{\lambda j : j \in S\}$ and $S + \lambda = \{j + \lambda : j \in S\}$. As before, when including all the elements in those sets as coordinates of a vector, we place them in increasing order. For example, $2\mathbb{Z}_8 = \{0,4,6,8\}$, $(\mathbb{Z}_4, \mathbb{Z}_4) = (0,1,2,3,0,1,2,3) \in \mathbb{Z}_4^8$ and $(\mathbb{Z}_2 \mid \mathbb{Z}_4 \mid 2\mathbb{Z}_8, 4\mathbb{Z}_8) = (0,1 \mid 0,1,2,3 \mid 0,2,4,6,0,4) \in \mathbb{Z}_2^2 \times \mathbb{Z}_4^4 \times \mathbb{Z}_8^6$.

Lemma 2.6. *Let $1 \leq i \leq 3$ and $j \in \{0,1,\ldots,i-1\}$.*

1. *If $\mu \in 2^j\mathbb{Z}_{2^i}$, then $2^j\mathbb{Z}_{2^i} + \mu = 2^j\mathbb{Z}_{2^i}$.*
2. *If $\mu \in 2^j\mathbb{Z}_{2^i}$, then $(2^j\mathbb{Z}_{2^i}, \overset{m}{\ldots}, 2^j\mathbb{Z}_{2^i}) + \mu\mathbf{1}$, where $m \geq 1$, is a permutation of the vector $(2^j\mathbb{Z}_{2^i}, \overset{m}{\ldots}, 2^j\mathbb{Z}_{2^i})$.*
3. *If $\mu \in 2\mathbb{Z}_{2^i}$, then $(\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}) + \mu = \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$.*
4. *If $\mu \in \mathbb{Z}_{2^i}$, then $(\mathbf{0}, \ldots, \mathbf{2^i - 1}) + (\mu, \overset{\ell \cdot 2^i}{\ldots}, \mu)$, where $\ell \geq 1$ and $\mathbf{k} = (k, \overset{\ell}{\ldots}, k)$ for $k \in \mathbb{Z}_{2^i}$, is a permutation of $(\mathbb{Z}_{2^i}, \overset{\ell}{\ldots}, \mathbb{Z}_{2^i})$.*

Proof. Item 1 follows from the fact that $\mathbb{Z}_{2^i}$ is a ring and $2^j\mathbb{Z}_{2^i}$ is an ideal of $\mathbb{Z}_{2^i}$. Item 2 follows from Item 1.

For Item 3, it $x \in (\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}) + \mu$, then $x - \mu \in \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$. Assume that $x \notin \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$, so $x \in 2\mathbb{Z}_{2^i}$. Since $2\mathbb{Z}_{2^i}$ is an ideal of $\mathbb{Z}_{2^i}$, we have that $x - \mu \in 2\mathbb{Z}_{2^i}$, which is a contradiction. Thus, $x \in \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$ and hence $(\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}) + \mu \subseteq \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$. In the same way, $(\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}) - \mu \subseteq \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$. Hence, $\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i} \subseteq (\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}) + \mu$ and therefore $(\mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}) + \mu = \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$.

For Item 4, note that $(\mathbf{0}, \ldots, \mathbf{2^i - 1}) + (\mu, \overset{\ell \cdot 2^i}{\ldots}, \mu)$ is a permutation of

$$(\mathbb{Z}_{2^i}, \overset{\ell}{\ldots}, \mathbb{Z}_{2^i}) + (\mu, \overset{\ell \cdot 2^i}{\ldots}, \mu). \tag{17}$$

Since $\mathbb{Z}_{2^i} + \mu = \mathbb{Z}_{2^i}$, (17) is a permutation of $(\mathbb{Z}_{2^i}, \overset{\ell}{\ldots}, \mathbb{Z}_{2^i})$. □

Lemma 2.7. *Let $1 \leq i \leq 3$, $\lambda \in \mathbb{Z}_{2^i} \backslash 2\mathbb{Z}_{2^i}$, and $u \in \mathbb{Z}_{2^i}^n$. Then,*

$$(u, \overset{2^i}{\ldots}, u) + \lambda(\mathbf{0}, \ldots, \mathbf{2^i - 1})$$

is a permutation of $(\mathbb{Z}_{2^i}, \overset{n}{\ldots}, \mathbb{Z}_{2^i})$.

Proof. Since $\lambda \in \mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$, $\lambda(\mathbf{0},\ldots,\mathbf{2^i-1})$ is a permutation of $(\mathbf{0},\ldots,\mathbf{2^i-1})$ and we may consider $\lambda = 1$. Then, $(u,\ldots,u) + (\mathbf{0},\ldots,\mathbf{2^i-1})$ is a permutation of $(u_1+\mathbb{Z}_{2^i},\ldots,u_n+\mathbb{Z}_{2^i}) = (\mathbb{Z}_{2^i}, \overset{n}{\ldots}, \mathbb{Z}_{2^i})$, where $u=(u_1,\ldots,u_n)$. □

Lemma 2.8. *Let* $u = (\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4) \in \mathbb{Z}_4^{m+2n+2r}$, *where* $m,n,r \geq 0$ *and* $\mu \in \mathbb{Z}_4\backslash 2\mathbb{Z}_4 = \{1,3\}$. *Then,*

$$(u,u,u,u) + (\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$$

is a permutation of $(2\mathbb{Z}_4, \overset{4n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$.

Proof. By Items 1 and 3 of Lemma 2.6, $u+\mathbf{2}$ is a permutation of $(\mu+2, \overset{m}{\ldots}, \mu+2, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$. Let $\mathbf{k} = (\mu, \overset{m}{\ldots}, \mu)$. Since $\mu \in \{1,3\}$, we have that $(\mathbf{k},\mathbf{k},\mathbf{k},\mathbf{k}) + (\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$ is a permutation of $(\mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$. Therefore, $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$ is a permutation of $(2\mathbb{Z}_4, \overset{4n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$. □

Lemma 2.9. *Let* $u = (\mu', \overset{m'}{\ldots}, \mu', \mu'', \overset{m'}{\ldots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8) \in \mathbb{Z}_8^{2m'+4n'+4r'}$, *where* $m',n',r' \geq 0$ *and* $\mu,\mu' \in \mathbb{Z}_8\backslash 2\mathbb{Z}_8 = \{1,3,5,7\}$. *Then,*

1. $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ *is a permutation of* $(2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'+2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$;
2. $(u,u,u,u)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ *is a permutation of* $(\mu', \overset{4m'}{\ldots}, \mu', \mu'+4, \overset{4m'}{\ldots}, \mu'+4, 2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ *if* $\mu' = \mu''$ *or* $\mu' = \mu''+4$, *or a permutation of* $(2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'+2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ *otherwise.*

Proof. For Item 1, by Items 1 and 3 of Lemma 2.6, if $j \in \{0,2,4,6\}$, then $u+\mathbf{j}$ is a permutation of $(\mu'+j, \overset{m'}{\ldots}, \mu'+j, \mu''+j, \overset{m'}{\ldots}, \mu''+j, 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$. Let $\mathbf{k'} = (\mu', \overset{m'}{\ldots}, \mu', \mu'' \overset{m'}{\ldots}, \mu'')$. Since $\mu',\mu'' \in \{1,3,5,7\}$, we have that $(\mathbf{k'}, \overset{4}{\ldots}, \mathbf{k'}) + (\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(\mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ and hence $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'+2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$.

For item 2, we have $(\mathbf{k'}, \overset{4}{\ldots}, \mathbf{k'}) + (\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(\mu', \overset{4m'}{\ldots}, \mu', \mu'+4, \overset{4m'}{\ldots}, \mu'+4)$ if $\mu' = \mu''$ or $\mu' = \mu''+4$, or a permutation of $(\mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ otherwise. Therefore, $(u,u,u,u)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(\mu', \overset{4m'}{\ldots}, \mu', \mu'+4, \overset{4m'}{\ldots}, \mu'+4, 2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ if $\mu' = \mu''$ or $\mu' = \mu''+4$, or a permutation of $(2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'+2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ otherwise. □

Lemma 2.10. *Let* $u = (\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8) \in \mathbb{Z}_8^{m+2n+2r}$, *where* $m,n,r \geq 0$ *and* $\mu \in 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8 = \{2,6\}$. *Then,*

1. $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ *is a permutation of* $(2\mathbb{Z}_8, \overset{2r+2n+m}{\ldots}, 2\mathbb{Z}_8)$;
2. $(u,u,u,u)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ *is a permutation of* $(4\mathbb{Z}_8, \overset{4n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{4r+2m}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$.

Proof. By Item 1 of Lemma 2.6, if $j \in \{0,4\}$, then $u+\mathbf{j}$ is a permutation of $(\mu+j, \overset{m}{\ldots}, \mu+j, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. Similarly, if $j \in \{2,6\}$, then $u+\mathbf{j}$ is a permutation of $(\mu+j, \overset{m}{\ldots}, \mu+j, 4\mathbb{Z}_8, \overset{r}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{n}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. Let $\mathbf{k} = (\mu, \overset{m}{\ldots}, \mu)$.

For Item 1, since $\mu \in \{2,6\}$, we have that $(\mathbf{k}, \overset{4}{\ldots}, \mathbf{k}) + (\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{m}{\ldots}, 2\mathbb{Z}_8)$, and hence $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{2r+2n+m}{\ldots}, 2\mathbb{Z}_8)$.

For Item 2, we have $(\mathbf{k},\overset{4}{\ldots},\mathbf{k})+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(2\mathbb{Z}_8\backslash 4\mathbb{Z}_8,\overset{2m}{\ldots},2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. Therefore, $(u,u,u,u)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(4\mathbb{Z}_8,\overset{4n}{\ldots},4\mathbb{Z}_8,2\mathbb{Z}_8\backslash 4\mathbb{Z}_8,\overset{4r+2m}{\ldots},2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. □

Let $t_1\geq 1, t_2\geq 0$, and $t_3\geq 1$ be integers. Let $\mathcal{G}^{t_1,t_2,t_3}$ be the set of all codewords of the code generated by the matrix obtained from A^{t_1,t_2,t_3} after removing the row $(\mathbf{1}\mid\mathbf{2}\mid\mathbf{4})$.

Lemma 2.11. *Let $t_1\geq 1$ be an integer. Let*

$$\mathbf{z}=(u_1,u_1\mid x_1,u_2,u_2,u_2,u_2\mid x_2,u_3,\overset{8}{\ldots},u_3)\in\mathcal{G}^{t_1+1,0,1},$$

where $\mathbf{u}=(u_1\mid u_2\mid u_3)\in\mathcal{G}^{t_1,0,1}$ and $x_{i-1}\in(2\mathbb{Z}_{2^i})^{2^{(i-1)t_1}}$ for $i\in\{2,3\}$. Then,

1. *if $o(\mathbf{z})=8$, then x_{i-1} is a permutation of $(2\mathbb{Z}_{2^i},\overset{2^{(i-1)(t_1-1)}}{\ldots},2\mathbb{Z}_{2^i})$ for $i\in\{2,3\}$.*
2. *if $o(\mathbf{z})=4$, then $x_1=\mathbf{0}$ and x_2 is a permutation of $(4\mathbb{Z}_8,\overset{2\cdot 4^{t_1-1}}{\ldots},4\mathbb{Z}_8)$.*
3. *if $o(\mathbf{z})=2$, then $x_1=\mathbf{0}$ and $x_2=\mathbf{0}$.*

Proof. Let $\mathbf{w}_j$, where $j\in\{1,\ldots,t_1+2\}$, be the jth row of $A^{t_1+1,0,1}$. Note that $\mathbf{w}_1=(\mathbf{1}\mid\mathbf{2}\mid\mathbf{4})$, and $\mathbf{w}_2,\ldots,\mathbf{w}_{t_1+2}$ are the rows of order 8, where $\mathbf{w}_{t_1+2}=(\mathbf{0},\mathbf{1}\mid\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\mid\mathbf{1},\mathbf{0},\ldots,\mathbf{7})$. Since any element of $\mathcal{G}^{t_1+1,0,1}$ can be written as $\mathbf{z}+\lambda\mathbf{w}_{t_1+2}$, where $\lambda\in\mathbb{Z}_8$, then $\mathbf{z}=\sum_{j=2}^{t_1+1}r_j\mathbf{w}_j=(u_1,u_1\mid x_1,u_2,u_2,u_2,u_2\mid x_2,u_3,\overset{8}{\ldots},u_3)$, where $r_j\in\mathbb{Z}_8$. By construction, x_1 and x_2 are generated by the rows of $M_1'=\{\mathbf{z}^T:\mathbf{z}\in\{0,2\}^{t_1}\}$ and $M_2'=\{\mathbf{z}^T:\mathbf{z}\in\{0,2,4,6\}^{t_1}\}$, respectively. Thus, $x_1=\mathbf{0}$ or x_1 is a permutation of $(2\mathbb{Z}_4,\overset{2^{t_1-1}}{\ldots},2\mathbb{Z}_4)$, and $x_2=\mathbf{0}$ or x_2 is a permutation of $(2\mathbb{Z}_8,\overset{4^{t_1-1}}{\ldots},2\mathbb{Z}_8)$ or $(4\mathbb{Z}_8,\overset{2\cdot 4^{t_1-1}}{\ldots},4\mathbb{Z}_8)$.

For Item 1, there exists at least one $j\in\{2,\ldots,t_1+1\}$ such that $r_j\in\{1,3,5,7\}$. Therefore, by Item 1 of Lemma 2.6, x_{i-1} is a permutation of $(2\mathbb{Z}_{2^i},\overset{2^{(i-1)(t_1-1)}}{\ldots},2\mathbb{Z}_{2^i})$ for $i\in\{2,3\}$.

For Item 2, we have that $r_j\in 2\mathbb{Z}_8$ for all $j\in\{2,\ldots,t_1+1\}$ and there exist at least one $j\in\{2,\ldots,t_1+1\}$ such that $r_j\in\{2,6\}$. Therefore, $x_1=\mathbf{0}$ and, by Item 1 of Lemma 2.6, x_2 is a permutation of $(4\mathbb{Z}_8,\overset{2\cdot 4^{t_1-1}}{\ldots},4\mathbb{Z}_8)$.

For Item 3, we have that $r_j\in 4\mathbb{Z}_8$ for all $j\in\{2,\ldots,t_1+1\}$ and there exist at least one $j\in\{2,\ldots,t_1+1\}$ such that $r_j=4$. Therefore, $x_1=\mathbf{0}$ and $x_2=\mathbf{0}$. □

Lemma 2.12. *Let $t_1\geq 1$ and $t_2\geq 0$ be integers. Let*

$$\mathbf{z}=(u_1,u_1\mid x_1,u_2,u_2,u_2,u_2\mid u_3,u_3,u_3,u_3)\in\mathcal{G}^{t_1,t_2+1,1},$$

where $\mathbf{u}=(u_1\mid u_2\mid u_3)\in\mathcal{G}^{t_1,t_2,1}$ and $x_1\in(2\mathbb{Z}_4)^{2^{t_1+t_2}}$. Then,

1. *if $o(\mathbf{z})=8$, then x_1 is a permutation of $(2\mathbb{Z}_4,\overset{2^{t_1+t_2-1}}{\ldots},2\mathbb{Z}_4)$.*
2. *if $o(\mathbf{z})=4$, then $x_1=\mathbf{0}$ if $u_1=\mathbf{0}$, and x_1 is a permutation of $(2\mathbb{Z}_4,\overset{2^{t_1+t_2-1}}{\ldots},2\mathbb{Z}_4)$ otherwise.*
3. *if $o(\mathbf{z})=2$, then $x_1=\mathbf{0}$.*

Proof. Let $\mathbf{w}_i$, where $i\in\{1,\ldots,t_1+t_2+2\}$, be the ith row of $A^{t_1,t_2+1,1}$. Note that $\mathbf{w}_1=(\mathbf{1}\mid\mathbf{2}\mid\mathbf{4})$, $\mathbf{w}_2,\ldots,\mathbf{w}_{t_1+1}$ are the rows of order 8, and $\mathbf{w}_{t_1+2},\ldots,\mathbf{w}_{t_1+t_2+2}$ are the rows of order 4, where $\mathbf{w}_{t_1+t_2+2}=(\mathbf{0},\mathbf{1}\mid\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\mid\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$. Since any element of $\mathcal{G}^{t_1,t_2+1,1}$ can be written as $\mathbf{z}+\lambda\mathbf{w}_{t_1+t_2+2}$, where $\lambda\in\{0,1,2,3\}$, then $\mathbf{z}=\sum_{i=2}^{t_1+t_2+1}r_i\mathbf{w}_i=(u_1,u_1\mid x_1,u_2,u_2,u_2,u_2\mid u_3,u_3,u_3,u_3)$, where $r_i\in\mathbb{Z}_8$ for $i\in\{2,\ldots,t_1+1\}$ and $r_i\in\{0,1,2,3\}$ for $i\in\{t_1+2,\ldots,t_1+t_2+1\}$. By construction, x_1 is generated by the rows of $M_1'=\{\mathbf{z}^T:\mathbf{z}\in\{0,2\}^{t_1+t_2}\}$. Thus, $x_1=\mathbf{0}$ or x_1 is a permutation of $(2\mathbb{Z}_4,\overset{2^{t_1+t_2-1}}{\ldots},2\mathbb{Z}_4)$.

For Item 1, there exists at least one $i \in \{2, \dots, t_1+1\}$ such that $r_i \in \{1,3,5,7\}$. Therefore, since x_1 is of order at most two, $x_1 \neq \mathbf{0}$.

For Item 2, we have that $r_i \in 2\mathbb{Z}_8$ for all $i \in \{2, \dots, t_1+1\}$ and $r_i \in \{0,1,2,3\}$ for all $i \in \{t_1+2, \dots, t_1+t_2+1\}$. Note that, since x_1 and u_1 are of order at most two, $x_1 \neq \mathbf{0}$ if and only if there exists at least one i for $i \in \{t_1+2, \dots, t_1+t_2+1\}$ such that $r_i \in \{1,3\}$, or equivalently, if and only if $u_1 \neq \mathbf{0}$.

For Item 3, we have that $r_i \in 4\mathbb{Z}_8 = \{0,4\}$ for all $i \in \{2, \dots, t_1+1\}$ and $r_i \in \{0,2\}$ for all $i \in \{t_1+2, \dots, t_1+t_2+1\}$. Therefore, since x_1 is of order at most two, $x_1 = \mathbf{0}$. □

Lemma 2.13. *Let $t_1 \geq 1$ be an integer. Let $\mathcal{H}^{t_1,0,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, 0, 1)$ generated by $A^{t_1,0,1}$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,0,1}$. Then,*

1. *if $o(\mathbf{u}) = 8$, then u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of $(\mu, \overset{m}{\dots}, \mu, 2\mathbb{Z}_4, \overset{n}{\dots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\dots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ for some integers $m, n, r \geq 0$ and $\mu \in \{1,3\}$, and u_3 is a permutation of $(\mu', \overset{m'}{\dots}, \mu', \mu'', \overset{m'}{\dots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\dots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\dots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ for some integers $m', n', r' \geq 0$ and μ, $\mu' \in \{1,3,5,7\}$.*
2. *if $o(\mathbf{u}) = 4$, then $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2) = 4^{t_1-1}$ times and $\frac{\alpha_2}{2} - \frac{\alpha_1}{4} = 4^{t_1-1} - 2^{t_1-1}$ times the element 0, and u_3 is a permutation of $(\mu, \overset{m}{\dots}, \mu, 4\mathbb{Z}_8, \overset{n}{\dots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\dots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $m, n, r \geq 0$ and $\mu \in \{2,6\}$.*
3. *if $o(\mathbf{u}) = 2$, then $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $\frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2 + 2\alpha_3) = 8^{t_1-1}$ times and $\frac{\alpha_3}{2} - \frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2) = 8^{t_1-1} - 4^{t_1-1}$ times the element 0.*

Proof. We perform a proof by induction on $t_1 \geq 1$. If $t_1 = 1$, then by Lemma 2.1, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 1$, and $\mathcal{G}^{1,0,1} = \langle (0,1 \mid 1 \mid 1) \rangle$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{1,0,1}$. Then, $\mathbf{u} = \lambda(0,1 \mid 1 \mid 1)$, where $\lambda \in \mathbb{Z}_8$. Thus, we have that $u_1 = \lambda(0,1)$, $u_2 = (\lambda)$, and $u_3 = (\lambda)$. If $o(\mathbf{u}) = 8$, then $\lambda \in \mathbb{Z}_8\backslash 2\mathbb{Z}_8$. Therefore, $\mathbf{u}$ satisfies Property 1. If $o(\mathbf{u}) = 4$, then $\lambda \in \{2,6\}$. In this case, $u_1 = (0,0)$, $u_2 = (2)$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $1 = \frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2)$ time and $0 = \frac{\alpha_2}{2} - \frac{\alpha_1}{4}$ times the element 0, and $u_3 = (\lambda)$. Thus, $\mathbf{u}$ satisfies Property 2. If $o(\mathbf{u}) = 2$, then $\lambda = 4$. In this case, $u_1 = (0,0)$, $u_2 = (0)$, and $u_3 = (4)$ contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $1 = \frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2 + 2\alpha_3)$ time and $0 = \frac{\alpha_3}{2} - \frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2)$ times the element 0. Thus, $\mathbf{u}$ satisfies Property 3. Therefore, the lemma holds for $t_1 = 1$.

Assume now that the lemma holds for the code $\mathcal{H}^{t_1,0,1}$ of type $(\alpha_1, \alpha_2, \alpha_3; t_1, 0, 1)$ with $t_1 \geq 1$. By Lemma 2.1, we have that

$$2^{t_1} = \alpha_1, 4^{t_1} = \alpha_1 + 2\alpha_2, \text{ and } 8^{t_1} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{18}$$

We must show that the lemma is also true for the code $\mathcal{H}^{t_1+1,0,1}$.

Let $\mathbf{v} = (v_1 \mid v_2 \mid v_3) \in \mathcal{G}^{t_1+1,0,1}$. We can write

$$\mathbf{v} = \mathbf{z} + \lambda \mathbf{w},$$

where $\mathbf{z} = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid x_2, u_3, \overset{8}{\dots}, u_3)$, $\mathbf{w} = (\mathbf{0}, \mathbf{1} \mid \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3} \mid \mathbf{1}, \mathbf{0}, \dots, \mathbf{7})$, $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,0,1}$, $\lambda \in \mathbb{Z}_8$, $x_1 \in (2\mathbb{Z}_4)^{2^{t_1}}$ such that either $x_1 = \mathbf{0}$ or x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1-1}}{\dots}, 2\mathbb{Z}_4)$, and $x_2 \in (2\mathbb{Z}_8)^{4^{t_1}}$ such that either $x_2 = \mathbf{0}$ or x_2 is a permutation of $(2\mathbb{Z}_8, \overset{4^{t_1-1}}{\dots}, 2\mathbb{Z}_8)$ or $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\dots}, 4\mathbb{Z}_8)$. Then,

$v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$ and, for $i \in \{2, 3\}$,

$$v_i = (x_{i-1}, u_i, \overset{2^i}{\ldots}, u_i) + \lambda(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{2^i - 1}). \tag{19}$$

If $\mathbf{z} = \mathbf{0}$, then $\mathbf{v} = \lambda\mathbf{w}$ and it is easy to see that $\mathbf{v}$ satisfies Property 1 if $\lambda \in \mathbb{Z}_8\backslash 2\mathbb{Z}_8 = \{1, 3, 5, 7\}$, Property 2 if $\lambda \in \{2, 6\}$, and Property 3 if $\lambda = 4$. Therefore, we focus on the case when $\mathbf{z} \neq \mathbf{0}$.

Case 1. Assume that $o(\mathbf{v}) = 8$. We have two subcases: when $o(\mathbf{z})$ is arbitrary and $\lambda \in \mathbb{Z}_8\backslash 2\mathbb{Z}_8$, and when $o(\mathbf{z}) = 8$ and $\lambda \in 2\mathbb{Z}_8$. In both subcases, note that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. For the first subcase, we have that $(u_i, \overset{2^i}{\ldots}, u_i) + \lambda(\mathbf{0}, \ldots, \mathbf{2^i - 1})$, for $i \in \{2, 3\}$, is a permutation of $(\mathbb{Z}_{2^i}, \overset{\alpha_i}{\ldots}, \mathbb{Z}_{2^i})$ by Lemma 2.7. Thus, from (19), v_i is a permutation of $(x_{i-1} + \lambda\mathbf{1}, \mathbb{Z}_{2^i}, \overset{\alpha_i}{\ldots}, \mathbb{Z}_{2^i})$. Since either $x_{i-1} + \lambda\mathbf{1} = \lambda\mathbf{1}$, or $x_{i-1} + \lambda\mathbf{1}$ is a permutation of $(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}, \overset{2^{(i-1)(t_1-1)}}{\ldots}, \mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})$, $\mathbf{v}$ satisfies Property 1.

For the second subcase when $o(\mathbf{v}) = 8$, that is, when $o(\mathbf{z}) = 8$ and $\lambda \in 2\mathbb{Z}_8$, we have that $o(\mathbf{u}) = 8$ and, by Item 1 of Lemma 2.11, x_{i-1} is a permutation of $(2\mathbb{Z}_{2^i}, \overset{2^{(i-1)(t_1-1)}}{\ldots}, 2\mathbb{Z}_{2^i})$ for $i \in \{2, 3\}$. By induction hypothesis, $\mathbf{u}$ satisfies Property 1 and then u_2 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4),$$

where $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of

$$(\mu', \overset{m'}{\ldots}, \mu', \mu'', \overset{m'}{\ldots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8),$$

where $m', n', r' \geq 0$ and $\mu', \mu'' \in \{1, 3, 5, 7\}$. From (19), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$. If $\lambda \in \{0, 4\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2)$ in $\mathbf{v}$ satisfies the same property as u_2 in $\mathbf{u}$; that is, Property 1. If $\lambda \in \{2, 6\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$. By Item 1 of Lemma 2.6, we have that $x_1 + \mathbf{2}$ is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1-1}}{\ldots}, 2\mathbb{Z}_4)$. Thus, by Lemma 2.8, v_2 is a permutation of

$$(2\mathbb{Z}_4, \overset{4n+2^{t_1-1}}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4).$$

Therefore, for $\lambda \in 2\mathbb{Z}_8$, v_2 satisfies Property 1. Now, we consider the coordinates in v_3. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{7})$. By Item 1 of Lemma 2.6, we have that, for $\lambda \in 2\mathbb{Z}_8$, $x_2 + \lambda\mathbf{1}$ is a permutation of $(2\mathbb{Z}_8, \overset{4^{t_1-1}}{\ldots}, 2\mathbb{Z}_8)$. If $\lambda = 0$, it is easy to see that v_3 satisfies Property 1. Note that $\lambda(\mathbf{0}, \ldots, \mathbf{7})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ if $\lambda \in \{2, 6\}$, and a permutation of $(\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ if $\lambda = 4$. Thus, by Lemma 2.9, v_3 satisfies Property 1. Therefore, if $o(\mathbf{v}) = 8$, then $\mathbf{v}$ satisfies Property 1.

Case 2. Assume that $o(\mathbf{v}) = 4$. We have two subcases: when $o(\mathbf{z}) = 4$ and $\lambda \in 2\mathbb{Z}_8$, and when $o(\mathbf{z}) = 2$ and $\lambda \in \{2, 6\}$. For the first subcase, since $o(\mathbf{z}) = 4$, we have that $o(\mathbf{u}) = 4$. Moreover, $x_1 = \mathbf{0}$ and x_2 is a permutation of $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 4\mathbb{Z}_8)$ by Item 2 of Lemma 2.11. By induction hypothesis, $\mathbf{u}$ satisfies Property 2. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly 4^{t_1-1} times and $4^{t_1-1} - 2^{t_1-1}$ times the element 0, and u_3 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$$

for some integers $m, n, r \geq 0$ and $\mu \in \{2, 6\}$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in 2\mathbb{Z}_8$, we have that $v_1 = \mathbf{0}$. From (19), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$. If $\lambda \in \{0, 4\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2)$. Since $x_1 = \mathbf{0}$ is of length 2^{t_1}, it is easy to see that v_2 in $\mathbf{v}$ satisfies the same property as u_2 in $\mathbf{u}$; that is, Property 2. If $\lambda \in \{2, 6\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$, where

$x_1 = \mathbf{0}$ is of length 2^{t_1}. Note that $u_2 + \mathbf{2}$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ as many times as u_2 contains the element 0, and the element 0 as many times as u_2 contains the element 2. Thus, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $2^{t_1}+2(4^{t_1-1})+2(4^{t_1-1}-2^{t_1-1}) = 4^{t_1}$ times and $2(4^{t_1-1})+2(4^{t_1-1}-2^{t_1-1}) = 4^{t_1}-2^{t_1}$ times the element 0. Therefore, for $\lambda \in 2\mathbb{Z}_8$, v_2 satisfies Property 2. Now, we consider the coordinates in v_3. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{7})$. If $\lambda = 0$, it is easy to see that v_3 satisfies Property 2. For $\lambda = 4$, $x_2 + \lambda\mathbf{1}$ is a permutation of $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 4\mathbb{Z}_8)$, and for $\lambda \in \{2, 6\}$, it is a permutation of

$$(2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8).$$

Note that $\lambda(\mathbf{0}, \ldots, \mathbf{7})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ if $\lambda \in \{2, 6\}$, and a permutation of $(\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ if $\lambda = 4$. Hence, by Lemma 2.10, v_3 also satisfies Property 2, and so does $\mathbf{v}$.

Now, we consider the second subcase, when $o(\mathbf{z}) = 2$ and $\lambda \in \{2, 6\}$. Since $o(\mathbf{z}) = 2$, we have that $o(\mathbf{u}) = 2$. Then, by Item 3 of Lemma 2.11, $x_1 = \mathbf{0}$ and $x_2 = \mathbf{0}$. By induction hypothesis, $\mathbf{u}$ satisfies Property 3, so $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $m = 8^{t_1-1}$ times and $m' = 8^{t_1-1} - 4^{t_1-1}$ times the element 0. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{2, 6\}$, we have that $v_1 = \mathbf{0}$. From (19), $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$. Since $x_1 = \mathbf{0}$ and $u_2 = \mathbf{0}$, of length α_1 and α_2, respectively, we have that $v_2 = (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$. Therefore, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\alpha_1 + 2\alpha_2 = 4^{t_1}$ times and $2\alpha_2 = 4^{t_1} - 2^{t_1}$ times the element 0, by (18). Therefore, v_2 satisfies Property 2. Now, we consider the coordinates in v_3. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{7})$. Since $x_2 = \mathbf{0}$, $x_2 + \lambda\mathbf{1} = (\lambda, \overset{4^{t_1}}{\ldots}, \lambda)$. Note that u_3 is a permutation of

$$(4, \overset{m-m'}{\ldots}, 4, 4\mathbb{Z}_8, \overset{m'}{\ldots}, 4\mathbb{Z}_8).$$

Moreover, since $\lambda \in \{2, 6\}$, $\lambda(\mathbf{0}, \ldots, \mathbf{7})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Thus, by Item 1 of Lemma 2.6, $(u_3, \overset{8}{\ldots}, u_3) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of

$$(2\mathbb{Z}_8, \overset{2(m-m')+4m'}{\ldots}, 2\mathbb{Z}_8).$$

Thus, v_3 is a permutation of $(\lambda, \overset{4^{t_1}}{\ldots}, \lambda, 2\mathbb{Z}_8, \overset{2(m-m')+4m'}{\ldots}, 2\mathbb{Z}_8)$ with $\lambda \in \{2, 6\}$, and hence v_3 also satisfies Property 2 and so does $\mathbf{v}$. Therefore, if $o(\mathbf{v}) = 4$, then $\mathbf{v}$ satisfies Property 2.

Case 3. Assume that $o(\mathbf{v}) = 2$. Then, $o(\mathbf{z}) = 2$ and $\lambda \in \{0, 4\}$. Since $o(\mathbf{z}) = 2$, then $o(\mathbf{u}) = 2$. Moreover, $x_1 = \mathbf{0}$ and $x_2 = \mathbf{0}$ by Item 3 of Lemma 2.11. By induction hypothesis, $\mathbf{u}$ satisfies Property 3, and then $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly 8^{t_1-1} times and $8^{t_1-1} - 4^{t_1-1}$ times the element 0. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $v_1 = \mathbf{0}$. From (19), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ and $u_2 = \mathbf{0}$, so $v_2 = \mathbf{0}$. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{7})$, where $x_2 = \mathbf{0}$ is of length 4^{t_1}. If $\lambda = 0$, it is easy to see that v_3 satisfies Property 3. If $\lambda = 4$, $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + (\mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$. Note that $u_3 + \mathbf{4}$ contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ as many times as u_3 contains the element 0, and the element 0 as many times as u_3 contains the element 4. Then, v_3 contains the element 4 exactly $4^{t_1} + 4(8^{t_1-1}) + 4(8^{t_1-1} - 4^{t_1-1}) = 8^{t_1}$ times and $4(8^{t_1-1}) + 4(8^{t_1-1} - 4^{t_1-1}) = 8^{t_1} - 4^{t_1}$ the element 0. Therefore, $\mathbf{v}$ satisfies Property 3. This completes the proof. □

Lemma 2.14. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. Let $\mathcal{H}^{t_1,t_2,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ generated by $A^{t_1,t_2,1}$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,t_2,1}$.*

1. *If $o(\mathbf{u}) = 8$, then $\mathbf{u}$ has the following property:*
   (a) *u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of $(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ for some integers $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of $(\mu', \overset{m'}{\ldots}, \mu', \mu'', \overset{m'}{\ldots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ for some integers $m', n', r' \geq 0$ and $\mu, \mu' \in \{1, 3, 5, 7\}$.*
2. *If $o(\mathbf{u}) = 4$, then $\mathbf{u}$ has one of the following properties:*
   (a) *$u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2) = 4^{t_1+t_2-1}$ times and $\frac{\alpha_2}{2} - \frac{\alpha_1}{4} = 4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $m, n, r \geq 0$ and $\mu \in \{2, 6\}$.*
   (b) *u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of $(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ for some integers $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{t}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{t'}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $t, t' \geq 0$.*
3. *If $o(\mathbf{u}) = 2$, then $\mathbf{u}$ has one of the following properties:*
   (a) *$u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $\frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2 + 2\alpha_3) = 8^{t_1-1}4^{t_2}$ times and $\frac{\alpha_3}{2} - \frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2) = 8^{t_1-1}4^{t_2} - 4^{t_1+t_2-1}$ times the element 0.*
   (b) *$u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2) = 4^{t_1+t_2-1}$ times and $\frac{\alpha_2}{2} - \frac{\alpha_1}{4} = 4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{m}{\ldots}, 4\mathbb{Z}_8)$ for some $m \geq 0$.*

Proof. We prove this lemma by induction on $t_2 \geq 0$. The lemma holds for the code $\mathcal{H}^{t_1,0,1}$ by Lemma 2.13. Assume that the lemma holds for the code $\mathcal{H}^{t_1,t_2,1}$ of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ with $t_1 \geq 1$ and $t_2 \geq 0$. By Lemma 2.1, we have that

$$2^{t_1+t_2} = \alpha_1, 4^{t_1+t_2} = \alpha_1 + 2\alpha_2, \text{ and } 8^{t_1}4^{t_2} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{20}$$

We must show that the lemma is also true for the code $\mathcal{H}^{t_1,t_2+1,1}$.

Let $\mathbf{v} = (v_1 \mid v_2 \mid v_3) \in \mathcal{G}^{t_1,t_2+1,1}$. We can write

$$\mathbf{v} = \mathbf{z} + \lambda\mathbf{w},$$

where $\mathbf{z} = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid u_3, u_3, u_3, u_3)$, $\mathbf{w} = (\mathbf{0}, \mathbf{1} \mid \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3} \mid \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$, $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,t_2,1}$, $\lambda \in \{0, 1, 2, 3\}$, and $x_1 \in (2\mathbb{Z}_4)^{2^{t_1+t_2}}$ such that either $x_1 = \mathbf{0}$ or a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$. Then,

$$\begin{aligned} v_1 &= (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1}), \\ v_2 &= (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}), \\ v_3 &= (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}). \end{aligned} \tag{21}$$

If $\mathbf{z} = \mathbf{0}$, then $\mathbf{v} = \lambda\mathbf{w}$. It is easy to see that $\mathbf{v}$ satisfies Property 2b if $\lambda \in \{1, 3\}$ and Property 3b if $\lambda = 2$. Therefore, we focus on the case when $\mathbf{z} \neq \mathbf{0}$.

Case 1. Assume that $o(\mathbf{v}) = 8$. Then, $o(\mathbf{z}) = 8$ and $\lambda \in \{0, 1, 2, 3\}$. We have that $o(\mathbf{u}) = 8$ and, by Item 1 of Lemma 2.12, x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$. By induction hypothesis, $\mathbf{u}$ satisfies Property 1a. Then, u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of

$$(\mu, \overset{m}{\dots}, \mu, 2\mathbb{Z}_4, \overset{n}{\dots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\dots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4),$$

where $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of

$$(\mu', \overset{m'}{\dots}, \mu', \mu'', \overset{m'}{\dots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\dots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\dots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8),$$

where $m', n', r' \geq 0$ and $\mu', \mu'' \in \{1, 3, 5, 7\}$. First, since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, v_1 contains every element of $\mathbb{Z}_2$ the same number of times, for any $\lambda \in \{0, 1, 2, 3\}$. Second, from (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$. If $\lambda = 0$, then v_2 clearly satisfies 1a. If $\lambda \in \{1, 3\}$, then we have that $(u_2, u_2, u_2, u_2) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ is a permutation of $(\mathbb{Z}_4, \overset{\alpha_2}{\dots}, \mathbb{Z}_4)$ by Lemma 2.7. For $\lambda \in \{1, 3\}$, since $x_1 + \lambda\mathbf{1}$ is a permutation of $(\mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\dots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ by Item 3 of Lemma 2.6, we have that v_2 satisfies Property 1a. If $\lambda = 2$, $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$. By Item 1 of Lemma 2.6, we have that $x_1 + \mathbf{2}$ is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\dots}, 2\mathbb{Z}_4)$. Therefore, by Lemma 2.8, v_2 is a permutation of $(2\mathbb{Z}_4, \overset{4n+2^{t_1+t_2-1}}{\dots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\dots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ and then v_2 satisfies Property 1a. Finally, we consider the coordinates in v_3. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. If $\lambda = 0$, then v_3 clearly satisfies 1a. Note that $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}) = (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ if $\lambda = 2$ and $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ if $\lambda \in \{1, 3\}$. Therefore, by Lemma 2.9, v_3 satisfies Property 1a, and so does $\mathbf{v}$.

Case 2. Assume that $o(\mathbf{v}) = 4$. We have two subcases: when $o(\mathbf{z}) = 4$ and $\lambda \in \{0, 1, 2, 3\}$, and when $o(\mathbf{z}) = 2$ and $\lambda \in \{1, 3\}$. For the first subcase, since $o(\mathbf{z}) = 4$, $o(\mathbf{u}) = 4$. By induction hypothesis, $\mathbf{u}$ satisfies Property 2a or 2b. Assume that $\mathbf{u}$ satisfies Property 2a. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $4^{t_1+t_2-1}$ times and $4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of

$$(\mu, \overset{m}{\dots}, \mu, 4\mathbb{Z}_8, \overset{n}{\dots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\dots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$$

for some integers $m, n, r \geq 0$ and $\mu \in \{2, 6\}$. Note that, in this case, $x_1 = \mathbf{0}$ by Item 2 of Lemma 2.12. If $\lambda = 0$, then it is easy to see that $\mathbf{v}$ satisfies Property 2a. If $\lambda = 2$, we show that $\mathbf{v}$ satisfies Property 2a. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda = 2$, we have that $v_1 = \mathbf{0}$. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. Note that $u_2 + \mathbf{2}$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ as many times as u_2 contains the element 0, and the element 0 as many times as u_2 contains the element 2. Thus, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $2^{t_1+t_2} + 2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2}$ times and $2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2} - 2^{t_1+t_2}$ times the element 0, so v_2 satisfies Property 2a. From (21), $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$. By Item 2 of Lemma 2.10, v_3 is a permutation of

$$(4\mathbb{Z}_8, \overset{4n}{\dots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{4r+2m}{\dots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8).$$

Therefore, for $\lambda = 2$, $\mathbf{v}$ satisfies Property 2a. Finally, if $\lambda \in \{1, 3\}$, we show that $\mathbf{v}$ satisfies Property 2b. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{1, 3\}$, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. Since $\lambda \in \{1, 3\}$, by Lemma 2.7, we have that v_2 is a permutation of $(\lambda, \overset{2^{t_1+t_2}}{\dots}, \lambda, \mathbb{Z}_4, \overset{\alpha_2}{\dots}, \mathbb{Z}_4)$. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Note that, for $\lambda \in \{1, 3\}$, $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Thus, by Item 1 of Lemma 2.10, v_3 satisfies Property 2b, and so does $\mathbf{v}$. Therefore, if $o(\mathbf{u}) = 4$ and $\mathbf{u}$ satisfies Property 2a, we have that $\mathbf{v}$ satisfies either Property 2a or 2b.

We continue with the first subcase, when $o(\mathbf{z}) = 4$ and $\lambda \in \{0,1,2,3\}$. Again, we have that $o(\mathbf{u}) = 4$. Now, we assume that $\mathbf{u}$ satisfies Property 2b. Then, u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$$

for some integers $m, n, r \geq 0$ and $\mu \in \{1,3\}$, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{t}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{t'}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $t, t' \geq 0$. Note that, in this case, x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$ by Item 2 of Lemma 2.12. Now, we show that $\mathbf{v}$ satisfies Property 2b. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$ and u_1 contains every element of $\mathbb{Z}_2$ the same number of times, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times, for any $\lambda \in \{0,1,2,3\}$. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$. If $\lambda = 0$, it is clear that v_2 satisfies Property 2b. Note that $x_1 + \lambda\mathbf{1}$ is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$ if $\lambda = 2$, and a permutation of $(\mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ if $\lambda \in \{1,3\}$. If $\lambda = 2$, then by Lemma 2.8, $(u_2, u_2, u_2, u_2) + (\mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$ is a permutation of

$$(2\mathbb{Z}_4, \overset{4n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4).$$

If $\lambda \in \{1,3\}$, then by Lemma 2.7, $(u_2, u_2, u_2, u_2) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ is a permutation of $(\mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4)$. Therefore, v_2 satisfies Property 2b. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. If $\lambda = 0$, it is clear that v_3 satisfies Property 2b. Note that $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ if $\lambda \in \{1,3\}$, and $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}) = (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ if $\lambda = 2$. Therefore, by Lemma 2.10, v_3 satisfies Property 2b, and so does $\mathbf{v}$.

Now, we consider the second subcase when $o(\mathbf{v}) = 4$, that is, when $o(\mathbf{z}) = 2$ and $\lambda \in \{1,3\}$. Since $o(\mathbf{z}) = 2$, $o(\mathbf{u}) = 2$. By induction hypothesis, $\mathbf{u}$ satisfies Property 3a or 3b. Assume that $\mathbf{u}$ satisfies Property 3a. Then, $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $m = 8^{t_1-1}4^{t_2}$ times and $m' = 8^{t_1-1}4^{t_2} - 4^{t_1+t_2-1}$ times the element 0. By Item 3 of Lemma 2.12, we have that $x_1 = \mathbf{0}$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{1,3\}$, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. By Lemma 2.7, we have that v_2 is a permutation of

$$(\lambda, \overset{2^{t_1+t_2}}{\ldots}, \lambda, \mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4),$$

where $\lambda \in \{1,3\}$. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Note that u_3 is a permutation of $(4, \overset{m-m'}{\ldots}, 4, 4\mathbb{Z}_8, \overset{m'}{\ldots}, 4\mathbb{Z}_8)$ and, since $\lambda \in \{1,3\}$, $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Thus, by Item 1 of Lemma 2.6, $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{m+m'}{\ldots}, 2\mathbb{Z}_8)$, so v_3 satisfies Property 2b, and so does $\mathbf{v}$. Therefore, if $o(\mathbf{u}) = 2$ and $\mathbf{u}$ satisfies Property 3a, we have that $\mathbf{v}$ satisfies Property 2b.

We continue with the second subcase, when $o(\mathbf{z}) = 2$ and $\lambda \in \{1,3\}$. Again, we have that $o(\mathbf{u}) = 2$. Now, we assume that $\mathbf{u}$ satisfies Property 3b. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $4^{t_1+t_2-1}$ times and $4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{m}{\ldots}, 4\mathbb{Z}_8)$ for some $m \geq 0$. By Item 3 of Lemma 2.12, we have that $x_1 = \mathbf{0}$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{1,3\}$, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. By Lemma 2.7, we have that v_2 is a permutation of

$$(\lambda, \overset{2^{t_1+t_2}}{\ldots}, \lambda, \mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4),$$

where $\lambda \in \{1,3\}$. From (21), $v_3 = (u_3,u_3,u_3,u_3) + \lambda(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$. Since $\lambda \in \{1,3\}$, $\lambda(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$. Thus, by Item 1 of Lemma 2.6, $v_3 = (u_3,u_3,u_3,u_3)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{2m}{\ldots}, 2\mathbb{Z}_8)$. Therefore, v_3 satisfies Property 2b, and so does $\mathbf{v}$.

Case 3. Assume that $o(\mathbf{v}) = 2$. Then, $o(\mathbf{z}) = 2$ and $\lambda \in \{0,2\}$. Since $o(\mathbf{z}) = 2$, we have that $o(\mathbf{u}) = 2$ and, by Item 3 of Lemma 2.12, $x_1 = \mathbf{0}$. By induction hypothesis, $\mathbf{u}$ satisfies Property 3a or 3b. Assume that $\mathbf{u}$ satisfies Property 3a. Then, $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $m = 8^{t_1-1}4^{t_2}$ times and $m' = 8^{t_1-1}4^{t_2} - 4^{t_1+t_2-1}$ times the element 0. If $\lambda = 0$, then $\mathbf{v} = (\mathbf{0} \mid \mathbf{0} \mid v_3)$ satisfies Property 3a, since v_3 contains $4m$ times the element 4 and $4m'$ the element 0. Now, we assume that $\lambda = 2$. Since $v_1 = (u_1,u_1) + \lambda(\mathbf{0},\mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda = 2$, we have that $v_1 = \mathbf{0}$. From (21), $v_2 = (x_1,u_2,u_2,u_2,u_2) + (\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$ and $u_2 = \mathbf{0}$. Therefore, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\alpha_1 + 2\alpha_2 = 4^{t_1+t_2}$ times and $2\alpha_2 = 4^{t_1+t_2} - 2^{t_1+t_2}$ times the element 0, by (20). From (21), $v_3 = (u_3,u_3,u_3,u_3) + (\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$. Note that u_3 is a permutation of

$$(4, \overset{m-m'}{\ldots}, 4, 4\mathbb{Z}_8, \overset{m'}{\ldots}, 4\mathbb{Z}_8).$$

Thus, by Item 1 of Lemma 2.6, v_3 is a permutation of $(4\mathbb{Z}_8, \overset{2m+2m'}{\ldots}, 4\mathbb{Z}_8)$, so v_3 satisfies Property 3b, and so does $\mathbf{v}$. Therefore, if $o(\mathbf{u}) = 2$ and $\mathbf{u}$ satisfies Property 3a, we have that $\mathbf{v}$ satisfies Property 3b.

We continue with the case when $o(\mathbf{z}) = 2$ and $\lambda \in \{0,2\}$. Again, we have that $o(\mathbf{u}) = 2$ and $x_1 = \mathbf{0}$. Now, we assume that $\mathbf{u}$ satisfies Property 3b. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $4^{t_1+t_2-1}$ times and $4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{m}{\ldots}, 4\mathbb{Z}_8)$ for some $m \geq 0$. If $\lambda = 0$, then it is easy to see that $\mathbf{v}$ satisfies Property 3b. Now, we assume that $\lambda = 2$. Since $v_1 = (u_1,u_1) + \lambda(\mathbf{0},\mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda = 2$, we have that $v_1 = \mathbf{0}$. From (21), $v_2 = (x_1,u_2,u_2,u_2,u_2)+(\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. Note that $u_2+\mathbf{2}$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ as many times as u_2 contains the element 0, and the element 0 as many times as u_2 contains the element 2. Therefore, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $2^{t_1+t_2}+2(4^{t_1+t_2-1})+2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2}$ times and $2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2} - 2^{t_1+t_2}$ times the element 0. From (21), $v_3 = (u_3,u_3,u_3,u_3)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$. By Item 1 of Lemma 2.6, v_3 is a permutation of $(4\mathbb{Z}_8, \overset{4m}{\ldots}, 4\mathbb{Z}_8)$. Therefore, v_3 satisfies Property 3b, and so does $\mathbf{v}$. This completes the proof. □

From [6], related to the generalized Gray map (1) considered in this paper, we have the following results:

Lemma 2.15. [6] *Let $\lambda, \mu \in \mathbb{Z}_2$. Then, $\phi_s(\lambda\mu 2^{s-1}) = \lambda\phi_s(\mu 2^{s-1}) = \lambda\mu\phi_s(2^{s-1})$.*

Lemma 2.16. [6] *Let $u, v \in \mathbb{Z}_{2^s}$. Then, $\phi_s(2^{s-1}u + v) = \phi_s(2^{s-1}u) + \phi_s(v)$.*

Proposition 2.17. [15, 6] *Let $u, v \in \mathbb{Z}_{2^s}$. Then,*

$$d_H(\phi_s(u), \phi_s(v)) = \mathrm{wt}_H(\phi_s(u-v)).$$

By Proposition 2.17, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes obtained from the Gray map Φ are distance invariant, that is, the Hamming weight distribution is invariant under translation by a codeword. Therefore, their minimum distance coincides with the minimum weight.

Proposition 2.18. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{t_1,t_2,1}$, generated by $A^{t_1,t_2,1}$, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code.*

Proof. Let $\mathcal{H}^{t_1,t_2,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,1)$ and $H^{t_1,t_2,1}$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length N. We have that $N=\alpha_1+2\alpha_2+4\alpha_3$. The cardinality of $H^{t_1,t_2,1}$ is $8^{t_1}\cdot 4^{t_2}\cdot 2=2(\alpha_1+2\alpha_2+4\alpha_3)=2N$ by Lemma 2.1. By Proposition 2.17, the minimum distance of $H^{t_1,t_2,1}$ is equal to the minimum weight of $H^{t_1,t_2,1}$. Therefore, we need to prove that the minimum weight of $H^{t_1,t_2,1}$ is $N/2$.

We can write that $\mathcal{H}^{t_1,t_2,1}=\mathcal{G}^{t_1,t_2,1}\cup(\mathcal{G}^{t_1,t_2,1}+(\mathbf{1}\mid\mathbf{2}\mid\mathbf{4}))$. By Lemma 2.16, $H^{t_1,t_2,1}=\Phi(\mathcal{G}^{t_1,t_2,1})\cup(\Phi(\mathcal{G}^{t_1,t_2,1})+\mathbf{1})$. Let $\mathbf{u}=(u_1\mid u_2\mid u_3)\in\mathcal{H}^{t_1,t_2,1}\backslash\{\mathbf{0},(\mathbf{1}\mid\mathbf{2}\mid\mathbf{4})\}$. We show that $\mathrm{wt}_H(\Phi(\mathbf{u}))=N/2$. First, consider $\mathbf{u}\in\mathcal{G}^{t_1,t_2,1}\backslash\{\mathbf{0}\}$. If $o(\mathbf{u})=8$, then by Lemma 2.14, u_1 contains every element of $\mathbb{Z}_2$ the same number of times, and for $i\in\{2,3\}$, u_i contains every element of $2^{i-1}\mathbb{Z}_{2^i}$ exactly s_i times, $s_i\geq 0$, and the remaining α_i-2s_i coordinates of u_i are from $\mathbb{Z}_{2^i}\backslash 2^{i-1}\mathbb{Z}_{2^i}$. Thus, from the definition of Φ, we have that $\mathrm{wt}_H(\Phi(\mathbf{u}))=\alpha_1/2+2s_2+(\alpha_2-2s_2)\cdot 1+4s_3+(\alpha_3-2s_3)\cdot 2=\alpha_1/2+\alpha_2+2\alpha_3=N/2$. If $o(\mathbf{u})=4$, then $\mathbf{u}$ satisfies Property 2a or 2b given in Lemma 2.14. If $\mathbf{u}$ satisfies Property 2a, then u_3 contains every element of $4\mathbb{Z}_8$ exactly m times, $m\geq 0$, and the remaining coordinates of u_3 are from $\mathbb{Z}_8\backslash 4\mathbb{Z}_8$. Thus, $\mathrm{wt}_H(\Phi(\mathbf{u}))=\alpha_1/2+\alpha_2+4m+(\alpha_3-2m)\cdot 2=\alpha_1/2+\alpha_2+2\alpha_3=N/2$. Otherwise, if $\mathbf{u}$ satisfies Property 2b, then $\mathrm{wt}_H(\Phi(\mathbf{u}))=\alpha_1/2+2n+(\alpha_2-2n)\cdot 1+4t+(\alpha_3-2t)\cdot 2=N/2$. If $o(\mathbf{u})=2$, then $\mathbf{u}$ satisfies Property 3a or 3b given in Lemma 2.14. If $\mathbf{u}$ satisfies Property 3a, then $\mathrm{wt}_H(\Phi(\mathbf{u}))=\frac{1}{4}(\alpha_1/2+\alpha_2+2\alpha_3)\cdot 4=N/2$. Otherwise, if $\mathbf{u}$ satisfies Property 3b, then $\backslash=2\cdot\frac{1}{2}(\alpha_1/2+\alpha_2)+4m+(\alpha_3-2m)\cdot 2=N/2$.

Finally, note that $\mathrm{wt}_H(\Phi(\mathbf{u})+\mathbf{1})=N/2$. Therefore, we have that the weight of every element of $H^{t_1,t_2,1}\backslash\{\mathbf{0},\mathbf{1}\}$ is $N/2$, that is, the minimum weight of $H^{t_1,t_2,1}$ is $N/2$. □

Proposition 2.19. *Let $t_1\geq 1$, $t_2\geq 0$, and $t_3\geq 1$ be integers. If $\mathcal{H}^{t_1,t_2,t_3}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,t_3)$, then, by applying Construction (5), $\mathcal{H}^{t_1,t_2,t_3+1}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(2\alpha_1,2\alpha_2,2\alpha_3;t_1,t_2,t_3+1)$.*

Proof. By Construction (5), $\mathcal{H}^{t_1,t_2,t_3+1}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha'_1,\alpha'_2,\alpha'_3;t_1,t_2,t_3+1)$, where $\alpha'_1=2\alpha_1$, $\alpha'_2=2\alpha_2$, and $\alpha'_3=2\alpha_3$.

Since H^{t_1,t_2,t_3} is a Hadamard code of length $N=\alpha_1+2\alpha_2+4\alpha_3$, then its minimum distance is $N/2$ and $|H^{t_1,t_2,t_3}|=2N$. Note that H^{t_1,t_2,t_3+1} is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length $N'=\alpha'_1+2\alpha'_2+4\alpha'_3=2N$ and $|H^{t_1,t_2,t_3+1}|=8^{t_1}4^{t_2}2^{t_3+1}=2|H^{t_1,t_2,t_3}|=2\cdot 2N=2N'$. By Proposition 2.17, the minimum distance of H^{t_1,t_2,t_3+1} is equal to the minimum weight of H^{t_1,t_2,t_3+1}. We only have to prove that the minimum weight of H^{t_1,t_2,t_3+1} is $N'/2$. Let $\mathcal{H}^{t_1,t_2,t_3}=(\mathcal{H}_1\mid\mathcal{H}_2\mid\mathcal{H}_3)$. Note that

$$\mathcal{H}^{t_1,t_2,t_3+1}=\bigcup_{\lambda\in\{0,1\}}((\mathcal{H}_1,\mathcal{H}_1\mid\mathcal{H}_2,\mathcal{H}_2\mid\mathcal{H}_3,\mathcal{H}_3)+\lambda(\mathbf{0},\mathbf{1}\mid\mathbf{0},\mathbf{2}\mid\mathbf{0},\mathbf{4})).$$

By Lemmas 2.15 and 2.16,

$$\begin{aligned} H^{t_1,t_2,t_3+1} &= \bigcup_{\lambda\in\{0,1\}}(\Phi(\mathcal{H}_1,\mathcal{H}_1\mid\mathcal{H}_2,\mathcal{H}_2\mid\mathcal{H}_3,\mathcal{H}_3)+\lambda(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1})) \\ &= A_0\cup A_1, \end{aligned} \tag{22}$$

where $A_\lambda=\Phi(\mathcal{H}_1,\mathcal{H}_1\mid\mathcal{H}_2,\mathcal{H}_2\mid\mathcal{H}_3,\mathcal{H}_3)+\lambda(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1})$, $\lambda\in\{0,1\}$. Next, we show that the minimum weight of A_λ is $N'/2$. Any element in A_λ is of the form $\Phi(u_1,u_1\mid u_2,u_2\mid u_3,u_3)+\lambda(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1})$, for $\mathbf{u}=(u_1\mid u_2\mid u_3)\in(\mathcal{H}_1\mid\mathcal{H}_2\mid\mathcal{H}_3)$.

Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)\backslash\{\mathbf{0}\}$. When $\lambda = 0$, we have that $\mathrm{wt}_H(\Phi(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3)) = 2\mathrm{wt}_H(\Phi(\mathbf{u}))$. Thus, the minimum weight of A_0 is $2 \cdot N/2 = N'/2$. Otherwise, when $\lambda = 1$, we have that $\mathrm{wt}_H(\Phi(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3) + (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})) = \mathrm{wt}_H(\Phi(\mathbf{u})) + \alpha_1 - \mathrm{wt}_H(u_1) + 2\alpha_2 - \mathrm{wt}_H(\Phi_2(u_2)) + 4\alpha_3 - \mathrm{wt}_H(\Phi_3(u_3)) = \mathrm{wt}_H(\Phi(\mathbf{u})) + \alpha_1 + 2\alpha_2 + 4\alpha_3 - \mathrm{wt}_H(\Phi(\mathbf{u})) = N = N'/2$. Thus, the minimum weight of A_1 is $N'/2$. Therefore, from (22), the minimum weight of H^{t_1,t_2,t_3+1} is $N'/2$. □

Next, from Proposition 2.18 and Proposition 2.19, one can derive the result below.

Theorem 2.20. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{t_1,t_2,t_3}$, generated by A^{t_1,t_2,t_3}, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code.*

To illustrate, we present an example.

Example 2.2. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{1,0,1}$ generated by $A^{1,0,1}$, given in (2), is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(2, 1, 1; 1, 0, 1)$. We can write $\mathcal{H}^{1,0,1} = \bigcup_{\alpha\in\mathbb{Z}_2}(\mathcal{A} + \alpha\mathbf{1})$, where $\mathcal{A} = \{\lambda(0, 1 \mid 1 \mid 1) : \lambda \in \mathbb{Z}_8\}$. Thus, $H^{1,0,1} = \Phi(\mathcal{H}^{1,0,1}) = \bigcup_{\alpha\in\mathbb{Z}_2}(\Phi(\mathcal{A}) + \alpha\mathbf{1})$, where $\Phi(\mathcal{A})$ consists of all the rows of the Hadamard matrix

$$H(2,4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $\Phi(\mathcal{A})$ is linear and the minimum distance of $\Phi(\mathcal{A})$ is 4, so $H^{1,0,1}$ is a binary linear Hadamard code of length 8.

Proposition 2.21. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let H^{t_1,t_2,t_3} be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of length 2^t. Then, $t + 1 = 3t_1 + 2t_2 + t_3$.*

Proof. Since H^{t_1,t_2,t_3} is a binary Hadamard code of length 2^t, we have that $|H^{t_1,t_2,t_3}| = 2 \cdot 2^t = 2^{t+1}$. Note that $|H^{t_1,t_2,t_3}| = 2^{3t_1+2t_2+t_3}$, and hence $t+1 = 3t_1+2t_2+t_3$. □

Now, we recall Theorem 2.22 in order to compare the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$) with the $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$) of the same length. Also recall that the type of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code can be given as $(\alpha_1, \alpha_2, 0; 0, t_2, t_3)$ if we see the code as a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code with $\alpha_3 = 0$, or directly $(\alpha_1, \alpha_2; t_2, t_3)$. Note that there are no $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes neither with only $\alpha_1 = 0$ nor with only $\alpha_2 = 0$ [25, 34].

Theorem 2.22. [29] *Let $t \geq 3$ and $t_2 \in \{0, \dots, \lfloor t/2 \rfloor\}$. Let H^{t_2,t_3} be the nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t and type $(\alpha_1, \alpha_2; t_2, t_3)$, where $\alpha_1 = 2^{t-t_2}$, $\alpha_2 = 2^{t-1} - 2^{t-t_2-1}$, and $t_3 = t + 1 - 2t_2$. Then,*

$$\mathrm{rank}(H^{t_2,t_3}) = t_3 + 2t_2 + \binom{t_2}{2} \quad \textit{and} \quad \ker(H^{t_2,t_3}) = t_2 + t_3.$$

We also recall the construction of the $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$ studied in [17], and Theorem 2.23 given in [19], in order to compare these codes with the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$) of the same length. Let $T_i = \{j \cdot 2^{i-1} : j \in \{0, 1, \dots, 2^{s-i+1}-1\}\}$ for all $i \in \{1, \dots, s\}$. Note that $T_1 = \{0, \dots, 2^s-1\}$. Let $t_1, t_2, \dots, t_s$ be non-negative integers with $t_1 \geq 1$. Consider the matrix $\bar{A}^{t_1,\dots,t_s}$ whose columns are exactly all the vectors of the form $\mathbf{z}^T$, $\mathbf{z} \in \{1\} \times T_1^{t_1-1} \times T_2^{t_2} \times \cdots \times T_s^{t_s}$. Let $\bar{\mathcal{H}}^{t_1,\dots,t_s}$ be the $\mathbb{Z}_{2^s}$-additive code of type $(n; t_1, \dots, t_s)$ generated by $\bar{A}^{t_1,\dots,t_s}$. Let $\bar{H}^{t_1,\dots,t_s} = \Phi_s(\bar{\mathcal{H}}^{t_1,\dots,t_s})$ be the corresponding $\mathbb{Z}_{2^s}$-linear Hadamard code.

Theorem 2.23. [19] *Let $\bar{H}^{t_1,\dots,t_s}$ be the $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$ and $t_s \geq 1$. Then, for all $\ell \in \{1, \dots, t_s\}$, $\bar{H}^{t_1,\dots,t_s}$ is permutation equivalent to the $\mathbb{Z}_{2^{s+\ell}}$-linear Hadamard code $\bar{H}^{1,\mathbf{0}^{\ell-1},t_1-1,t_2,\dots,t_{s-1},t_s-\ell}$.*

For $5 \leq t \leq 11$, Tables 1 and 3 given in [17] show all possible values of $(t_1, \dots, t_s)$ corresponding to nonlinear $\mathbb{Z}_{2^s}$-linear Hadamard codes, with $s \geq 2$, of length 2^t. For each of them, the values (r, k) are shown, where r is the rank, and k is the dimension of the kernel. Note that if two codes have different values (r, k), they are not equivalent. The following example shows that all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17], with $s \geq 2$, of the same length 2^{11}.

Example 2.3. Consider $t = 11$. By solving equation $t + 1 = 3t_1 + 2t_2 + t_3$ given in Proposition 2.21, all $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are the ones in

$$T = \{H^{1,0,9}, H^{1,1,7}, H^{1,2,5}, H^{1,3,3}, H^{1,4,1}, H^{2,0,6}, H^{2,1,4}, H^{2,2,2}, H^{3,0,3}, H^{3,1,1}\}.$$

By using the computer algebra system Magma [13], their corresponding values of (r, k), where r is the rank and k is the dimension of the kernel, are $(12, 12)$, $(14, 9)$, $(17, 8)$, $(21, 7)$, $(26, 6)$, $(17, 8)$, $(22, 7)$, $(28, 6)$, $(28, 6)$, and $(37, 5)$, respectively. The code $H^{1,0,9}$ is the only linear code in T since $r = k = 12$. Using Magma, we can check that the following codes in each pair are nonequivalent to each other: $(H^{1,2,5}, H^{2,0,6})$, $(H^{2,2,2}, H^{3,0,3})$. Therefore, the codes in T are not equivalent to each other.

Let $\bar{T} = T \setminus \{H^{1,0,9}\}$. Similarly, by solving equation $t + 1 = 2t_2 + t_3$ given in Theorem 2.22, all nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of length 2^{11} are $H^{2,8}$, $H^{3,6}$, $H^{4,4}$ and $H^{5,2}$, and by Theorem 2.22, their corresponding values of (r, k) are $(13, 10)$, $(15, 9)$, $(18, 8)$, and $(22, 7)$, respectively. Using Magma, we can check that $H^{2,1,4}$ and $H^{5,2}$ are nonequivalent. Therefore, all the codes in $\bar{T}$ are nonequivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^{11}.

Finally, note that all the codes in $\bar{T}$, except $H^{1,1,7}$ and $H^{2,1,4}$, are not equivalent to any $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$, of length 2^{11}, since they have different values of (r, k). The $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^{11}, having the same values $(r, k) = (14, 9)$ as $H^{1,1,7}$, are $\bar{H}^{2,0,6}$, $\bar{H}^{1,1,0,5}$, $\bar{H}^{1,0,1,0,4}$, $\bar{H}^{1,0,0,0,1,0,2}$, and $\bar{H}^{1,0,0,0,0,0,1,0,0}$, which are equivalent to each other by Theorem 2.23. The $\mathbb{Z}_4$-linear Hadamard code $\bar{H}^{6,0}$ is the only $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^{11}, having the same values $(r, k) = (22, 7)$ as $H^{2,1,4}$. However, using Magma, we can check that the following codes in each pair are nonequivalent to each other: $(H^{1,1,7}, \bar{H}^{2,0,6})$, $(H^{2,1,4}, \bar{H}^{6,0})$.

Therefore, all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17], with $s \geq 2$, of the same length 2^{11}.

Finally, the following example shows that other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes can not be constructed by Construction (3). However, in the next section, we also show that other constructions of these codes do generate equivalent codes.

Example 2.4. Consider the matrix

$$B = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & 1111 & 0646 & 11113333 \\ 00 & 11 & 31 & 0123 & 1771 & 01234725 \end{array} \right).$$

Using Magma, we can check that the code generated by B is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of type $(4, 6, 12; 2, 0, 1)$, and it is nonequivalent to the code $H^{2,0,1}$ generated by $A^{2,0,1}$ given in (6).

3. **Same type equivalent $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes.** In this section, we see that if we consider other specific starting matrices, instead of the matrix $A^{1,0,1}$ given in (2), and apply the same recursive Construction (3), (4) and (5), or new constructions more general than (3) and (4), and the same Construction (5), we also obtain $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Indeed, the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes, after applying the Gray map Φ, are equivalent to the codes $\Phi(\mathcal{H}^{t_1,t_2,t_3})$ of the same type constructed in Section 2.

Let $\mathbb{Z}_{2^i}^*$ be the group of units of $\mathbb{Z}_{2^i}$ for $i \in \{2, 3\}$. Then, $\mathbb{Z}_4^* = \{1, 3\}$ and $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$.

Proposition 3.1. *Let $\bar{a}_1 = (a_1)$ and $\bar{b}_1 = (b_1)$, where $a_1 \in \mathbb{Z}_4^*$ and $b_1 \in \mathbb{Z}_8^*$. Then, the code generated by*

$$\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & a_1 & b_1 \end{array} \right), \tag{23}$$

denoted by $\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(2, 1, 1; 1, 0, 1)$. Moreover, the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $\Phi(\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1})$ is permutation equivalent to $\Phi(\mathcal{H}^{1,0,1})$.

Proof. Let $\overline{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ be the matrix obtained from $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ by applying the row operation described in Table 1, depending on the values of $a_1 \in \mathbb{Z}_4^*$ and $b_1 \in \mathbb{Z}_8^*$. Note that $\overline{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ is also a generator matrix of $\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$. After permuting the first two columns of $\overline{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$, if necessary, we obtain $A^{1,0,1}$. Thus, $\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ and $\mathcal{H}^{1,0,1}$ are permutation equivalent, and so are the codes $\Phi(\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1})$ and $\Phi(\mathcal{H}^{1,0,1})$. $\square$

Theorem 3.2. *Let $\ell \geq 1$. Let $\bar{a}_\ell = (a_1, \ldots, a_\ell) \in (\mathbb{Z}_4^*)^\ell$, $\bar{b}_\ell = (b_1, \ldots, b_\ell) \in (\mathbb{Z}_8^*)^\ell$, $\mathbf{a}_i = (a_i, \overset{2^{i-1}}{\ldots}, a_i)$, and $\mathbf{b}_i = (b_i, \overset{4^{i-1}}{\ldots}, b_i)$, $1 \leq i \leq \ell$. Let $\hat{A}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$, with $t_1 \geq 1$, be the matrix obtained by using the following construction (instead of Construction (3)). We start with $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ given in (23). If we have $\hat{A}^{\ell-1,0,1}_{\bar{a}_{\ell-1},\bar{b}_{\ell-1}} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$, with $\ell \geq 2$, we may construct*

$$\hat{A}^{\ell,0,1}_{\bar{a}_\ell,\bar{b}_\ell} = \left(\begin{array}{cc|ccccc|ccccc} \hat{A}_1 & \hat{A}_1 & \hat{M}_1 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{M}_2 & \hat{A}_3 & \hat{A}_3 & \cdots & \hat{A}_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{a}_\ell & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{b}_\ell & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array} \right), \tag{24}$$

where $\hat{M}_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0, 2\}^{\ell-1}\}$, $\hat{M}_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0, 2, 4, 6\}^{\ell-1}\}$. We repeat Construction (24) until $\ell = t_1$. Then, the code generated by $\hat{A}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$, denoted by $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, 0, 1)$ with

$\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Moreover, the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $\Phi(\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}})$ is permutation equivalent to $\Phi(\mathcal{H}^{t_1,0,1})$.

Proof. It is enough to show that $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent. We prove this by induction on $t_1 \geq 1$. By Proposition 3.1, this is true for $t_1 = 1$. Assume that $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent. Let $\hat{A}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$ and $A^{t_1,0,1} = (A_1 \mid A_2 \mid A_3)$. By Construction (24), we have

$$\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}} = \left(\begin{array}{cc|ccccc|ccccc} \hat{A}_1 & \hat{A}_1 & \hat{M}_1 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{M}_2 & \hat{A}_3 & \hat{A}_3 & \cdots & \hat{A}_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{a}_{t_1+1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{b}_{t_1+1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right),$$

where $\hat{M}_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1}\}$, $\hat{M}_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0,2,4,6\}^{t_1}\}$, $\mathbf{a}_{t_1+1} = (a_{t_1+1}, \overset{2^{t_1}}{\ldots}, a_{t_1+1})$, and $\mathbf{b}_{t_1+1} = (b_{t_1+1}, \overset{4^{t_1}}{\ldots}, b_{t_1+1})$. By Construction (3), we have

$$A^{t_1+1,0,1} = \left(\begin{array}{cc|ccccc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & M_2 & A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right),$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1}\}$, $M_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0,2,4,6\}^{t_1}\}$. Let $\hat{\mathcal{H}}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and $\mathcal{H}^{t_1+1,0,1}$ be the codes generated by $\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and $A^{t_1+1,0,1}$, respectively.

Since $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent, there exist some row operations and column permutations so that after applying these operations on $(\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$, we obtain $(A_1 \mid A_2 \mid A_3)$. First, we apply the same row operations to $\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and the corresponding column permutations to each submatrix

$$\begin{pmatrix} \hat{A}_i \\ \mathbf{k_i} \end{pmatrix},$$

for $i \in \{1,2,3\}$, $k_i \in \mathbb{Z}_{2^i}$. Thus, for $i \in \{1,2,3\}$, $\hat{A}_i$ becomes A_i. Then, we change the last row by applying the row operation described in Table 1, depending on the values of $a_{t_1+1} \in \mathbb{Z}_4^*$ and $b_{t_1+1} \in \mathbb{Z}_8^*$. After that, we permute the blocks of the form

$$\begin{pmatrix} A_1 \\ \mathbf{k_1} \end{pmatrix}, \begin{pmatrix} A_2 \\ \mathbf{k_2} \end{pmatrix} \text{ and } \begin{pmatrix} A_3 \\ \mathbf{k_3} \end{pmatrix},$$

for $k_i \in \mathbb{Z}_{2^i}$, so that we obtain the submatrices

$$\begin{pmatrix} A_1 & A_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} A_2 & A_2 & A_2 & A_2 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix} \text{ and } \begin{pmatrix} A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{pmatrix},$$

respectively. Let $\overline{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ be the matrix obtain from $\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ after applying all these operations. Let M_1' and M_2' be the matrices $\hat{M}_1$ and $\hat{M}_2$, respectively, after all these operations. Finally, after a suitable permutation of the columns corresponding to the blocks of the form

$$\begin{pmatrix} M_1' \\ \mathbf{1} \end{pmatrix} \text{ and } \begin{pmatrix} M_2' \\ \mathbf{1} \end{pmatrix}$$

in $\overline{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$, we obtain $A^{t_1+1,0,1}$. Thus, the codes $\hat{\mathcal{H}}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and $\mathcal{H}^{t_1+1,0,1}$ are permutation equivalent, and so are the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes. This completes the proof. $\square$

a_i	b_i	*row operation*
1	1	$r_{i+1} \leftarrow r_{i+1}$
1	3	$r_{i+1} \leftarrow r_1 - r_{i+1}$
1	7	$r_{i+1} \leftarrow 5r_1 - 5r_{i+1}$
1	5	$r_{i+1} \leftarrow 5r_{i+1}$
3	1	$r_{i+1} \leftarrow r_1 - 3r_{i+1}$
3	3	$r_{i+1} \leftarrow 3r_{i+1}$
3	5	$r_{i+1} \leftarrow r_1 + r_{i+1}$
3	7	$r_{i+1} \leftarrow 7r_{i+1}$

TABLE 1. Row operations depending on the values of $a_i \in \mathbb{Z}_4^*$ and $b_i \in \mathbb{Z}_8^*$.

Example 3.1. Let $\bar{a}_2 = (1, 3)$ and $\bar{b}_2 = (3, 5)$. Then,

$$\hat{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2} = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & \mathbf{a}_1 & 0246 & \mathbf{b}_1 \\ 00 & 11 & \mathbf{a}_2 & 0123 & \mathbf{b}_2 & 01234567 \end{array}\right)$$
$$= \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & 1111 & 0246 & 33333333 \\ 00 & 11 & 33 & 0123 & 5555 & 01234567 \end{array}\right),$$

which is obtained by using Construction (24), starting with the matrix $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ given in (23).

First, note that we have

$$\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \end{array}\right) \text{ and } A^{1,0,1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right).$$

Therefore, by using Table 1, $A^{1,0,1} = (A_1 \mid A_2 \mid A_3)$ can be obtained from $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$ by applying the row operation $r_2 \leftarrow r_1 - r_2$ and the column permutation $(1, 2)$. No column permutation is performed on the submatrices $\hat{A}_2$ and $\hat{A}_3$. Then, we apply the same row operation $r_2 \leftarrow r_1 - r_2$ to $\hat{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2}$ and the column permutation $(1, 2)$ to each submatrix $\begin{pmatrix} \hat{A}_1 \\ \mathbf{k_1} \end{pmatrix}$, for $k_1 \in \mathbb{Z}_2$. Thus, for $i \in \{1, 2, 3\}$, $\hat{A}_i$ becomes A_i. Then, we apply $r_3 \leftarrow r_1 + r_3$, described in Table 1, so we obtain

$$\left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 20 & 1111 & 4206 & 11111111 \\ 11 & 00 & 11 & 2301 & 1111 & 45670123 \end{array}\right).$$

After that, we permute the blocks of the form $\begin{pmatrix} A_i \\ \mathbf{k_i} \end{pmatrix}$, for $k_i \in \mathbb{Z}_{2^i}$ and $i \in \{1, 2, 3\}$, so that we obtain

$$\overline{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2} = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 10 & 10 & 20 & 1111 & 4206 & 11111111 \\ 00 & 11 & 11 & 0123 & 1111 & 01234567 \end{array}\right).$$

Finally, after applying a suitable column permutation to the submatrices,

$$\begin{pmatrix} 22 \\ 20 \\ 11 \end{pmatrix} \text{ and } \begin{pmatrix} 4444 \\ 4206 \\ 1111 \end{pmatrix}$$

in $\overline{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2}$, we can obtain $A^{2,0,1}$. Thus, the codes $\hat{\mathcal{H}}^{2,0,1}_{\bar{a}_2,\bar{b}_2}$ and $\mathcal{H}^{2,0,1}$ are permutation equivalent, which is equivalent to say that the codes $\Phi(\hat{\mathcal{H}}^{2,0,1}_{\bar{a}_2,\bar{b}_2})$ and $\Phi(\mathcal{H}^{2,0,1})$ are permutation equivalent.

Theorem 3.3. *Let $t_1 \geq 1$, $t_2 \geq 0$ and $\ell \geq 0$. Let $\bar{a}_{t_1+\ell} = (a_1, \ldots, a_{t_1+\ell}) \in (\mathbb{Z}_4^*)^{t_1+\ell}$, $\bar{b}_{t_1} = (b_1, \ldots, b_{t_1}) \in (\mathbb{Z}_8^*)^{t_1}$, and $\mathbf{a}_{t_1+i} = (a_{t_1+i}, \overset{2^{t_1+i-1}}{\ldots}, a_{t_1+i})$ for $0 \leq i \leq \ell$. Let $\hat{A}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ be the matrix obtained by using the following construction (instead of Construction (4)), starting with $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ given in (23). If we have $\hat{A}^{t_1,\ell-1,1}_{\bar{a}_{t_1+\ell-1},\bar{b}_{t_1}} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$, with $\ell \geq 1$, we may construct*

$$\hat{A}^{t_1,\ell,1}_{\bar{a}_{t_1+\ell},\bar{b}_{t_1}} = \left(\begin{array}{cc|ccccc|cccc} \hat{A}_1 & \hat{A}_1 & \hat{M}_1 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_3 & \hat{A}_3 & \hat{A}_3 & \hat{A}_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{a}_{t_1+\ell} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array} \right), \tag{25}$$

where $\hat{M}_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1+\ell-1}\}$. We repeat Construction (25) until $\ell = t_2$. Then, the code generated by $\hat{A}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, denoted by $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3;\ t_1, t_2, 1)$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Moreover, the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $\Phi(\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}})$ is permutation equivalent to $\Phi(\mathcal{H}^{t_1,t_2,1})$.

Proof. It is enough to show that $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,1}$ are permutation equivalent. We prove this by induction on $t_2 \geq 0$. By Theorem 3.2, $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent. Assume that $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,1}$, generated by $\hat{A}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $A^{t_1,t_2,1}$, respectively, are permutation equivalent. We have that $a_{t_1+t_2+1} \in \mathbb{Z}_4^* = \{1,3\}$. Let $\hat{\mathcal{H}}^{t_1,t_2+1,1}_{\bar{a}_{t_1+t_2+1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2+1,1}$ be the codes generated by $\hat{A}^{t_1,t_2+1,1}_{\bar{a}_{t_1+t_2+1},\bar{b}_{t_1}}$ using Construction (25) and $A^{t_1,t_2+1,1}$ using Construction (4), respectively. Then, by the same arguments as in the proof of Theorem 3.2 and applying the row operation $r_{t_1+t_2+1} \leftarrow -r_{t_1+t_2+1}$ if $a_{t_1+t_2+1} = 3$, the codes $\hat{\mathcal{H}}^{t_1,t_2+1,1}_{\bar{a}_{t_1+t_2+1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2+1,1}$ are permutation equivalent, and so are the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes. This completes the proof. □

Theorem 3.4. *$t_1 \geq 1$, $t_2 \geq 0$ and $t_3 \geq 1$. Let $\bar{a}_{t_1+t_2} = (a_1, \ldots, a_{t_1+t_2}) \in (\mathbb{Z}_4^*)^{t_1+t_2}$, $\bar{b}_{t_1} = (b_1, \ldots, b_{t_1}) \in (\mathbb{Z}_8^*)^{t_1}$. Let $\hat{A}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ be the matrix obtained by using Construction (5) in the following way, starting with $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ given in (23). If we have $\hat{A}^{t_1,t_2,\ell-1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, $\ell \geq 2$, we may construct $\hat{A}^{t_1,t_2,\ell}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ by Construction (5). We repeat Construction (5) until $\ell = t_3$. Then, the code generated by $\hat{A}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, denoted by $\hat{\mathcal{H}}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Moreover, the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $\Phi(\hat{\mathcal{H}}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}})$ is permutation equivalent to $\Phi(\mathcal{H}^{t_1,t_2,t_3})$.*

Proof. We have that that $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,1}$ are permutation equivalent by Theorem 3.3. Then, by the same arguments as in the proof of Theorem 3.2, the codes $\hat{\mathcal{H}}^{t_1,t_2,t_3+1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,t_3+1}$ are permutation equivalent and the result follows. □

4. **Conclusions and further research.** In this paper, we give several recursive constructions of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. We show that they all allow us to construct the same family of codes since they generate permutation equivalent codes. Moreover, from Example 2.3, we see that all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17], with $s \geq 2$, of the same length 2^{11}. Therefore, we have that some nonlinear Hadamard codes, without any known structure, now can be seen as the Gray map image of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. As further research, it would be interesting to generalize this result, given only for 2^{11}, to any length 2^t.

Another further research could be to generalize the given construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes to $\mathbb{Z}_2\mathbb{Z}_4 \dots \mathbb{Z}_{2^s}$-linear Hadamard codes with $\alpha_1, \dots, \alpha_s$ different to zero, or even to $\mathbb{Z}_p\mathbb{Z}_{p^2} \dots \mathbb{Z}_{p^s}$-linear generalized Hadamard codes with p prime. The study of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes may represent an important step to study the general case, and other papers [2, 14] have also focused on this particular case. However, the generalizations are not feasible using the same techniques employed in this paper.

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