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Linearity and classification of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear generalized Hadamard codes ☆,☆☆

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The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive codes are subgroups of $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, and can be seen as linear codes over $\mathbb{Z}_p$ when $\alpha_2 = 0$, $\mathbb{Z}_{p^2}$-additive codes when $\alpha_1 = 0$, or $\mathbb{Z}_2\mathbb{Z}_4$-additive codes when $p = 2$. A $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear generalized Hadamard (GH) code is a GH code over $\mathbb{Z}_p$ which is the Gray map image of a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code. Recursive constructions of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$ are known. In this paper, we generalize some known results for $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $p = 2$ to any $p \geq 3$ prime when $\alpha_1 \neq 0$, and then we compare them with the ones obtained when $\alpha_1 = 0$. First, we show for which types the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes are nonlinear over $\mathbb{Z}_p$. Then, for these codes, we compute the kernel and its dimension, which allow us to classify them completely. Moreover, by computing the rank of some of these codes, we show that, unlike $\mathbb{Z}_4$-linear Hadamard codes, the $\mathbb{Z}_{p^2}$-linear GH codes are not included in the family of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ when $p \geq 3$ prime. Indeed, there are some families with infinite nonlinear $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear

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GH codes, where the codes are not equivalent to any $\mathbb{Z}_{p^s}$-linear GH code with $s \geq 2$.

1. Introduction

Let $\mathbb{Z}_{p^s}$ be the ring of integers modulo p^s, where p is a prime. Let $\mathbb{Z}_{p^s}^n$ denote the set of all n-tuples over $\mathbb{Z}_{p^s}$. In this paper, the elements of $\mathbb{Z}_{p^s}^n$ will also be called vectors of length n. A code over $\mathbb{Z}_p$ of length n is a nonempty subset of $\mathbb{Z}_p^n$, and it is linear if it is a subspace of $\mathbb{Z}_p^n$. Similarly, a nonempty subset of $\mathbb{Z}_{p^s}^n$ is a $\mathbb{Z}_{p^s}$-additive code if it is a subgroup of $\mathbb{Z}_{p^s}^n$. A $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code is a subgroup of $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$. Note that a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code is a linear code over $\mathbb{Z}_p$ when $\alpha_2 = 0$, a $\mathbb{Z}_{p^2}$-additive code when $\alpha_1 = 0$, or a $\mathbb{Z}_2\mathbb{Z}_4$-additive code when $p = 2$. The order of a vector $u \in \mathbb{Z}_{p^s}^n$, denoted by $o(u)$, is the smallest positive integer m such that $mu = (0, \dots, 0)$. Also, the order of $\mathbf{u} \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, denoted by $o(\mathbf{u})$, is the smallest positive integer m such that $m\mathbf{u} = (0, \dots, 0 \mid 0, \dots, 0)$.

Two codes C_1 and C_2 over $\mathbb{Z}_p$ of length n are said to be monomially equivalent (or just equivalent) provided there is a monomial matrix M such that $C_2 = \{cM : c \in C_1\}$. Recall that a monomial matrix is a square matrix with exactly one nonzero entry in each row and column. Both codes are said to be permutation equivalent if there is a permutation matrix P such that $C_2 = \{cP : c \in C_1\}$. Recall that a permutation matrix is a square matrix with exactly one 1 in each row and column and zeros elsewhere.

The Hamming weight of a vector $u \in \mathbb{Z}_p^n$, denoted by $\mathrm{wt}_H(u)$, is the number of nonzero coordinates of u. The Hamming distance of two vectors $u, v \in \mathbb{Z}_p^n$, denoted by $d_H(u, v)$, is the number of coordinates in which they differ. Note that $d_H(u, v) = \mathrm{wt}_H(u - v)$. The minimum distance of a code C over $\mathbb{Z}_p$ is $d(C) = \min\{d_H(u, v) : u, v \in C, u \neq v\}$.

In [3], a Gray map from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is defined as $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$ and $\phi(3) = (1, 0)$. There exist different generalizations of this Gray map, which go from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$ [4–8]. The one given in [7] can be defined in terms of the elements of a Hadamard code [8], and Carlet's Gray map [4] is a particular case of the one given in [8] satisfying $\sum \lambda_i \phi(2^i) = \phi(\sum \lambda_i 2^i)$ [9]. In this paper, we focus on a generalization of Carlet's Gray map from $\mathbb{Z}_{p^s}$ to $\mathbb{Z}_p^{p^{s-1}}$, also denoted by ϕ, which is a particular case of the one given in [10]. Let $\Phi : \mathbb{Z}_{p^s}^n \to \mathbb{Z}_p^{np^{s-1}}$ be the extension of the Gray map ϕ given by $\Phi(y') = (\phi(y'_1), \dots, \phi(y'_n))$, for any $y' = (y'_1, \dots, y'_n) \in \mathbb{Z}_{p^s}^n$. We also denote by Φ the extension $\Phi : \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2} \to \mathbb{Z}_p^{\alpha_1 + p\alpha_2}$, given by

$$\Phi((y \mid y')) = (y, \phi(y'_1), \dots, \phi(y'_{\alpha_2})),$$

for any $y \in \mathbb{Z}_p^{\alpha_1}$ and $y' = (y'_1, \dots, y'_{\alpha_2}) \in \mathbb{Z}_{p^2}^{\alpha_2}$.

Let $\mathcal{C} \subseteq \mathbb{Z}_{p^s}^n$ be a $\mathbb{Z}_{p^s}$-additive code. We say that its Gray map image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_{p^s}$-linear code of length np^{s-1}. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_{p^s}^n$, it is isomorphic to an abelian structure $\mathbb{Z}_{p^s}^{t_1} \times \mathbb{Z}_{p^{s-1}}^{t_2} \times \cdots \times \mathbb{Z}_p^{t_s}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(n; t_1, \ldots, t_s)$. Note that $|\mathcal{C}| = p^{st_1} p^{(s-1)t_2} \cdots p^{t_s}$. Similarly, if $\mathcal{C} \subseteq \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code, we say that its Gray map image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code of length $\alpha_1 + p\alpha_2$. Since $\mathcal{C}$ can be seen as a subgroup of $\mathbb{Z}_{p^2}^{\alpha_1+\alpha_2}$, it is isomorphic to $\mathbb{Z}_{p^2}^{t_1} \times \mathbb{Z}_p^{t_2}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(\alpha_1, \alpha_2; t_1, t_2)$. Note that $|\mathcal{C}| = p^{2t_1+t_2}$. Note that a $\mathbb{Z}_{p^2}$-linear code of type $(n; t_1, t_2)$ can also be seen as a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code of type $(0, n; t_1, t_2)$. Unlike linear codes over finite fields, linear codes over rings do not have a basis, but there exist generator matrices for these codes having minimum number of rows.

Two structural properties of codes over $\mathbb{Z}_p$ are the rank and dimension of the kernel. The rank of a code C over $\mathbb{Z}_p$ is simply the dimension of the linear span, $\langle C \rangle$, of C. The kernel of a code C over $\mathbb{Z}_p$ is defined as $\mathrm{K}(C) = \{x \in \mathbb{Z}_p^n : x + C = C\}$ [11,12]. If the all-zero vector belongs to C, then $\mathrm{K}(C)$ is a linear subcode of C. Note also that if C is linear, then $K(C) = C = \langle C \rangle$. We denote the rank of C as $\mathrm{rank}(C)$ and the dimension of the kernel as $\ker(C)$. These parameters can be used to distinguish between nonequivalent codes, since equivalent ones have the same rank and dimension of the kernel.

A generalized Hadamard (GH) matrix $H(p, \lambda) = (h_{ij})$ of order $N = p\lambda$ over $\mathbb{Z}_p$ is a $p\lambda \times p\lambda$ matrix with entries from $\mathbb{Z}_p$ with the property that for every i, j, $1 \leq i < j \leq p\lambda$, each of the multisets $\{h_{is} - h_{js} : 1 \leq s \leq p\lambda\}$ contains every element of $\mathbb{Z}_p$ exactly λ times [13]. For $\mu \geq 1$, an ordinary Hadamard matrix of order 4μ corresponds to a GH matrix $H(2, \lambda)$ over $\mathbb{Z}_2$, where $\lambda = 2\mu$ [14]. Two GH matrices H_1 and H_2 of order N are said to be equivalent if one can be obtained from the other by a permutation of the rows and columns and adding the same element of $\mathbb{Z}_p$ to all the coordinates in a row or in a column. Then, we can always change the first row and column of a GH matrix into zeros and we obtain an equivalent GH matrix which is called normalized. From a normalized GH matrix H, we denote by F_H the code consisting of the rows of H. We define $C_H = \cup_{\alpha \in \mathbb{Z}_p} (F_H + \alpha\mathbf{1})$, where $F_H + \alpha\mathbf{1} = \{h + \alpha\mathbf{1} : h \in F_H\}$ and $\mathbf{1}$ denotes the all-one vector. The code C_H over $\mathbb{Z}_p$ is called generalized Hadamard (GH) code [15]. Note that C_H is generally a nonlinear code over $\mathbb{Z}_p$. Moreover, if it is of length N, it has pN codewords and $d(C_H) = N(p-1)/p$.

The $\mathbb{Z}_{p^s}$-additive (resp. $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive) codes such that after the Gray map Φ give GH codes are called $\mathbb{Z}_{p^s}$-additive (resp. $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive) GH codes and the corresponding images are called $\mathbb{Z}_{p^s}$-linear (resp. $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear) GH codes. The classification of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of length 2^t with $\alpha_1 = 0$ and $\alpha_1 \neq 0$ is given in [16,17], showing that there are $\lfloor (t-1)/2 \rfloor$ and $\lfloor t/2 \rfloor$ such nonequivalent codes, respectively. Moreover, in [18], it is shown that each $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 = 0$ is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$, so indeed there are only $\lfloor t/2 \rfloor$ nonequivalent $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of length 2^t. Later, in [19,9,20,21], an iterative construction for $\mathbb{Z}_{p^s}$-linear GH codes is described, the linearity is established, and a partial classification is obtained, giving the exact amount of nonequivalent nonlinear such codes for some parameters. The $\mathbb{Z}_{p^s}$-additive codes have also been studied in [22,23] as two-weight codes over $\mathbb{Z}_{p^s}$ by considering the homogeneous weight.

This paper is focused mainly on $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ and $p \geq 3$ prime, generalizing some results given for $p = 2$ in [17] related to the linearity, kernel, rank and classification of such codes. These codes are also compared with the $\mathbb{Z}_{p^s}$-linear GH codes studied in [19]. This paper is organized as follows. In Section 2, we recall the definition of the Gray map considered in this paper and some of its properties. Then, we recall the constructions of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0$ and p prime and some results related to that. In Sections 3 and 4, we establish for which types these codes are linear, and we give the kernel and its dimension whenever they are nonlinear. We also show that the dimension of the kernel is enough to classify completely the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ of a given length, providing the number of nonequivalent such codes, like it was proved for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes in [17]. In Section 5, we compute the rank of some families of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. Finally, in Section 6, we show that, unlike $\mathbb{Z}_4$-linear Hadamard codes, the $\mathbb{Z}_{p^2}$-linear GH codes are not included in the family of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ when $p \geq 3$ prime. Indeed, we prove that there are some families of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes, where the codes are not equivalent to any $\mathbb{Z}_{p^s}$-linear GH code with $s \geq 2$.

2. Preliminary results

In this section, we first give the definition of the Gray map considered in this paper for elements of $\mathbb{Z}_{p^s}$ and some of its properties used in the paper. Then, we recall the constructions of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with p prime when $\alpha_1 \neq 0$ and some results related to that.

We consider the following Gray map ϕ, given in [4,24]:

$$\begin{aligned} \phi : \mathbb{Z}_{p^s} &\longrightarrow \mathbb{Z}_p^{p^{s-1}} \\ u &\mapsto (u_{s-1}, \dots, u_{s-1}) + (u_0, \dots, u_{s-2})Y_{s-1}, \end{aligned} \tag{1}$$

where $u \in \mathbb{Z}_{p^s}$, $[u_0, u_1, \dots, u_{s-1}]_p$ is the p-ary expansion of u, that is, $u = \sum_{i=0}^{s-1} p^i u_i$ ($u_i \in \mathbb{Z}_p$), and Y_{s-1} is a matrix of size $(s-1) \times p^{s-1}$ whose columns are the elements of $\mathbb{Z}_p^{s-1}$. Without loss of generality, we assume that the columns of Y_{s-1} are ordered in ascending order, by considering the elements of $\mathbb{Z}_p^{s-1}$ as the p-ary expansions of the elements of $\mathbb{Z}_{p^{s-1}}$.

Let $u', v' \in \mathbb{Z}_{p^2}$ and $[u'_0, u'_1]_p$, $[v'_0, v'_1]_p$ be the p-ary expansions of u' and v', respectively, i.e. $u' = u'_0 + u'_1 p$ and $v' = v'_0 + v'_1 p$. We define the operation "$\odot_p$" between elements u' and v' in $\mathbb{Z}_{p^2}$ as $u' \odot_p v' = \xi_0 + \xi_1 p$, where

$$\xi_i = \begin{cases} 1 & \text{if} \quad u'_i + v'_i \geq p, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the p-ary expansion of $u' \odot_p v'$ is $[\xi_0, \xi_1]_p$, where $\xi_0, \xi_1 \in \{0, 1\}$. For $u, v \in \mathbb{Z}_p$, we define $u \odot_p v = 1$ if $u + v \geq p$ and 0 otherwise. We denote in the same way, "$\odot_p$", the component-wise operation. For $\mathbf{u} = (u \mid u'), \mathbf{v} = (v \mid v') \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, we denote $\mathbf{u} \odot_p \mathbf{v} = (u \odot_p v \mid u' \odot_p v')$. Note that $p(\mathbf{u} \odot_p \mathbf{v}) = (\mathbf{0} \mid p(u' \odot_p v'))$.

From [19], we have the following results:

Lemma 2.1. *[19] Let $u \in \mathbb{Z}_{p^2}$ and $\lambda \in \mathbb{Z}_p$. Then, $\phi(u + \lambda p) = \phi(u) + (\lambda, \dots, \lambda)$.*

Corollary 2.1. *[19] Let $\lambda, \mu \in \mathbb{Z}_p$. Then, $\phi(\lambda\mu p) = \lambda\phi(\mu p) = \lambda\mu\phi(p)$.*

Corollary 2.2. *[19] Let $u, v \in \mathbb{Z}_{p^2}$. Then, $\phi(u) + \phi(v) = \phi(u + v - p(u \odot_p v))$.*

Corollary 2.3. *[19] Let $u, v \in \mathbb{Z}_{p^2}$. Then, $\phi(pu + v) = \phi(pu) + \phi(v)$.*

Corollary 2.4. *[19] Let $u, v \in \mathbb{Z}_{p^2}$ and $[u_0, u_1]_p$, $[v_0, v_1]_p$ be the p-ary expansions of u and v, respectively. Then, $\phi(u + v) = \phi(u) + \phi(v) + (\xi_0, \dots, \xi_0)$, where $\xi_0 = 1$ if $u_0 + v_0 \geq p$ and 0 otherwise.*

Proposition 2.1. *[4,24,19] Let $u, v \in \mathbb{Z}_{p^2}$. Then, $d_H(\phi(u), \phi(v)) = \mathrm{wt}_H(\phi(u - v))$.*

By Proposition 2.1, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes obtained from the Gray map Φ are distance invariant, that is, the Hamming weight distribution is invariant under translation by a codeword. Therefore, their minimum distance coincides with the minimum weight.

Let $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p^2 - 1}$ be the vectors having the same element $0, 1, 2, \dots, p^2 - 1$ repeated in all its coordinates, respectively. Let

$$A_p^{1,1} = \left(\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & p & p & \cdots & p \\ 0 & 1 & \cdots & p-1 & 1 & 2 & \cdots & p-1 \end{array}\right). \tag{2}$$

Any matrix $A_p^{t_1,t_2}$ with $t_1 \geq 1, t_2 \geq 2$ or $t_1 \geq 2, t_2 \geq 1$ can be obtained by applying the following iterative constructions. First, if A is a generator matrix of a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code, that is, a subgroup of $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, then we denote by A_1 the submatrix of A consisting of the first α_1 columns over $\mathbb{Z}_p$, and A_2 the submatrix consisting of the last α_2 columns over $\mathbb{Z}_{p^2}$. We start with $A_p^{1,1}$. Then, if we have a matrix $A = A_p^{t_1,t_2}$ with $t_1, t_2 \geq 1$, we may construct the matrices

$$A_p^{t_1,t_2+1} = \left(\begin{array}{cccc|cccc} A_1 & A_1 & \cdots & A_1 & A_2 & A_2 & \cdots & A_2 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & p\cdot\mathbf{0} & p\cdot\mathbf{1} & \cdots & p\cdot(\mathbf{p-1}) \end{array}\right) \tag{3}$$

and

$$A_p^{t_1+1,t_2} = \left(\begin{array}{cccc|ccccccc} A_1 & A_1 & \cdots & A_1 & pA_1 & \cdots & pA_1 & A_2 & A_2 & \cdots & A_2 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p^2-1} \end{array}\right). \tag{4}$$

Example 2.1. Let

$$A_3^{1,1} = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right)$$

be the matrix described in (2) for $p = 3$. By using the constructions described in (3) and (4), we obtain $A_3^{1,2}$ and $A_3^{2,1}$, respectively, as follows:

$$A_3^{1,2} = \left(\begin{array}{ccccccccc|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 3 & 3 & 6 & 6 \end{array}\right),$$

$$A_3^{2,1} = \left(\begin{array}{ccccccccc|ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots & 3 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 3 & 6 & 0 & 3 & 6 & 1 & 2 & \cdots & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & \cdots & 8 & 8 \end{array}\right).$$

We can consider that the matrices $A_p^{t_1,t_2}$ are constructed recursively starting from $A_p^{1,1}$ in the following way. First, we add $t_1 - 1$ rows of order p^2 up to obtain $A_p^{t_1,1}$, and then we add $t_2 - 1$ rows of order p up to achieve $A_p^{t_1,t_2}$. Note that in the first row there is always the row $(\mathbf{1} \mid \mathbf{p})$.

Let $\mathcal{H}_p^{t_1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type $(\alpha_1, \alpha_2; t_1, t_2)$ generated by the matrix $A_p^{t_1,t_2}$, with $t_1, t_2 \geq 1$, p prime, and $\alpha_1 \neq 0$. Let $H_p^{t_1,t_2} = \Phi(\mathcal{H}_p^{t_1,t_2})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code. By Theorem 2.1, we have that $H_p^{t_1,t_2}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code.

Theorem 2.1. *[25] Let $t_1, t_2 \geq 1$ and p prime. Then, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code $H_p^{t_1,t_2}$ of type $(p^{t-t_1}, p^{t-1} - p^{t-t_1-1}; t_1, t_2)$ is a GH code over $\mathbb{Z}_p$ of length p^t, with $t = 2t_1 + t_2 - 1$.*

Let $\mathcal{H}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with p prime. Let $\mathcal{H}_1$ (respectively, $\mathcal{H}_2$) be the punctured code of $\mathcal{H}$ by deleting the last α_2 coordinates over $\mathbb{Z}_{p^2}$ (respectively, the first α_1 coordinates over $\mathbb{Z}_p$).

Remark 2.1. [25] Let $\mathcal{H} = \mathcal{H}_p^{t_1,t_2}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$ and p prime. Let $H = \Phi(\mathcal{H}_p^{t_1,t_2})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length $\alpha_1 + p\alpha_2$. Then, since H is a GH code, its minimum weight is

$$\frac{(p-1)(\alpha_1 + p\alpha_2)}{p}.$$

Note that, by construction, $\mathcal{H}_1$ is a GH code over $\mathbb{Z}_p$ of length α_1 and minimum weight $(p-1)\alpha_1/p$.

Remark 2.2. [25] Since the length of the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $\Phi(\mathcal{H}_p^{1,1})$ is p^2, its minimum weight is $(p-1)p^2/p = p(p-1)$ by Remark 2.1.

Remark 2.3. [25] The above constructions (3) and (4) give always $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_2 \neq 0$ since the starting matrix $A_p^{1,1}$ has $\alpha_2 \neq 0$. If $\alpha_2 = 0$, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes coincide with the codes obtained from a Sylvester GH matrix, so they are always linear and of type $(p^{t_2-1}, 0; 0, t_2)$ [15]. Therefore, in this paper, we only focus on the ones with $\alpha_2 \neq 0$.

Finally, we give some notations and recall two results proved in [25], which are used in the next sections.

When we include all the elements of $\mathbb{Z}_p$ (resp. $\mathbb{Z}_{p^2}$) as coordinates of a vector, we place them in increasing order. We denote by N_p the set $\{0, 1, \dots, p-1\} \subset \mathbb{Z}_{p^2}$ and $N_p^- = N_p \setminus \{0\}$. As before, when including all the elements in those sets as coordinates of a vector, we place them in increasing order. For a set $S \subseteq \mathbb{Z}_{p^2}$ and $\lambda \in \mathbb{Z}_{p^2}$, we define $\lambda S = \{\lambda j : j \in S\}$. For example, $N_3 = \{0, 1, 2\} \subset \mathbb{Z}_9$, $N_3^- = \{1, 2\} \subset \mathbb{Z}_9$, $2N_3^- = \{2, 4\}$, $3\mathbb{Z}_9 = \{0, 3, 6\}$, $(\mathbb{Z}_3, \mathbb{Z}_3) = (0, 1, 2, 0, 1, 2) \in \mathbb{Z}_3^6$ and $(\mathbb{Z}_3 \mid \mathbb{Z}_9, 2N_3^-) = (0, 1, 2 \mid 0, 1, 2, 3, 4, 5, 6, 7, 8, 2, 4) \in \mathbb{Z}_3^3 \times \mathbb{Z}_9^{11}$.

Lemma 2.2. *[25] Let $\mathcal{H}_p^{t_1,1}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code generated by the matrix $A_p^{t_1,1}$ with $t_1 \geq 2$ and p prime. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{t_1}$ be the row vectors of $A_p^{t_1,1}$ of order p^2. Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_p^{t_1,1}$ such that $\mathbf{v} = \sum_{i=1}^{t_1} \lambda_i \mathbf{v}_i$, where $\lambda_i \in N_p$ and at least one $\lambda_i \neq 0$. Then, v' contains every element of $p\mathbb{Z}_{p^2}$ the same number of times and one of the following conditions is satisfied:*

1. *There exists $\lambda \in N_p^-$ such that v' contains every element of λN_p^- the same number of times and every element of $\mathbb{Z}_{p^2} \setminus (p\mathbb{Z}_{p^2} \cup \lambda N_p^-)$ zero times.*
2. *There exists $\lambda \in N_p^-$ such that v' contains every element of λN_p^- the same number of times and every element of $\mathbb{Z}_{p^2} \setminus (p\mathbb{Z}_{p^2} \cup \lambda N_p^-)$ the same number of times.*
3. *Every element of $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$ appears in v' the same number of times.*

Corollary 2.5. *[25] Let $\mathcal{H}_p^{t_1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code generated by the matrix $A_p^{t_1,t_2}$ with $t_1 \geq 2$, $t_2 \geq 1$, and p prime. Let $\mathbf{u} = (u \mid u') \in \mathcal{H}_p^{t_1,t_2}$ such that $o(\mathbf{u}) = p^2$. Then, u' contains every element of $p\mathbb{Z}_{p^2}$ the same number of times and the remaining coordinates are from $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$.*

3. Linearity of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$

As it is mentioned in Remark 2.3, since the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_2 = 0$ are linear, we only need to focus on the codes with $\alpha_2 \neq 0$. The linearity of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 = 0$ was studied in [16,17] for $p = 2$ and [19] for $p \geq 3$ prime. When $\alpha_1 \neq 0$, they are the ones constructed from matrices $A_p^{t_1,t_2}$ given in Section 2. In [17], it is shown that the $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of type $(\alpha_1, \alpha_2; 1, t_2)$ with $t_2 \geq 1$ are the only ones which are linear, when $\alpha_1 \neq 0$. The next result shows that there are no $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$, with $\alpha_1 \neq 0$, $t_1, t_2 \geq 1$ and $p \geq 3$ prime, which are linear. Note that this result for $p \geq 3$ does not coincide with the known result for $p = 2$ if $t_1 = 1$.

Theorem 3.1. *Let $\mathcal{H}_p^{t_1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0$, $t_1, t_2 \geq 1$ and $p \geq 3$ prime. Then, $H_p^{t_1,t_2} = \Phi(\mathcal{H}_p^{t_1,t_2})$ is nonlinear.*

Proof. First, we prove that $H_p^{1,1}$ is nonlinear. Since $\mathbf{u} = (0, 1, \ldots, p-1 \mid 1, 2, \ldots, p-1) \in \mathcal{H}_p^{1,1}$, then $(p-1)\mathbf{u} = (0, (p-1)\cdot 1, \ldots, (p-1)\cdot(p-1) \mid (p-1)\cdot 1, (p-1)\cdot 2, \ldots, (p-1)\cdot(p-1)) \in \mathcal{H}_p^{1,1}$. Next, we see that $\Phi(\mathbf{u}) + \Phi((p-1)\mathbf{u}) \notin H_p^{1,1}$. Since $\phi(1) + \phi(p-1) = \mathbf{0}$ by the definition of ϕ, then the first $2p$ coordinates of the vector $\Phi(\mathbf{u}) + \Phi((p-1)\mathbf{u})$ of length p^2 are zero. Therefore, $\mathrm{wt}_H(\Phi(\mathbf{u}) + \Phi((p-1)\mathbf{u})) \leq p^2 - 2p = p(p-2) < p(p-1)$. Moreover, it is easy to see that $\Phi(\mathbf{u}) + \Phi((p-1)\mathbf{u}) \neq \mathbf{0}$. Hence, $\Phi(\mathbf{u}) + \Phi((p-1)\mathbf{u}) \notin H_p^{1,1}$, since the minimum weight of $H_p^{1,1}$ is $p(p-1)$ by Remark 2.2. Therefore, $H_p^{1,1}$ is nonlinear.

Second, we prove that if $H_p^{t_1-1,t_2}$ is nonlinear for $t_1 \geq 2, t_2 \geq 1$, then $H_p^{t_1,t_2}$ is also nonlinear. Assume that $H_p^{t_1,t_2}$ is linear. Then, by the iterative construction defined in (4), for any $\mathbf{u} = (u \mid u')$, $\mathbf{v} = (v \mid v') \in \mathcal{H}_p^{t_1-1,t_2}$, we have that $\bar{\mathbf{u}}$, $\bar{\mathbf{v}} \in \mathcal{H}_p^{t_1,t_2}$, where

$$\bar{\mathbf{u}} = (u, \overset{p}{\ldots}, u \mid pu, \overset{p-1}{\ldots}, pu, u', \overset{p^2}{\ldots}, u'),$$
$$\bar{\mathbf{v}} = (v, \overset{p}{\ldots}, v \mid pv, \overset{p-1}{\ldots}, pv, v', \overset{p^2}{\ldots}, v').$$

Moreover, since $H_p^{t_1,t_2}$ is linear, $\Phi(\bar{\mathbf{u}}) + \Phi(\bar{\mathbf{v}}) \in H_p^{t_1,t_2}$. Again, by construction (4), we have that $\Phi(\bar{\mathbf{u}}) + \Phi(\bar{\mathbf{v}}) = \Phi((a, \overset{p}{\ldots}, a \mid pa, \overset{p-1}{\ldots}, pa, a', \overset{p^2}{\ldots}, a') + \lambda(\mathbf{0}, \mathbf{1}, \ldots, \mathbf{p-1} \mid \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p-1}, \mathbf{0}, \mathbf{1}, \ldots, \mathbf{p^2-1})) \in H_p^{t_1,t_2}$, for some $\mathbf{a} = (a \mid a') \in \mathcal{H}_p^{t_1-1,t_2}$ and $\lambda \in \mathbb{Z}_{p^2}$. Considering the coordinates in positions 1 and $2p$ of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$, we have that $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) = \Phi(\mathbf{a}) \in H_p^{t_1-1,t_2}$, and then $H_p^{t_1-1,t_2}$ is linear, which is a contradiction.

Finally, if $H_p^{t_1,t_2-1}$ is nonlinear, then as above we can show that $H_p^{t_1,t_2}$ is also nonlinear, and hence the result follows. $\square$

Example 3.1. Let $\mathcal{H}_3^{1,1}$ be the $\mathbb{Z}_3\mathbb{Z}_9$-additive GH code of type $(3, 2; 1, 1)$ generated by $A_3^{1,1}$ given in Example 2.1. We have that

$$\begin{aligned}\Phi(0,1,2 \mid 1,2) + \Phi(0,2,1 \mid 2,4) &= (0,1,2,0,1,2,0,2,1) + (0,2,1,0,2,1,1,2,0)\\ &= (0,0,0,0,0,0,1,1,1) \notin H_3^{1,1} = \Phi(\mathcal{H}_3^{1,1}),\end{aligned}$$

since $H_3^{1,1}$ has minimum weight 6 by Remark 2.2. Therefore, $H_3^{1,1}$ is nonlinear.

Example 3.2. Considering all the integer solutions with $t_1, t_2 \geq 1$ of the equation $6 = 2t_1 + t_2 - 1$, we have that the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ of length p^6, $p \geq 3$ prime, are the following: $H_p^{1,5}$, $H_p^{2,3}$ and $H_p^{3,1}$. By Theorem 3.1, all of them are nonlinear. The types of these codes for $p = 3$ and $p = 5$ can be found in the row corresponding to $t = 6$ in Tables 1 and 2, respectively.

Table 1
Type and parameters (r, k) of $\mathbb{Z}_9$-linear and $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes.

t	$\mathbb{Z}_9$-linear		$\mathbb{Z}_3\mathbb{Z}_9$-linear	
	$(n; t_1, t_2)$	(r, k)	$(\alpha_1, \alpha_2; t_1, t_2)$	(r, k)
2	(3;1,1)	(3,3)	$(9, 0; 0, 3)$	(3,3)
			$(3, 2; 1, 1)$	(4,2)
3	$(9; 1, 2)$	(4,4)	$(27, 0; 0, 4)$	(4,4)
	$(9; 2, 0)$	(5,2)	$(9, 6; 1, 2)$	(5,3)
4	$(27; 1, 3)$	(5,5)	$(81, 0; 0, 5)$	(5,5)
	$(27; 2, 1)$	(6,3)	$(27, 18; 1, 3)$	(6,4)
			$(9, 24; 2, 1)$	(10,3)
5	$(81; 1, 4)$	(6,6)	$(243, 0; 0, 6)$	(6,6)
	$(81; 2, 2)$	(7,4)	$(81, 54; 1, 4)$	(7,5)
	$(81; 3, 0)$	(11,3)	$(27, 72; 2, 2)$	(11,4)
6	$(243; 1, 5)$	(7,7)	$(729, 0; 0, 7)$	(7,7)
	$(243; 2, 3)$	(8,5)	$(243, 162; 1, 5)$	(8,6)
	$(243; 3, 1)$	(12,4)	$(81, 216; 2, 3)$	(12,5)
			$(27, 234; 3, 1)$	(20,4)
7	$(729; 1, 6)$	(8,8)	$(2187, 0; 0, 8)$	(8,8)
	$(729; 2, 4)$	(9,6)	$(729, 486; 1, 6)$	(9,7)
	$(729; 3, 2)$	(13,5)	$(243, 648; 2, 4)$	(13,6)
	$(729; 4, 0)$	(21,4)	$(81, 702; 3, 2)$	(21,5)
8	$(2187; 1, 7)$	(9,9)	$(6561, 0; 0, 9)$	(9,9)
	$(2187; 2, 5)$	(10,7)	$(2187, 1458; 1, 7)$	(10,8)
	$(2187; 3, 3)$	(14,6)	$(729, 1944; 2, 5)$	(14,7)
	$(2187; 4, 1)$	(22,5)	$(243, 2106; 3, 3)$	(22,6)
			$(81, 2160; 4, 1)$	(35,5)

Table 2
Type and parameters (r, k) of $\mathbb{Z}_{25}$-linear and $\mathbb{Z}_5\mathbb{Z}_{25}$-linear GH codes.

t	$\mathbb{Z}_{25}$-linear		$\mathbb{Z}_5\mathbb{Z}_{25}$-linear	
	$(n; t_1, t_2)$	(r, k)	$(\alpha_1, \alpha_2; t_1, t_2)$	(r, k)
2	(5;1,1)	$(3, 3)$	$(25, 0; 0, 3)$	(3,3)
			(5,4;1,1)	$(5, 2)$
3	$(25; 1, 2)$	$(4, 4)$	$(125, 0; 0, 4)$	$(4, 4)$
	$(25; 2, 0)$	$(7, 2)$	$(25, 20; 1, 2)$	$(6, 3)$
4	$(125; 1, 3)$	$(5, 5)$	$(625, 0; 0, 5)$	$(5, 5)$
	$(125; 2, 1)$	$(8, 3)$	$(125, 100; 1, 3)$	$(7, 4)$
			$(25, 120; 2, 1)$	(18,3)
5	$(625; 1, 4)$	$(6, 6)$	$(3125, 0; 0, 6)$	$(6, 6)$
	$(625; 2, 2)$	$(9, 4)$	$(625, 500; 1, 4)$	$(8, 5)$
	$(625; 3, 0)$	$(22, 3)$	$(125, 600; 2, 2)$	$(19, 4)$
6	$(3125; 1, 5)$	$(7, 7)$	$(15625, 0; 0, 7)$	$(7, 7)$
	$(3125; 2, 3)$	$(10, 5)$	$(3125, 2500; 1, 5)$	$(9, 6)$
	$(3125; 3, 1)$	$(23, 4)$	$(625, 3000; 2, 3)$	$(20, 5)$
			$(125, 3100; 3, 1)$	(50,4)
7	$(15625; 1, 6)$	$(8, 8)$	$(78125, 0; 0, 8)$	$(8, 8)$
	$(15625; 2, 4)$	$(11, 6)$	$(15625, 12500; 1, 6)$	$(10, 7)$
	$(15625; 3, 2)$	$(24, 5)$	$(3125, 15000; 2, 4)$	$(21, 6)$
	$(15625; 4, 0)$	$(57, 4)$	$(625, 15500; 3, 2)$	$(51, 5)$
8	$(78125; 1, 7)$	(9,9)	$(390625, 0; 0, 9)$	(9,9)
	$(78125; 2, 5)$	(12,7)	$(78125, 62500; 1, 7)$	(11,8)
	$(78125; 3, 3)$	(25,6)	$(15625, 75000; 2, 5)$	(22,7)
	$(78125; 4, 1)$	(58,5)	$(3125, 77500; 3, 3)$	(52,6)
			$(625, 78000; 4, 1)$	(?,5)

4. Kernel of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$

Again, we focus on $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_2 \neq 0$, because otherwise the codes are linear. The kernel of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 = 0$ and its dimension was studied in [16,17] for $p = 2$ and [19] for $p \geq 3$ prime. The kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and its dimension are given in [17]. In this section, we generalize these results for $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ and $p \geq 3$ prime. First, we determine the kernel, and then we establish a basis of the kernel, which gives us its dimension.

Specifically, we prove that the dimension of the kernel of a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$, with $\alpha_1 \neq 0$, $t_1, t_2 \geq 1$ and $p \geq 3$ prime, is $t_1 + t_2$. Again, note that this result for $p \geq 3$ does not coincide with the known result for $p = 2$ when $t_1 = 1$, since in this last case the codes are linear, so the dimension of kernel is $2t_1 + t_2$.

Theorem 4.1. *Let $\mathcal{H}_p^{t_1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0$, $t_1, t_2 \geq 1$ and $p \geq 3$ prime. Let $\mathcal{H}_p$ be the subcode of $\mathcal{H}_p^{t_1,t_2}$ which contains all the codewords of order at most p. Then, $K(\Phi(\mathcal{H}_p^{t_1,t_2})) = \Phi(\mathcal{H}_p)$.*

Proof. Let $\mathcal{H} = \mathcal{H}_p^{t_1,t_2}$ and $H = \Phi(\mathcal{H})$. By Corollary 2.3, for all $\mathbf{b} = (b \mid b') \in \mathcal{H}_p$ and $\mathbf{u} = (u \mid u') \in \mathcal{H}$, we have that $\Phi(\mathbf{b})+\Phi(\mathbf{u}) = (b+u, \Phi(b')+\Phi(u')) = (b+u, \Phi(b'+u')) \in H$ and, therefore, $\Phi(\mathcal{H}_p) \subseteq K(H)$.

Now, let $\Phi(\mathbf{u}) \in K(H)$, where $\mathbf{u} = (u \mid u') \in \mathcal{H} \setminus \{0\}$. We prove that $o(\mathbf{u}) = p$ and thus $K(H) \subseteq \Phi(\mathcal{H}_p)$. Assume that $o(\mathbf{u}) = p^2$, and consider two cases: when $t_1 = 1$ and $t_1 \geq 2$. In both cases, to obtain a contradiction, we just need to find an element $\tilde{\mathbf{u}} \in \mathcal{H}$ such that $\Phi(\mathbf{u}) + \Phi(\tilde{\mathbf{u}}) \notin H$. First, let $t_1 = 1$. In this case, $\mathcal{H} = \mathcal{H}_p^{1,t_2}$. Recall that $N_p^- = \{1, \dots, p-1\} \subset \mathbb{Z}_{p^2}$. By construction, the row vector of $A_p^{t_1,t_2}$ of order p^2 is

$$\mathbf{v} = (\mathbb{Z}_p, \overset{\alpha_1/p}{\dots}, \mathbb{Z}_p \mid N_p^-, \overset{\alpha_2/(p-1)}{\dots}, N_p^-).$$

Since $o(\mathbf{u}) = p^2$, $\mathbf{u}$ can be expressed as $\mathbf{u} = \lambda\mathbf{v} + \mathbf{w}$, where $\lambda \in N_p^-$ and $\mathbf{w}$ is a codeword of order at most p. Note that for $\lambda \in N_p^-$, $\{\lambda j \bmod p : j \in N_p^-\} = N_p^-$. Therefore, we also have that $\lambda\mathbf{v} = \mathbf{v}_1+\mathbf{w}_1$, where $\mathbf{v}_1 = (\mathbf{0} \mid v')$, v' contains every element of N_p^- exactly $\alpha_2/(p-1)$ times, and $\mathbf{w}_1$ is a vector of order at most p. By Corollary 2.3, $\Phi(\mathbf{u}) = \Phi(\mathbf{v}_1)+\Phi(\mathbf{w}_1+\mathbf{w})$. Now, by Corollaries 2.2 and 2.3, $\Phi(\mathbf{u})+\Phi(\lambda\mathbf{v}) = \Phi(\lambda\mathbf{v}+\lambda\mathbf{v}+\mathbf{w}-p(\mathbf{v}_1 \odot_p \mathbf{v}_1))$. If we prove that $p(\mathbf{v}_1 \odot_p \mathbf{v}_1) \notin \mathcal{H}$, then $\Phi(\mathbf{u}) + \Phi(\lambda\mathbf{v}) \notin H$ since $\lambda\mathbf{v}$, $\mathbf{w} \in \mathcal{H}$. Note that, from the definition of $\odot_p$, for $a \in \mathbb{Z}_{p^2}$, $p(a \odot_p a) = 0$ if and only if $a \bmod p < p/2$. Since v' contains an element a such that $a \bmod p \geq p/2$, $p(v' \odot_p v') \neq \mathbf{0}$, so $p(\mathbf{v}_1 \odot_p \mathbf{v}_1) \neq \mathbf{0}$. We have that the number of ones in v' is $\alpha_2/(p-1)$. Since $\phi(p(1 \odot_p 1)) = (0, \overset{p}{\dots}, 0)$, $\mathrm{wt}_H(\Phi(p(\mathbf{v}_1 \odot_p \mathbf{v}_1))) = \mathrm{wt}_H(\Phi(p(v' \odot_p v'))) \leq \alpha_2 p - \alpha_2 p/(p-1) < (p-1)\alpha_2$. By Remark 2.1, the minimum weight of $\Phi(\mathcal{H})$ is $d = (p-1)/p(\alpha_1 + p\alpha_2)$. Since $\alpha_1/p \geq 1$, $(p-1)\alpha_2 < d$, and thus $p(\mathbf{v}_1 \odot_p \mathbf{v}_1) \notin \mathcal{H}$. Therefore, we have that $\Phi(\mathbf{u}) \notin K(H)$, which is a contradiction, so $o(\mathbf{u}) = p$.

Second, let $t_1 \geq 2$. Since $o(\mathbf{u}) = p^2$, $\mathbf{u}$ can be expressed as $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} = (v, \dots, v \mid v', \dots, v') \in \mathcal{H}$, $(v \mid v') \in \mathcal{H}_p^{t_1,1}$, and $\mathbf{w} \in \mathcal{H}$ is of order at most p.

By Corollary 2.3, $\Phi(\mathbf{u}) = \Phi(\mathbf{v}) + \Phi(\mathbf{w})$. Now, $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) = \Phi(\mathbf{w}) + \Phi(\mathbf{v}) + \Phi(\mathbf{v}) = \Phi(\mathbf{w}) + \Phi(\mathbf{v} + \mathbf{v} - p(\mathbf{v} \odot_p \mathbf{v})) = \Phi(\mathbf{v} + \mathbf{v} + \mathbf{w} - p(\mathbf{v} \odot_p \mathbf{v}))$ by Corollaries 2.2 and 2.3. If we prove that $p(\mathbf{v} \odot_p \mathbf{v}) \notin \mathcal{H}$, then $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) \notin H$ since $\mathbf{v}$, $\mathbf{w} \in \mathcal{H}$. By Lemma 2.2, there exist $\lambda \in N_p^-$ such that v' contains every element of λN_p^- the same number of times. Note that $\{\lambda j \bmod p : j \in N_p^-\} = N_p^-$, and hence λN_p^- always contains an element a such that $a \bmod p > p/2$. Therefore, we have that $p(v' \odot_p v') \neq \mathbf{0}$, so $p(\mathbf{v} \odot_p \mathbf{v}) \neq \mathbf{0}$. Now, by Corollary 2.5, $\bar{v}' = (v', \dots, v')$ contains every element of $p\mathbb{Z}_{p^2}$ exactly m times, $m > 0$, and the remaining $\alpha_2 - pm$ coordinates are from $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$. Thus, $\mathrm{wt}_H(\Phi(p(\mathbf{v} \odot_p \mathbf{v}))) = \mathrm{wt}_H(\Phi(p(\bar{v}' \odot_p \bar{v}'))) \leq (\alpha_2 - pm)(p-1) < \alpha_2(p-1)$, and hence $p(\mathbf{v} \odot_p \mathbf{v}) \notin \mathcal{H}$ by Remark 2.1. Therefore, we have that $\Phi(\mathbf{u}) \notin K(H)$, which is a contradiction, so $o(\mathbf{u}) = p$. $\square$

Corollary 4.1. *Let $\mathcal{H}_p^{t_1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0$, $t_1, t_2 \geq 1$ and $p \geq 3$ prime. Let $\mathbf{w}_k$ be the kth row of $A_p^{t_1,t_2}$ and $Q = \{(o(\mathbf{w}_k)/p)\mathbf{w}_k\}_{k=1}^{t_1+t_2}$. Then, $\Phi(Q)$ is a basis of $K(\Phi(\mathcal{H}_p^{t_1,t_2}))$ and*

$$\ker(\Phi(\mathcal{H}_p^{t_1,t_2})) = t_1 + t_2.$$

Example 4.1. Let $\mathcal{H}_3^{1,2}$ be the $\mathbb{Z}_3\mathbb{Z}_9$-additive GH code of type $(9, 6; 1, 2)$ generated by $A_3^{1,2}$ given in Example 2.1. By Corollary 4.1, we have that $\ker(H_3^{1,2}) = 1 + 2 = 3$. Also by Corollary 4.1, we can construct $K(H_3^{1,2})$ from a basis. We have that $Q = \{(\mathbf{1} \mid \mathbf{3}), (\mathbf{0} \mid 3, 6, 3, 6, 3, 6), (0, 0, 0, 1, 1, 1, 2, 2, 2 \mid 0, 0, 3, 3, 6, 6)\}$. Thus,

$$K(H_3^{1,2}) = \langle \Phi(\mathbf{1} \mid \mathbf{3}), \Phi(\mathbf{0} \mid 3, 6, 3, 6, 3, 6), \Phi(0, 0, 0, 1, 1, 1, 2, 2, 2 \mid 0, 0, 3, 3, 6, 6) \rangle.$$

More generally, if $\mathcal{H}_p^{1,2}$ is the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code generated by $A_p^{1,2}$ with $p \geq 3$ prime, then we have that

$$K(H_p^{1,2}) = \langle \Phi(\mathbf{1} \mid \mathbf{p}), \Phi(\mathbf{0} \mid u'), \Phi(v \mid v') \rangle,$$

where u' is the p-fold replication of $(p, 2p, \dots, (p-1)p)$, $v = (\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1})$ with $\mathbf{i} = (i, \overset{p}{\dots}, i)$ for $i \in \mathbb{Z}_p$, and $v' = (\mathbf{0}, p \cdot \mathbf{1}, \dots, p \cdot (\mathbf{p-1}))$ with $\mathbf{j} = (j, \overset{p-1}{\dots}, j)$ for $j \in N_p$. Therefore, $\ker(H_p^{1,2}) = 3$. Note that $\ker(H_2^{1,2}) = 4$, since $H_2^{1,2}$ is linear [17].

Example 4.2. Let $\mathcal{H}_3^{2,1}$ be the $\mathbb{Z}_3\mathbb{Z}_9$-additive GH code of type $(9, 24; 2, 1)$ generated by $A_3^{2,1}$ given in Example 2.1. By Corollary 4.1, we have that $\ker(H_3^{2,1}) = 1 + 2 = 3$. Also by Corollary 4.1, we can construct $K(H_3^{2,1})$ from a basis. We have that $Q = \{(\mathbf{1} \mid \mathbf{3}), (\mathbf{0} \mid 0, 0, 0, 0, 0, 0, u), (\mathbf{0} \mid 3, 3, 3, 6, 6, 6, v)\}$, where u is a 9-fold replication of $(3, 6)$ and v is a 3-fold replication of $(0, 0, 3, 3, 6, 6)$. Thus,

$$K(H_3^{2,1}) = \langle \Phi(\mathbf{1} \mid \mathbf{3}), \Phi((\mathbf{0} \mid 0, 0, 0, 0, 0, 0, u)), \Phi((\mathbf{0} \mid 3, 3, 3, 6, 6, 6, v)) \rangle.$$

More generally, if $\mathcal{H}_p^{2,1}$ is the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code generated by $A_p^{2,1}$ with $p \geq 3$ prime, then we have that

$$K(H_p^{2,1}) = \langle \Phi(\mathbf{1} \mid \mathbf{p}), \Phi(\mathbf{0} \mid 0, \overset{p^2-p}{\ldots}, 0, u), \Phi(\mathbf{0} \mid w, v) \rangle,$$

where u is the p^2-fold replication of $(p, 2p, \ldots, (p-1)p)$, $w = (p \cdot \mathbf{1}, p \cdot \mathbf{2}, \ldots, p \cdot (\mathbf{p-1}))$ with $\mathbf{i} = (i, \overset{p}{\ldots}, i)$ for $i \in N_p^-$, and $v = (p \cdot \mathbf{0}, p \cdot \mathbf{1}, \ldots, p \cdot (\mathbf{p^2 - 1}))$ with $\mathbf{j} = (j, \overset{p-1}{\ldots}, j)$ for $j \in \mathbb{Z}_{p^2}$. Therefore, $\ker(H_p^{2,1}) = 3$. Note that, in this case, it is also true for $p = 2$, since $\ker(H_2^{2,1}) = 3$ as shown in [17].

The dimension of the kernel is an invariant for the equivalence relation of codes over $\mathbb{Z}_p$, which allows us to give a complete classification of the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ of length p^t, as shown in the next result.

Corollary 4.2. *For any $t \geq 2$ and $p \geq 3$ prime, there are exactly $\lfloor t/2 \rfloor$ nonequivalent nonlinear $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ of length p^t by using the recursive constructions given in (3) and (4).*

Proof. Considering all the integer solutions (t_1, t_2) with $t_1, t_2 \geq 1$ of the equation $t+1 = 2t_1+t_2$, we have that all the nonlinear $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of length p^t are $H_p^{t_1, t-2t_1+1}$, where $1 \leq t_1 \leq \lfloor t/2 \rfloor$, by Theorem 3.1. Then, by Corollary 4.1, $\ker(H_p^{t_1,t-2t_1+1}) = t - t_1 + 1$, which gives different values for distinct values of t_1. Therefore, they are all pairwise nonequivalent codes. Therefore, the result follows. □

Note that Corollary 4.2 is also true for the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes obtained from more general recursive constructions given in [25], since all of them are equivalent to the constructions given in (3) and (4).

Example 4.3. Let $t = 6$ and $p = 3$. All the nonlinear $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes with $\alpha_1 \neq 0$ of length 3^6 are $H_3^{1,5}$, $H_3^{2,3}$ and $H_3^{3,1}$ having a kernel of dimension 6, 5 and 4, respectively, as it is also shown in Table 1. Since these values are all different, all these codes are pairwise nonequivalent. We have only one linear $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code of length 3^6, which is of type $(3^6, 0; 0, 7)$. Therefore, there are exactly $\lfloor 6/2 \rfloor + 1 = 4$ nonequivalent $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes of length 3^6.

5. Rank of some families of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes

In this section, we establish some results about the rank of some families of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. These results are used in Section 6 to show that these codes are not equivalent to the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 = 0$ (also called just $\mathbb{Z}_{p^2}$-linear GH codes) constructed in [19]. First, we prove some technical lemmas.

Lemma 5.1. *Let $\mathbf{v} = (\mathbb{Z}_p \mid N_p^-) \in \mathbb{Z}_p^p \times \mathbb{Z}_{p^2}^{p-1}$. Then, $\Phi(\lambda\mathbf{v}) + \Phi((p-\lambda)\mathbf{v}) = \Phi(p\mathbf{v}) + (p-1)(\mathbf{0}, \mathbf{1}, \mathbf{1}, \ldots, \mathbf{1})$ for any $\lambda \in \{1, 2, \ldots, (p-1)/2\}$.*

Proof. For the first p coordinates over $\mathbb{Z}_p$, it is clearly true, since the Gray map Φ is the identity. For $\lambda \in \{1, 2, \ldots, (p-1)/2\}$ and $i \in N_p^-$, we can write that $i\lambda = u_0 + pu_1 \in \mathbb{Z}_{p^2}$, where $u_0, u_1 \in N_p$. Then, $i(p-\lambda) = (p-u_0) + p(i-u_1-1)$. By Corollary 2.4, $\phi(i\lambda) + \phi(i(p-\lambda)) = \phi(ip) - (1, 1, \ldots, 1)$ since $u_0 + p - u_0 \geq p$. Therefore, $\Phi(\lambda\mathbf{v}) + \Phi((p-\lambda)\mathbf{v}) = \Phi(p\mathbf{v}) + (p-1)(\mathbf{0}, \mathbf{1}, \mathbf{1}, \ldots, \mathbf{1})$. $\square$

Lemma 5.2. *Let* $\mathbf{v} = (\mathbb{Z}_p \mid N_p^-) \in \mathbb{Z}_p^p \times \mathbb{Z}_{p^2}^{p-1}$. *Then,* $\Phi(\lambda\mathbf{v}) + \Phi((p-\lambda)\mathbf{v}) = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{p-2})$ *for any* $\lambda \in \{1, 2, \ldots, (p-1)/2\}$.

Proof. Straightforward from Lemmas 2.1 and 5.1. $\square$

Lemma 5.3. *Let* $p \geq 3$ *prime and* $a \in \{1, \ldots, p-1\}$. *Then,*

$$|\{ia \bmod p \leq a : i \in \{1, \ldots, (p-1)/2\}\}| = \lfloor (a-1)/2 \rfloor + 1.$$

Proof. Let $A = \{ia \bmod p : i \in \{1, \ldots, (p-1)/2\}\}$ and $B = \{ia \bmod p \leq a : i \in \{1, \ldots, (p-1)/2\}\}$. Note that, for $p = 3$, $A = \{a\} = B$, and hence the lemma holds. Now, we assume that $p > 3$ prime, and consider two cases: when $a \leq 2$ and when $a > 2$. First, if $a \leq 2$, then $A = \{a, 2a, \ldots, (p-1)a/2\}$ and $B = \{a\}$, so the lemma holds. Second, assume $a > 2$. Let β be the biggest integer such that $\frac{(p-1)a}{2} > \beta p$, so $\beta = \lfloor \frac{(p-1)a}{2p} \rfloor$. Note that $\frac{(p-1)a}{2} < (\beta+1)p$. In this case, we consider a partition of A into disjoints subsets, $A = \cup_{b=0}^{\beta} A_b$, where

$$A_b = \{ia \bmod p : i \in \{1, \ldots, (p-1)/2\}, bp < ia \leq (b+1)p\}$$

for $b \in \{0, \ldots, \beta\}$. Note that $\beta \geq 1$ since $a > 2$. Then, we have that

$$A_b = \{\Big(\Big\lfloor \frac{bp}{a} \Big\rfloor + 1\Big)a - bp, \Big(\Big\lfloor \frac{bp}{a} \Big\rfloor + 2\Big)a - bp, \ldots, \Big\lfloor \frac{(b+1)p}{a} \Big\rfloor a - bp\},$$

for $b \in \{0, 1, \ldots, \beta - 1\}$, and

$$A_\beta = \{\Big(\Big\lfloor \frac{\beta p}{a} \Big\rfloor + 1\Big)a - \beta p, \Big(\Big\lfloor \frac{\beta p}{a} \Big\rfloor + 2\Big)a - \beta p, \ldots, \frac{(p-1)a}{2} - \beta p\}.$$

Note that $B = A \cap B = \cup_{b=0}^{\beta}(A_b \cap B)$. For $b \in \{0, \ldots, \beta\}$, we have that every element in A_b is smaller than p by definition, and we see that $|A_b \cap B| = 1$. First, it is easy to see that $A_0 \cap B = \{a\}$. Next, for all $b \in \{1, 2, \ldots, \beta - 1\}$, we see that, for $j \in \{1, \ldots, \lfloor \frac{(b+1)p}{a} \rfloor - \lfloor \frac{bp}{a} \rfloor\}$, $(\lfloor \frac{bp}{a} \rfloor + j)a - bp \leq a$ if and only if $j = 1$. We can write $bp = ka + r$, where $k = \lfloor \frac{bp}{a} \rfloor$ and $0 \leq r < a$. Then, $(\lfloor \frac{bp}{a} \rfloor + j)a - bp = ka + ja - ka - r = ja - r$. Since $j \geq 1$ and $0 < a - r \leq a$, we have that $ja - r \leq a$ if and only if $j = 1$ and hence $|A_b \cap B| = 1$. Similarly, for $j \in \{1, \ldots, \frac{(p-1)}{2} - \lfloor \frac{\beta p}{a} \rfloor\}$, we have that $(\lfloor \frac{\beta p}{a} \rfloor + j)a - \beta p \leq a$ if and only if $j = 1$, and hence $|A_\beta \cap B| = 1$

Finally, since $B = \cup_{b=0}^{\beta}(A_b \cap B)$, we have that $|B| = \sum_{b=0}^{\beta} |A_b \cap B| = \beta + 1 = \lfloor \frac{(p-1)a}{2p} \rfloor + 1$. Now, note that $a - 1 < \frac{(p-1)a}{p} < a$, so $\frac{a-1}{2} < \frac{(p-1)a}{2p} < \frac{a}{2}$, and hence $\lfloor \frac{(p-1)a}{2p} \rfloor = \lfloor \frac{a-1}{2} \rfloor$. Therefore, $|B| = \lfloor \frac{a-1}{2} \rfloor + 1$. $\square$

Lemma 5.4. *Let $p \geq 3$ prime and $a \in \{1, \dots, p-1\}$. Then,*

$$|\{a + (ia \bmod p) \geq p : i \in \{1, \dots, (p-1)/2\}\}| = \lfloor a/2 \rfloor.$$

Proof. Note that $|\{ia \bmod p : i \in \{1, \dots, p-1\}\}| = p - 1$, and so

$$|\{ia \bmod p \geq p - a : i \in \{1, \dots, p-1\}\}| = a. \tag{5}$$

Let $i, a \in \{1, 2 \dots, (p-1)/2\}$. We can write that $ia = u_0 + pu_1 \in \mathbb{Z}_{p^2}$, where $u_0, u_1 \in N_p$. Then, $(p - i)a = (p - u_0) + p(a - u_1 - 1)$ since $i < p$. We have that $u_0 \leq a$ if and only if $p - u_0 \geq p - a$ and, hence, $ia \bmod p \leq a$ if and only if $(p - i)a \bmod p \geq p - a$. Therefore, $|\{ia \bmod p \geq p - a : i \in \{(p+1)/2, \dots, p-1\}\}| = |\{ia \bmod p \leq a : i \in \{(1, \dots, (p-1)/2)\}\}| = \lfloor \frac{a-1}{2} \rfloor + 1$ by Lemma 5.3. Finally, from (5), we have that $|\{ia \bmod p \geq p - a : i \in \{1, \dots, (p-1)/2\}\}| = a - (\lfloor \frac{a-1}{2} \rfloor + 1) = \lfloor \frac{a}{2} \rfloor$, which completes the proof. $\square$

Proposition 5.1. *Let $\mathcal{H}_p^{1,1}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p, p-1; 1, 1)$ with $p \geq 3$ prime, and $H_p^{1,1} = \Phi(\mathcal{H}_p^{1,1})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length p^2. Then,* $\mathrm{rank}(H_p^{1,1}) = 3 + (p-1)/2$.

Proof. Let $\mathbf{u} = (1, \overset{p}{\dots}, 1 \mid p, \overset{p-1}{\dots}, p)$ and $\mathbf{v} = (\mathbb{Z}_p \mid N_p^-) \in \mathbb{Z}_p^p \times \mathbb{Z}_{p^2}^{p-1}$. If $\mathbf{x} \in \mathcal{H}_p^{1,1}$, then $\mathbf{x}$ can be expressed as $\mathbf{x} = \lambda\mathbf{v} + \mu\mathbf{u}$, where $\lambda \in \mathbb{Z}_{p^2}$ and $\mu \in N_p \subset \mathbb{Z}_{p^2}$. By Corollary 2.3, we have that $\Phi(\mathbf{x}) = \Phi(\lambda\mathbf{v}) + \Phi(\mu\mathbf{u})$ and also $\Phi(\lambda\mathbf{v}) = \Phi(\lambda_0\mathbf{v}) + \Phi(\lambda_1 p\mathbf{v})$, where $\lambda = \lambda_0 + \lambda_1 p$, and $\lambda_0, \lambda_1 \in N_p$. By Corollary 2.1, $\Phi(\lambda_1 p\mathbf{v}) = \lambda_1\Phi(p\mathbf{v})$ and $\Phi(\mu\mathbf{u}) = \mu\Phi(\mathbf{u})$. Therefore, $\Phi(\mathbf{x}) = \Phi(\lambda_0\mathbf{v}) + \lambda_1\Phi(p\mathbf{v}) + \mu\Phi(\mathbf{u})$.

By Corollary 2.4, for all $i \in \{1, \dots, (p-1)/2\}$, we have that

$$\Phi((i+1)\mathbf{v}) = \Phi(\mathbf{v}) + \Phi(i\mathbf{v}) + v_i, \tag{6}$$

where $v_i = (\mathbf{0}, v_{i,1}, v_{i,2}, \dots, v_{i,p-1})$, $v_{i,a} \in \{\mathbf{0}, \mathbf{1}\}$ for $a \in \{1, \dots, p-1\}$, and $\mathbf{0}, \mathbf{1}$ are of length p. Consider the vector $v_{(p-1)/2}$. It is easy to see that

$$\frac{p-1}{2} a \bmod p = \begin{cases} \frac{p-a}{2} & \text{if } \ a \text{ is odd,} \\ p - \frac{a}{2} & \text{if } \ a \text{ is even.} \end{cases}$$

Then, if a is odd, $a + (p-a)/2 = (p+a)/2 < p$; and if a is even, $a + p - a/2 = p + a/2 \geq p$. Therefore, $v_{(p-1)/2} = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \dots, \mathbf{0}, \mathbf{1})$.

Next, we show that $\Phi(\lambda_0\mathbf{v})$, for $\lambda_0 \in N_p$, is a linear combination of $\Phi(\mathbf{v})$ and $\{v_i\}_{1\leq i\leq (p-1)/2}$. First, for $\lambda_0 \in \{2,3,\ldots,(p+1)/2\}$, $\Phi(\lambda_0\mathbf{v}) = \lambda_0\Phi(\mathbf{v}) + \sum_{i=1}^{\lambda_0-1} v_i$ by applying recursively (6). Second, from Lemma 5.4,

$$\sum_{i=1}^{(p-1)/2} v_i = (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\ldots,\frac{\mathbf{p-1}}{\mathbf{2}}-\mathbf{1},\frac{\mathbf{p-1}}{\mathbf{2}}-\mathbf{1},\frac{\mathbf{p-1}}{\mathbf{2}}), \tag{7}$$

so $(\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{2},\ldots,\mathbf{p-2}) = 2(\sum_{i=1}^{(p-1)/2} v_i) - v_{(p-1)/2}$ since we have that $v_{(p-1)/2} = (\mathbf{0},\mathbf{0},\mathbf{1},\ldots,\mathbf{0},\mathbf{1})$. Then, for $\lambda_0 \in \{(p+1)/2+1,\ldots,p-1\}$, $\Phi(\lambda_0\mathbf{v})$ is a linear combination of $\Phi(\mathbf{v})$ and $\{v_i\}_{1\leq i\leq (p-1)/2}$ by Lemma 5.2. Therefore, $\Phi(\mathbf{x})$ is generated by $\Phi(\mathbf{u}), \Phi(\mathbf{v}), \Phi(p\mathbf{v})$ and $\{v_i\}_{1\leq i\leq (p-1)/2}$. In other words, we can take a matrix G whose row vectors are $\Phi(\mathbf{u}), \Phi(\mathbf{v}), \Phi(p\mathbf{v})$ and $\{v_i\}_{1\leq i\leq (p-1)/2}$, as a generator matrix of $\langle H_p^{1,1}\rangle$.

The vectors $\Phi(\mathbf{u}), \Phi(\mathbf{v})$ and $\Phi(p\mathbf{v})$ are clearly linearly independent over $\mathbb{Z}_p$. We consider the vectors v_i, $i \in \{1,\ldots,(p-1)/2\}$, as the rows of a matrix V. Let S_a, $a \in \{1,\ldots,p-1\}$, be the set of p coordinate positions of $v_{1,a}$ in v_1. There exists a set of $(p-1)/2$ linearly independent columns of V by taking one column of each S_a with $a \in \{2,4,\ldots,p-1\}$, since the number of ones in each one of these columns is different by (7). Therefore, the vectors $\{v_i\}_{1\leq i\leq (p-1)/2}$ are linearly independent. In fact, it is easy to see that all the rows of G are linearly independent. Therefore, $\mathrm{rank}(H_p^{1,1}) = 3 + (p-1)/2$. □

Example 5.1. Let $\mathcal{H}_7^{1,1}$ be the $\mathbb{Z}_7\mathbb{Z}_{49}$-additive GH code of type $(7,6;1,1)$ and $H_7^{1,1} = \Phi(\mathcal{H}_7^{1,1})$ be the corresponding $\mathbb{Z}_7\mathbb{Z}_{49}$-linear GH code of length 49. Let $\mathbf{u} = (1,1,1,1,1,1,1 \mid 7,7,7,7,7,7,7)$ and $\mathbf{v} = (0,1,\ldots,7 \mid 1,2,\ldots,7)$. By the proof of Proposition 5.1, $\langle H_7^{1,1}\rangle$ can be generated by $\Phi(\mathbf{u}), \Phi(\mathbf{v}), \Phi(7\mathbf{v})$, and $\{v_i\}_{1\leq i\leq 3}$, where

$$\begin{aligned} v_1 &= (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{1}),\\ v_2 &= (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1}),\\ v_3 &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1}),\end{aligned}$$

and the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length 7. Since these vectors are linearly independent, we have that $\mathrm{rank}(H_7^{1,1}) = 6$.

Example 5.2. Let $\mathcal{H}_{11}^{1,1}$ be the $\mathbb{Z}_{11}\mathbb{Z}_{121}$-additive GH code of type $(11,10;1,1)$ and $H_{11}^{1,1} = \Phi(\mathcal{H}_{11}^{1,1})$ be the corresponding $\mathbb{Z}_{11}\mathbb{Z}_{121}$-linear GH code of length 121. Let $\mathbf{u} = (1,1,1,1,1,1,1,1,1,1,1 \mid 11,11,11,11,11,11,11,11,11,11)$ and $\mathbf{v} = (0,1,\ldots,10 \mid 1,2,\ldots,10)$. By the proof of Proposition 5.1, $\langle H_{11}^{1,1}\rangle$ can be generated by $\Phi(\mathbf{u}), \Phi(\mathbf{v})$, $\Phi(11\mathbf{v})$, and $\{v_i\}_{1\leq i\leq 5}$, where

$$v_1 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}),$$
$$v_2 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}),$$
$$v_3 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{1}),$$
$$v_4 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}),$$
$$v_5 = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}),$$

and the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length 11. Since these vectors are linearly independent, we have that $\mathrm{rank}(H_{11}^{1,1}) = 8$.

Let $\mathcal{H}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with p prime. Recall that $\mathcal{H}_1$ (respectively, $\mathcal{H}_2$) is the punctured code of $\mathcal{H}$ by deleting the last α_2 coordinates over $\mathbb{Z}_{p^2}$ (respectively, the first α_1 coordinates over $\mathbb{Z}_p$). Then, we can write that $\mathcal{H} = (\mathcal{H}_1 \mid \mathcal{H}_2)$. Let $(\mathcal{H}_i, \ldots, \mathcal{H}_i)$ be the code having the following set of codewords $\{(h_i, \ldots, h_i) : h_i \in \mathcal{H}_i\}$ for $i \in \{1, 2\}$.

Theorem 5.1. *Let $\mathcal{H}_p^{1,t-1}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p^{t-1}, (p-1)p^{t-2}; 1, t-1)$ with $t \geq 2$ and $p \geq 3$ prime, and $H_p^{1,t-1} = \Phi(\mathcal{H}_p^{1,t-1})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length p^t. Then,*

$$\mathrm{rank}(H_p^{1,t-1}) = 1 + t + (p-1)/2.$$

Proof. We prove this theorem by induction on t. First, note that the result is true for $t = 2$, by Proposition 5.1. We assume that the result is true for a given $t \geq 2$. Let $\mathcal{H}_p^{1,t-1} = (\mathcal{H}_1 \mid \mathcal{H}_2)$ and $H_p^{1,t} = \Phi(\mathcal{H}_p^{1,t})$. Then, by the recursive construction (3), we can write

$$\mathcal{H}_p^{1,t} = \bigcup_{\lambda \in N_p} ((\mathcal{H}_1, \overset{p}{\ldots}, \mathcal{H}_1 \mid \mathcal{H}_2, \overset{p}{\ldots}, \mathcal{H}_2) + \lambda(\mathbf{0}, \ldots, \mathbf{p-1} \mid p \cdot \mathbf{0}, \ldots, p \cdot \mathbf{p-1})).$$

By Corollary 2.3, we have that

$$H_p^{1,t} = \bigcup_{\lambda \in N_p} (\Phi(\mathcal{H}_1, \overset{p}{\ldots}, \mathcal{H}_1 \mid \mathcal{H}_2, \overset{p}{\ldots}, \mathcal{H}_2) + \Phi(\lambda(\mathbf{0}, \ldots, \mathbf{p-1} \mid p \cdot \mathbf{0}, \ldots, p \cdot \mathbf{p-1}))),$$

so $\mathrm{rank}(H_p^{1,t}) = 1 + \mathrm{rank}(H_p^{1,t-1})$. Therefore, by induction hypothesis,

$$\mathrm{rank}(H_p^{1,t}) = 1 + (1+t) + (p-1)/2. \quad \square$$

Example 5.3. Let $\mathcal{H}_7^{1,2}$ be the $\mathbb{Z}_7\mathbb{Z}_{49}$-additive GH code of type $(49, 42; 1, 2)$ and $H_7^{1,2} = \Phi(\mathcal{H}_7^{1,2})$ be the corresponding $\mathbb{Z}_7\mathbb{Z}_{49}$-linear GH code of length 143. Let $\mathbf{u} = (u \mid u')$ and $\mathbf{v} = (v \mid v')$ and $v_i = (\mathbf{0}, v_i')$, $1 \leq i \leq 3$ be the vectors given in Example 5.1. By applying construction (3) to $A_7^{1,1}$, we have that $\bar{\mathbf{u}} = (u, \overset{7}{\ldots}, u \mid u', \overset{7}{\ldots}, u')$, $\bar{\mathbf{v}} = (v, \overset{7}{\ldots}, v \mid v', \overset{7}{\ldots}, v')$ and $\bar{\mathbf{w}} = (\mathbf{0}, \mathbf{1}, \ldots, \mathbf{6} \mid \mathbf{0}, \mathbf{7}, \ldots, \mathbf{42})$ are the rows of the generator matrix $A_7^{1,2}$. Therefore, by Example 5.1 and the proof of Theorem 5.1, we have that $\langle H_7^{1,2}\rangle$ can be generated by $\Phi(\bar{\mathbf{u}}), \Phi(\bar{\mathbf{v}}), \Phi(7\bar{\mathbf{v}}), \Phi(\bar{\mathbf{w}})$ and $\{\bar{v}_i\}_{1\le i\le 3}$, where $\bar{v}_i = (\mathbf{0}, \overset{7}{\ldots}, \mathbf{0}, v_i', \overset{7}{\ldots}, v_i')$, $1 \le i \le 3$. Since these vectors are linearly independent, we have that $\mathrm{rank}(H_7^{1,2}) = 7$.

Proposition 5.2. *Let $\mathcal{H}_p^{2,1}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p^2, (p-1)(p+p^2); 2, 1)$ with $p \ge 3$ prime, and $H_p^{2,1} = \Phi(\mathcal{H}_p^{2,1})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code of length p^4. Then,* $\mathrm{rank}(H_p^{2,1}) > p + 3$.

Proof. Let $\mathbf{u} = (1, \overset{p^2}{\ldots}, 1 \mid p, \overset{p^3-p}{\ldots}, p)$, and $\mathbf{v}, \mathbf{w}$ be the rows of $A_p^{2,1}$ of order p^2, that is,

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \left(\begin{array}{cccc|cccccc} \mathbb{Z}_p & \mathbb{Z}_p & \cdots & \mathbb{Z}_p & p\mathbb{Z}_{p^2} & \cdots & p\mathbb{Z}_{p^2} & N_p^- & \cdots & N_p^- \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{0} & \cdots & \mathbf{p^2-1} \end{array}\right).$$

By Corollary 2.4 and the proof of Proposition 5.1, for all $i \in \{1, \ldots, (p-1)/2\}$, we have that

$$\Phi((i+1)\mathbf{w}) = \Phi(\mathbf{w}) + \Phi(i\mathbf{w}) + w_i,$$

where $w_i = (\mathbf{0}, w_{i,1}, w_{i,2}, \ldots, w_{i,p-1}, *, \ldots, *)$, $w_{i,a} \in \{\mathbf{0}, \mathbf{1}\}$ for $a \in \{1, \ldots, p-1\}$, $\sum_{i=1}^{(p-1)/2} w_i = (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \ldots, \mathbf{\frac{p-1}{2}} - \mathbf{1}, \mathbf{\frac{p-1}{2}} - \mathbf{1}, \mathbf{\frac{p-1}{2}}, *, \ldots, *)$, and $\mathbf{j} = (j, \overset{p^2}{\ldots}, j)$ for $j \in \{0, 1, \ldots, (p-1)/2\}$. The symbol $*$ in a coordinate means that it can be any element of $\mathbb{Z}_p$.

If we consider the vectors w_i, $i \in \{1, \ldots, (p-1)/2\}$, as the rows of a matrix, then we have a set of $(p-1)/2$ linearly independent columns by taking one column of each $w_{i,a}$ with $a \in \{2, 4, \ldots, p-1\}$, since the number of ones in each one of these columns is different. Therefore, the vectors $\{w_i\}_{1\le i\le (p-1)/2}$ are linearly independent.

Again, by Corollary 2.4 and the proof of Proposition 5.1, for all $i \in \{1, \ldots, (p-1)/2\}$, we have that

$$\Phi((i+1)\mathbf{v}) = \Phi(\mathbf{v}) + \Phi(i\mathbf{v}) + v_i,$$

where $v_i = (0, \overset{p^3}{\ldots}, 0, \bar{v}_i, \overset{p^2}{\ldots}, \bar{v}_i)$, $\bar{v}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,p-1})$, $v_{i,a} \in \{\mathbf{0}, \mathbf{1}\}$ for $a \in \{1, \ldots, p-1\}$, $\sum_{i=1}^{(p-1)/2} v_i = (0, \overset{p^3}{\ldots}, 0, \bar{x}, \overset{p^2}{\ldots}, \bar{x})$, $\mathbf{k} = (k, \overset{p}{\ldots}, k)$ for $k \in \{0, 1, \ldots, (p-1)/2\}$, and $\bar{x} = (\mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \ldots, \mathbf{\frac{p-1}{2}} - \mathbf{1}, \mathbf{\frac{p-1}{2}} - \mathbf{1}, \mathbf{\frac{p-1}{2}})$.

We consider the vectors v_i, $i \in \{1, \ldots, (p-1)/2\}$, as the rows of a matrix V. Let S_a, $a \in \{1, \ldots, p-1\}$, be the set of coordinate positions of $v_{1,a}$ in the first copy of $\bar{v}_1$ in v_1. There exists a set of $(p-1)/2$ linearly independent columns of V by taking one column of each S_a with $a \in \{2, 4, \ldots, p-1\}$, since the number of ones in each one of these columns is different. Therefore, the vectors $\{v_i\}_{1\le i\le (p-1)/2}$ are linearly independent.

Note that $\Phi(\mathbf{u}), \Phi(\mathbf{v}), \Phi(p\mathbf{v}), \Phi(\mathbf{w}), \Phi(p\mathbf{w}), \{v_i\}_{1\le i\le (p-1)/2}, \{w_i\}_{1\le i\le (p-1)/2}$ are all linearly independent. Therefore, $\mathrm{rank}(H_p^{2,1}) > p + 3$. $\square$

Example 5.4. Let $\mathcal{H}_3^{2,1}$ be the $\mathbb{Z}_3\mathbb{Z}_9$-additive GH code of type $(9, 24; 2, 1)$ and $H_3^{2,1} = \Phi(\mathcal{H}_3^{2,1})$ be the corresponding $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code of length 81. Let $\mathbf{u} = (\mathbf{1} \mid \mathbf{3})$, $\mathbf{v} = (0, 1, 2, 0, 1, 2, 0, 1, 2 \mid 0, 3, 6, 0, 3, 6, v, \overset{9}{\ldots}, v)$, where $v = (1, 2)$, and $\mathbf{w} = (0, 0, 0, 1, 1, 1, 2, 2, 2 \mid 1, 1, 1, 2, 2, 2, 0, 0, \ldots, 8, 8)$ be the rows of the generator matrix $A_3^{2,1}$ given in Example 2.1.

We have that $\Phi(2\mathbf{v}) = 2\Phi(\mathbf{v}) + v_1$ and $\Phi(2\mathbf{w}) = 2\Phi(\mathbf{w}) + w_1$, where

$$
\begin{aligned}
v_1 &= (0, \overset{27}{\ldots}, 0, \bar{v}_1, \overset{9}{\ldots}, \bar{v}_1), \\
w_1 &= (0, \overset{18}{\ldots}, 0, 1, \overset{9}{\ldots}, 1, \bar{w}_1, \bar{w}_1, \bar{w}_1),
\end{aligned}
$$

$\bar{v}_1 = (0, 0, 0, 1, 1, 1)$ and $\bar{w}_1 = (0, \overset{12}{\ldots}, 0, 1, \overset{6}{\ldots}, 1)$. We also have that $\Phi(3\mathbf{v}) = (0, \overset{27}{\ldots}, 0, x, \overset{9}{\ldots}, x)$, where $x = (1, 1, 1, 2, 2, 2)$, and $\Phi(3\mathbf{w}) = (0, \overset{9}{\ldots}, 0, 1, \overset{9}{\ldots}, 1, 2, \overset{9}{\ldots}, 2, y, y, y)$, where $y = (0, \overset{6}{\ldots}, 0, 1, \overset{6}{\ldots}, 1, 2, \overset{6}{\ldots}, 2)$. Note that the set S generated by $\Phi(\mathbf{u})$, $\Phi(\mathbf{v})$, $\Phi(\mathbf{w})$, $\Phi(3\mathbf{v})$, $\Phi(3\mathbf{w})$, v_1 and w_1, is a subspace of $\langle H_3^{2,1} \rangle$. Since these vectors are linearly independent, we have that $\mathrm{rank}(H_3^{2,1}) \geq \dim(S) = 7 > 6$.

Theorem 5.2. *Let $\mathcal{H}_p^{2,t-3}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p^{t-2}, (p-1)(p^{t-3} + p^{t-2}); 2, t-3)$ with $t \geq 4$ and $p \geq 3$ prime, and $H_p^{2,t-3} = \Phi(\mathcal{H}_p^{2,t-3})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length p^t. Then,*

$$\mathrm{rank}(H_p^{2,t-3}) > p + t - 1.$$

Proof. We prove this theorem by induction on t. First, note that the result is true for $t = 4$, by Proposition 5.2. We assume that the result is true for a given $t \geq 4$. By following the same argument as in the proof of Theorem 5.1, we obtain that $\mathrm{rank}(H_p^{2,t-2}) = 1 + \mathrm{rank}(H_p^{2,t-3})$. Therefore, by induction hypothesis, $\mathrm{rank}(H_p^{2,t-2}) > 1 + p + t - 1 = p + t$, and the result follows. □

6. Classification results

The classification of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ of length 2^t, for any $t \geq 3$, using the rank or the dimension of the kernel is shown in [16,17]. For $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 = 0$ (that is, $\mathbb{Z}_4$-linear Hadamard codes), the classification is also shown in [16,17]. Some partial results on the classification of $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t, for any $t \geq 3$ and $s > 2$, are proved in [9,21]; and in general for $\mathbb{Z}_{p^s}$-linear GH codes of length p^t, for any $t \geq 2$, $s \geq 2$, and $p \geq 3$ prime, in [19,26]. For any $t \geq 2$, the full classification of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of length p^t, with $\alpha_1 \neq 0$ and $p \geq 3$ prime, is given by Corollary 4.2 in Section 4, by using just the dimension of the kernel. In this section, we compare the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes having $\alpha_1 \neq 0$ with the $\mathbb{Z}_{p^s}$-linear GH codes.

First, we recall the construction given in [19] of $\mathbb{Z}_{p^s}$-linear GH codes with $s \geq 2$ and p prime. We also recall for which types these codes are linear and what is the kernel and its dimension for these codes whenever they are nonlinear. Then, we compute some results about the rank of some families of $\mathbb{Z}_{p^2}$-linear GH codes, in order to determine whether these codes are equivalent or not to the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ considered in this paper. Finally, we see that, unlike for the case $p = 2$, the $\mathbb{Z}_{p^2}$-linear GH codes with $p \geq 3$ prime are not equivalent to the codes $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. There are at least two infinite families of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ and $p \geq 3$ prime, such that their codes are not equivalent to any $\mathbb{Z}_{p^2}$-linear GH code of the same length p^t. Indeed, we prove that they are not equivalent to any $\mathbb{Z}_{p^s}$-linear GH code with $s \geq 2$ and not only for $s = 2$.

First, we describe the construction given in [19] of $\mathbb{Z}_{p^s}$-linear GH codes with $s \geq 2$ and p prime. Let $T_i = \{j \cdot p^{i-1} \ : \ j \in \{0, 1, \ldots, p^{s-i+1} - 1\}\}$ for all $i \in \{1, \ldots, s\}$. Note that $T_1 = \{0, \ldots, p^s - 1\}$. Let t_1, t_2,..., t_s be non-negative integers with $t_1 \geq 1$. Consider the matrix $\bar{A}_p^{t_1,\ldots,t_s}$ whose columns are exactly all the vectors of the form $\mathbf{z}^T$, $\mathbf{z} \in \{1\} \times T_1^{t_1-1} \times T_2^{t_2} \times \cdots \times T_s^{t_s}$.

Example 6.1. For $p = 3$ and $s = 2$, we have the following matrices:

$$\bar{A}_3^{1,1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \end{pmatrix}, \quad \bar{A}_3^{1,2} = \begin{pmatrix} 111 & 111 & 111 \\ 036 & 036 & 036 \\ 000 & 333 & 666 \end{pmatrix},$$

$$\bar{A}_3^{2,1} = \begin{pmatrix} 111111111 & 111111111 & 111111111 \\ 012345678 & 012345678 & 012345678 \\ 000000000 & 333333333 & 666666666 \end{pmatrix}.$$

Any matrix $\bar{A}_p^{t_1,\ldots,t_s}$ can also be obtained by applying the following recursive construction. We start with $\bar{A}_p^{1,0,\ldots,0} = (1)$. Then, if we have a matrix $\bar{A} = \bar{A}_p^{t_1,\ldots,t_s}$, for any $i \in \{1, \ldots, s\}$, we may construct the matrix

$$\bar{A}_i = \begin{pmatrix} \bar{A} & \bar{A} & \cdots & \bar{A} \\ 0 \cdot \mathbf{p^{i-1}} & 1 \cdot \mathbf{p^{i-1}} & \cdots & (p^{s-i+1} - 1) \cdot \mathbf{p^{i-1}} \end{pmatrix}. \tag{8}$$

Finally, permuting the rows of $\bar{A}_i$, we obtain a matrix $\bar{A}_p^{t'_1,\ldots,t'_s}$, where $t'_j = t_j$ for $j \neq i$ and $t'_i = t_i + 1$. Note that any permutation of columns of $\bar{A}_i$ gives also a matrix $\bar{A}_p^{t'_1,\ldots,t'_s}$.

We consider that the matrices $\bar{A}_p^{t_1,\ldots,t_s}$ are constructed recursively starting from $\bar{A}_p^{1,0,\ldots,0}$ in the following way. First, we add $t_1 - 1$ rows of order p^s, up to obtain $\bar{A}_p^{t_1,0,\ldots,0}$; then t_2 rows of order p^{s-1} up to generate $\bar{A}_p^{t_1,t_2,\ldots,0}$; and so on, until we add t_s rows of order p to achieve $\bar{A}_p^{t_1,\ldots,t_s}$. See [19] for examples.

Let $\bar{\mathcal{H}}_p^{t_1,\ldots,t_s}$ be the $\mathbb{Z}_{p^s}$-additive code of type $(n; t_1, \ldots, t_s)$ generated by the matrix $\bar{A}_p^{t_1,\ldots,t_s}$, where $t_1, \ldots, t_s$ are non-negative integers with $t_1 \geq 1$ and p prime. Let $\bar{H}_p^{t_1,\ldots,t_s} = \Phi(\bar{\mathcal{H}}_p^{t_1,\ldots,t_s})$ be the corresponding $\mathbb{Z}_{p^s}$-linear code.

Let $\mathbf{w}_i^{(s)}$ be the ith row of $\bar{A}_p^{t_1,\dots,t_s}$, $1 \leq i \leq t_1 + \cdots + t_s$. By construction, $\mathbf{w}_1^{(s)} = \mathbf{1}$ and $o(\mathbf{w}_i^{(s)}) \leq o(\mathbf{w}_j^{(s)})$ if $i > j$. We define $\sigma \in \{1,\dots,s\}$ as the integer such that $o(\mathbf{w}_2) = p^{s+1-\sigma}$. For $\bar{\mathcal{H}}_p^{1,0,\dots,0}$, we define $\sigma = s$. Note that $\sigma = 1$ if $t_1 > 1$, and $\sigma = \min\{i : t_i > 0, i \in \{2,\dots,s\}\}$ if $t_1 = 1$.

Theorem 6.1. *[19] Let $t_1,\dots,t_s$ be non-negative integers with $s \geq 2$ and $t_1 \geq 1$. The $\mathbb{Z}_{p^s}$-linear code $\bar{H}_p^{t_1,\dots,t_s}$ of type $(n; t_1,\dots,t_s)$ is a GH code over $\mathbb{Z}_p$ of length $N = p^t$, with $t = (\sum_{i=1}^{s}(s-i+1)\cdot t_i) - 1$ and $n = p^{t-s+1}$.*

Theorem 6.2. *[19] The $\mathbb{Z}_{p^s}$-linear GH codes $\bar{H}_p^{1,0,\dots,0,t_s}$, with $p \geq 3$ prime, $s \geq 2$ and $t_s \geq 0$, are the only $\mathbb{Z}_{p^s}$-linear GH codes which are linear.*

Theorem 6.3. *[19] Let $\bar{H}_p^{t_1,\dots,t_s}$ be the $\mathbb{Z}_{p^s}$-linear GH code of type $(n; t_1, \dots, t_s)$ with $p \geq 3$ prime, $s \geq 2$, $t_1 \geq 1$, and $t_i \geq 0$ for $i \in \{2,\dots,s\}$. Then,*

$$\ker(\bar{H}_p^{t_1,\dots,t_s}) = (\sum_{i=1}^{s} t_i) + \sigma - 1.$$

Now, we establish some results about the rank of a family of $\mathbb{Z}_{p^2}$-linear GH codes, in order to determine whether these codes are equivalent or not to the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ considered in this paper.

Proposition 6.1. *Let $\bar{\mathcal{H}}_p^{2,0}$ be the $\mathbb{Z}_{p^2}$-additive GH code of type $(p^2; 2, 0)$ with $p \geq 3$ prime, and $\bar{H}_p^{2,0} = \Phi(\bar{\mathcal{H}}_p^{2,0})$ be the corresponding $\mathbb{Z}_{p^2}$-linear GH code of length p^3. Then,* $\mathrm{rank}(\bar{H}_p^{2,0}) = p + 2$.

Proof. Let $\mathbf{u} = (1, \overset{p^2}{\dots}, 1)$ and $\mathbf{v} = (0, 1, \dots, p^2 - 1)$ be the rows of $\bar{A}_p^{2,0}$. Let $\lambda, \mu \in \mathbb{Z}_{p^2}$. We can write $\lambda = \lambda_0 + \lambda_1 p$ and $\mu = \mu_0 + \mu_1 p$, where $\lambda_0, \lambda_1, \mu_0, \mu_1 \in N_p$. By Corollaries 2.1 and 2.3, we have that

$$\Phi(\lambda\mathbf{u} + \mu\mathbf{v}) = \Phi(\lambda_0\mathbf{u} + \mu_0\mathbf{v}) + \lambda_1\Phi(p\mathbf{u}) + \mu_1\Phi(p\mathbf{v}). \tag{9}$$

Now, we consider $\Phi(\lambda_0\mathbf{u} + \mu_0\mathbf{v})$ for all $\lambda_0, \mu_0 \in N_p$. First, if $\mu_0 = 1$, then, by Corollary 2.4, we have that $\Phi(\lambda_0\mathbf{u} + \mathbf{v}) = \Phi(\lambda_0\mathbf{u}) + \Phi(\mathbf{v}) + v_{\lambda_0}$, where

$$v_{\lambda_0} = (\bar{v}_{\lambda_0}, \overset{p}{\dots}, \bar{v}_{\lambda_0}), \quad \bar{v}_{\lambda_0} = (\mathbf{0}, \overset{p-\lambda_0}{\dots}, \mathbf{0}, \mathbf{1}, \overset{\lambda_0}{\dots}, \mathbf{1}), \tag{10}$$

and the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length p. Second, if $\mu_0 \in \{2,\dots,p-1\}$, we have that $\Phi(\lambda_0\mathbf{u} + \mu_0\mathbf{v}) = \Phi(\lambda_0\mathbf{u}) + \Phi(\mu_0\mathbf{v}) + w_{\lambda_0,\mu_0}$, where

$$w_{\lambda_0,\mu_0} = (\bar{w}_{\lambda_0,\mu_0}, \overset{p}{\dots}, \bar{w}_{\lambda_0,\mu_0}), \quad \bar{w}_{\lambda_0,\mu_0} = (\mathbf{0}, w_{\lambda_0,\mu_0,1}, \dots, w_{\lambda_0,\mu_0,p-1}),$$

$w_{\lambda_0,\mu_0,a} \in \{\mathbf{0},\mathbf{1}\}$ for $a \in \{1,\dots,p-1\}$, and the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length p. By the definition of Φ, since $\lambda_0 \in N_p$, it is easy to see that $\Phi(\lambda_0\mathbf{u}) = \lambda_0\Phi(\mathbf{u})$. For $\mu_0 \in \{2,\dots,p-1\}$, we have that $\Phi(\mu_0\mathbf{v}) = \Phi(\mathbf{v}) + \Phi((\mu_0-1)\mathbf{v}) + x_{\mu_0}$, where

$$x_{\mu_0} = (\bar{x}_{\mu_0}, \overset{p}{\ldots}, \bar{x}_{\mu_0}), \quad \bar{x}_{\mu_0} = (\mathbf{0}, x_{\mu_0,1},\dots,x_{\mu_0,p-1}),$$

$x_{\mu_0,a} \in \{\mathbf{0},\mathbf{1}\}$ for $a \in \{1,\dots,p-1\}$, and the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length p.

Note that $\{(0, \overset{p-i-1}{\ldots}, 0, 1, \overset{i}{\ldots}, 1)\}_{1\le i\le p-1}$ is a basis of the vector space $\mathbb{Z}_p^{p-1}$. Thus, we have that w_{λ_0,μ_0} and x_{μ_0} can be written as a linear combination of the vectors $\{v_i\}_{1\le i\le p-1}$ given in (10). Moreover, $\Phi(p\mathbf{v}) = \Phi((pN_p, \overset{p}{\ldots}, pN_p)) = \sum_{i=1}^{p-1} v_i$. Therefore, from (9), $\bar{H}_p^{2,0}$ can be generated by $\Phi(\mathbf{u}), \Phi(p\mathbf{u}), \Phi(\mathbf{v})$ and $\{v_i\}_{1\le i\le p-1}$. Since these vectors are all linearly independent, we obtain that $\operatorname{rank}(\bar{H}_p^{2,0}) = p+2$. $\square$

Example 6.2. Let $\bar{\mathcal{H}}_3^{2,0}$ be the $\mathbb{Z}_9$-additive GH code of type $(9;2,0)$ and $\bar{H}_3^{2,0} = \Phi(\bar{\mathcal{H}}_3^{2,0})$ be the corresponding $\mathbb{Z}_9$-linear GH code of length 27. Let $\mathbf{u} = (\mathbf{1})$ and $\mathbf{v} = (0,1,2,3,4,5,6,7,8)$ be the rows of the generator matrix $\bar{A}_3^{2,0}$. Let $\lambda = \lambda_0 + 3\lambda_1$ and $\mu = \mu_0 + 3\mu_1$, where $\lambda_0,\lambda_1,\mu_0,\mu_1 \in \{0,1,2\} \subseteq \mathbb{Z}_9$. Note that

$$\Phi(\lambda\mathbf{u} + \mu\mathbf{v}) = \Phi(\lambda_0\mathbf{u} + \mu_0\mathbf{v}) + \lambda_1\Phi(3\mathbf{u}) + \mu_1\Phi(3\mathbf{v}). \tag{11}$$

Now, for $\lambda_0 \in \{1,2\}$, we have that $\Phi(\lambda_0\mathbf{u} + \mathbf{v}) = \Phi(\lambda_0\mathbf{u}) + \Phi(\mathbf{v}) + v_{\lambda_0}$, where

$$\begin{aligned} v_1 &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{1}),\\ v_2 &= (\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1}),\end{aligned}$$

and the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length 3. Moreover, we have that

$$\begin{aligned} \Phi(3\mathbf{v}) &= v_1 + v_2,\\ \Phi(2\mathbf{u}) &= 2\Phi(\mathbf{u}),\\ \Phi(2\mathbf{v}) &= 2\Phi(\mathbf{v}) + v_1,\\ \Phi(\mathbf{u}+2\mathbf{v}) &= \Phi(\mathbf{u}) + 2\Phi(\mathbf{v}) + v_2,\\ \Phi(2\mathbf{u}+2\mathbf{v}) &= 2\Phi(\mathbf{u}) + 2\Phi(\mathbf{v}) + v_1 + v_2.\end{aligned} \tag{12}$$

Therefore, from (11) and (12), $\langle \bar{H}_3^{2,0}\rangle$ can be generated by $\Phi(\mathbf{u})$, $\Phi(\mathbf{v})$, $\Phi(3\mathbf{u})$, v_1 and v_2. Since these vectors are linearly independent, $\operatorname{rank}(\bar{H}_3^{2,0}) = 5$.

Theorem 6.4. *Let $\bar{\mathcal{H}}_p^{2,t-3}$ be the $\mathbb{Z}_{p^2}$-additive GH code of type $(p^{t-1};2,t-3)$ with $t \ge 3$ and $p \ge 3$ prime, and $\bar{H}_p^{2,t-3} = \Phi(\bar{\mathcal{H}}_p^{2,t-3})$ be the corresponding $\mathbb{Z}_{p^2}$-linear GH code of length p^t. Then,*

$$\operatorname{rank}(\bar{H}_p^{2,t-3}) = p + t - 1.$$

Proof. We prove this theorem by induction on t. First, note that the result is true for $t = 3$, by Proposition 6.1. We assume that the result is true for a given $t \geq 3$. By following the same argument as in the proof of Theorem 5.1, we obtain that $\mathrm{rank}(\bar{H}_p^{2,t-2}) = 1+\mathrm{rank}(\bar{H}_p^{2,t-3})$. Therefore, by induction hypothesis, $\mathrm{rank}(\bar{H}_p^{2,t-2}) = 1+p+t-1 = p+t$, and the result follows. $\square$

Then, we present two corollaries that show that there are some families of the new $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes presented in this paper, containing an infinite number of codes, which are not equivalent to the $\mathbb{Z}_{p^s}$-linear GH codes presented in [19]. The first one uses just the dimension of the kernel to determine that the codes are not equivalent, and the second one uses both invariants, the rank and the dimension of the kernel.

Corollary 6.1. *For any $t \geq 2$ and $p \geq 3$ prime, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $H_p^{1,t-1}$ of type $(p^{t-1}, (p-1)p^{t-2}; 1, t-1)$ is nonequivalent to a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of any other type, having $\alpha_1 \neq 0$ and the same length p^t. Moreover, it is also nonequivalent to any $\mathbb{Z}_{p^s}$-linear GH code with $s \geq 2$ of length p^t.*

Proof. From the proof of Corollary 4.2, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $H_p^{1,t-1}$ is nonequivalent to any other $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code with $\alpha_1 \neq 0$ of length p^t, because the dimensions of the kernels are different.

By Corollary 4.1, we have that $\ker(H_p^{1,t-1}) = 1 + t - 1 = t$. Let $\bar{H}_p^{t_1,\dots,t_s}$ be the $\mathbb{Z}_{p^s}$-linear GH code of length p^t. By Theorem 6.3, we have that $\ker(\bar{H}_p^{t_1,\dots,t_s}) = (\sum_{i=1}^{s} t_i) + \sigma - 1$. Now, to complete the proof, we just need to show that $\ker(H_p^{1,t-1}) \neq \ker(\bar{H}_p^{t_1,\dots,t_s})$. Assume $\ker(H_p^{1,t-1}) = \ker(\bar{H}_p^{t_1,\dots,t_s})$, that is, $t = (\sum_{i=1}^{s} t_i) + \sigma - 1$. By Theorem 6.1, we have that $st_1 + (s-1)t_2 + \cdots + 2t_{s-1} + t_s = t+1$. Then, $st_1 + (s-1)t_2 + \cdots + 2t_{s-1} + t_s = (\sum_{i=1}^{s} t_i) + \sigma$, that is,

$$(s-1)t_1 + (s-2)t_2 + \cdots + t_{s-1} = \sigma. \tag{13}$$

From the definition of σ, we have that either $\sigma = 1$ when $t_1 \geq 2$, or $2 \leq \sigma \leq s$ when $t_1 = 1$, $t_2 = \cdots = t_{\sigma-1} = 0$ and $t_\sigma \geq 1$. For the first case, we obtain a contradiction from (13). For the second case, (13) becomes $(s-1) + (s-\sigma)t_\sigma + \lambda = \sigma$, where $\lambda = (s-\sigma-1)t_{\sigma+1} + \cdots + t_{s-1}$. Thus, if $\sigma < s$, we have that $t_\sigma + 1 = \frac{1-\lambda}{s-\sigma}$, which is a contradiction, since $t_\sigma + 1 \geq 2$, $\lambda \geq 0$, and $s - \sigma > 0$. If $s = \sigma$, $s - 1 = s$, which is also a contradiction. This completes the proof. $\square$

Example 6.3. Let $t = 6$ and $p = 3$. By the proof of Corollary 4.2 or from Example 4.3, the $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code $H_3^{1,5}$ is nonequivalent to any other $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code with $\alpha_1 \neq 0$ of the same length 3^6.

Now, all the nonlinear $\mathbb{Z}_{3^s}$-linear GH codes, with $s \geq 2$, of length 3^6 are $\bar{H}_3^{3,1}$, $\bar{H}_3^{2,3}$, $\bar{H}_3^{1,2,0}$, $\bar{H}_3^{2,0,1}$, $\bar{H}_3^{1,1,2}$, $\bar{H}_3^{1,1,0,0}$, $\bar{H}_3^{1,0,1,1}$, and $\bar{H}_3^{1,0,0,1,0}$ having a kernel of dimension 4, 5, 4, 3, 5, 3, 5, and 5, respectively, as it is shown in Table 4 given in [19]. All the linear $\mathbb{Z}_{3^s}$-linear GH codes, with $s \geq 2$, of length 3^6 are $\bar{H}_3^{1,5}$, $\bar{H}_3^{1,0,4}$, $\bar{H}_3^{1,0,0,3}$, $\bar{H}_3^{1,0,0,0,2}$,

$\bar{H}_3^{1,0,0,0,0,1}$ and $\bar{H}_3^{1,0,0,0,0,0,0}$. Therefore, since $H_3^{1,5}$ is nonlinear and $\ker(H_3^{1,5}) = 1+5 = 6$ by Corollary 4.1, $H_3^{1,5}$ is nonequivalent to any $\mathbb{Z}_{3^s}$-linear GH codes, with $s \geq 2$, of the same length 3^6.

Theorem 6.5. *[26] Let $\bar{H}_p^{t_1,\dots,t_s}$ be a $\mathbb{Z}_{p^s}$-linear GH code with $t_s \geq 1$. Then, $\bar{H}_p^{t_1,\dots,t_s}$ is equivalent to the $\mathbb{Z}_{p^{s+\ell}}$-linear GH code $\bar{H}_p^{1,0,\stackrel{\ell-1}{\dots},0,t_1-1,t_2,\dots,t_{s-1},t_s-\ell}$, for all $\ell \in \{1,\dots,t_s\}$.*

Corollary 6.2. *For any $t \geq 4$ and $p \geq 3$ prime, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $H_p^{2,t-3}$ of type $(p^{t-2},(p-1)(p^{t-3}+p^{t-2});2,t-3)$ is nonequivalent to a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of any other type, having $\alpha_1 \neq 0$ and the same length p^t. Moreover, it is also nonequivalent to any $\mathbb{Z}_{p^s}$-linear GH code with $s \geq 2$ of length p^t.*

Proof. From the proof of Corollary 4.2, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $H_p^{2,t-3}$ is nonequivalent to any other $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code with $\alpha_1 \neq 0$ of length p^t, because the dimensions of the kernels are different.

Let $\bar{H}_p^{t_1,\dots,t_s}$ be the $\mathbb{Z}_{p^s}$-linear GH code of length p^t. By Theorem 6.1, we have that

$$st_1 + (s-1)t_2 + \dots + 2t_{s-1} + t_s = t+1. \tag{14}$$

By Theorem 6.5, $\bar{H}_p^{2,t-3}$, $\bar{H}_p^{1,1,t-4}$, $\bar{H}_p^{1,0,1,t-5}$, $\bar{H}_p^{1,0,0,1,t-6}$, $\dots$, $\bar{H}_p^{1,0,\stackrel{t-4}{\dots},0,1,0}$ are all pairwise equivalent codes over $\mathbb{Z}_p$ of length p^t. Let

$$F = \{\bar{H}_p^{2,t-3}, \bar{H}_p^{1,1,t-4}, \bar{H}_p^{1,0,1,t-5}, \bar{H}_p^{1,0,0,1,t-6}, \dots, \bar{H}_p^{1,0,\stackrel{t-4}{\dots},0,1,0}\}$$

and $T = \{(2,t-3),(1,1,t-4),(1,0,1,t-5),(1,0,0,1,t-6),\dots,(1,0,\stackrel{t-4}{\dots},0,1,0)\}$. By Theorem 6.4, $\mathrm{rank}(\bar{H}_p^{2,t-3}) = p+t-1$, and hence $\bar{H}_p^{2,t-3}$ is nonequivalent to the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $H_p^{2,t-3}$, since $\mathrm{rank}(H_p^{2,t-3}) > p+t-1$ by Theorem 5.2. Therefore, all the codes in F are nonequivalent to $H_p^{2,t-3}$.

By Corollary 4.1, we have that $\ker(H_p^{2,t-3}) = 2+t-3 = t-1$. By Theorem 6.3, we have that $\ker(\bar{H}_p^{t_1,\dots,t_s}) = (\sum_{i=1}^s t_i) + \sigma - 1$. Now, to complete the proof, we just need to show that $\ker(H_p^{2,t-3}) \neq \ker(\bar{H}_p^{t_1,\dots,t_s})$ for all $\bar{H}_p^{t_1,\dots,t_s} \notin F$. Assume $\ker(H_p^{2,t-3}) = \ker(\bar{H}_p^{t_1,\dots,t_s})$, that is, $t-1 = (\sum_{i=1}^s t_i) + \sigma - 1$, where $(t_1,\dots,t_s) \notin T$. Then, from (14), we have that $st_1 + (s-1)t_2 + \dots + 2t_{s-1} + t_s = (\sum_{i=1}^s t_i) + \sigma + 1$, that is, we have that

$$(s-1)t_1 + (s-2)t_2 + \dots + t_{s-1} = \sigma + 1, \tag{15}$$

where $(t_1,\dots,t_s) \notin T$. From the definition of σ, we have that either $\sigma = 1$ when $t_1 \geq 2$, or $2 \leq \sigma \leq s$ when $t_1 = 1$, $t_2 = \dots = t_{\sigma-1} = 0$ and $t_\sigma \geq 1$.

For the first case, note that if $t_1 = 2$ and $s = 2$, then $\bar{H}_p^{2,t-3}$ is the only $\mathbb{Z}_{p^2}$-linear GH code of type $(p^{t-1};2,t_2)$ by Theorem 6.1. If $t_1 \geq 2$ and $s > 2$, or $t_1 > 2$ and $s = 2$, we obtain a contradiction from (15).

For the second case, if $\sigma = s$, then (15) becomes $\sigma - 1 = \sigma + 1$, which is a contradiction. Then, we consider that $2 \leq \sigma < s$. We can write (15) as $(s-1) + (s-\sigma)t_\sigma + \lambda = \sigma + 1$, where $\lambda = (s-\sigma-1)t_{\sigma+1} + \cdots + t_{s-1}$. Thus,

$$(s-\sigma)(t_\sigma + 1) = 2 - \lambda, \tag{16}$$

where $(t_1, \dots, t_s) \notin T$. Since $2 \leq \sigma < s$ and $t_\sigma + 1 \geq 2$, we have that $\lambda = 0$. Then, (16) becomes

$$(s-\sigma)(t_\sigma + 1) = 2. \tag{17}$$

Again, since $t_\sigma + 1 \geq 2$, from (17), we have that $s - \sigma = 1$. Then, $t_\sigma = t_{s-1} = 1$, and from (14), we have that $s + 2 + t_s = t + 1$, that is,

$$t - t_s = s + 1. \tag{18}$$

Now, since $(t_1, \dots, t_s) = (1, 0, \dots, 0, 1, t_s) \notin T$, we have that $t_s \notin \{t-4, t-5, \dots, 0\}$, that is, $t_s \geq t - 3$. Therefore, from (18), $t - s - 1 \geq t - 3$, that is, $s = 2$, since $s \geq 2$. Thus, $\sigma = s - 1 = 1$, and we obtain a contradiction, since $2 \leq \sigma < s$. This completes the proof. $\square$

Example 6.4. Let $t = 6$ and $p = 3$. By the proof of Corollary 4.2 or from Example 4.3, the $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code $H_3^{2,3}$ is nonequivalent to any other $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code with $\alpha_1 \neq 0$ of same length 3^6.

Now, all the linear $\mathbb{Z}_{3^s}$-linear GH codes, with $s \geq 2$, of length 3^6 are $\bar{H}_3^{1,5}$, $\bar{H}_3^{1,0,4}$, $\bar{H}_3^{1,0,0,3}$, $\bar{H}_3^{1,0,0,0,2}$, $\bar{H}_3^{1,0,0,0,0,1}$ and $\bar{H}_3^{1,0,0,0,0,0,0}$. These codes are nonequivalent to $H_3^{2,3}$, since $H_3^{2,3}$ is a nonlinear code of length 3^6. All the nonlinear $\mathbb{Z}_{3^s}$-linear GH codes, with $s \geq 2$, of length 3^6 are

$$\bar{H}_3^{3,1}, \bar{H}_3^{2,3}, \bar{H}_3^{1,2,0}, \bar{H}_3^{2,0,1}, \bar{H}_3^{1,1,2}, \bar{H}_3^{1,1,0,0}, \bar{H}_3^{1,0,1,1}, \text{and } \bar{H}_3^{1,0,0,1,0}$$

having the following parameters (r, k), where r is the rank and k the dimension of the kernel: $(12, 4)$, $(8, 5)$, $(12, 4)$, $(14, 3)$, $(8, 5)$, $(14, 3)$, $(8, 5)$ and $(8, 5)$, respectively, as it is shown in Table 4 given in [19]. By Corollary 4.1, $\ker(H_3^{2,3}) = 5$. Therefore, $\bar{H}_3^{3,1}$, $\bar{H}_3^{1,2,0}$, $\bar{H}_3^{2,0,1}$ and $\bar{H}_3^{1,1,0,0}$ are nonequivalent to $H_3^{2,3}$, since they all have a kernel of dimension less than 5. Let $F = \{\bar{H}_3^{2,3}, \bar{H}_3^{1,1,2}, \bar{H}_3^{1,0,1,1}, \bar{H}_3^{1,0,0,1,0}\}$. Since all the codes in F have the same rank 8, they all are nonequivalent to $H_3^{2,3}$, since $\mathrm{rank}(H_3^{2,3}) > 3 + 6 - 1 = 8$ by Theorem 5.2.

7. Conclusions and further research

In this paper, we study the linearity of the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ constructed in Section 2, and found out that they are always nonlinear. We also determine the dimension of the kernel for these nonlinear codes, and prove that this invariant can be used to see that they are pairwise not equivalent for any given length. This generalizes some results given for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes and $\mathbb{Z}_{p^2}$-linear GH codes given in [17] and [19], respectively. Finally, the rank of some infinite families of codes has been computed, and this invariant has been used to prove that, unlike for $p = 2$, for $p \geq 3$ prime, there are $\mathbb{Z}_{p^2}$-linear GH codes which are not equivalent to any $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. Actually, there are infinite families of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ such that their codes are not equivalent to any $\mathbb{Z}_{p^s}$-linear GH codes with $s \geq 2$. Therefore, we show that some nonlinear GH codes, without any known structure, now can be seen as the Gray map image of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive codes.

As a further research, it would be interesting to prove that the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$ and $p \geq 3$ prime are always not equivalent to any $\mathbb{Z}_{p^2}$-linear GH code of the same length p^t, as it can be seen for $p \in \{3, 5\}$ and $2 \leq t \leq 8$, from Tables 1 and 2. More generally, it may be proved that indeed they are not equivalent to any $\mathbb{Z}_{p^s}$-linear GH codes with $s \geq 2$ of same length p^t, as it can be checked for $p = 3$ and $2 \leq t \leq 8$ by looking at Tables 4 and 5 from [19].

Data availability

No data was used for the research described in the article.

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