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Bielliptic quotient modular curves of $X_0(N)$

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Abstract

Let $N \geq 1$ be a non-square free integer and let W_N be a nontrivial subgroup of the group of the Atkin-Lehner involutions of $X_0(N)$ such that the modular curve $X_0(N)/W_N$ has genus at least two. We determine all pairs (N, W_N) such that $X_0(N)/W_N$ is a bielliptic curve and the pairs (N, W_N) such that $X_0(N)/W_N$ has an infinite number of quadratic points over \mathbb{Q} .

1 Introduction

Let X be a smooth projective curve of genus $g_X \geq 2$ defined over a number field K . We also assume that X has at least one K -rational point. (All curves we investigate in this paper have this property.)

The curve X is called **bielliptic** if it has an involution v such that X/v has genus 1. More precisely, X is called bielliptic over L for an extension L of K if v is defined over L . In this case X/v inherits an L -rational point from X and hence is an elliptic curve over L .

Although for every finite extension L of K the set $X(L)$ of L -rational points of X is finite by a famous theorem by Faltings, the set

$$\Gamma_2(X, K) := \bigcup_{[L:K] \leq 2} X(L)$$

of quadratic points over K can be infinite. This happens for example if X is bielliptic over K and the elliptic curve X/v has positive rank over K , because then over each of the infinitely many K -rational points of X/v there is at least one quadratic point on X . By a similar argument $\Gamma_2(X, K)$ is infinite if X is hyperelliptic, because then the hyperelliptic involution u is defined over K and $X/u \cong \mathbb{P}^1$.

Less obvious is that the converse also holds, i.e. $\Gamma_2(X, K)$ is infinite if and only if X is hyperelliptic or X is bielliptic over K with elliptic quotient curve of positive K -rank. For details see [Bar18], also for the more complicated situation where $X(K) = \emptyset$. Note that the similar statement in [HS91] is slightly weaker in that (at least for $g_X \leq 5$) it is not stated over which field the bielliptic involution is defined.

Therefore an investigation which curves of a certain type or in a certain family have infinitely many quadratic points often starts with determining the bielliptic ones, the determination of the hyperelliptic ones usually having been done decades earlier.

This paper is the final installment in a series that investigates the two problems, biellipticity and infinitely many quadratic points over \mathbb{Q} , for curves $X_0(N)/W_N$ where $X_0(N)$ is a modular

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curve of Hecke type and W_N is a subgroup of the group $B(N)$ of Atkin-Lehner involutions of $X_0(N)$. Note that the AL-involutions are defined over \mathbb{Q} , and so $X_0(N)/W_N$ inherits a \mathbb{Q} -rational cusp from $X_0(N)$.

For the trivial subgroup $W_N = \{1\}$ it was already [Bar99] which determined all bielliptic curves $X_0(N)$ and those with infinite $\Gamma_2(X_0(N), \mathbb{Q})$. Later [Jeo18] solved the same problem for the curves $X_0^+(N) = X_0(N)/w_N$ where w_N is the Fricke involution. Then [BG19] and [BG20] treated the curves $X_0^*(N) = X_0(N)/B(N)$ for square-free resp. non-square-free N .

After the case of proper subgroups W_N was settled in [BGK20] for square-free N , we now deal with proper subgroups W_N when N is not square-free.

In Section 3, specializing a result by Harris and Silverman we note that $X_0(N)/W_N$ only has a chance of being bielliptic when $X_0^*(N)$ is bielliptic, hyperelliptic or of genus at most 1. So with [BG20], [Has97] and [GL98] we can restrict the candidates to a finite, explicit set of N , albeit with usually 7 curves for each such N .

González and the first author in [BG20] provided a computational method by use of a theorem of Petri to decide whether a concrete curve is bielliptic or not, for modular curves after computing the Jacobian and the Galois conjugation basis. But the computations are tedious and not transparent to the reader. Therefore we first use several old and new criteria (see Section 4.1) to quickly discard as many candidates as possible.

A bielliptic curve of genus smaller than 6 might have several bielliptic involutions, and, worse, it can happen that none of these is defined over \mathbb{Q} . But if N is square-free, then all automorphisms of $X_0(N)/W_N$ are known to be defined over \mathbb{Q} , which is quite helpful. For non-square-free N this does not necessarily hold.

On the other hand, most non-square-free candidates N are divisible by 4 or 9, and in these cases there exist additional involutions of $X_0(N)$ coming from the normalizer of $\Gamma_0(N)$ in $GL_2(\mathbb{R})$. This gives us more candidates for bielliptic involutions. Indeed, for practically all bielliptic $X_0(N)/W_N$ with N not square-free we can write down explicit bielliptic involutions, as matrices if one wants to.

Our first main result is

Theorem 1.1. *Let $N > 1$ be a non square-free integer not a power of a prime. Assume that the genus of the modular curve $X_0(N)/W_N$ is at least 2 for a nontrivial subgroup W_N of $B(N)$ different from $\langle w_N \rangle$. The curve $X_0(N)/W_N$, denoted by the pair (N, W_N) is a bielliptic curve if and only if it appears below:*

(i) *It is a pair (N, W_N) such that $|W_N| = 2$ and N is in the set*

$$\{40, 48, 52, 63, 68, 72, 75, 76, 80, 96, 98, 99, 100, 108, 124, 188\},$$

or it is a pair (N, W_N) such that $|W_N| = 4$ and N is in the set

$$\{84, 90, 120, 126, 132, 140, 150, 156, 220\}.$$

All these quotient modular curves are bielliptic over \mathbb{Q} with an elliptic quotient given by $X_0^(N)$ of genus 1,*

(ii) or it is one of the following 29 pairs, ordered by genus

Genus	(N, W_N)
2	$(44, \langle w_4 \rangle), (60, \langle w_{20} \rangle), (60, \langle w_4, w_3 \rangle)$
3	$(56, \langle w_8 \rangle), (60, \langle w_4 \rangle)$
4	$(60, \langle w_3 \rangle), (60, \langle w_5 \rangle), (112, \langle w_7 \rangle), (168, \langle w_3, w_{56} \rangle)$
5	$(84, \langle w_4 \rangle), (88, \langle w_{11} \rangle), (90, \langle w_9 \rangle)$ $(117, \langle w_9 \rangle), (120, \langle w_{15} \rangle), (126, \langle w_{63} \rangle), (168, \langle w_8, w_7 \rangle),$ $(168, \langle w_7, w_{24} \rangle), (180, \langle w_4, w_9 \rangle), (184, \langle w_{23} \rangle), (252, \langle w_4, w_{63} \rangle)$
6	$(104, \langle w_8 \rangle), (168, \langle w_8, w_3 \rangle)$
7	$(120, \langle w_{24} \rangle), (136, \langle w_8 \rangle), (252, \langle w_9, w_7 \rangle)$
9	$(126, \langle w_9 \rangle), (171, \langle w_9 \rangle), (252, \langle w_4, w_9 \rangle)$
10	$(176, \langle w_{16} \rangle)$

Remark 1.2. It turns out that almost all of the curves listed in Theorem 1.1 are bielliptic over \mathbb{Q} . The only exceptions are the isomorphic curves $X_0(126)/w_{63}$ and $X_0(252)/\langle w_4, w_{63} \rangle$ which are bielliptic over $\mathbb{Q}(\sqrt{-3})$.

For more information see the tables in Appendix B where we list bielliptic involutions and the conductor of the corresponding elliptic quotient curve as well as the splitting of the Jacobian of $X_0(N)/W_N$.

The hyperelliptic $X_0(N)/W_N$ are already known (see [FH99]). Checking which of the non-hyperelliptic ones satisfy the condition from the theorem discussed at the beginning of this Introduction we obtain our second main result.

Theorem 1.3. Let $N > 1$ be a non square-free integer not a power of a prime. Assume that the genus of the modular curve $X_0(N)/W_N$ is at least 2 for a nontrivial subgroup W_N of $B(N)$ different from $\langle w_N \rangle$. Then the set

$$\Gamma_2(X_0(N)/W_N, \mathbb{Q})$$

is infinite if and only if (N, W_N) appears in the following list:

- (i) It is a pair (N, W_N) that is an hyperelliptic curve, determined in [FH99] that we reproduce for the convenience of the readers:

Genus	(N, W_N)
2	$(40, \langle w_8 \rangle), (40, \langle w_5 \rangle), (44, \langle w_4 \rangle), (48, \langle w_{16} \rangle), (48, \langle w_3 \rangle), (52, \langle w_4 \rangle), (54, \langle w_2 \rangle),$ $(60, \langle w_{20} \rangle), (60, \langle w_4, w_3 \rangle), (60, \langle w_5, w_{12} \rangle), (72, \langle w_8 \rangle), (84, \langle w_4, w_3 \rangle), (84, \langle w_4, w_{21} \rangle)$ $(84, \langle w_3, w_{28} \rangle), (84, \langle w_7, w_{12} \rangle), (90, \langle w_9, w_5 \rangle), (90, \langle w_9, w_{10} \rangle), (90, \langle w_2, w_{45} \rangle)$ $(90, \langle w_5, w_{18} \rangle), (100, w_4), (120, \langle w_8, w_{15} \rangle), (120, \langle w_{24}, w_{40} \rangle), (126, \langle w_2, w_{63} \rangle)$ $(126, \langle w_{18}, w_{14} \rangle), (132, \langle w_4, w_{11} \rangle), (140, \langle w_4, w_{35} \rangle), (150, \langle w_6, w_{50} \rangle)$ $(156, \langle w_4, w_{39} \rangle)$
3	$(56, \langle w_8 \rangle), (60, \langle w_4 \rangle), (60, \langle w_{60} \rangle), (63, \langle w_9 \rangle), (72, \langle w_9 \rangle), (120, \langle w_5, w_{24} \rangle),$ $(126, \langle w_9, w_7 \rangle), (126, \langle w_9, w_{14} \rangle),$
4	$(60, \langle w_{12} \rangle), (168, \langle w_{24}, w_{56} \rangle),$
5	$(92, \langle w_4 \rangle).$

- (ii) or for bielliptic curves that are not hyperelliptic corresponding to a pair $(99, W_N)$ with $|W_N| = 2$.

We observe that among the elliptic curves $X_0^*(N)$ the 38 ones with square-free N all have positive rank, whereas the 25 ones with non-square-free N with the exception of $X_0^*(99)$ all have rank 0.

We used also codes implemented in [Math] and [Magma] for obtaining and supporting above results. Such codes are available to math community at Quotient Modular Curves folders in

<https://github.com/FrancescBars>

2 Notation

Let $N > 1$ be an integer. We fix once and for all the following notation.

- (i) We denote by $B(N)$ the group of the Atkin-Lehner involutions of $X_0(N)$. So, the order of $B(N)$ is $2^{\omega(N)}$, where $\omega(N)$ is the number of different primes dividing N .
- (ii) For $N'|N$, with $(N', N/N') = 1$, $B(N')$ denotes the subgroup of $B(N)$ formed by the Atkin-Lehner involutions w_d such that $d|N'$ and $(d, N/N') = 1$. In general, W_N denotes a nontrivial subgroup of $B(N)$.
- (iii) The integers g_N , g_{W_N} and g_N^* are the genus of $X_0(N)$, $X_0(N)/W_N$ and $X_0^*(N) = X_0(N)/B(N)$ respectively.
- (iv) We denote by New_N the set of normalized newforms in $S_2(\Gamma_0(N))$. The sets $\text{New}_N^{W_N}$ and $S_2(N)^{W_N}$ are the subsets of New_N and $S_2(\Gamma_0(N))$ formed by the cusp forms invariant under the action of the group W_N .
- (v) $J_0(N)$ and $J_0(N)^{W_N}$ are the Jacobians $\text{Jac}(X_0(N))$ and $\text{Jac}(X_0(N)/W_N)$ respectively.
- (vi) Let $h \in S_2(\Gamma_0(N))$ be an eigenform of the form $\sum_{d|N/M} c_d f(q^d)$ for some $f \in \text{New}_M$ with $M|N$ and $c_d \in \mathbb{Z}$. Since for every divisor d of N/M there is a morphism B_d from $J_0(M)$ to $J_0(N)$ defined over \mathbb{Q} sending every cusp form $g \in S_2(M)$ to $g(q^d) \in S_2(N)$, the morphism $\sum_{d|N/M} c_d B_d$ provides an abelian variety A_h defined over \mathbb{Q} attached to h and \mathbb{Q} -isogenous to the abelian variety A_f attached by Shimura to f . This abelian variety can be defined as the optimal quotient of $J_0(N)$ such that the pullback of $\Omega_{A_h/\mathbb{Q}}^1$ is the vector space generated by the Galois conjugates of $h(q) dq/q$ with rational q -expansion. This definition determines the \mathbb{Q} -isomorphism class of A_h , although we are only interested in its \mathbb{Q} -isogeny class.
- (vii) Given two abelian varieties A and B defined over the number field K , the notation $A \stackrel{K}{\sim} B$ stands for A and B are isogenous over K .
- (viii) For an integer $m \geq 1$ and $f \in \text{New}_N$, $a_m(f)$ is the m -th Fourier coefficient of f .
- (ix) As usual, ψ denotes the Dedekind psi function. That is, $\psi(N) = N \prod_{p|N} (1 + p^{-1})$, where the product is extended to all primes p dividing N .
- (x) We write $\#(w, X)$ for the number of fixed points of the automorphism w on the curve X .
- (xi) We write $S_d = \begin{pmatrix} 1 & 1/d \\ 0 & 1 \end{pmatrix}$, and by $w_d^{(kd)} \in GL_2(\mathbb{Z})$ a matrix which corresponds to a lifting on level kd of the Atkin-Lehner involution w_d of $X_0(d)$ with $d \geq 2$ and $k \geq 2$ integers, (thinking $w_d \in GL_2(\mathbb{Z})$ with determinant d).

- (xii) To be able to simultaneously describe the Atkin-Lehner involutions of different curves (for example in the tables in Appendix A) we use the notation $\varpi_i = w_{p_i^{e_i}}$ where $N = \prod_{i=1}^s p_i^{e_i}$ and $p_1 < p_2 < \dots < p_s$ are the different primes dividing N .

Recall, if $X_0(N)/W_N$ is bielliptic, there is an involution $u \in \text{Aut}(X_0(N)/W_N)$, called bielliptic involution, which is unique if $g_{W_N} \geq 6$ [HS91]) such that $(X_0(N)/W_N)/u$ is a genus 1 curve defined over a number field K (over \mathbb{Q} if u is unique). Since $X_0(N)/W_N(\mathbb{Q})$ is not empty, the genus 1 curve has a rational point and, therefore, it is an elliptic curve E over K , called a bielliptic quotient of $X_0(N)/W_N$. In particular, such an elliptic curve is a K -isogeny factor for $J_0(N)^{W_N}$.

3 Selecting candidate bielliptic curves $X_0^*(N)$

The starting point of our selection is based on the following result, which follows from [HS91, Proposition 1] by considering the natural projection map $X_0(N)/W_N \rightarrow X_0^*(N)$.

Lemma 3.1. *Let N be an integer, and take $X_0(N)/W_N$ of $g_W \geq 2$. If it is bielliptic, then $X_0^*(N)$ is bielliptic, hyperelliptic or has genus at most one.*

We assume once and for all N not square-free and N not a power of a prime.

Theorem 3.2. *Consider $X_0(N)^*$ with N a non-square free level not a power of a prime, then*

- (i) (González-Lario, [GL98]) $g_N^* = 0$ if and only if $N \in \{12, 18, 20, 24, 28, 36, 44, 45, 50, 54, 56, 60, 92\}$
- (ii) (González-Lario, [GL98]) $g_N^* = 1$ if and only if $N \in \{40, 48, 52, 63, 68, 72, 75, 76, 80, 84, 90, 96, 98, 99, 100, 108, 120, 124, 126, 132, 140, 150, 156, 188, 220\}$
- (iii) (Hasegawa, [Has95]) $g_N^* = 2$ if and only if $N \in \{88, 104, 112, 116, 117, 135, 147, 153, 168, 180, 184, 198, 204, 276, 284, 380\}$
- (iv) (Hasegawa, [Has97]) $g_N^* > 2$ and $X_0^*(N)$ is hyperelliptic if and only if N appears next:

g_N^*	N
3	136; 171; 207; 252; 315;
4	176;
5	279.

- (v) (Bars-González, [BG20]) $X_0^*(N)$ is bielliptic if and only if N appears in the following table

g_N^*	N
2	88; 112; 116; 153; 180; 184; 198; 204; 276; 284; 380;
3	144; 152; 164; 189; 196; 207; 234; 236; 240; 245; 248; 252; 294; 312; 315; 348; 420; 476;
4	148; 160; 172; 200; 224; 225; 228; 242; 260; 264; 275; 280; 300; 306; 342;
5	364; 444; 495;
7	558.

4 Selecting possible bielliptic quotient curves

4.1 General criteria to discard bielliptic curves

Here we collect some general criteria that allow to prove comparatively easily that certain curves are not bielliptic.

Lemma 4.1. *(special form of the Castelnuovo inequality [Acc94, Theorem 3.5]) Let $\phi : X \rightarrow Y$ be a morphism of degree d of curves. If X has a bielliptic involution v , then*

$$g(X) \leq dg(Y) + d + 1$$

or the morphism ϕ factors over X/v .

In particular: A hyperelliptic curve of genus $g \geq 4$ cannot be bielliptic. A trigonal curve of genus strictly bigger than 4 cannot be bielliptic. A curve of genus $g \geq 6$ has at most one bielliptic involution.

Proposition 4.2. *[JKS20, Proposition 3.2] Let X be a bielliptic curve of genus $g \geq 6$ defined over a field K of characteristic 0. Then the bielliptic involution is unique, defined over K and lies in the center of $\text{Aut}(X)$.*

Proof. The uniqueness was just mentioned in Lemma 4.1. Acting on this bielliptic involution with $\text{Aut}(X)$ by conjugation resp. with $\text{Gal}(\overline{K}/K)$, the uniqueness implies the other properties. \square

The following Lemmas 4.3 and 4.4 appeared in [Sch01] and [JKS20] respectively for genus ≥ 6 . Here we present a slightly different proof for general genus.

Lemma 4.3. *Let w be an involution of X with more than 8 fixed points. Then either w is a bielliptic involution or X is not bielliptic*

Proof. Let g and h be the genera of X and X/w . Since w has more than 8 fixed points, by the Hurwitz formula we have $g > 2h + 3$. If v is a bielliptic involution, from Castelnuovo we get the contradiction that g cannot be bigger than $2h + 3$, unless the two involutions factor over a common curve, i.e. are the same. \square

Lemma 4.4. *Let X be a curve of genus g with a bielliptic involution v and let G be a subgroup of $\text{Aut}(X)$ such that the curve $Y = X/G$ has genus $h \geq 2$.*

- (a) *If the map $\phi : X \rightarrow Y$ is ramified, i.e. if $g - 1 > |G|(h - 1)$, and $g \geq 6$, then Y must be hyperelliptic and v induces the hyperelliptic involution on Y .*
- (b) *(unramified covering criterion) If Y is not hyperelliptic, then it must be bielliptic and the map $\phi : X \rightarrow Y$ must be unramified, i.e. $g - 1 = |G|(h - 1)$.*

Proof. (a) Obviously $v \notin G$ because $h > 1$. Since $g \geq 6$, we know from Proposition 4.2 that v is central. So v induces an involution \tilde{v} on Y , and G induces a group \tilde{G} (isomorphic to G) of automorphisms on X/v . If ϕ is ramified, at least one nontrivial element of G has at least one fixed point. So the same holds for \tilde{G} . Hence the Hurwitz formula for the covering from X/v to $(X/v)/\tilde{G} = Y/\tilde{v}$ shows that the latter curve has genus 0.

(b) Since Y is not hyperelliptic we have $h \geq 3$. Assume that ϕ is ramified. Then the Hurwitz formula implies $g \geq 6$, and part (a) leads to a contradiction. \square

If $X_0(N)/W_N$ is bielliptic over \mathbb{Q} and $p \nmid N$, then the morphism reduces modulo p and we have that

$$|X_0(N)/W_N(\mathbb{F}_{p^n})| \leq 2|E(\mathbb{F}_{p^n})| \quad (4.1)$$

for all $n \geq 1$, which can be computed. In particular we reproduce similar results as [BG19, Lemmas 5,7]:

Lemma 4.5. *Assume $X_0(N)/W_N$ is bielliptic over \mathbb{Q} , and $p \nmid N$. Then the following equality holds: $\frac{\psi(N)}{|W_N|} \leq 12 \cdot \frac{2|E(\mathbb{F}_{p^2})| - 1}{p - 1}$. In particular we have, $\frac{\psi(N)}{|W_N|} \leq 12 \cdot \frac{2(p+1)^2 - 1}{p - 1}$.*

Proof. Assume $p \nmid N$. We generalize the argument used by Ogg in [Ogg74]. Indeed, $X_0(N)(\mathbb{F}_{p^2})$ contains $2^{\omega(N)}$ cusps and at least $(p-1)\frac{\psi(N)}{12}$ many supersingular points (cf. [BGGP05, Lemma 3.20 and 3.21]). Since there is a nonconstant morphism defined over \mathbb{Q} from $X_0(N)$ to an elliptic quotient E of $X_0(N)/W$ which has degree $2 \cdot |W_N|$, we get $|X_0(N)(\mathbb{F}_{p^2})| \leq 2 \cdot |W_N| |E(\mathbb{F}_{p^2})|$. \square

Similarly, for optimal quotients with conductor $M = N$ for elliptic quotient (related with the strong Weil parametrization) it is easy to derive the following lemma.

Lemma 4.6. *Let E' be the optimal elliptic curve in the \mathbb{Q} -isogeny class of the bielliptic quotient E of $X_0(N)/W_N$ with conductor $M = N$. Then the degree D of the modular parametrization $\pi_N: X_0(N) \rightarrow E'$ divides $2 \cdot |W_N|$. Note that the degree D can be found in [Cre17, Table 5].*

Theorem 4.7. *Let X be a curve over a field of characteristic 0. Then*

- (a) *The stabilizer in $\text{Aut}(X)$ of any point of X is cyclic.*
- (b) *Distinct involutions in $\text{Aut}(X)$ have disjoint fixed points.*

Proof. (a) is proved in [FK80, Corollary III.7.7], and (b) follows immediately from (a). \square

Proposition 4.8. *Let X be a curve of genus g at least 6. Assume that $\text{Aut}(X)$ has a subgroup H of order 2^t such that 2^t does not divide $2(g-1)$. Then either the bielliptic involution of X is contained in H or X is not bielliptic.*

Proof. Let v (outside H) be the bielliptic involution. Since v commutes with the elements of H , H acts on the $2(g-1)$ fixed points of v . If $|H|$ does not divide $2(g-1)$, there must be an orbit whose length is not $|H|$. So there must be an involution in H that fixes a point in this orbit. But v also fixes this point, contradicting Theorem 4.7. \square

The curves $X_0(N)/W_N$ always have such a subgroup H (isomorphic to $B(N)/W_N$), and we can discard directly even genus quotient curves when $B(N)/W_N$ is of order at least 4.

4.2 Criteria for genus 5 bielliptic curves

For curves of genus $g \leq 5$ it can be difficult to decide whether they are bielliptic, because there might be several bielliptic involutions, but none of them is guaranteed to be defined over \mathbb{Q} . The following results will be helpful for several of the curves we will encounter.

Lemma 4.9. [Acc94, p.50] *A genus 5 curve that is a degree two covering of a hyperelliptic genus 3 curve is either hyperelliptic or bielliptic.*

Lemma 4.10. [KMV11, Lemma 2.3] *Let X a bielliptic curve of genus 5. Then it has 1, 2, 3 or 5 bielliptic involutions, and these involutions commute and generate a group of exponent 2 and order 2, 4, 8 and 16 respectively. Moreover, the product of any two bielliptic involutions is an involution whose quotient curve is of genus 3 and hyperelliptic.*

Lemma 4.11. *Let X be a bielliptic genus 5 curve with an involution u , such that X/u has genus 2.*

- (a) *If X has an odd number of bielliptic involutions, then there is a unique one among them, call it v , that commutes with u . If moreover u is defined over \mathbb{Q} , then v is also defined over \mathbb{Q} .*
- (b) *If X has exactly 2 bielliptic involutions, neither one of them will commute with u .*

Proof. The involution u acts on the bielliptic involutions by conjugation, so they come in pairs of conjugated ones plus the ones which commute with u . This already shows that there is at least one such v (resp. two or none if X has exactly 2 bielliptic involutions).

Now we show that there can be at most one such v . Since v commutes with u and the map $X \rightarrow X/u$ is ramified, we can argue as in the proof of Theorem 4.4 (a) with the role of G being played by $\langle u \rangle$. Again we obtain that v must induce the hyperelliptic involution on X/u . So the only possibility for a bielliptic involution other than v that commutes with u would be uv . But they cannot both be bielliptic, because then $u = (uv)v$ would have quotient genus 3 by Lemma 4.10.

If u is defined over \mathbb{Q} and commutes with v , then from the action of the absolute Galois group of \mathbb{Q} it also commutes with all Galois conjugates of v , which by the uniqueness must be equal to v . \square

Lemma 4.12. *Let X be a curve of genus 5 with an involution w such that X/w is of genus 3 and non-hyperelliptic. If X is bielliptic, then X has exactly one or exactly 3 bielliptic involutions.*

Proof. First we treat the case where X has 5 bielliptic involutions. Then by [KMV11, Lemma 2.3 and Remark 3.2] the bielliptic involutions generate a group H of order 16 in which all other elements are involutions which have a quotient that is hyperelliptic of genus 3. In particular, w is not in H . Pick a bielliptic involution v_1 that does not commute with w and define v_2 as its conjugate under w . If w commutes with all bielliptic involutions, just pick any two v_1 and v_2 . In either case w commutes with the involution v_1v_2 , which has quotient genus 3. By a classical result (see for example [Acc94, Lemma 5.10]) a genus 3 curve that is a degree 2 cover of a genus 2 curve must be hyperelliptic. So since X/w is not hyperelliptic, $X/\langle v_1v_2, w \rangle$ cannot have genus 0 or 2, and hence has genus 1. From

$$g(X/w) + g(X/v_1v_2) + g(X/v_1v_2w) = g(X) + 2g(X/\langle v_1v_2, w \rangle) = 7$$

(see for example [Acc94, p.49]) we see that X/v_1v_2w also has genus 1. Therefore v_1v_2w is in H . With v_1v_2 in H we get the contradiction $w \in H$. So we have proved that X has at most 3 bielliptic involutions.

Now assume that X has exactly 2 bielliptic involutions. Call them v_1 and v_2 . By the same argument as before we get a third bielliptic involution. \square

Corollary 4.13. *Let X be a curve of genus 5 over \mathbb{Q} that has involutions u and w as in the two lemmas. with u defined over \mathbb{Q} . If X is bielliptic, then it has one or three bielliptic involutions and at least one bielliptic involution is defined over \mathbb{Q} . More precisely, if it has 3 bielliptic involutions, then there is a unique one that commutes with u , and that one is defined over \mathbb{Q} . (The other two might be defined over \mathbb{Q} or not.)*

4.3 Formulas for the genus of a quotient curve by certain involutions

The most natural way to prove that a curve is bielliptic is to exhibit a bielliptic involution. We will do this for practically all the bielliptic curves $X_0(N)/W_N$. The most obvious candidates are the Atkin-Lehner involutions outside W_N . If N is divisible by 4 or 9, then $X_0(N)$ has additional involutions, which under certain conditions induce involutions on $X_0(N)/W_N$. See below for details.

When v is an involution of $X_0(N)$ that induces an involution \tilde{v} on $X_0(N)/W_N$, it is usually quicker not to bother about the fixed points of \tilde{v} on $X_0(N)/W_N$ but to determine the genus of $(X_0(N)/W_N)/v$ from the fixed points of the elements of $G = \langle W_N, v \rangle$ on $X_0(N)$ by applying the Hurwitz formula to the covering $X_0(N) \rightarrow X_0(N)/G$.

In the situations we encounter the group G is of the form $G \cong \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. Then by Theorem 4.7 the stabilizer of a point is trivial or $\mathbb{Z}/2$. Moreover, all involutions in G have disjoint fixed points, and the Hurwitz formula takes the following simple shape.

Lemma 4.14. *Let G be a subgroup of $\text{Aut}(X_0(N))$ in which all the nontrivial elements are involutions. Then the fixed points of these involutions are disjoint and the genus of $X_0(N)/G$ is obtained by the formula*

$$|G|(2g(X_0(N)/G) - 2) + \sum_{w \in G} \#(w, X_0(N)) = 2g(X_0(N)) - 2.$$

Formulas for the number of fixed points of the Atkin-Lehner involutions on $X_0(N)$ are in [Ogg74]. For the other involutions we will review and expand the known results in this section.

Recall $S_k = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix}$. The following result is well-known (see [FH99] or [Bar99]).

Proposition 4.15. *Let $N = 2^\alpha M$ with $\alpha \geq 2$ and M odd.*

- (a) *Then S_2 is an involution of $X_0(N)$, defined over \mathbb{Q} , and commutes with all Atkin-Lehner involutions w_r for which r is odd. Hence, $V_2 = S_2 w_{2^\alpha} S_2$ also is an involution of $X_0(N)$, defined over \mathbb{Q} , and commutes with all w_r for which $r \nmid M$.*
- (b) *If $\alpha \geq 3$, then V_2 also commutes with w_{2^α} . So $V_2 w_{2^\alpha}$ is an involution, and consequently $S_2 w_{2^\alpha}$ has order 4. In fact, $\langle S_2, w_{2^\alpha} \rangle \cong D_4$.*
- (c) *If $\alpha = 2$, then $\langle S_2, w_4 \rangle$ is non-abelian of order 6 with $V_2 = S_2 w_4 S_2 = w_4 S_2 w_4$ being the third involution and $S_2 w_4$ and $w_4 S_2$ having order 3.*

Lemma 4.16. [Bar99, Proposition 3.5] *If $N = 2^\alpha M$ with $\alpha \geq 2$ and M odd, then*

$$X_0(N)/w_{2^\alpha} S_2 w_{2^\alpha} = X_0(N/2).$$

Proof. An easy calculation shows that $w_{2^\alpha} S_2 w_{2^\alpha}$ lies in $\Gamma_0(N/2)$ but not in $\Gamma_0(N)$. □

Lemma 4.17. *Let u and v be two commuting involutions on a curve X . Then uv is also an involution and*

$$\#(uv, X) = 2\#(u, X/v) - \#(u, X).$$

Proof. By Theorem 4.7, the fixed points of the three involutions u , v and uv are disjoint, and each fixed point has ramification index 2 in the degree 4 covering $X \rightarrow X/\langle u, v \rangle$.

This implies that every fixed point of u or uv must be ramified in $X/v \rightarrow X/\langle u, v \rangle$. Conversely, if a point of X/v is ramified in $X/v \rightarrow X/\langle u, v \rangle$, then the two points of X lying above it must have ramification group $\langle u \rangle$ or $\langle uv \rangle$. So, all in all $\#(u, X) + \#(uv, X) = 2\#(u, X/v)$. □

Lemma 4.18. *Let $N = 2^\alpha M$ with $\alpha \geq 2$ and M odd. Also let $r \parallel M$.*

- (a) *[Bar99, Proposition 3.9] $\#(V_2, X_0(N)) = \#(w_{2^\alpha}, X_0(N))$ and $\#(V_2 w_r, X_0(N)) = \#(w_{2^\alpha} w_r, X_0(N))$.*
- (b) *$\#(S_2, X_0(N)) = \#(w_{2^\alpha} S_2 w_{2^\alpha}, X_0(N)) = (2g(X_0(N)) - 2) - 2(2g(X_0(N/2)) - 2)$.*
- (c) *[Bar99, Proposition 3.6] $\#(S_2 w_r, X_0(N)) = \#(w_{2^\alpha} S_2 w_{2^\alpha} w_r, X_0(N)) = 2\#(w_r, X_0(N/2)) - \#(w_r, X_0(N))$.*
- (d) *[Bar99, Proposition 3.10] If $\alpha \geq 3$, then*

$$\#(V_2 w_{2^\alpha}, X_0(N)) = 2\#(S_2, X_0(N/2)) - \#(S_2, X_0(N)) \text{ and}$$

$$\#(V_2 w_{2^\alpha} w_r, X_0(N)) = 2\#(S_2 w_r, X_0(N/2)) - \#(S_2 w_r, X_0(N)).$$

Proof. Note that conjugate involutions have the same number of fixed points. This proves (a) and together with Lemma 4.16 also (b). The other formulas are special cases of Lemma 4.17, also using Lemma 4.16. \square

Lemma 4.19. *Let $9 \parallel N$ and $S_3 = \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$.*

- (a) *S_3 normalizes $\Gamma_0(N)$ and induces an automorphism of $X_0(N)$ of order 3 defined over $\mathbb{Q}(\sqrt{-3})$. Its Galois conjugate is S_3^2 . Moreover, S_3 commutes with the Atkin-Lehner involutions w_r with $r \equiv 1 \pmod{3}$, whereas for $r \equiv 2 \pmod{3}$ we have $w_r S_3 = S_3^2 w_r$ and $w_9 S_3$ has order 3.*
- (b) *$V_3 = S_3 w_9 S_3^2$ is an involution of $X_0(N)$. With respect to Atkin-Lehner involutions we have $w_r V_3 = \begin{cases} V_3 w_r & \text{if } r \equiv 1 \pmod{3} \text{ or } r = 9 \text{ and} \\ V_3 w_9 w_r & \text{if } r \equiv 2 \pmod{3} \end{cases}$*
Moreover, if $r \equiv 2 \pmod{3}$ then $\langle V_3, w_r \rangle \cong D_4$ and $V_3 w_r$ has order 4 with $(V_3 w_r)^2 = w_9$.
- (c) *V_3 as an involution of $X_0(N)$ is defined over $\mathbb{Q}(\sqrt{-3})$. Its $\text{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$ -conjugate is $V_3 w_9$. In particular, V_3 and $V_3 w_9$ have the same number of fixed points on $X_0(N)$.*
- (d) *More generally we have*

$$\#(V_3 w_9, X_0(N)) = \#(V_3, X_0(N)) = \#(w_9, X_0(N))$$

and for $r \equiv 1 \pmod{3}$ also

$$\#(V_3 w_9 w_r, X_0(N)) = \#(V_3 w_r, X_0(N)) = \#(w_9 w_r, X_0(N)).$$

- (e) *V_3 as an involution of $X_0(N)/W$ is defined over \mathbb{Q} if and only if $w_9 \in W$.*

Proof. Most of this (and some more) is already in [Bar99], [FH99] or [Bar08]. So we only prove (c).

$V_3 = S_3 w_9 S_3^2$ is defined over $\mathbb{Q}(\sqrt{-3})$ because S_3 is. Now let σ be the nontrivial Galois automorphism of $\mathbb{Q}(\sqrt{-3})$. Then $\sigma(V_3) = \sigma(S_3 w_9 S_3^2) = S_3^2 w_9 S_3$. So $V_3 \sigma(V_3) = w_9 (w_9 S_3)^3 = w_9$ by part (a). \square

Remark 4.20. *From Lemma 4.19(e), we obtain the result in [BG20] that V_3 as an involution of $X_0^*(N)$ is always defined over \mathbb{Q} .*

4.4 Useful isomorphisms on non-square free quotient curves

For later use we refine [FH99, Proposition 4].

Proposition 4.21. *Suppose $4||N$ and write $N = 4M$. Let W' be a subgroup of $B(N)$ generated by $w_4, w_{m_1}, \dots, w_{m_s}$ with $m_i||M$. Then we have*

$$X_0(N)/W' \cong X_0(N)/\langle S_2 w_4 S_2, w_{m_1}, \dots, w_{m_s} \rangle = X_0(N)/\langle w_4 S_2 w_4, w_{m_1}, \dots, w_{m_s} \rangle = X_0(2M)/\langle w_{m_1}, \dots, w_{m_s} \rangle.$$

Hence if $A \in GL_2(\mathbb{R})$ is a bielliptic involution of $X_0(2M)/\langle w_{m_1}, \dots, w_{m_s} \rangle$, then $S_2 A S_2$ normalizes $\langle \Gamma_0(N), W' \rangle$ and induces a bielliptic involution on $X_0(N)/W'$.

Proof. The isomorphism comes from conjugating with S_2 , the first equality from part (c) of Proposition 4.15 and the final equality from Lemma 4.16. \square

Proposition 4.22. [FH99, Proposition 5] *Assume $9||N$. Let W' be a subgroup of $B(N)$ generated by w_{n_1}, \dots, w_{n_t} ($n_i||N$) and let $W'' = \langle \{w_{n_i} w_9^{e(n_i)}\}_{i \in \{1, \dots, t\}} \rangle$ where $e(m) = 0$ if $m \equiv 1 \pmod{3}$ or if $9||m$ and $m/9 \equiv 1 \pmod{3}$, and $e(m) = 1$ otherwise. Then V_3 induces an isomorphism*

$$X_0(N)/W' \cong X_0(N)/W''.$$

5 Jacobian decomposition, field of bielliptic involutions, Petri theorem.

5.1 On the field where bielliptic involutions may be defined

Let $X_0(N)/W_N$ be a quotient curve. We want to control if the automorphism, or more concretely if a candidate to bielliptic involution is defined over \mathbb{Q} or a number field. In order to control the number field K (when $g_{W_N} \leq 5$), we have the following results in [BG20]:

Proposition 5.1. *Let A be a modular abelian variety defined over \mathbb{Q} such that $A \stackrel{\mathbb{Q}}{\sim} \prod_{i=1}^m A_{f_i}^{n_i}$ for some $f_i \in \text{New}_{N_i}$, where A_{f_i} are pairwise non-isogenous over \mathbb{Q} . All endomorphisms of A are defined over \mathbb{Q} if, and only if, for every nontrivial quadratic Dirichlet character χ , the newform $f_i \otimes \chi$ is different from any Galois conjugates of f_j for all i and j .*

Remark 5.2. *If χ is the quadratic Dirichlet character attached to the quadratic number field $K = \mathbb{Q}(\sqrt{D})$, then there is an isogeny between the abelian varieties A_f and $A_{f \otimes \chi}$ defined over K .*

Also, the following result specific for modular forms non-corresponding to elliptic curves [Pyl04] clarifies the possible elliptic quotient that could appear:

Proposition 5.3. *When $\dim A_f > 1$ and f does not have complex multiplication (CM), i.e. $f \neq f \otimes \chi$ for all quadratic Dirichlet characters, a necessary condition for A_f to have an elliptic quotient over $\overline{\mathbb{Q}}$ is $a_p(f)^2 \in \mathbb{Z}$ (the p -th Fourier coefficient of the modular form f) for all primes p .*

5.2 On the Jacobian decomposition of quotient modular curves

We recall that the \mathbb{Q} -decomposition for $J_0(N)$ has the form

$$J_0(N) \stackrel{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f \in \text{New}_M / G_{\mathbb{Q}}} A_f^{n_f},$$

where n_f is the number of positive divisors of N/M and $G_{\mathbb{Q}}$ denotes the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Each newform $f \in \text{New}_M$ provides an n_f -dimensional vector subspace of $S_2(N)$ generated by $\{f(q^d) : 1 \leq d|N/M\}$.

To determine the \mathbb{Q} -decomposition for $J_0(N)^{W_N}$

$$J_0(N)^{W_N} \stackrel{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f \in \text{New}_M / G_{\mathbb{Q}}} A_f^{m_f},$$

we need to control which A_f appears in this decomposition and the precise exponent $0 \leq m_f \leq n_f$, and next results allow us, once fixed (N, W_N) , to determine a basis of $S_2(N)^{W_N}$ and, in particular, the splitting of $J_0(N)^{W_N}$ (see [BG20, Lemma 2.1, Prop.2.2]).

Lemma 5.4. *Let M and N be positive integers such that $M|N$. Let M_1 be a positive divisor of M such that $\gcd(M, M/M_1) = 1$ and let d be a positive divisor of N/M such that $\gcd(M_1 d, N/(M_1 d)) = 1$. If $f \in S_2(\Gamma_0(M))$ is an eigenvector of the Atkin-Lehner involution w_{M_1} with eigenvalue $\varepsilon(f)$ and $\varepsilon \in \{-1, 1\}$, then $f(q) + \varepsilon d f(q^d) \in S_2(\Gamma_0(N))$ is an eigenvector of the Atkin-Lehner involution $w_{M_1 d}$ with eigenvalue $\varepsilon(f) \cdot \varepsilon$.*

Proposition 5.5. *Assume that $N = p^k \cdot M$, where $k \geq 1$, p is a prime and M is an integer coprime to p . For $0 \leq i < k$, let $f \in S_2(\Gamma_0(p^i \cdot M))^W$ be such that $w_{p^i}(f) = \varepsilon \cdot f$ with $W \leq B(M)$ (clearly $\varepsilon = 1$ when $i = 0$). Let \mathcal{S} be the vector subspace of $S_2(\Gamma_0(p^k \cdot M))^W$ generated by the $k - i + 1$ linearly independent, \mathbb{Q} -isogenous to f , eigenforms $\{f, B_p(f), \dots, B_p^{k-i}(f)\}$. Then,*

(i) *The following normalized eigenforms*

$$g_0 = (1 + pB_p)^{k-i} f, \dots, g_j = (1 + pB_p)^{k-i-j} (1 - pB_p)^j f, \dots, g_{k-i} = (1 - pB_p)^{k-i} f,$$

are a basis of \mathcal{S} (recall B_p is the morphism sending a modular form $g(q)$ to $g(q^p)$).

(ii) *Every g_j is an eigenvector of w_{p^k} with eigenvalue $(-1)^j \varepsilon$.*

Consider the \mathbb{Q} -decomposition for $J_0(N)^{W_N}$

$$J_0(N)^{W_N} \stackrel{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f \in \text{New}_M / G_{\mathbb{Q}}} A_f^{m_f}.$$

Now, for $f \in \text{New}_M$ if $m_f > 0$ then f is necessarily fixed by the Atkin-Lehner involutions $w_d \in W_N$, with $d|M$, and f provides m_f -eigenforms $g_i \in S_2(N)^{W_N}$ lying in the vector space generated by $\{f(q^d) : 1 \leq d|N/M\}$. The integer m_f is determined by using Lemma 5.4 and Proposition 5.5. Read readme.md file in

<https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves>

Jacobian decomposition allows us to compute $|X_0(N)/W_N(\mathbb{F}_{p^n})|$ for all $p \nmid N$ thanks to the Eichler-Shimura congruence, see a MAGMA function code `FpnpointsQuotientCurve` in `funcions.m` and examples in the `Readme.md` in the github folder

<https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves>

5.3 The application of a result of Petri

If $X_0(N)/W_N$ is hyperelliptic, we know an equation (see [FH99], [Has95]) and MAGMA computes the automorphism group over \mathbb{Q} . In this case, if $X_0(N)/W_N$ has a non-hyperelliptic involution over \mathbb{Q} , we can compute the genus of the quotient curve by using [Ogg74, Proposition 1] and determine if $X_0(N)/W_N$ is bielliptic or not over \mathbb{Q} . For finite extensions of K we need to deal with the decomposition of the Jacobian and study the endomorphism algebra following Proposition 5.1 and to compute it in such number field, which is a quadratic field. MAGMA computes such automorphism group over quadratic fields. Thus for hyperelliptic quotient modular curves we can decide if they are bielliptic or not.

Consider a non-hyperelliptic curve X of genus $g \geq 3$ defined over a subfield K of the complex field \mathbb{C} . For a fixed basis $\omega_1, \dots, \omega_g$ of $\Omega_{X/K}^1$ and an integer $i \geq 2$, we denote by \mathcal{L}_i the K -vector space formed by the homogenous polynomials $Q \in K[x_1, \dots, x_g]$ of degree i such that $Q(\omega_1, \dots, \omega_g) = 0$.

By using a theorem of Petri, [BG19, Lemma 13] characterizes the existence of a bielliptic involution of $X_0^*(N)$ with N square-free for non-hyperelliptic curves. Later, [BG20, Proposition 2.6] generalizes this result to any non-hyperelliptic curve of genus > 2 .

Proposition 5.6. *With the above notation, assume that $\text{Jac}(X) \stackrel{K}{\sim} E^m \times A$, where E is an elliptic curve and A an abelian variety such that does not have E as a quotient defined over K . Denote by $I_{g-m} \in M_{g-m}(\mathbb{Q})$ the identity matrix. Take the basis $\{\omega_i\}$ such that $\omega_1, \dots, \omega_m$ and $\omega_{m+1}, \dots, \omega_g$ are bases of the pullback of $\Omega_{E^m/K}^1$ and $\Omega_{A/\mathbb{Q}}^1$ respectively. Then, E is K -isogenous to the Jacobian of a bielliptic quotient of X over K if, and only if, there exists a matrix $\mathcal{A} \in \text{GL}_m(K)$ that satisfies*

$$Q((-x_1, x_2, \dots, x_g) \cdot \mathcal{B}) \in \mathcal{L}'_i \text{ for all } Q \in \mathcal{L}_i \text{ and for all } i \geq 2, \quad (5.1)$$

where \mathcal{B} is the matrix $\left(\begin{array}{c|c} \mathcal{A} & 0 \\ \hline 0 & I_{g-m} \end{array} \right) \in \text{GL}_g(K)$ and $\mathcal{L}'_i = \{Q((x_1, x_2, \dots, x_g) \cdot \mathcal{B}) : Q \in \mathcal{L}_i\}$.

Remark 5.7. *The K -vector space \mathcal{L}'_i is the set of homogenous polynomials in $K[x_1, \dots, x_g]$ of degree i such that $Q(\omega'_1, \dots, \omega'_m, \omega_{m+1}, \dots, \omega_g) = 0$, where $(\omega'_1, \dots, \omega'_m) = \mathcal{A}^{-1}(\omega_1, \dots, \omega_m)$.*

Remark 5.8. *We recall that if $g = 3$, then $\dim \mathcal{L}_4 = 1$ and the condition (5.1) can be restricted to $i = 4$. When $g > 3$, $\dim \mathcal{L}_2 = (g-3)(g-1)/2$. In this case, it suffices to check (5.1) only for $i = 2, 3$ and, in the particular case that X is neither a smooth quintic plane curve ($g = 6$) nor a trigonal curve, we can restrict the condition to $i = 2$.*

As in [BG19], for $j \leq g$ we introduce the K -vector space

$$\mathcal{L}_{2,j} = \{Q \in \mathcal{L}_2 : Q(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n) \in \mathcal{L}_2\}.$$

By using that the polynomials in \mathcal{L}_2 are irreducible, in [BG19] it is proved that

$$\mathcal{L}_{2,j} = \{Q \in \mathcal{L}_2 : Q(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n) = Q(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n)\}.$$

and a modular form corresponding to a dimension one abelian variety of genus > 3 with associated differential ω_i , is bielliptic if and only if $\dim \mathcal{L}_2 = \dim \mathcal{L}_{2,i}$ (here we are assuming not a smooth plane quintic curve, nor a trigonal curve).

A similar result is obtained when $g = 3$ and we replace $\mathcal{L}_{2,j}$ with $\mathcal{L}_{4,j}$.

Remark 5.9. We have $J_0(N)^{W_N} \cong \prod_{i=1}^s A_{f_i}^{n_i}$, for some $f_i \in \text{New}_{M_i}$ with $M_i | N$ and the abelian varieties A_{f_i} are pairwise non-isogenous over \mathbb{Q} . Any f_i determines n_i normalized eigenforms g_j in $S_2(N)^{W_N}$ such that $J_0(N)^{W_N} \cong \prod_{j=1}^r A_{g_j}$, where $r = \sum_{i=1}^s n_i$ and g_1, \dots, g_r are all of these eigenforms. The basis of the Galois conjugates of the newforms f_i together the exponents n_i allow us to compute $|X_0/W_N(\mathbb{F}_{p^n})|$ for all primes $p \nmid N$, thanks to the Eichler-Shimura congruence. The basis of the regular differentials formed by all Galois conjugates of $g_j(q) dq/q$ allows us to compute equations for X_0/W_N by use of a theorem of Petri in the non-hyperelliptic case.

Thus for any Atkin-Lehner quotient of a modular curve we can carry out all these computations and decide if is bielliptic or not. For examples, see the proofs of Lemmas 7.5, 7.6, 8.7 and 8.16 and the github folder

<https://github.com/FrancescBars/Mathematica-files-on-Quotient-Modular-Curves>

for computations (and their source files) done by MATHEMATICA applying the above results from Petri's theorem.

6 Bielliptic quotients with $g_N^* = 0$

From the tables in the Appendix we see that for these levels most quotients have genus 1 or even 0. So we only have to examine the following 13 curves.

- $X_0(44)/w_4 \cong X_0(22)$ by Proposition 4.21. Since $X_0(22)$ has the two bielliptic involutions w_2 and w_{22} by [Bar99], from Proposition 4.21 we also see that $S_2 w_2^{(22)} S_2$ and $S_2 w_{22}^{(22)} S_2$ are bielliptic involutions of $X_0(44)/w_4$.
- $X_0(54)/w_2$ of genus 2, has Jacobian decomposition $E27a \times E54a$, so all automorphisms are defined over \mathbb{Q} . By MAGMA the automorphism group is $\mathbb{Z}/2\mathbb{Z}$, so the only involution is the hyperelliptic one and the curve is not bielliptic.
- $X_0(56)/w_8$ is bielliptic. Namely, by Lemma 4.18 the involution $V_2 w_7$ has 8 fixed points on $X_0(56)$, so it has at least 4 fixed points on $X_0(56)/w_8$. Thus it is a bielliptic involution or the hyperelliptic involution. But the hyperelliptic involution obviously is w_7 .
- $X_0(92)/w_4$ of genus 5 and $X_0(92)/w_{92}$ of genus 4 are both hyperelliptic, so by the Castelnuovo inequality they cannot be bielliptic.
- $X_0(60)/w_{12}$, by the same argument, is hyperelliptic of genus 4 and thus not bielliptic.
- $X_0(60)/w_4 \cong X_0(30)$ and $X_0(60)/\langle w_4, w_3 \rangle \cong X_0(30)/w_3$ again by Proposition 4.21. As before, the bielliptic involutions (compare [Bar99] and [BGK20]) conjugate back to the bielliptic involutions w_5 , $S_2 w_6^{(30)} S_2$ and $S_2 w_{30}^{(30)} S_2$ of $X_0(60)/w_4$ resp. $S_2 w_2^{(30)} S_2$ and $S_2 w_{10}^{(30)} S_2$ of $X_0(60)/\langle w_4, w_3 \rangle$.
- $X_0(60)/w_4$ and $X_0(60)/w_{60}$ each have a bielliptic Atkin-Lehner involution because they map of degree 2 to the elliptic curve $X_0(60)/\langle w_4, w_5 \rangle$ resp. $X_0(60)/\langle w_3, w_{20} \rangle$. (See also the previous item.) Noting that $X_0(60)/\langle w_3, w_5 \rangle$ also has genus 1 we see that each of $X_0(60)/w_3$, $X_0(60)/w_5$ and $X_0(60)/w_{20}$ maps to two of these three elliptic curves and hence has two bielliptic AL-involutions.
- $X_0(60)/\langle w_5, w_{12} \rangle$ has genus 2. Its Jacobian decomposition over \mathbb{Q} is $E20a \times E30a$, therefore all automorphisms are defined over \mathbb{Q} . By MAGMA its automorphism group over \mathbb{Q} is $\mathbb{Z}/2\mathbb{Z}$, so there is no bielliptic involution.

7 Bielliptic quotient curves when $X_0^*(N)$ has genus 1

Part (i) of Theorem 1.1 is almost self-evident, as these are exactly the values of N and W_N for which $X_0^*(N)$ has genus 1 and there is a degree 2 map from $X_0(N)/W_N$ to it. From the tables in the Appendix we see that with the exception of the elliptic curves $X_0^+(40)$, $X_0^+(48)$, $X_0^+(63)$ and $X_0^+(75)$ these curves $X_0(N)/W_N$ do have genus at least 2.

In the remainder of this section we finish the case $g_N^* = 1$ by deciding the curves $X_0(N)/w_d$ where $X_0^*(N)$ is elliptic and N has 3 different prime divisors.

Lemma 7.1. *The following 48 quotient curves $X_0(N)/w_d$ are not bielliptic*

N	(N, w_d)
84	$(84, w_7); (84, w_{28}); (84, w_{21})$
90	$(90, w_2); (90, w_{10}); (90, w_{18})$
120	$(120, w_5); (120, w_8); (120, w_3); (120, w_{40})$
126	$(126, w_2); (126, w_7); (126, w_{18})$
132	$(132, w_d), d \parallel 132$
140	$(140, w_d), d \parallel 140$
150	$(150, w_d), d \parallel 150$
156	$(156, w_d), d \parallel 156$
220	$(220, w_d), d \parallel 220$

Proof. From the tables in the Appendix we see that for each such curve $X_0(N)/w_d$ there exists a suitable Atkin-Lehner involution w_m such that $g(X_0(N)/w_d) \geq 2g(X_0(N)/\langle w_d, w_m \rangle)$ and $X_0(N)/\langle w_d, w_m \rangle$ is not subhyperelliptic. So $X_0(N)/w_d$ cannot be bielliptic by Lemma 4.4. \square

Remark 7.2. *There are several other methods by which one could prove a large subset of the 48 curves in the previous lemma to be not bielliptic.*

Note that all these curves have a map of degree 4 to the genus 1 curve $X_0^(N)$. So if $g(X_0(N)/w_d) > 10$ a hypothetical bielliptic map would by the Castelnuovo inequality have to factor over a common quotient curve with this degree 4 map, i.e. the bielliptic involution would have to be an Atkin-Lehner involution, which by the tables does not exist.*

Alternatively, again because $\text{Aut}(X_0(N)/w_d)$ has a subgroup of order 4, Proposition 4.8 shows that those with even genus $g \geq 6$ are not bielliptic.

Finally, one could also use Lemma 4.5 with $p = 3$ to exclude all curves with $N = 220$.

Lemma 7.3. *The curve $X_0(126)/w_9$ is bielliptic with bielliptic involution V_3w_7 . The genus 5 curve $X_0(126)/w_{63}$ has at least two bielliptic involutions, namely V_3 and V_3w_9 , both defined over $\mathbb{Q}(\sqrt{-3})$. The two genus 7 curves $X_0(126)/w_{14}$ and $X_0^+(126)$ are isomorphic and not bielliptic.*

Proof. From Lemma 4.19 we see that V_3 has the same number of fixed points as w_9 , namely none. And V_3w_9 , being a Galois conjugate of V_3 , also has the same number of fixed points. See Lemma 4.19.

By exactly the same arguments each of the involutions w_{63} , V_3w_7 and V_3w_{63} has 16 fixed points. Now with Lemma 4.14 one easily checks that the modular curves $X_0(126)/\langle w_9, V_3w_7 \rangle$, $X_0(126)/\langle w_{63}, V_3 \rangle$ and $X_0(126)/\langle w_{63}, V_3w_7 \rangle$ have genus 1.

By Proposition 4.22 the curves $X_0(126)/w_{14}$ and $X_0^+(126)$ are isomorphic, and by [Jeo18] the latter one is not bielliptic. \square

Lemma 7.4. *The involution V_2w_{40} induces on each of the curves $X_0(120)/w_{15}$, $X_0(120)/w_{24}$ and $X_0^+(120)$ a bielliptic involution. Moreover, $X_0(120)/w_{15}$ has exactly two more bielliptic involutions, namely S_2 and $w_8S_2w_8$.*

Proof. From Lemma 4.18 we obtain the following table:

v	$\#(v, X_0(120))$	v	$\#(v, X_0(120))$
id	-	V_2	0
w_8	0	V_2w_8	0
w_3	0	V_2w_3	8
w_5	0	V_2w_5	0
w_{24}	8	V_2w_{24}	0
w_{40}	0	V_2w_{40}	16
w_{15}	16	V_2w_{15}	8
w_{120}	8	V_2w_{120}	0

Now the genus of $X_0(120)/\langle w_d, V_2w_{40} \rangle$ can be easily calculated using Lemma 4.14.

Since $X_0(120)/w_{15}$ maps with degree 2 to the elliptic curve $X_0(60)/w_{15}$ isomorphic to $X_0(120)/\langle w_8S_2w_8, w_{15} \rangle$ (compare Lemma 4.16), $w_8S_2w_8$ is also a bielliptic involution of $X_0(120)/w_{15}$. Furthermore, S_2 is conjugate to $w_8S_2w_8$ in the automorphism group of $X_0(120)/w_{15}$, and hence also a bielliptic involution. Finally, since $S_2(w_8S_2w_8)V_2w_{40} = w_5$ and $X_0(120)/\langle w_{15}, w_5 \rangle$ is non-hyperelliptic of genus 3, by [KMV11, Remark 3.2] there are no further bielliptic involutions of $X_0(120)/w_{15}$. □

Lemma 7.5. V_3w_{10} is a bielliptic involution of $X_0(90)/w_9$. And $X_0^+(90)$ actually has (at least) two, namely V_3 and V_3w_9 . The curves $X_0(90)/w_5$ and $X_0(90)/w_{45}$ on the other hand are not bielliptic.

Proof. As before we get from Lemma 4.19 that each of w_9 , V_3 and V_3w_9 has 4 fixed points, and each of w_{90} , V_3w_{10} , V_3w_{90} has 8. Then we use Lemma 4.14 to check the genus of the quotient curves in question.

We have

$$\begin{aligned} J_0(90)^{\langle w_5 \rangle} &\sim_{\mathbb{Q}} (E30a)^2 \times (E45a)^2 \times E90b \\ J_0(90)^{\langle w_{45} \rangle} &\sim_{\mathbb{Q}} (E15a)^2 \times E30a \times E90b \times E90c \end{aligned}$$

For $J_0(90)^{\langle w_{45} \rangle}$ there is a quadratic twist $E30a \sim_{\mathbb{Q}(\sqrt{-3})} E90c$. Over the rationals we have for $p = 11$ and $E = E30a$ or $E90c$ does not satisfy (4.1) because $|\#X_0(90)/w_{45}(\mathbb{F}_{11}) - 2*\#E(\mathbb{F}_{11})| = 2$, and $\dim \mathcal{L}_{2,E90b} < \dim \mathcal{L}_2$. So remains if $E15a$ is or not a bielliptic quotient over \mathbb{Q} , but is not possible because there does not exist any matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \text{GL}_2(\mathbb{Q})$ satisfying the condition:

$$Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, x_4, x_5) = Q_2(-a_1x_1 + a_2x_2, -b_1x_1 + b_2x_2, x_3, x_4, x_5) \quad (7.1)$$

for all $Q_2 \in \mathcal{L}_2$.

Now for $\mathbb{Q}(\sqrt{-3})$ (by the quadratic twist) we have the Jacobian decomposition

$$J_0(90)^{\langle w_{45} \rangle} \sim_{\mathbb{Q}(\sqrt{-3})} (E15a)^2 \times (E30a)^2 \times E90b$$

but there does not exist any matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \text{GL}_2(\mathbb{Q}(\sqrt{-3}))$ satisfying the condition (7.1) and

$$Q_2(x_1, x_2, a_1x_3 + a_2x_4, b_1x_3 + b_2x_4, x_5) = Q_2(x_1, x_2, -a_1x_3 + a_2x_4, -b_1x_3 + b_2x_4, x_5) \quad (7.2)$$

for all $Q_2 \in \mathcal{L}_2$.

For $J_0(90)^{\langle w_5 \rangle}$ we obtain that there are no quadratic twists, thus any automorphism of the curve is defined over \mathbb{Q} . Lemma 4.6 discards $E90b$ as elliptic quotient. Applying Proposition 5.6 we obtain that there does not exist bielliptic quotient. Similarly because there is no matrix $\mathcal{A} \in \mathrm{GL}_2(\mathbb{Q})$ as above that satisfies (7.1), thus $E30a$ is not a bielliptic quotient, and similarly we discard $E45a$ because no matrix \mathcal{A} as above that satisfies equation (7.2).

After we observe that $J_0(90)^{\langle w_{45} \rangle} \sim_{\mathbb{Q}(\sqrt{-3})} J_0(90)^{\langle w_5 \rangle}$ because $E15a \sim_{\mathbb{Q}(\sqrt{-3})} E45a$. Recall that $X_0(90)/w_5$ and $X_0(90)/w_{45}$ are isomorphic by use of V_3 .

See all computation details in name files related to above quotient modular curves in the folder

<https://github.com/FrancescBars/Mathematica-files-on-Quotient-Modular-Curves>

□

Lemma 7.6. *The curve $X_0(84)/w_4$ has $S_2 w_{14}^{(42)} S_2$ as a bielliptic involution. $X_0^+(84)$ is also bielliptic. But $X_0(84)/w_3$ and $X_0(84)/w_{12}$ are not bielliptic.*

Proof. This follows from Proposition 4.21 and [Bar99] resp. from [Jeo18].

For the other two curves we use Proposition 5.6. The Jacobian decomposition over \mathbb{Q} is

$$\begin{aligned} J_0(84)^{\langle w_3 \rangle} &\sim (E14a)^2 \times (E42a)^2 \times E84b \\ J_0(84)^{\langle w_{12} \rangle} &\sim (E14a)^2 \times (E21a) \times (E42a) \times (E84a) \end{aligned}$$

From the Jacobian decomposition of $J_0(84)^{\langle w_3 \rangle}$ and $J_0(84)^{\langle w_{12} \rangle}$ all endomorphisms are defined over \mathbb{Q} (no quadratic twist in the elliptic curves involved and is the same decomposition in the algebraic closure of the rationals), see Proposition 5.1. The factors with power 1 are discarded because $\dim \mathcal{L}_{2,i} < \dim \mathcal{L}$, thus are not bielliptic quotients. The bielliptic quotient $E14a$ is not possible because there does not exist any matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ satisfying (7.1) for all $Q_2 \in \mathcal{L}_2$. Similarly there does not exist such a matrix for $E42a$ for $J_0(84)^{\langle w_3 \rangle}$ satisfying (7.2) for all $Q_2 \in \mathcal{L}_2$. Thus by Proposition 5.6, they are not bielliptic.

See all computation details in

<https://github.com/FrancescBars/Mathematica-files-on-Quotient-Modular-Curves>

□

8 Quotient modular curves of level N with $X_0^*(N)$ hyperelliptic

8.1 Quotient modular curves with $X_0^*(N)$ hyperelliptic and N having two prime divisors

In this subsection we discuss the candidates for which $X_0^*(N)$ is hyperelliptic and N is only divisible by two different primes. So we treat the 15 values

$$N = 88, 104, 112, 116, 117, 135, 147, 153, 184, 284, 136, 171, 207, 176, 279.$$

We emphasize that the curves $X_0^+(N)$ have already been treated in [Jeo18] and are not listed in our Theorem 1.1. So we largely ignore them and only mention a few bielliptic ones among them in passing.

Lemma 8.1. *The curves $X_0(284)/w_d$ are not bielliptic.*

Proof. This follows easily from Lemma 4.5 by counting \mathbb{F}_9 -rational points. \square

Lemma 8.2. *The curves $X_0(184)/w_8$, $X_0(207)/w_9$, $X(279)/w_9$ and $X_0(279)/w_{31}$ are not bielliptic.*

Proof. These curves have an involution whose quotient is the hyperelliptic curve $X_0^*(N)$. By the Hurwitz formula the number of fixed points is $2g(X_0(N)/w_d) - 4g(X_0^*(N)) + 2 > 8$ (see the tables in the Appendix). So by Lemma 4.3 they are not bielliptic. \square

We quickly decide some more curves.

Lemma 8.3. (a) *The curves $X_0(88)/w_{11}$, $X_0(112)/w_7$ and $X_0(184)/w_{23}$ are bielliptic.*

(b) *On the other hand, $X_0(116)/w_4$, $X_0(153)/w_{17}$ and $X_0(207)/w_{23}$ are not bielliptic.*

Proof. (a) These curves map of degree 2 to the elliptic curves $X_0(44)/w_{11}$, $X_0(56)/w_7$ and $X_0(92)/w_{23}$, respectively (see the table for $g_N^* = 0$ in the Appendix). So by Lemma 4.16 a bielliptic involution is given by $w_8 S_2 w_8$ (resp. $w_{16} S_2 w_{16}$ for $X_0(112)/w_7$). Moreover, S_2 is a conjugate bielliptic involution.

(b) The curve $X_0(116)/w_4$ is by Proposition 4.21 isomorphic to $X_0(58)$, and hence not bielliptic by [Bar99]. Also, $X_0(153)/w_{17}$ and $X_0(207)/w_{23}$ are by Proposition 4.22 isomorphic to $X_0^+(153)$ resp. $X_0^+(207)$ and so not bielliptic by [Jeo18]. \square

Lemma 8.4. *Let $N \in \{104, 117, 136, 171, 176\}$ and $i = 2$ resp. 3 depending on whether N is even or odd. Then $V_i w_N$ is a bielliptic involution of $X_0(N)/\varpi_1$ and of $X_0^+(N)$, whereas $X_0(N)/\varpi_2$ is not bielliptic. Here we are using the notation ϖ_1 and ϖ_2 from Section 2 (xii), which allows us to simultaneously describe the Atkin-Lehner involutions for different N .*

Proof. Thanks to Lemmas 4.18 respectively 4.19 we can calculate the number of fixed points of every involution $V_i w_d$. We only carry this out for $N = 176$, for which we obtain the following table

v	$\#(v, X_0(176))$	v	$\#(v, X_0(176))$
id	-	V_2	0
w_{16}	0	$V_2 w_{16}$	4
w_{11}	0	$V_2 w_{11}$	12
w_{176}	12	$V_2 w_{176}$	24

With Lemma 4.14 we can thus check that $X_0(176)/w_{11}$ has no bielliptic involution in $\langle B(N), V_2 \rangle$ whereas $V_2 w_{176}$ is a bielliptic involution for the other two curves.

To finish the proof we note that by Lemma 4.4 a bielliptic involution v on $X_0(176)/w_{11}$ would induce the hyperelliptic involution on $X_0^*(176)$, which according to the table above is V_2 . So v could only be V_2 or $V_2 w_{16}$. \square

Lemma 8.5. *The curve $X_0(112)/w_{16}$ is not bielliptic.*

Proof. By exactly the same arguments as in the previous lemma we can show that $\langle B(112), V_2 \rangle$ contains no bielliptic involution for this curve. However, now the problem is that we don't know the hyperelliptic involution of $X_0^*(112)$. (Actually, V_2 does not help because it is a bielliptic involution of $X_0^*(112)$.) But Proposition 4.8 guarantees that the bielliptic involution of $X_0(112)/w_{16}$ would have to be contained in $\langle w_7, V_2 \rangle$. \square

Lemma 8.6. *The genus 7 curve $X_0(116)/w_{29}$ is not bielliptic.*

Proof. The involutions w_4 and S_2 , both defined over \mathbb{Q} , generate a group $H \cong S_3$ of automorphisms of $X = X_0(116)/w_{29}$. Each of the three (conjugate) involutions has 8 fixed points on X .

If v is a bielliptic involution of X , then H induces an isomorphic group \tilde{H} of automorphisms on X/v . Under the action of v on the 8 fixed points of an involution in H these can at worst fall together in pairs. So each involution in \tilde{H} has at least 4 fixed points on X/v . By the Hurwitz formula the curve $(X/v)/\tilde{H}$ thus has genus 0 and the automorphisms of order 3 in \tilde{H} have no fixed points on X/v . So the genus 1 curve X/v has a fixed point free automorphism of order 3, defined over \mathbb{Q} , and hence it must have a \mathbb{Q} -rational 3-torsion point. But the only elliptic curves over \mathbb{Q} of level M properly dividing 116 have $M = 58$ and no 3-torsion by [Cre17]. For $M = 116$ the modular degrees of the optimal elliptic curves 116a1, 116b1, 116c1 are 120, 8 and 15, and hence too big by Lemma 4.6. \square

Lemma 8.7. *The remaining six curves, that is the quotient curves $X_0(88)/w_8$, $X_0(135)/w_{27}$, $X_0(135)/w_5$, $X_0(147)/w_3$, $X_0(147)/w_{49}$, $X_0(153)/w_9$, are not bielliptic.*

Proof. Ordering by genus, the Jacobian decompositions over \mathbb{Q} are as follows:

$$\begin{array}{ll} J_0(88)^{\langle w_8 \rangle} & \sim (E11a)^2 \times E44a \times E88a \\ J_0(147)^{\langle w_3 \rangle} & \sim E49a \times E147a \times E147c \times A_f; \dim A_f = 2, f \in \text{New}(147) \\ \hline J_0(135)^{\langle w_5 \rangle} & \sim E27a \times (E45a)^2 \times E135a \times A_f; \dim A_f = 2, f \in \text{New}(135) \\ J_0(147)^{\langle w_{49} \rangle} & \sim E21a \times E147b \times A_f \times A_g; \dim(A_f) = \dim(A_g) = 2; f, g \in \text{New}(147) \\ J_0(135)^{\langle w_{27} \rangle} & \sim (E15a)^2 \times E45a \times E135a \times E135b \times A_f; \dim A_f = 2; f \in \text{New}(135) \\ J_0(153)^{\langle w_9 \rangle} & \sim (E17a)^2 \times E51a \times A_f \times E153a \times E153b; \dim A_f = 2; f \in \text{New}(51) \end{array}$$

For the genus 4 cases there are no quadratic twists, thus all endomorphisms are defined over \mathbb{Q} by Proposition 5.1. By Remark 5.8 we consider \mathcal{L}_2 and \mathcal{L}_3 , see further details in [BG20, §6]. We compute a nonzero polynomial $Q_2 \in \mathcal{L}_2$ which generates \mathcal{L}_2 : $48t^2 + x^2 + xy - 7y^2 - 16z^2$. For $E44a$ and $E88a$ corresponding to z, t we have that $Q_2(\omega_1, \omega_2, \omega_3, \omega_4) = Q_2(\omega_1, \omega_2, -\omega_3, \omega_4)$ and similar for ω_4 , but

$$\frac{Q_3(\dots, \omega_i, \dots) - Q_3(\dots, -\omega_i, \dots)}{x_i} \notin \mathcal{L}_2 \quad (8.1)$$

where $Q_3 \in \mathcal{L}_3$ that is not a multiple of \mathcal{L}_2 , thus not bielliptic. For $E11a$, we make a change of variables w_1, w_2 taking $Q_2(x, y, z, t) = 48t^2 - 176x^2 + y^2 - 16z^2$ but this does not satisfy the condition on \mathcal{L}_3 of equation (8.1) thus is not bielliptic. See the computational details of this example and the remaining ones in files related with such quotient modular curves in the folder

<https://github.com/FrancescBars/Mathematica-files-on-Quotient-Modular-Curves>

We only recall that the bielliptic involution is defined over \mathbb{Q} for genus ≥ 6 (in such cases it is unique) thus we only need to worry if a bielliptic involution exists and is not defined over \mathbb{Q} for the genus 5 curve $X_0(147)/w_3$. The dimension two factor does not have any elliptic quotient by use of Proposition 5.3, and there only appears an inert twist $E49a \sim_{\mathbb{Q}(\sqrt{-7})} E49a$, therefore the bielliptic involution, if it exists, is defined over $\mathbb{Q}(\sqrt{-7})$. But because over $\mathbb{Q}(\sqrt{-7})$ we have the same Jacobian decomposition, we conclude as we did over \mathbb{Q} that there is no bielliptic involution.

Thus for all remaining situations, any dimension one factor with power 1 in the Jacobian we only need to observe that $\dim \mathcal{L}_{2,E} < \dim \mathcal{L}_2$ and for higher power, a similar argument used in equation (7.1) in the previous section, with a correct choice of the variables, shows that no bielliptic involution appears by Proposition 5.6. \square

8.2 Quotient modular curves with $X_0^*(N)$ hyperelliptic and N having 3 prime divisors

In this part we deal with the cases where $X_0^*(N)$ is hyperelliptic and N is divisible by 3 different primes. So N is one of

168, 180, 198, 204, 276, 380 (all with $g_N^* = 2$) or 252, 315 (both with $g_N^* = 3$).

Proposition 8.8. *Let N be one of the eight numbers just listed. Then none of the curves $X_0(N)/w_d$ is bielliptic.*

Proof. From [Jeo18] we know that none of the eight curves $X_0^+(N)$ is bielliptic. So we can assume $d \neq N$. We consider the covering $X_0(N)/w_d \rightarrow X_0(N)/\langle w_d, w_N \rangle$. Since w_N always has fixed points, by Lemma 4.4 a bielliptic involution of $X_0(N)/w_d$ would induce a hyperelliptic involution on $X_0(N)/\langle w_d, w_N \rangle$. But by the tables in the Appendix none of these curves is hyperelliptic. \square

Now we investigate the curves $X_0(N)/W$ with $|W| = 4$.

Lemma 8.9. *$V_3 w_7$ is a bielliptic involution of the curves $X_0(252)/\langle w_4, w_9 \rangle$, $X_0(252)/\langle w_9, w_7 \rangle$ and $X_0(252)/\langle w_4, w_{63} \rangle$. Moreover V_3 is a second bielliptic involution of $X_0(252)/\langle w_4, w_{63} \rangle$.*

By contrast, $X_0(252)/\langle w_4, w_7 \rangle$, $X_0(252)/\langle w_9, w_{28} \rangle$, $X_0(252)/\langle w_7, w_{36} \rangle$ and $X_0(252)/\langle w_{36}, w_{28} \rangle$ are not bielliptic.

Proof. Using Lemma 4.19 we obtain the following table with number of fixed points.

v	$\#(v, X_0(252))$	v	$\#(v, X_0(252))$
id	-	V_3	0
w_4	8	$V_3 w_4$	0
w_9	0	$V_3 w_9$	0
w_7	0	$V_3 w_7$	24
w_{36}	0	$V_3 w_{36}$	0
w_{28}	0	$V_3 w_{28}$	8
w_{63}	24	$V_3 w_{63}$	24
w_{252}	8	$V_3 w_{252}$	8

With Lemma 4.14 we can then easily check three things, namely that $V_3 w_7$ is a bielliptic involution for the first three curves (and V_3 as well for $X_0(252)/\langle w_4, w_{63} \rangle$), that the other four curves have no bielliptic involution in $\langle B(N), V_3 \rangle$, and that V_3 induces the hyperelliptic involution on $X_0^*(N)$.

Combining the second and third fact proves that the four curves are not bielliptic, because a bielliptic involution would by Lemma 4.4 induce the hyperelliptic involution on $X_0^*(N)$, and hence would be in $\langle B(N), V_3 \rangle$. \square

Lemma 8.10. *$V_2 w_{168}$ is a bielliptic involution of the curves $X_0(168)/\langle w_8, w_3 \rangle$, $X_0(168)/\langle w_8, w_7 \rangle$, $X_0(168)/\langle w_3, w_{56} \rangle$ and $X_0(168)/\langle w_7, w_{24} \rangle$.*

Moreover, $V_2 w_8$ is a second bielliptic involution of $X_0(168)/\langle w_3, w_{56} \rangle$.

The remaining curves $X_0(168)/\langle w_3, w_7 \rangle$, $X_0(168)/\langle w_8, w_{21} \rangle$ and $X_0(168)/\langle w_{24}, w_{56} \rangle$ are not bielliptic.

Proof. Apart from using the involutions $V_2 w_d$ the proof is completely analogous to that of Lemma 8.9, except for the case of $X_0(168)/\langle w_{24}, w_{56} \rangle$, in which case we cannot apply Lemma 4.4 because the genus is 4. But being hyperelliptic this curve therefore cannot also be bielliptic by the Castelnuovo inequality (Lemma 4.1). \square

Unfortunately the method we just used does not really work for the six other values of N , as for them we don't know the hyperelliptic involution of $X_0^*(N)$. For example, V_3 induces a bielliptic involution on $X_0^*(180)$, $X_0^*(198)$ and $X_0^*(315)$.

Lemma 8.11. *None of the curves $X_0(380)/W$ with $|W| = 4$ is bielliptic.*

Proof. Assume that such a curve is bielliptic and let E be the elliptic curve it covers. Then \overline{E} (the reduction of E modulo 3) would by Lemma 4.5 have the maximally possible number of 16 \mathbb{F}_9 -rational points. So \overline{E} must necessarily be supersingular. This is equivalent to $j(\overline{E}) = 0$. This in turn is equivalent to the condition that for the coefficients a_i of the global minimal model of E the expression $a_1^2 + a_2$ is divisible by 3. With this one can quickly exclude almost all candidates for E in [Cre17]. Only the isogeny classes 190a and 380a remain. But the curves in 190a have 7 rational points over \mathbb{F}_3 and hence the same number over \mathbb{F}_9 . And for 380a the degree of the strong Weil uniformization is 24 for the elliptic curve 380a1 and 240 for 380b1, so both too big by Lemma 4.6. \square

In the previous lemma the four curves of genus bigger than 10 could also have been quickly excluded using the following method.

Lemma 8.12. *The curves $X_0(276)/W$ with W any one of $\langle w_4, w_3 \rangle$, $\langle w_4, w_{69} \rangle$, $\langle w_3, w_{92} \rangle$, $\langle w_{12}, w_{92} \rangle$, as well as $X_0(204)/\langle w_4, w_3 \rangle$ and $X_0(315)/\langle w_9, w_7 \rangle$ are all not bielliptic.*

Proof. For all these curves the map to the (hyperelliptic) curve $X_0^*(N)$ is given by an involution with more than 8 fixed points. So by Lemma 4.3 they are not bielliptic. \square

Lemma 8.13. *None of the curves $X_0(315)/W$ with $|W| = 4$ is bielliptic.*

Proof. The curve $X_0(315)/\langle w_9, w_7 \rangle$ has already been excluded in the last lemma. So let X be one of the other six curves and assume it is bielliptic with corresponding elliptic curve E . A necessary condition for this is that for the reductions modulo 2 we have

$$|\overline{X}(\mathbb{F}_{2^k})| \leq 2|\overline{E}(\mathbb{F}_{2^k})|$$

for all k . With MAGMA one can calculate $|\overline{X}(\mathbb{F}_{2^k})|$ for small values of k . It turns out that $|\overline{X}(\mathbb{F}_4)| = 18$ for all six curves. Hence \overline{E} must have the maximally possible number of 9 rational points over \mathbb{F}_4 . So it also only has 9 rational points over \mathbb{F}_{16} . This excludes the curves $X_0(315)/\langle w_9, w_5 \rangle$ and $X_0(315)/\langle w_9, w_{35} \rangle$ because they both have $|\overline{X}(\mathbb{F}_{16})| = 26$.

Moreover, $|\overline{E}(\mathbb{F}_4)| = 9$ implies $j(\overline{E}) = 0$. This is equivalent to the coefficient a_1 of the global minimal model of E being even. So with one glance at [Cre17] one can exclude that E has conductor 15, 21, 45, 63 or 105. Only conductor 35 or 315 is possible. But for 315 the degree of the strong Weil uniformization is too big. And an elliptic curve with conductor 35 does not appear in the Jacobian of $X_0(315)/\langle w_7, w_{45} \rangle$ and $X_0(315)/\langle w_{45}, w_{63} \rangle$.

$$J(X_0(315)/\langle w_7, w_{45} \rangle) \sim_{\mathbb{Q}} (E15a) \times (E21a)^2 \times A_{35, x^2+z-4} \times (E63a) \times A_{315, x^2+2x-1}$$

$$J(X_0(315)/\langle w_{45}, w_{63} \rangle) \sim_{\mathbb{Q}} (E15a) \times (E21a) \times (E105a) \times (E315a) \times A_{315, x^2+2x-1} \times A_{315, x^2-5}.$$

The other two curves of genus 8 are isomorphic to the above two by Lemma 4.22, so also not bielliptic. \square

From Proposition 4.21 we obtain the following isomorphisms over \mathbb{Q} .

Lemma 8.14. $X_0(180)/\langle w_4, w_9 \rangle \cong X_0(90)/w_9$, so by Lemma 7.5 and Proposition 4.21 it has $S_2 V_3 w_{10}^{(90)} S_2$ as a bielliptic involution.

The following curves are not bielliptic by Section 7 resp. [BG20] resp. [BGK20].

$$\begin{aligned} X_0(180)/\langle w_4, w_5 \rangle &\cong X_0(90)/w_5; \\ X_0(180)/\langle w_4, w_{45} \rangle &\cong X_0(90)/w_{45}; \\ X_0(198)/\langle w_9, w_{11} \rangle &\cong X_0^*(396); \\ X_0(204)/\langle w_4, w_{17} \rangle &\cong X_0(102)/w_{17}; \\ X_0(204)/\langle w_4, w_{51} \rangle &\cong X_0(102)/w_{51}; \\ X_0(276)/\langle w_4, w_{23} \rangle &\cong X_0(138)/w_{23}. \end{aligned}$$

Lemma 8.15. The curve $X_0(204)/\langle w_3, w_{68} \rangle$ is not bielliptic.

Proof. By [HS06] this curve is trigonal. If it also were bielliptic, then by the Castelnuovo inequality its genus could be at most 4. But it has genus 5. \square

Lemma 8.16. The remaining next 15 curves are not bielliptic.

$X_0(180)/\langle w_9, w_5 \rangle,$	$X_0(180)/\langle w_9, w_{20} \rangle,$	$X_0(180)/\langle w_5, w_{36} \rangle,$
$X_0(180)/\langle w_{36}, w_{20} \rangle,$	$X_0(198)/\langle w_2, w_9 \rangle,$	$X_0(198)/\langle w_2, w_{11} \rangle,$
$X_0(198)/\langle w_2, w_{99} \rangle,$	$X_0(198)/\langle w_9, w_{22} \rangle,$	$X_0(198)/\langle w_{11}, w_{18} \rangle,$
$X_0(198)/\langle w_{18}, w_{22} \rangle,$	$X_0(204)/\langle w_3, w_{17} \rangle,$	$X_0(204)/\langle w_{17}, w_{12} \rangle,$
$X_0(204)/\langle w_{12}, w_{51} \rangle,$	$X_0(276)/\langle w_3, w_{23} \rangle,$	$X_0(276)/\langle w_{23}, w_{12} \rangle$

Proof. First we consider the genus 5 curves. For that we need to study quadratic twists because the possible bielliptic involution (or automorphisms) could be not defined over \mathbb{Q} .

$$\begin{aligned} J_0(180)^{\langle w_9, w_{20} \rangle} &\sim_{\mathbb{Q}} E15a \times E30a \times E36a \times E90a \times E90b \\ J_0(180)^{\langle w_5, w_{36} \rangle} &\sim_{\mathbb{Q}} E20a \times E45a \times (E30a)^2 \times E90b \\ J_0(180)^{\langle w_{36}, w_{20} \rangle} &\sim_{\mathbb{Q}} E15a \times E30a \times E90b \times E90c \times E180a \\ \hline J_0(198)^{\langle w_2, w_{99} \rangle} &\sim_{\mathbb{Q}} E11a \times E33a \times E66a \times E99a \times E198a \\ J_0(198)^{\langle w_{11}, w_{18} \rangle} &\sim_{\mathbb{Q}} E66a \times E66b \times E99a \times E99b \times E99d \\ \hline J_0(204)^{\langle w_{12}, w_{51} \rangle} &\sim_{\mathbb{Q}} E17a \times E34a \times E102a \times E102b \times E204b \\ \hline J_0(276)^{\langle w_{23}, w_{12} \rangle} &\sim_{\mathbb{Q}} E69a \times E92a \times E138a \times E138b \times E138c \end{aligned}$$

We find the following quadratic twists (we are only interested in a fixed Jacobian, thus we do not compare here quadratic twist for modular forms between different Jacobians): $E90a \sim_{\mathbb{Q}(\sqrt{-3})} E90b$, $E30a \sim_{\mathbb{Q}(\sqrt{-3})} E90c$ and $E36a \sim_{\mathbb{Q}(\sqrt{-3})} E36a$.

Anyway, we know by V_3 the following modular curves are isomorphic over $\mathbb{Q}(\sqrt{-3})$: $X_0(180)/\langle w_5, w_{36} \rangle \cong X_0(180)/\langle w_{36}, w_{20} \rangle$, $X_0(198)/\langle w_2, w_{99} \rangle \cong X_0(198)/\langle w_{11}, w_{18} \rangle$ and $X_0(198)/\langle w_2, w_{11} \rangle \cong X_0(198)/\langle w_{18}, w_{22} \rangle$. These isomorphism induce isogenous Jacobians, by the following twists: $E15a \sim_{\mathbb{Q}(\sqrt{-3})} E45a$, $E180a \sim_{\mathbb{Q}(\sqrt{-3})} E20a$, $E11a \sim_{\mathbb{Q}(\sqrt{-3})} E99d$, $E33a \sim_{\mathbb{Q}(\sqrt{-3})} E99b$, $E198a \sim_{\mathbb{Q}(\sqrt{-3})} E66b$, $E66a \sim_{\mathbb{Q}(\sqrt{-3})} E198b$, $E66c \sim_{\mathbb{Q}(\sqrt{-3})} E198e$, and $E99a \sim_{\mathbb{Q}(\sqrt{-3})} E99c$. Thus, over $\mathbb{Q}(\sqrt{-3})$:

$$\begin{aligned} J_0(180)^{\langle w_9, w_{20} \rangle} &\sim_{\mathbb{Q}(\sqrt{-3})} E15a \times E30a \times E36a \times (E90b)^2 \\ J_0(180)^{\langle w_{36}, w_{20} \rangle} &\sim_{\mathbb{Q}(\sqrt{-3})} E15a \times (E30a)^2 \times E90b \times E180a \end{aligned}$$

In the above situation $\dim \mathcal{L}_2 = 3$ and for each elliptic curve with power one we obtain $\dim \mathcal{L}_{2,E} < 3$, thus they are not bielliptic quotients. For the E^2 -factors with E an elliptic curve we prove that there is no $\mathcal{A} \in GL_2(K)$ where $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-3})$ respectively of the Jacobian decomposition satisfying Proposition 5.6. Thus none of these curves is bielliptic.

See the complete computation details on each curve and for the following ones in this proposition in name files related with such quotient modular curves in the folder

<https://github.com/FrancescBars/Mathematica-files-on-Quotient-Modular-Curves>

Now consider the genus ≥ 6 curves. It is enough to study them over the rationals (we are not interested where the general endomorphism are defined for the problem on biellipticity). The Jacobian decomposition is as follow:

$$\begin{array}{l}
J_0(180)^{\langle w_9, w_5 \rangle} \sim_{\mathbb{Q}} (E20a)^2 \times (E30a)^2 \times E36a \times (E90b)^2 \\
\hline
J_0(198)^{\langle w_2, w_{11} \rangle} \sim_{\mathbb{Q}} (E66a)^2 \times E99a \times E99b \times E99d \times E198e \\
J_0(198)^{\langle w_{18}, w_{22} \rangle} \sim_{\mathbb{Q}} E11a \times E33a \times E66a \times E66c \times E99c \times E198b \\
J_0(198)^{\langle w_2, w_9 \rangle} \sim_{\mathbb{Q}} (E11a)^2 \times E33a \times E66a \times E99a \times E99c \times E198d \\
J_0(198)^{\langle w_9, w_{22} \rangle} \sim_{\mathbb{Q}} (E11a)^2 \times E33a \times E66a \times E66c \times E99a \times E99c \\
\hline
J_0(204)^{\langle w_3, w_{17} \rangle} \sim_{\mathbb{Q}} (E34a)^2 \times A_{f,68} \times (E102a)^2 \times E(204a) \\
J_0(204)^{\langle w_{17}, w_{12} \rangle} \sim_{\mathbb{Q}} (E34a)^2 \times E51a \times A_{f,68} \times E102a \times E102c \\
\hline
J_0(276)^{\langle w_3, w_{23} \rangle} \sim_{\mathbb{Q}} E92a \times (E138a)^2 \times (E138c)^2 \times A_{f,276}
\end{array}$$

where $\dim(A_{f,68}) = \dim(A_{f,68}) = \dim(A_{f,276}) = 2$ and the number next to f_i is the level where it appears as newform. By Theorem 5.3 they do not give any elliptic quotient. Moreover, for each non-repeated factor of dimension one in the Jacobian we have $\dim(\mathcal{L}_{2,E}) < \dim(\mathcal{L})$ and for the terms E^2 with E an elliptic curve, there is no matrix $\mathcal{A} \in \text{GL}_2(\mathbb{Q})$ satisfying Proposition 5.6. All in all we obtain that none is a bielliptic quotient. \square

9 Quotient modular curves of level N with $X_0^*(N)$ not subhyperelliptic

Proposition 9.1. *Let N be non-square free such that $X_0^*(N)$ is not subhyperelliptic. Then the only bielliptic curve $X_0(N)/W_N$ where W_N is a proper subgroup of $B(N)$ is $X_0(144)/\langle w_{144} \rangle$.*

Proof. Let $X_0(N)/W_N$ be bielliptic and $X_0^*(N)$ not subhyperelliptic. Then $X_0^*(N)$ must be bielliptic by Lemma 3.1. So we are dealing with a subset of the levels in case (v) of Theorem 3.2. Furthermore, by Lemma 4.4 (b) the covering $X_0(N)/W_N \rightarrow X_0^*(N)$ must be totally unramified.

Next we point out that every Atkin-Lehner involution w_d that has a fixed point on $X_0(N)$ must be contained in W_N . Otherwise it would induce an involution with fixed points on $X_0(N)/W_N$, contradicting the fact that the map to $X_0^*(N)$ is totally unramified. In particular, the full Atkin-Lehner involution w_N , which by [Ogg74, p.454] always has fixed points, must be contained in W_N .

Now we are ready to settle $N = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$, the only case in this paper where N has more than by 3 different primes divisors. As just said, $w_{420} \in W_{420}$. And as w_4 fixes a cusp by [Ogg74, Proposition 3], also $w_4 \in W_{420}$. Moreover, w_{20} , w_{35} , w_{84} and w_{140} also all have fixed points by [Ogg74, p.453]. So they also all must be contained in W_{420} , leading to the contradiction $W_{420} = B(420)$.

Next we treat the cases where N is divisible by 3 different primes. Thus, N is one of the following:

N	g_N^*
234, 240, 252, 294, 312, 315, 348, 476	3
228, 260, 264, 280, 300, 306, 342	4
364, 444, 495	5
558	7

Obviously it suffices to prove that there are no such bielliptic curves with $|W_N| = 4$. As necessarily $w_N \in W_N$, there are at most three possible choices for W_N , namely $\langle \varpi_1, \varpi_2 * \varpi_3 \rangle$, $\langle \varpi_2, \varpi_1 * \varpi_3 \rangle$ and $\langle \varpi_3, \varpi_1 * \varpi_2 \rangle$. Moreover, the covering $X_0(N)/W_N \rightarrow X_0^*(N)$ is unramified if and only if $X_0(N)/W_N$ has genus $2g_N^* - 1$. By the tables in the Appendix the only curves surviving this test are $X_0(260)/\langle w_4, w_{65} \rangle$ and $X_0(300)/\langle w_4, w_{75} \rangle$, both of genus 7. But they are also not bielliptic. By Proposition 4.21 they are isomorphic to $X_0(130)/\langle w_{65} \rangle$ and $X_0(150)/\langle w_{75} \rangle$, respectively. For the second curve we have shown in Lemma 7.1 that it is not bielliptic. The exact same proof works for $N = 130$, or alternatively (since 130 is square-free) we could invoke [BGK20].

Finally, if N has only two different prime divisors it belongs to one of the following sets:

N	g_N^*
144, 152, 164, 189, 196, 236, 245, 248	3
148, 160, 172, 200, 224, 225, 242, 275	4

As discussed earlier, then necessarily $W_N = \langle w_N \rangle$. But by [Jeo18, Theorem 1.1] $X_0^+(144)$ is the only such curve that is bielliptic. □

10 Quadratic points

In this section we prove our second main result, Theorem 1.3.

Let X be a curve of genus at least 2 which has a \mathbb{Q} -rational point. Then by [Bar18, Theorem 2.14] X has infinitely many quadratic points if and only if X is hyperelliptic or X has an involution v , defined over \mathbb{Q} , such that X/v is an elliptic curve E with positive rank over \mathbb{Q} .

Now we specialize to X being of the form $X(N)/W_N$. The hyperelliptic ones have already been determined in [FH99]. On the other hand, if the bielliptic involution v is defined over \mathbb{Q} , then the conductor of the elliptic curve E will be a divisor M of N . So our main tool will be [Cre17].

We start with the cases where $g_N^* = 1$ and N has 3 different prime divisors. Among these, for $N = 84, 90, 120, 126, 132, 140, 150$ there are no divisors M for which there exist elliptic curves of positive rank. For $N = 156$ and 220 the only possibility is $M = N$. So in that case the map from $X_0(N)$ to E of degree $2|W_N| \leq 8$ must factor through the strong Weil parametrization. But by [Cre17] for the elliptic curves of positive rank the degree of that parametrization is 12 for $N = 156$ and 36 for $N = 220$.

We also mention as a warning that if M is a proper divisor of N it seems that one cannot expect any help from the strong Weil parametrization from $X_0(M)$ to E . For example, $X_0(22)$ has two bielliptic involutions defined over \mathbb{Q} , namely w_2 and w_{22} . As there are no elliptic curves with conductor 22, the quotient curves must be isogenous to the elliptic curve $X_0(11)$. But the bielliptic maps do not factor through the canonical map to $X_0(11)$, which has degree 3.

By the same token, for the cases where $g_N^* = 1$ and N has only 2 different prime divisors, we can exclude all levels N except 99 and 124.

Indeed $Jac(X_0^*(99)) \sim_{\mathbb{Q}} E_{99a}$, which has rank 1, furnishing the curves $X_0(99)/\langle w_d \rangle$ with infinitely many quadratic points. For $N = 124$ again the degree of the strong Weil parametrization (here 6) is too small.

By exactly the same method one can exclude all remaining bielliptic curves in the table of Theorem 1.1 except for the following two constellations: $N = 171$ with $M = 57$ and $N = 176$ with $M = 88$.

So assume that there is a map of degree 2, defined over \mathbb{Q} , from $X_0(171)/\langle w_9 \rangle$ to the elliptic curve E with conductor 57 and positive rank over \mathbb{Q} . For the reduction modulo 2 one easily verifies $\#E(\mathbb{F}_2) = 5$. And with the general formula

$$\#E(\mathbb{F}_{q^2}) = \#E(\mathbb{F}_q)(2q + 2 - \#E(\mathbb{F}_q))$$

we see that E does not acquire more points over \mathbb{F}_4 . This leads to a contradiction in Lemma 4.5.

The exact same proof works for $N = 176$ if we reduce the elliptic curve with conductor 88 modulo 3 and count the \mathbb{F}_9 -rational points.

Finally we provide some more information on the two curves that are bielliptic but not over \mathbb{Q} .

Lemma 10.1. *The genus 5 curve $X_0(126)/w_{63}$ has exactly two bielliptic involutions, namely V_3 and V_3w_9 , both defined over $\mathbb{Q}(\sqrt{-3})$. This curve has only finitely many points that are quadratic over $\mathbb{Q}(\sqrt{-3})$.*

The same holds for the curve $X_0(252)/\langle w_4, w_{63} \rangle$, which is isomorphic to $X_0(126)/w_{63}$ by Proposition 4.21. Its two bielliptic involutions are V_3 and V_3w_7 , both defined over $\mathbb{Q}(\sqrt{-3})$.

Proof. In Lemma 7.3 we exhibited the two bielliptic involutions of $X_0(126)/w_{63}$. Since the curve $X_0(126)/\langle w_{63}, w_2 \rangle$ has genus 2, we can apply Lemma 4.11. It tells us that if $X_0(126)/w_{63}$ had further bielliptic involutions, it would have one defined over \mathbb{Q} . So one checks with Petri that $X_0(126)/w_{63}$ has no bielliptic involution over \mathbb{Q} . More computationally, one could also show from the splitting of the Jacobian that all automorphisms of $X_0(126)/w_{63}$ are defined over $\mathbb{Q}(\sqrt{-3})$ and then use Petri to determine the bielliptic involutions over that field.

Finally one checks that the elliptic quotients of $X_0(126)/w_{63}$ by V_3 resp. V_3w_9 , namely the base change of $E14a$ to $\mathbb{Q}(\sqrt{-3})$ has rank 0 over $\mathbb{Q}(\sqrt{-3})$. \square

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A Computations of the genus

We list the genus of $X_0(N)/W_N$ with $W_N \leq B(N)$ such that N is non-square-free and not a power of a prime, and $X_0^*(N)$ is of genus ≤ 1 , or is hyperelliptic or is bielliptic. These levels N are listed in Theorem 3.2. For such list we omit: $N = 420$ (a product involving four primes, see the results of §9 to discard $N = 420$), and $N = 12, 18, 20, 24$ because $g_N \leq 1$.

Recall the definition of ϖ_i from Section 2 (xii).

A.1 N a product of two primes

N	$Primefactor$	$g_{X_0(N)}$	$g_{X_0(N)/\langle \varpi_1 \rangle}$	$g_{X_0(N)/\langle \varpi_2 \rangle}$	$g_{X_0(N)/\langle \varpi_1 * \varpi_2 \rangle} = g_{X_0^+(N)}$
$g_{X_0^*(N)} = 0$					
28	$2^2 * 7$	2	1	0	1
44	$2^2 * 11$	4	2	1	1
45	$3^2 * 5$	3	1	1	1
50	$2 * 5^2$	2	1	1	0
54	$2 * 3^3$	4	2	1	1
56	$2^3 * 7$	5	3	1	1
92	$2^2 * 23$	10	5	1	4
$g_{X_0^*(N)} = 1$					
40	$2^3 * 5$	3	2	2	1
48	$2^4 * 3$	3	2	2	1
52	$2^2 * 13$	5	2	3	2
63	$3^2 * 7$	5	3	3	1
68	$2^2 * 17$	7	3	4	2
72	$2^3 * 3^2$	5	2	3	2
75	$3 * 5^2$	5	3	3	1
76	$2^2 * 19$	8	4	3	3
80	$2^4 * 5$	7	3	4	2
96	$2^5 * 3$	9	3	5	3
98	$2 * 7^2$	7	4	3	2
99	$3^2 * 11$	9	5	3	3
100	$2^2 * 5^2$	7	2	4	3
108	$2^2 * 3^3$	10	4	4	4
124	$2^2 * 31$	14	7	3	6
188	$2^2 * 47$	22	11	4	9

N	$Primefactor$	$g_{X_0(N)}$	$g_{X_0(N)/\langle \varpi_1 \rangle}$	$g_{X_0(N)/\langle \varpi_2 \rangle}$	$g_{X_0(N)/\langle \varpi_1 * \varpi_2 \rangle} = g_{X_0^+(N)}$
$g_{X_0^*(N)} = 2$					
88	$2^3 * 11$	9	4	5	4
104	$2^3 * 13$	11	6	6	3
112	$2^4 * 7$	11	6	4	5
116	$2^2 * 29$	13	6	7	4
117	$3^2 * 13$	11	5	6	4
135	$3^3 * 5$	13	7	6	4
147	$3 * 7^2$	11	5	6	4
153	$3^2 * 17$	15	7	6	6
184	$2^3 * 23$	21	11	5	9
284	$2^2 * 71$	34	17	7	14
$g_{X_0^*(N)} = 3$					
136	$2^3 * 17$	15	7	8	6
144	$2^4 * 3^2$	13	7	7	5
152	$2^3 * 19$	17	8	9	6
164	$2^2 * 41$	19	9	10	6
171	$3^2 * 19$	17	9	9	5
189	$3^3 * 7$	19	8	10	7
196	$2^2 * 7^2$	17	7	9	7
207	$3^2 * 23$	21	11	8	8
236	$2^2 * 59$	28	14	10	10
245	$5 * 7^2$	21	10	9	8
248	$2^3 * 31$	29	15	9	11
$g_{X_0^*(N)} = 4$					
148	$2^2 * 37$	17	8	9	8
160	$2^5 * 5$	17	9	9	7
172	$2^2 * 43$	20	10	9	9
176	$2^4 * 11$	19	10	10	7
200	$2^3 * 5^2$	19	10	10	7
224	$2^5 * 7$	25	13	11	9
225	$3^2 * 5^2$	19	9	10	8
242	$2 * 11^2$	22	11	10	9
275	$5^2 * 11$	25	13	11	9
$g_{X_0^*(N)} = 5$					
279	$3^2 * 31$	29	15	15	9

In the previous tables **N** means that $X_0^*(N)$ is an hyperelliptic curve, and **k** with k an integer means that the quotient modular curve with such genus is an hyperelliptic curve.

A.2 N a product of three primes

N	60	84	90	120	126	132	140	150	156
$Factor(N)$	$2^2 * 3 * 5$	$2^2 * 3 * 7$	$2 * 3^2 * 5$	$2^3 * 3 * 5$	$2 * 3^2 * 7$	$2^2 * 3 * 11$	$2^2 * 5 * 7$	$2 * 3 * 5^2$	$2^2 * 3 * 13$
$W_N = \{id\}$	7	11	11	17	17	19	19	19	23
$\langle \varpi_1 \rangle$	3	5	6	9	9	9	9	10	11
$\langle \varpi_2 \rangle$	4	5	5	9	9	10	10	10	11
$\langle \varpi_3 \rangle$	4	6	5	9	9	7	10	10	12
$\langle \varpi_1 \varpi_2 \rangle$	4	5	6	7	9	10	8	9	11
$\langle \varpi_1 \varpi_3 \rangle$	2	6	6	9	7	7	10	7	12
$\langle \varpi_2 \varpi_3 \rangle$	1	6	5	5	5	10	7	7	6
$\langle \varpi_1 \varpi_2 \varpi_3 \rangle$	3	4	4	7	7	8	7	8	10
$\langle \varpi_1, \varpi_2 \rangle$	2	2	3	4	5	5	4	5	5
$\langle \varpi_1, \varpi_3 \rangle$	1	3	3	5	4	2	5	4	6
$\langle \varpi_2, \varpi_3 \rangle$	1	3	2	3	3	4	4	4	3
$\langle \varpi_1, \varpi_2 \varpi_3 \rangle$	0	2	2	2	2	4	2	3	2
$\langle \varpi_2, \varpi_1 \varpi_3 \rangle$	1	2	2	4	3	3	4	3	5
$\langle \varpi_3, \varpi_1 \varpi_2 \rangle$	2	2	2	3	4	3	3	4	5
$\langle \varpi_1 \varpi_2, \varpi_1 \varpi_3 \rangle$	0	3	3	2	2	4	3	2	3
$B(N)$	0	1	1	1	1	1	1	1	1

N	220	168	180	198	204	276	380	234	240
$Factor(N)$	$2^2 * 5 * 11$	$2^3 * 3 * 7$	$2^2 * 3^2 * 5$	$2 * 3^2 * 11$	$2^2 * 3 * 17$	$2^2 * 3 * 23$	$2^2 * 5 * 19$	$2 * 3^2 * 13$	$2^4 * 3 * 5$
$W_N = \{id\}$	31	25	25	29	31	43	55	35	37
$\langle \varpi_1 \rangle$	15	13	11	14	15	21	27	18	19
$\langle \varpi_2 \rangle$	16	13	13	15	16	22	28	17	19
$\langle \varpi_3 \rangle$	13	13	13	12	16	13	25	18	19
$\langle \varpi_1 \varpi_2 \rangle$	16	11	11	14	16	22	28	18	19
$\langle \varpi_1 \varpi_3 \rangle$	13	9	11	15	12	19	25	15	15
$\langle \varpi_2 \varpi_3 \rangle$	10	13	13	12	13	22	16	16	15
$\langle \varpi_1 \varpi_2 \varpi_3 \rangle$	14	11	11	13	13	18	24	15	17
$\langle \varpi_1, \varpi_2 \rangle$	8	6	5	7	8	11	14	9	10
$\langle \varpi_1, \varpi_3 \rangle$	5	5	5	6	6	5	11	8	8
$\langle \varpi_2, \varpi_3 \rangle$	4	7	7	5	7	7	7	8	8
$\langle \varpi_1, \varpi_2 \varpi_3 \rangle$	4	6	5	5	5	9	6	7	7
$\langle \varpi_2, \varpi_1 \varpi_3 \rangle$	6	4	5	7	5	8	11	6	7
$\langle \varpi_3, \varpi_1 \varpi_2 \rangle$	6	5	5	5	7	5	11	8	9
$\langle \varpi_1 \varpi_2, \varpi_1 \varpi_3 \rangle$	4	4	5	6	5	10	7	7	6
$B(N)$	1	2	2	2	2	2	2	3	3

N	252	294	312	315	348	476	228	260	264
$Factor(N)$	$2^2 * 3^2 * 7$	$2 * 3 * 7^2$	$2^3 * 3 * 13$	$3^2 * 5 * 7$	$2^2 * 3 * 29$	$2^2 * 7 * 17$	$2^2 * 3 * 19$	$2^2 * 5 * 13$	$2^3 * 3 * 11$
$W_N = \{id\}$	37	41	49	41	55	67	35	37	41
$\langle \varpi_1 \rangle$	17	21	25	21	27	33	17	17	19
$\langle \varpi_2 \rangle$	19	21	25	19	28	34	17	19	21
$\langle \varpi_3 \rangle$	19	21	25	21	28	34	18	19	21
$\langle \varpi_1 \varpi_2 \rangle$	19	21	25	19	28	34	17	19	19
$\langle \varpi_1 \varpi_3 \rangle$	19	17	19	21	22	30	18	19	21
$\langle \varpi_2 \varpi_3 \rangle$	13	17	17	17	19	19	18	19	21
$\langle \varpi_1 \varpi_2 \varpi_3 \rangle$	17	17	23	17	25	29	16	15	17
$\langle \varpi_1, \varpi_2 \rangle$	9	10	13	9	14	17	8	9	9
$\langle \varpi_1, \varpi_3 \rangle$	9	9	10	11	11	15	9	9	10
$\langle \varpi_2, \varpi_3 \rangle$	7	9	9	8	10	10	9	10	11
$\langle \varpi_1, \varpi_2 \varpi_3 \rangle$	5	8	8	7	8	7	8	7	8
$\langle \varpi_2, \varpi_1 \varpi_3 \rangle$	9	7	9	8	10	13	8	8	9
$\langle \varpi_3, \varpi_1 \varpi_2 \rangle$	9	9	12	8	13	15	8	8	8
$\langle \varpi_1 \varpi_2, \varpi_1 \varpi_3 \rangle$	7	7	6	8	7	8	9	10	10
$B(N)$	3	3	3	3	3	3	4	4	4

N	280	300	306	342	364	444	495	558
$Factor(N)$	$2^3 * 5 * 7$	$2^2 * 3 * 5^2$	$2 * 3^2 * 17$	$2 * 3^2 * 19$	$2^2 * 7 * 13$	$2^2 * 3 * 37$	$3^2 * 5 * 11$	$2 * 3^2 * 31$
$W_N = \{id\}$	41	43	47	53	51	71	65	89
$\langle \varpi_1 \rangle$	21	19	23	26	25	35	33	45
$\langle \varpi_2 \rangle$	21	22	23	27	26	35	33	45
$\langle \varpi_3 \rangle$	21	22	22	27	26	36	29	45
$\langle \varpi_1 \varpi_2 \rangle$	19	22	23	26	26	35	33	45
$\langle \varpi_1 \varpi_3 \rangle$	17	22	24	24	24	36	29	41
$\langle \varpi_2 \varpi_3 \rangle$	21	19	22	21	23	24	33	33
$\langle \varpi_1 \varpi_2 \varpi_3 \rangle$	19	19	20	24	23	32	25	41
$\langle \varpi_1, \varpi_2 \rangle$	10	10	11	13	13	17	17	23
$\langle \varpi_1, \varpi_3 \rangle$	9	10	11	12	12	18	13	21
$\langle \varpi_2, \varpi_3 \rangle$	11	10	10	11	12	12	15	17
$\langle \varpi_1, \varpi_2 \varpi_3 \rangle$	10	7	9	9	10	10	13	15
$\langle \varpi_2, \varpi_1 \varpi_3 \rangle$	8	10	10	11	11	16	11	19
$\langle \varpi_3, \varpi_1 \varpi_2 \rangle$	9	10	9	12	12	16	11	21
$\langle \varpi_1 \varpi_2, \varpi_1 \varpi_3 \rangle$	8	10	11	9	11	12	15	15
$B(N)$	4	4	4	4	5	5	5	7

The double vertical lines separate the different values of g_0^* . A red entry for the genus means that the curve is hyperelliptic.

B Table of bielliptic quotient curves

We consider (N, W_N) with N non-square free of genus $g_W \geq 2$ which is a bielliptic curve. We assume that $W_N \neq B(N)$ and $W_N \neq \langle w_N \rangle$, and nontrivial. Let F be the field where are defined all automorphism elements of $End(J_0(N)^{W_N})$, which is \mathbb{Q} or a quadratic field K . Such a field is determined from the \mathbb{Q} -isogeny decomposition of the Jacobian of (N, W_N) by using mainly Propositions 5.1 and 5.3. We indicate the elliptic factors through K -isogeny if the field K is not \mathbb{Q} (when $g_{W_N} \leq 5$). By (w, E) a couple where w indicates a bielliptic involution as an element of $Aut(X_0(N)/W_N) \bmod W_N$, and E denotes the corresponding bielliptic quotient modulo \mathbb{Q} -isogeny (not \mathbb{Q} -isomorphism!) and (w, E) a couple as before but defined all over K in particular E as K -isogeny. We add \dots in the tables to indicate that the modular curve $X_0(N)/W_N$ could have more bielliptic involutions.

For all bielliptic (N, W_N) that are not hyperelliptic, such that no Atkin-Lehner appears as a bielliptic involution (or $g_{W_N} \geq 6$ or that are listed in Theorem 1.1), we determine all bielliptic involutions by use of computations related to Petri's theorem, (in the situations where Petri's theorem applies). This procedure can be done also for the rest of quotient curves where we can apply the Petri methodology. One may consult all the computations done in MATHEMATICA in:

<https://github.com/FrancescBars/Mathematica-files-on-Quotient-Modular-Curves>

For (N, W_N) hyperelliptic and bielliptic, with hyperelliptic involution u , the bielliptic involutions can be computed by MAGMA or SAGE by using the explicit equation given by in [Has95] and [FH99]. In such situations we do not make explicit the bielliptic involutions except if one corresponds to an Atkin-Lehner involution.

(N, W_N)	(w, E)	$\mathbb{Q} - \text{Jacobiandecomposition}$
$(40, \langle w_8 \rangle)$	$(w_5, E20a \sim X_0^*(40)), (u \circ w_5, E40a)$	$E20a \times E40a$
$(40, \langle w_5 \rangle)$	$(w_8, E20a), (V_2, E20a), (S_2, E20a), (w_8 S_2 w_8, E20a)$	$(E20a)^2$
$(44, \langle w_4 \rangle)$	$(S_2 w_2^{(22)} S_2, E11a), (S_2 w_{22}^{(22)} S_2, E11a)$	$(E11a)^2$
$(48, \langle w_{16} \rangle)$	$(w_3, 24a \sim X_0^*(48)), (u \circ w_3, 48a)$	$E24a \times E48a$
$(48, \langle w_3 \rangle)$	$(w_{16}, E24a), (V_2, E24a), (S_2, E24a), (S_2 w_{16} S_3, E24a)$	$E24a \sim_{\mathbb{Q}(\sqrt{-1})} E48a$
$(52, \langle w_4 \rangle)$	$(w_{13}, E26b \sim X_0^*(52)), (u \circ w_{13}, E26a)$	$(E24a)^2$
$(60, \langle w_{20} \rangle)$	$(w_3, E15a), (w_3 \circ u = w_4, E30a)$	$E26a \times E26b$
$(60, \langle w_4, w_3 \rangle)$	$(S_2 w_{10}^{(30)} S_2, E15a), (S_2 w_2^{(30)} S_2, E30a)$	$E15a \times E30a$
$(72, \langle w_8 \rangle)$	$(w_9, E36a \sim X_0^*(72)), (u \circ w_9, E72a)$	$E15a \times E30a$
$(84, \langle w_4, w_3 \rangle)$	$(w_7, E42a \sim X_0^*(84)), (u \circ w_7, E14a)$	$E36a \times E72a$
$(84, \langle w_4, w_{21} \rangle)$	$(w_7, E42a), (u \circ w_7, E14a)$	$E14a \times E42a$
$(84, \langle w_3, w_{28} \rangle)$	$(w_7, E42a), (u \circ w_7, E14a)$	$E14a \times E42a$
$(84, \langle w_7, w_{12} \rangle)$	$(w_3, E42a \sim X_0^*(84)), (u \circ w_3, E21a)$	$E14a \times E42a$
$(90, \langle w_9, w_5 \rangle)$	$(w_9, E30a \sim X_0^*(90)), (u \circ w_9, E90b)$	$E21a \times E42a$
$(90, \langle w_2, w_{45} \rangle)$	$(w_9, E30a), (u \circ w_9, E15a)$	$E30a \times E90b$
$(90, \langle w_9, w_{10} \rangle)$	$(w_5, E30a), (u \circ w_5, E15a)$	$E15a \times E30a$
$(90, \langle w_5, w_{18} \rangle)$	$(w_9, E30a), (u \circ w_9, E45a)$	$E15a \times E30a$
$(100, \langle w_4 \rangle)$	$(w_{25}, E50b \sim X_0^*(100)), (w_{25} \circ u, E50a)$	$E30a \times E45a$
$(120, \langle w_8, w_{15} \rangle)$	$(w_3, E20a \sim X_0^*(120)), (u \circ w_3, E40a)$	$E50a \times E50b$
$(120, \langle w_{15}, w_{40} \rangle)$	$(w_3, E20a), (u \circ w_3, E120a)$	$E50a \sim_{\mathbb{Q}(\sqrt{5})} E50b$
$(126, \langle w_2, w_{63} \rangle)$	$(w_9, E21a \sim X_0^*(126)), (u \circ w_9, E14a)$	$E20a \times E40a$
$(126, \langle w_{14}, w_{18} \rangle)$	$(w_9, E21a), (u \circ w_9, E126a)$	$E20a \times E120a$
$(132, \langle w_4, w_{11} \rangle)$	$(w_3, E66b \sim X_0^*(132)), (u \circ w_3, E66c)$	$E14a \times E21a$
$(140, \langle w_4, w_{35} \rangle)$	$(w_5, E70a \sim X_0^*(140)), (u \circ w_5, E14a)$	$E21a \times E126a$
$(150, \langle w_6, w_{50} \rangle)$	$(w_2, E15a \sim X_0^*(150)), (u \circ w_2, E150a)$	$E66b \times E66c$
$(156, \langle w_4, w_{39} \rangle)$	$(w_{13}, E26b \sim X_0^*(156)), (u \circ w_{13}, E26a)$	$E14a \times E70a$
		$E15a \times E150a$
		$E26a \times E26b$

Table 1. Case $g_{W_N} = 2$

For genus 3 table below we list first the three hyperelliptic ones. We remind that the automorphism group can be computed in MAGMA and here we do not make it explicit if is not an Atkin-Lehner involution.

For example, two of the bielliptic involutions of $X_0(63)/w_9$ that are not defined over \mathbb{Q} are simply listed as $(**, \tilde{E}), (*, \tilde{E})$. Their elliptic quotient \tilde{E} is $\mathbb{Q}(\sqrt{-3})$ -isogenous to $Y^2 = +1 + 6\sqrt{-3}X - 27X^2 - (26 + 6\sqrt{-3})X^3$.

(N, W_N)	(w, E)	$\mathbb{Q} - \text{Jacobiandecomp.}$
$(56, \langle w_8 \rangle)$	$(V_2 w_8, E14a)$	$(E14a)^2 \times E56b$
$(60, \langle w_4 \rangle)$	$(S_2 w_5^{(30)} S_2, E30a), (S_2 w_6^{(30)} S_2, E15a),$ $(S_2 w_{30}^{(30)} S_2, E15a).$	$(E15a)^2 \times E30a$
$(63, \langle w_9 \rangle)$	$(w_7, E21a \sim X_0^*(63))$ $(**, \tilde{E}), (*, \tilde{E})$	$E21a \times A_{f,63},$ $A_{f,63} \sim_{\mathbb{Q}(\sqrt{-3})} \tilde{E}^2$
$(52, \langle w_7 \rangle)$	$(w_8, E26b \sim X_0^*(52)), \dots$	$(E26b)^2 \times E52a$
$(63, \langle w_7 \rangle)$	$(w_9, E21a), \dots$	$(E21a)^2 \times E63a,$ $E21a \sim_{\mathbb{Q}(\sqrt{-3})} E63a$
$(68, \langle w_4 \rangle)$	$(w_{17}, E34a \sim X_0^*(68)), \dots$	$(E17a)^2 \times E34a$
$(72, \langle w_9 \rangle)$	$(w_8, E36a \sim X_0^*(72)), \dots$	$E24a \times (E36a)^2$
$(75, \langle w_3 \rangle)$	$(w_{25}, E15a \sim X_0^*(75)), \dots$	$(E15a)^2 \times E75a$
$(75, \langle w_{25} \rangle)$	$(w_3, E15a), \dots$	$E15a \times E75b \times E75c,$ $E15a \sim_{\mathbb{Q}(\sqrt{5})} E75b$
$(76, \langle w_{19} \rangle)$	$(w_4, E38b \sim X_0^*(76)), \dots$	$(E38b)^2 \times E76a$
$(80, \langle w_{16} \rangle)$	$(w_5, E20a \sim X_0^*(80)), \dots$	$E20a \times E40a \times E80a$ $E40a \sim_{\mathbb{Q}(\sqrt{-1})} E80a$
$(84, \langle w_4, w_7 \rangle)$	$(w_3, E42a \sim X_0^*(84)), \dots$	$(E21a)^2 \times E42a$
$(84, \langle w_3, w_7 \rangle)$	$(w_4, E42a), \dots$	$(E42a)^2 \times E84b$
$(84, \langle w_{12}, w_{21} \rangle)$	$(w_3, E42a), \dots$	$E14a \times E42a \times E84a$
$(90, \langle w_2, w_9 \rangle)$	$(w_5, E30a \sim X_0^*(90)), \dots$	$E15a \times E30a \times E90a$
$(90, \langle w_2, w_5 \rangle)$	$(w_9, E30a), \dots$	$(E30a)^2 \times E45a$
$(90, \langle w_{10}, w_{18} \rangle)$	$(w_5, E30a), \dots$	$E15a \times E30a \times E90c$ $E30a \sim_{\mathbb{Q}(\sqrt{-3})} E90c$
$(96, \langle w_{32} \rangle)$	$(w_3, E24a \sim X_0^*(96)), \dots$	$E24a \times E48a \times E96a$ $E24a \sim_{\mathbb{Q}(\sqrt{-1})} E48a$
$(98, \langle w_{49} \rangle)$	$(w_2, E14a \sim X_0^*(98)), \dots$	$E14a \times A_{f,98}$
$(99, \langle w_{11} \rangle)$	$(w_9, E99a \sim X_0^*(99)), \dots$	$E99a \times E99b \times E99d$
$(120, \langle w_3, w_5 \rangle)$	$(w_8, E20a \sim X_0^*(120)), \dots$	$(E20a)^2 \times E24a$
$(120, \langle w_5, w_{24} \rangle)$	$(w_8, E20a), \dots$	$(E20a)^2 \times E30a$
$(124, \langle w_{31} \rangle)$	$(w_4, E62a \sim X_0^*(124)), \dots$	$(E62a)^2 \times E124b$
$(126, \langle w_9, w_7 \rangle)$	$(w_2, E21a \sim X_0^*(126)), \dots$	$(E21a)^2 \times E42a$
$(126, \langle w_9, w_{14} \rangle)$	$(w_2, E21a \sim X_0^*(126)), \dots$	$(E21a) \times A_{f,63}$
$(132, \langle w_3, w_{44} \rangle)$	$(w_7, E66b \sim X_0^*(132)), \dots$	$E11a \times E33a \times E66b$
$(132, \langle w_{11}, w_{12} \rangle)$	$(w_7, E66b), \dots$	$E44a \times E66a \times E66b$
$(140, \langle w_7, w_{20} \rangle)$	$(w_4, E70a \sim X_0^*(140)), \dots$	$A_{f,35} \times E70a$
$(140, \langle w_{35}, w_{20} \rangle)$	$(w_4, E70a), \dots$	$E14a \times E70a \times E140a$
$(150, \langle w_2, w_{75} \rangle)$	$(w_{25}, E15a \sim X_0^*(150)), \dots$	$E15a \times E30a \times E50a$
$(150, \langle w_3, w_{50} \rangle)$	$(w_{25}, E15a), \dots$	$(E15a)^2 \times E75a$
$(156, \langle w_3, w_{13} \rangle)$	$(w_4, E26b \sim X_0^*(156)), \dots$	$(E26b)^2 \times E52a$
$(156, \langle w_{39}, w_{12} \rangle)$	$(w_4, E26b), \dots$	$E26a \times E26b \times E156b$

Table 2. Case $g_{W_N} = 3$

(N, W_N)	(w, E)	$\mathbb{Q} - \text{Jacobiandecomposition}$
$(60, \langle w_3 \rangle)$	$(w_5, E20a), (w_{20}, E15a), \dots$	$(E15a)^3 \times E20a$
$(60, \langle w_5 \rangle)$	$(w_4, E30a), (w_3, E20a), \dots$	$(E20a)^2 \times (E30a)^2$
$(68, \langle w_{17} \rangle)$	$(w_4, E34a \sim X_0^*(68)), \dots$	$(E34a)^2 \times A_{f, x^2-2x-2}, \dim(A_f) = 2$
$(76, \langle w_4 \rangle)$	$(w_{19}, E38b \sim X_0^*(76)), \dots$	$(E19a)^2 \times E38a \times E38b$
$(80, \langle w_5 \rangle)$	$(w_{16}, E20a \sim X_0^*(80)), \dots$	$(E20a)^3 \times E80b$
$(98, \langle w_2 \rangle)$	$(w_{49}, E14a \sim X_0^*(98)), \dots$	$E20a \sim_{\mathbb{Q}(\sqrt{-1})} E80b$ $(E14a)^2 \times E49a \times E98a$
$(100, \langle w_{25} \rangle)$	$(w_4, E50b \sim X_0^*(100)), \dots$	$E14a \sim_{\mathbb{Q}(\sqrt{-7})} E98a$ $E20a \times (E50b)^2 \times E100a$
$(108, \langle w_4 \rangle)$	$(w_{27}, E54b \sim X_0^*(108)), \dots$	$E20a \sim_{\mathbb{Q}(\sqrt{5})} E100a$ $(E27a)^2 \times E54a \times E54b$
$(108, \langle w_{27} \rangle)$	$(w_4, E54b), \dots$	$E54a \sim_{\mathbb{Q}(\sqrt{-3})} E54b$ $E36a \times (E54b)^2 \times E108a$
$(112, \langle w_7 \rangle)$	$(S_2, E56a), (w_{16}S_2w_{16}, E56a)$	$(E56a)^2 \times E112a \times E112c$
$(120, \langle w_8, w_3 \rangle)$	$(w_5, E20a \sim X_0^*(120)) \dots$	$(E15a)^2 \times E20a \times E40a$
$(120, \langle w_3, w_{40} \rangle)$	$(w_5, E20a) \dots$	$(E15a)^2 \times E20a \times E24a$
$(126, \langle w_2, w_7 \rangle)$	$(w_9, 21a \sim X_0^*(126)) \dots$	$(E21a)^2 \times E63a \times E126b$
$(126, \langle w_7, w_{18} \rangle)$	$(w_9, E21a) \dots$	$E21a \sim_{\mathbb{Q}(\sqrt{-3})} E126b$ $(E21a)^2 \times E42a \times E63a$
$(132, \langle w_3, w_{11} \rangle)$	$(w_4, E66b \sim X_0^*(132)), \dots$	$E21a \sim_{\mathbb{Q}(\sqrt{-3})} E63a$ $E44a \times (E66b)^2 \times E132b$
$(132, \langle w_4, w_{33} \rangle)$	$(w_3, E66b), \dots$	$(E11a)^2 \times E66b \times E66c$
$(132, \langle w_{12}, w_{33} \rangle)$	$(w_4, E66b), \dots$	$E11a \times E66b \times E66c \times E132a$
$(140, \langle w_4, w_5 \rangle)$	$(w_7, E70a \sim X_0^*(140)), \dots$	$E14a \times (E35a)^2 \times E70a$
$(140, \langle w_5, w_7 \rangle)$	$(w_4, E70a), \dots$	$E20a \times (E70a)^2 \times E140b$
$(140, \langle w_5, w_{28} \rangle)$	$(w_4, E70a), \dots$	$E14a \times E20a \times E35a \times E70a$
$(150, \langle w_2, w_{25} \rangle)$	$(w_3, E15a \sim X_0^*(150)), \dots$	$E15a \times E30a \times E75b \times E75c$
$(150, \langle w_3, w_{25} \rangle)$	$(w_2, E15a), \dots$	$E15a \sim_{\mathbb{Q}(\sqrt{5})} E75b$ $(E15a)^2 \times E50b \times E150c$
$(150, \langle w_{25}, w_6 \rangle)$	$(w_2, E15a), \dots$	$E15a \times E50b \times E75b \times E75c$
$(168, \langle w_3, w_{56} \rangle)$	$(V_2w_{168}, E14a), (V_2w_8, E24a)$	$E15a \sim_{\mathbb{Q}(\sqrt{5})} E75b$ $E14a \times E24a \times E42a \times E84b$
$(188, \langle w_{47} \rangle)$	$(w_4, E94a \sim X_0^*(188)) \dots$	$(E94a)^2 \times A_{f, x^2-x-3} \dim(A_f) = 2$
$(220, \langle w_5, w_{11} \rangle)$	$(w_4, E110b \sim X_0^*(220)), \dots$	$E20a \times E44a \times (E110b)^2$
$(220, \langle w_4, w_{55} \rangle)$	$(w_{11}, E110b), \dots$	$(E11a)^2 \times E110c \times E110b$
$(220, \langle w_{44}, w_{55} \rangle)$	$(w_{11}, E110b), \dots$	$E11a \times E110a \times E110b \times E220b$

Table 3, case $g_{W_N} = 4$

(N, W_N)	(w, E)	$\mathbb{Q} - \text{Jacobiandecomp.}$
$(84, \langle w_4 \rangle)$	$(S_2 w_{14}^{(42)} S_2, E21a)$	$(E14a)^2 \times (E21a)^2 \times E42a$
$(88, \langle w_{11} \rangle)$	$(S_2, E44a), (w_8 S_2 w_8, E44a)$	$(E44a)^2 \times E88a \times A_{f,88,x^2-x+4}$
$(90, \langle w_9 \rangle)$	$(V_3 w_{10}, E15a)$	$(E15a)^2 \times E30a \times E90a \times E90b$
		$E90a \sim_{\mathbb{Q}(\sqrt{-3})} E90b$
$(96, \langle w_3 \rangle)$	$(w_{32}, E24a = X_0^*(96))...$	$(E24a)^3 \times E32a \times E96b$
$(99, \langle w_9 \rangle)$	$(w_{11}, E99a = X_0^*(99))...$	$(E11a)^2 \times E33a \times E99a \times E99c$
		$E99a \sim_{\mathbb{Q}(\sqrt{-3})} E99c$
$(117, \langle w_9 \rangle)$	$(V_3 w_{117}, E39a)$	$E39a \times A_{f_1} \times A_{f_2}$
		$A_{f_2} \sim_{\mathbb{Q}(\sqrt{-3})} \tilde{E}^2$
$(120, \langle w_{15} \rangle)$	$(V_2 w_{40}, E24a), (w_8 S_2 w_8, E20a), (S_2, E20a)$	$(E20a)^2 \times E24a \times E40a \times E120a$
$(120, \langle w_8, w_5 \rangle)$	$(w_3, E20a = X_0^*(120))...$	$(E20a)^2 \times (E30a)^2 \times E120b$
$(126, \langle w_{63} \rangle)$	$(V_3, E14a), (V_3 w_9, E14a)$	$E14a \times (E21a)^2 \times E42a \times E126a$
		$E14a \sim_{\mathbb{Q}(\sqrt{-3})} E126a$
$(126, \langle w_2, w_9 \rangle)$	$(w_7, E21a = X_0^*(126)), ...$	$(E14a)^2 \times E21a \times A_{f,63,x^2-3}$
		$A_f \sim_{\mathbb{Q}(\sqrt{-3})} \tilde{E}^2$
$(132, \langle w_4, w_3 \rangle)$	$(w_{11}, E33a = X_0^*(132))...$	$(E11a)^2 \times (E33a)^2 \times E66b$
$(140, \langle w_4, w_7 \rangle)$	$(w_5, E70a = X_0^*(140))...$	$A_{f,35}^2 \times E70a$
$(150, \langle w_2, w_3 \rangle)$	$(w_{25}, E15a = X_0^*(150))...$	$(E15a)^2 \times E50a \times E75a \times E150b$
$(156, \langle w_4, w_3 \rangle)$	$(w_{13}, E26b = X_0^*(156))...$	$(E26a) \times (E26b) \times (E39a)^2 \times E78a$
$(156, \langle w_3, w_{56} \rangle)$	$(w_4, E26b)...$	$E26a \times E26b \times E39a \times E78a \times E156a$
$(156, \langle w_{13}, w_{12} \rangle)$	$(w_4, E26b)...$	$(E26b)^2 \times A_f \times E52a$
$(168, \langle w_8, w_7 \rangle)$	$(V_2 w_{168}, E21a)$	$(E21a)^2 \times E42a \times E84b \times E168a$
$(168, \langle w_7, w_{24} \rangle)$	$(V_2 w_{168}, E21a)$	$(E21a)^2 \times E42a \times E56a \times E84b$
$(180, \langle w_4, w_9 \rangle)$	$(S_2 V_3 w_{10}^{(90)} S_2, E15a)$	$(E15a)^2 \times E30a \times E90a \times E90b$
		$E90a \sim_{\mathbb{Q}(\sqrt{-3})} E90b$
$(184, \langle w_{23} \rangle)$	$(S_2, E92a), (w_8 S_2 w_8, E92a)$	$(E92a)^2 \times E184b \times A_{f,184,x^2+x-4}$
$(220, \langle w_4, w_{11} \rangle)$	$(w_5, E110b = X_0^*(220))...$	$(E55a)^2 \times E110b \times A_f$
$(252, \langle w_4, w_{63} \rangle)$	$(V_3, E14a), (V_3 w_7, E14a)$	$E14a \times (E21a)^2 \times E42a \times E126a$
		$E14a \sim_{\mathbb{Q}(\sqrt{-3})} E126a$

Table 4, Case $g_{W_N} = 5$

g_{W_N}	(N, W_N)	(w, E)	$\mathbb{Q} - \text{Jacobiandecomp.}$
6	$(104, \langle w_8 \rangle)$	$(V_2 w_{104}, E26a)$	$(E26a)^2 \times E26b \times E52a \times A_{f,104}$
	$(156, \langle w_4, w_{13} \rangle)$	$(w_3, E26b = X_0^*(156))$	$(E26b)^2 \times A_{f,39}^2$
	$(168, \langle w_8, w_3 \rangle)$	$(V_2 w_{168}, E14a)$	$(E14a)^2 \times E42a \times E56b \times E84b \times E168b$
	$(220, \langle w_5, w_{44} \rangle)$	$(w_4, E110b = X_0^*(220))$	$E11a \times E20a \times A_f \times E110b \times E110c$
	$(220, \langle w_{11}, w_{20} \rangle)$	$(w_4, E110b)$	$E44a \times E55a \times E110b \times A_f \times E220a$
7	$(120, \langle w_{24} \rangle)$	$(V_2 w_{40}, E15a)$	$(E15a)^2 \times (E20a)^2 \times E30a \times E40a \times E120a$
	$(124, \langle w_4 \rangle)$	$(w_{31}, E62a = X_0^*(124))$	$(A_{f_1,31})^2 \times E62a \times A_{f_3,62}$
	$(136, \langle w_8 \rangle)$	$(V_2 w_{136}, E17a)$	$(E17a)^2 \times E34a \times A_{f_3,64} \times A_{f_4,136}$
	$(252, \langle w_9, w_7 \rangle)$	$(V_3 w_7, E36a)$	$(E21a)^3 \times E36a \times (E42a)^2 \times E84b$
8	$(220, \langle w_4, w_5 \rangle)$	$(w_{11}, E110b = X_0^*(220))$	$(E11a)^2 \times A_f^2 \times E110b \times E110c$
9	$(126, \langle w_9 \rangle)$	$(V_3 w_7, E14a)$	$(E14a)^2 \times (E21a)^2 \times E42a \times (A_{f,63})^2$
	$(171, \langle w_9 \rangle)$	$(V_3 w_{171}, E19a)$	$(E19a)^2 \times E57a \times E57b \times E57c \times A_{f,171}, \dim(A_f) = 4$
	$(252, w_9, \langle w_4 \rangle)$	$(V_3 w_7, E14a)$	$(E14a)^2 \times (E21a)^2 \times E42a \times (A_{f,63})^2$
10	$(176, \langle w_{16} \rangle)$	$(V_3 w_{176}, E11a)$	$(E11a)^3 \times E44a \times E88a \times A_{f_1,88} \times E176a \times A_{f_2,176}$
11	$(188, \langle w_4 \rangle)$	$(w_{47}, X_0^*(188) = E94a)$	$A_{f_1}^2 \times E94a \times A_{f_3}, \dim(A_{f_1}) = 4$

Table 5, $g_{W_N} \geq 6$

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