Journal of Mathematical Behavior 70 (2023) 101040

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Does collaborative and experiential work influence the solution of real-context estimation problems? A study with prospective teachers

C. Segura [a,1], I. Ferrando [a,2], L. Albarracín [b,*,3]

[a] *Universitat de Valencia, Spain*
[b] *Universitat Autònoma de Barcelona, Spain*

ARTICLE INFO

Keywords:
Mathematical modeling
Fermi problems
Estimation
Prospective primary school teachers
Real context
In-situ mathematical activities

ABSTRACT

Although it is a challenge for primary school teachers, real-context estimation problems can be used as an introduction to mathematical modeling. With this aim, we designed a two-phase activity: in the first phase, 224 prospective teachers developed individual action plans to solve a sequence of real-context estimation problems in the classroom; in the second phase, they completed the solution of the same problems working in groups in the real location where the four problems were contextualized. A comparative study showed that, in the second phase, prospective teachers were able to adapt their solutions to contextual features detected *in situ* that had not been anticipated in the action plans developed during the first phase. Two-phase modeling activities, which permit a comparison of different perspectives on problems, facilitate the experience of collaborative work. These activities could be incorporated into prospective teachers' initial training as a useful resource for improving their problem-solving expertise.

1. Introduction

Modeling tasks pose problems related to real-world situations that require the formulation, interpretation and resolution of a mathematical model (Blum, 2015). Furthermore, the answer must be validated both mathematically and within its own context. In recent years, there has been a growing interest in introducing mathematical modeling into the curricula at different educational levels (Vorhölter et al., 2014). But this emphasis on modeling activities comes up against a glaring difficulty: the introduction of mathematical modeling activities in primary school classrooms is a real challenge for teachers. Since it is necessary to work on the initiation to modeling as an effective practice in the classroom, it is important to know how prospective teachers solve tasks that lead to the introduction of mathematical modeling at this educational stage (Borromeo Ferri, 2018).

In this study we used real-context estimation problems known as Fermi problems, specifically those that consist of estimating a large number of elements in a delimited area, for example, how many people fit in a city' s main square. These are accessible tasks that invite students to link their mathematical knowledge to real-world phenomena (Ärlebäck, 2009). Previous research has shown that

* Corresponding author.
E-mail addresses: carlos.segura@uv.es (C. Segura), irene.ferrando@uv.es (I. Ferrando), lluis.albarracin@uab.cat (L. Albarracín).
[1] https://orcid.org/0000-0002-1457-5740
[2] https://orcid.org/0000-0003-3746-4581
[3] https://orcid.org/0000-0002-1387-5573

https://doi.org/10.1016/j.jmathb.2023.101040
Received 11 November 2021; Received in revised form 6 February 2023; Accepted 7 February 2023
Available online 20 February 2023

these problems can be solved by generating mathematical models (Peter-Koop, 2009; Albarracín & Gorgorió, 2014, 2019).

Given that it is not yet a common practice in Spain to introduce these types of activities into the classroom, prospective primary school teachers do not usually have any previous experience of real-context estimation problems from their time as school students. Our interest from the standpoint of teacher training was to provide them with mathematical knowledge of how to deal with these types of problems. This research focuses on promoting the development of prospective primary school teachers' problem-solving skills as part of what is known as specialized content knowledge within the Mathematical Knowledge for Teaching (MKT) framework developed by Ball et al. (2008). With the goal of identifying prospective teachers' specific learning needs and designing efficient training methods, we were interested in finding out how they approach Fermi problem-solving. In this work we considered studies that highlight the importance of working collaboratively to improve problem-solving skills (Chapman, 1999; Szydlik et al., 2003), particularly while solving Fermi problems (Ärlebäck, 2009). We have also considered some studies that have shown that prospective teachers' attitudes and involvement improve when they work on problem-solving outside the classroom, in real locations (Vale et al., 2019).

In this research we used a sequence of real-context estimation problems in the design of a two-phase activity. In the first phase, 224 prospective teachers were asked to devise an action plan (in the sense of Pólya, 1957) to achieve the required estimation, individually and in the classroom. In the second, the same participants, organized into 62 workgroups and working at the real location of the problems, complete the resolution of the same sequence. Thus, the individual work became collaborative work, and the classroom work became experiential work. Indeed, by proposing a sequence of real-context problems that the solvers had to tackle in the real location where they were contextualized, it was possible to work mathematically by making measurements and estimations to find the solutions.

In this study we aimed to analyze the evolution from action plans in the first phase to solutions in the second phase, while attempting to determine the influence of collaborative and experiential work on the solution of Fermi problems as modeling tasks. Knowing whether this two-phase type of activity design helps develop problem-solving skills is important because it has the potential to inform the development of practical training in modeling for prospective primary school teachers.

2. Theoretical framework

2.1. Real-context estimation problems as modeling tasks

The literature presents various ways of understanding mathematical modeling (Abassian et al., 2020; Blomhøj, 2009; Kaiser & Sriraman, 2006), but a widely shared conception is the vision of mathematical modeling as a problem-solving process linking the real world and mathematics. It involves mathematizing real-world situations and developing mathematical models to describe the phenomena studied, often conceptualized as the result of having engaged in a complex modeling process (Blum, 2015). To relate the characteristic elements of mathematical models to solvers' productions, in our study we rely upon Lesh and Harel's (2003) definition of mathematical models as a system consisting of mathematical concepts, symbolic representations of reality, relationships, regularities or schemes, as well as the procedures, mathematical or otherwise, associated with their use. From this definition, we understand that to create and develop mathematical models intended to abstractly describe or represent a certain phenomenon or reality, is a complex task.

One type of real context estimation problems are the so-called Fermi problems, defined by Ärlebäck (2009) as:

Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations. (p. 331).

One of the singularities of Fermi problems lies in their formulation: they always seem diffuse in their statement and offer little specific information on how to tackle the solving process (Efthimiou & Llewellyn, 2007). According to Sriraman and Knott (2009), Fermi problems are estimation problems whose purpose is to prompt students to make educated guesses. Only through a detailed analysis of the situation can the problem be decomposed into simpler problems in order to arrive at a reasoned solution (Carlson, 1997). This simplification process involves the synthesis of a model (Robinson, 2008). In this study we used Fermi problems because they are useful activities to promote mathematical modeling suitable for different educational levels, from primary education to university level (Ärlebäck, 2009; Czocher, 2016, 2018; Albarracín & Gorgorió, 2019). This research focused on those studies that show that working with Fermi problems enables primary education students to generate their own mathematical models from elements that they identify in the studied phenomenon, which they then connect with previous knowledge about the real world or other disciplines. Specifically, students learn to identify the most relevant elements shaping a mathematical model (Robinson, 2008) and generate mathematical models from their prior knowledge (Henze & Fritzlar, 2010; Peter-Koop, 2009) that are enriched by the dynamics of group work (Ärlebäck, 2009). Fermi problems encourage students to produce mathematical models that can be adapted to new situations (Albarracín et al., 2021), while developing their mathematical modeling skills (Haberzettl et al., 2018).

Specifically, we used a subset of real-context estimation problems (or Fermi problems), those that consist of estimating a large number of elements in a delimited area, for example, how many people can fit in a public square. Ferrando and Albarracín (2021) explained that this class of real-context estimation problems enables students to use different types of solving strategies based on mathematical models that are dependent on the specific context of each problem. Four types of solution were found: *counting*; *linearization* (solutions that distribute the elements by rows); *base unit* (solutions based on the procedure of dividing the total area by the area of an element taken as a unit); *density* (solutions based on the procedure of multiplying the total area by an estimated density).

2.2. Experiential and collaborative real-world problem solving

There are two ways of approaching a modeling problem: the first is to recreate the real context in the mind of the solver (Sriraman & Knott, 2009); the second is to experiment and work mathematically in the scene of the problem (Vale et al., 2019). In experiential work, solvers have to solve, *in situ*, a task that requires the estimation and measurement of variables, calculations, and the comparison of areas and volumes, thereby mathematizing and modeling the environment (Buchholtz, 2017). Given the complexity of executing the processes involved, this form of experiential problem-solving is often done collaboratively, in groups (Shoaf et al., 2004). Some studies have focused on helping prospective teachers develop their problem-solving competence by engaging them in activities that involve problem solving in groups (Chapman, 2009; Szydlik et al., 2003). On the other hand, Stacey (1992) warns that it is not always better to work in groups, especially in the case of some modeling problems that require deep reflection in order to develop an action plan (Pólya, 1957). But the combination of collaborative work and experimentation can successfully foster this reflection.

Prospective primary school teachers – and even prospective secondary school mathematics teachers, who have a broader mathematical background – have difficulties in solving real-world mathematical problems, particularly as regards clearly explaining their assumptions and choice of model made during modeling work (Widjaja, 2013). In order to overcome these difficulties, prospective teachers need to practice mathematical modeling individually in order to develop a greater knowledge of modeling and link their understanding of mathematical content to real-world situations (Cai et al., 2014).

2.3. Teachers' knowledge of problem-solving

Real-context problem solving, and in particular modeling, plays an essential role in educating citizens to understand the world they live in. It is therefore one of the aspects of teaching in schools that demands improvement (Lott, 2007). In order to improve the teaching of modeling, teachers – and primary school teachers in particular – need to be trained to guide the mathematical process in a variety of real-world situations. However, numerous difficulties have been detected when it comes to sustainably implementing this type of task (Artigue & Blomhøj, 2013).

Following on from Shulman's (1986) proposal, Ball et al. (2008) propose a characterization of Mathematical Knowledge for Teaching (MKT) based on the knowledge and skills that teachers need their students to learn. While keeping the focus fixed on knowledge for teaching, their proposition highlights the importance of mathematical content knowledge. In their work they identify a new domain of teacher's knowledge, "a less recognized domain of content knowledge for teaching that is not contained in pedagogical content knowledge, but yet – we hypothesize – is essential to effective teaching. We refer to this as *specialized content knowledge*" (Ball et al., 2008, p. 390). Indeed, Llinares and Krainer (2006) point out that mathematics education requires complex mathematical knowledge. And Chapman (2015) highlights the importance of teachers' knowledge of problem solving. Their knowledge of problem solving is part of the specialized content knowledge of the MKT model and involves understanding the processes associated with modeling. More can be said: for instance, the problem-solving proficiency of teachers plays an important role in interpreting students' responses or understanding the implications of using certain strategies when tackling a problem.

It is advisable to provide students with tasks that elicit high cognitive activation (Stein et al., 1996). However, Arbaugh et al. (2006) point out that when teachers choose complex problems for implementation in the classroom, there is often no guarantee that they will be carried out with a high level of cognitive demand due to the difficulties encountered by teachers in guiding their students during these practices. Several studies indicate that prospective teachers do not acquire these problem-solving skills during their initial training. For example, they show an inability to successfully relate their solutions to the real context when solving contextualized problems (Tirosh et al., 1991). Furthermore, they are often only aware of a narrow range of solution strategies and show little flexibility in their use, not changing their chosen strategy even when it is unproductive or ineffective (Chapman, 1999; Son & Lee, 2021). The results of these studies not only underscore certain limitations of teachers' knowledge of problem solving for teaching but also draw attention to the knowledge needed to develop their skills as problem solvers. In fact, some researchers suggest that teachers need to experience problem solving from the problem solver's perspective before they can adequately cope with teaching it (Thompson, 1985).

In our work we consider that specific content knowledge of mathematical modeling activities in primary education should focus on adaptive expertise. Verschaffel et al. (2009) define adaptive expertise as the conscious or unconscious selection and use of the most appropriate solution strategy for a given mathematical problem in a given context. Solvers also demonstrate adaptive expertise when they are able to use new and more efficient solving strategies for previously encountered problems (Hatano & Inagaki, 1986). Although determining the efficiency of a solving strategy for a given type of problem is a theoretical challenge (Heinze et al., 2009), in real-context estimation problems there are some characteristics (*e.g.*, the degree of order or disorder in the distribution of the elements to be estimated) that condition the choice of strategy (Ferrando et al., 2020).

While the works cited above understand the notion of adaptive expertise as an individual's problem-solving ability, Anthony et al. (2015) note that, for pre-service teachers, an important aspect of adaptive expertise involves the ability to interact and learn with others. In cases where the solvers, when working collaboratively and experientially, adapt their solutions to the real context of the problem, which they have already worked on individually in the classroom, the group members are forced to contrast their different individual strategies (Stasson et al., 1991). They should agree and adapt their individual proposals to find the most appropriate solution.

Thus, in this study we will be interested, on the one hand, in whether group and collaborative problem solving contributes to future teachers demonstrating adaptive expertise. Moreover, bearing in mind that a key aspect in the group solving process is the interaction that leads to choosing one solving strategy and discarding others (Szydlik et al., 2003; Zawojewski et al., 2003), we will also be interested in the types of agreements that students make during the second phase of the experience to reach a consensus on a solution for each problem (Stasson et al., 1991).

3. Research goals

In view of the need to adequately train future elementary school teachers to contend with mathematical modeling activities, we used Fermi problems as activities that can foster the solvers' adaptive expertise. We prepared a sequence of problems that prospective teachers first tackled individually and then solved in groups working in the real context in which the problem was posed.

The goal was to analyze how collaborative and experiential work influences a real-context estimation problem sequence that prospective teachers have already tackled individually in the classroom. Our research questions were as follows:

R1. Do prospective teachers demonstrate adaptive expertise, switching their individual action plans to other solutions better suited to the context, when they solve problems collaboratively and experientially?

R2. What types of consensus are reached among prospective teachers when they choose the type of solution collaboratively and experientially?

4. Experimental design

4.1. Sample

The sample consisted of 224 prospective teachers. It was a convenience sample that made up 25% of the total population: students belonging to six natural groups in the last year of the primary school education degree at the University of XXXX. The research was carried out in the 2017–2018 and 2018–2019 academic years and had two phases: in-class individual work, and collaborative, experiential work. In the two phases, prospective teachers were asked to solve the same sequence of four real-context estimation problems, set out as follows.

4.2. Real-context estimation problem sequences

According to Doerr and English (2003), a model development sequence is an instructional sequence of modeling tasks that aims to support the development of models by students that can then be applied to a wider range of contexts. Therefore, we designed a sequence of four real-context estimation problems that gave the solvers the opportunity to vary the mathematical models that can be useful for reaching an estimate. In this way, they were able to give greater importance to those elements that best fitted each aspect of the phenomenon under study. The idea for the sequence design, in line with Ärlebäck and Doerr (2015), is based on Ko and Marton's (2004) variation theory. Variation brings certain critical aspects into the foreground, while other aspects remain in or move into the background, which helps students to better discern the said aspects. Four types of variation can be distinguished: contrast, generalization, separation, and fusion (Marton et al., 2004). We relied on the first type of variation, contrast, in the design of the sequence of real-context estimation problems. Contrast variation is a way of discerning a new aspect of a learning situation by comparison with one that has been varied. To design the pattern of contrast variation in the sequence problems, we relied on the identification of relevant contextual features known to affect the solutions developed by the solvers (Ferrando et al., 2020). Since these problems ask the solver to estimate the number of elements that fit in a rectangular area, and this requires the solver to mentally reconstruct the two-dimensional space and distribute the elements, it is possible to identify which features of the context are involved in this process: the size and shape of the elements, the total area, and the way the elements are arranged in the area. These features take on a particular value within a possible set of values: size (large, medium, small), shape (regular, irregular) and arrangement (ordered, disordered). Table 1 shows the four real-context estimation problems selected for the sequence design, and the variation by contrast of contextual features. All four problems were situated in the surroundings of the Faculty of Education, a setting that the future teachers participating in the research were familiar with.

4.3. In-class individual work

We provided each prospective teacher with the written statements of the four problems in the sequence (see Table 1) and a small image for each problem. At the start of the activity the prospective teachers were told that they would have to tackle a sequence of four

Table 1
Task sequence and contextual features.

Statement	Contextual features			
	Element size	Element shape	Element arrangement	Area size
P1-People. How many students can stand on the faculty porch when it rains?	medium	irregular	disordered	medium
P2-Tiles. How many paving tiles are there between the education faculty building and the gymnasium?	medium	regular	ordered	large
P3-Grass. How many blades of grass are there in this space?	small	irregular	disordered	medium
P4-Cars. How many cars fit in the faculty car park?	large	regular	ordered	large

tasks. They then worked individually for 45 min to solve the sequence. The following aspects were highlighted: they had to prepare an action plan for each problem (in the sense of Pólya, 1957), indicating the measurements they would need to make to obtain an estimation; the work had to be done individually; they had to explain their procedures in writing and were allowed to use drawings or diagrams; and, finally, they were not expected to find a solution but rather to explain how to obtain the requested estimation.

4.4. Collaborative and experiential work

One week later, the sample of 224 prospective teachers was again presented with the same sequence of four estimation tasks, although the working conditions were different. The session lasted 90 min. The prospective teachers were randomly organized into 62 groups of three to five people. Each group was again given a file with the statements of the same four problems used in the first part of the activity; and the participants were redeployed to the real contexts of the problems in the surroundings of the Faculty of Education. They were told that they had to come up with a complete solution *in situ* for each of the problems. To this end, each group had to agree on a common solving strategy, and then make measurements, gather data, and carry out the necessary procedures and calculations to obtain a numerical estimation of the number of elements that could fit in each problem space. To do this, each group was provided with measuring instruments: several retractable tape measures and an odometer to measure longer distances. Zawojewski et al. (2003) suggest that students – when working in small groups and dealing with a problem situation that is significant and relevant to them – will have to invent, expand, and refine their own mathematical constructions to respond to the demands of the given problem. In particular, we wanted to analyze whether prospective teachers demonstrate adaptive expertise when doing collaborative and experiential work.

5. Analysis of individual and collaborative solutions

For the categorization of types of individual solutions (action plans) and group solutions, the three researchers used the categories proposed by Ferrando and Albarracín (2021): *counting*, *linearization*, *base unit* and *density*. Those proposals that did not provide enough details to obtain an estimate were categorized as *incomplete*. Triangulation was used by the researchers when describing and interpreting the students' work because intersubjective agreement ensures the reduction of bias (Denzin, 2009). The triangulation was carried out in pairs. In the first phase, two researchers independently categorized the 452 proposals of 113 prospective teachers. Categorization reliability was calculated using Cohen's kappa (Landis & Koch, 1977), giving a value of $k = 0.78$, which indicates a good level of agreement on the Landis and Koch (1977) scale. In the second phase, two researchers categorized the work of the remaining 111 prospective teachers, with a degree of reliability close to very good ($k = 0.80$). During both phases, any discrepancies were discussed with the third researcher in order to reach a consensus. Since the category system was reliable, the categorization of the solutions produced by the 62 groups was done qualitatively, with agreement being reached between the three researchers. Below we present examples of each of the categories, pointing out relevant aspects that emerged from the qualitative analysis.

5.1. Counting

This type of solution proposes an exact, direct counting procedure to obtain the estimation. There is no model of the real context to be mathematized, but the strategy consists of physically representing or simulating the real context. An example can be found in the following transcript of an action plan for the *P4-Cars* problem:

> "We fill the car park with cars, without leaving any gaps, take a photo from above the car park, and then we count the cars. That number will be the total number of cars that fit."

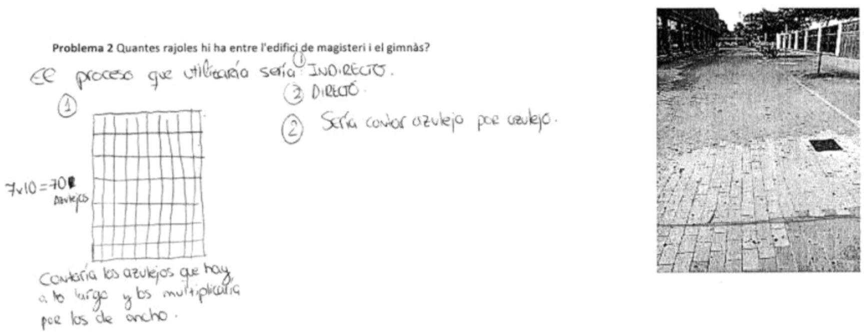


Fig. 1. Statement and picture of the P2-Tiles problem, with individual solution using linearization. "*The process I would use could be either (1) Indirect or (2) Direct. (1) I would count the tiles lengthwise and multiply by the number counted widthwise. (2) I would count each tile.*".

5.2. Linearization

Here, the estimation problem is reduced from two dimensions to one, and once the method of achieving a linear estimation has been clarified, the process required to obtain an estimation of the total number of elements on the surface only requires the calculation of a Cartesian product. As shown in Fig. 1, the solver developed a linear model, estimating how many tiles there are in a row and how many in a column, and then calculated the Cartesian product to obtain an estimate. As can be observed in the visual scheme supporting the action plan (Fig. 1), the solver conceived the area between the gymnasium and the Faculty of Education as a grid, explaining that he would "count how many paving tiles there were lengthwise and multiply the resulting figure by the number of tiles counted widthwise".

5.3. Base unit

The starting point of this type of solution is the estimation of the area occupied by an element in relation to the total surface area. The implicit question underlying this initial model is how to take into account the total surface using the surface area of the element as a measurement unit. The procedure is as follows: first the total surface area is calculated or measured, then the area occupied by an element is calculated or estimated, and finally the total surface area is divided by the area occupied by an element, with the result being precisely the number of elements that fit in the total area. An example is the action plan shown in Fig. 2, which was categorized as a *base unit* solution. The solver wrote that she would calculate the area of the car park, and drew a rectangle to represent such a delimited space, noting that it should measure both width and length. She then wrote that the area occupied by a car must also be calculated, and drew another rectangle representing this area, noting the width and length on the drawing. Finally, she proposed dividing the larger area by the smaller one to obtain the estimate.

5.4. Density

This solution is based on the idea of density, *i.e.* considering the number of elements in a predetermined smaller sub-area (for example, in one square meter). The solver chooses a smaller sub-area in his model to make it easier to estimate the number of elements (usually done by direct counting). Once the number of elements in a sub-area has been estimated, the solver calculates how many sub-areas make up the entire space. Finally, the solver has to multiply the number of elements in the sub-area by the number of sub-areas.

The following is an example of a *density* plan of action for the *P3-Grass* problem:

> "I would measure one centimeter lengthwise and one centimeter widthwise to delimit the sub-area, so that we can count how many blades there are in a square centimeter [sub-area]. Then I would measure the width and length of the entire space [a rectangular planter] to find the total area in square centimeters. Finally, I would multiply the number of blades of grass in 1 cm2 by the total area counted in square centimeters."

As regards the solutions to the *P1-People* problem, when they moved to the scene of the problem, the solvers discovered that there was an arrangement of large paving tiles on the porch and used it to obtain an estimate based on multiplying the number of people that fitted on a tile by the total number of tiles covering the whole floor. This mathematical model is explained by the picture shown in Fig. 3, in which the group divided the pavement into 304 tiles. Then they deducted 14 tiles (because they considered them unusable) and they were left with 290 free tiles. They made two estimations, one with one person per tile (290 people) and the other with two people per tile (580 people).

5.5. Incomplete

These were proposals that met three possible conditions: they did not come up with an answer; they did not develop a mathematical model of the situation or an estimation strategy; or they began to develop a model but did not explain the process well enough to determine whether the solver could make an estimation. This category therefore included all those proposals that did not meet the minimum requirements in order to merit consideration as either a solution or an action plan.

An example of an *incomplete* individual proposal can be found in the following transcript of a proposal for an action plan to tackle the P1-People problem:

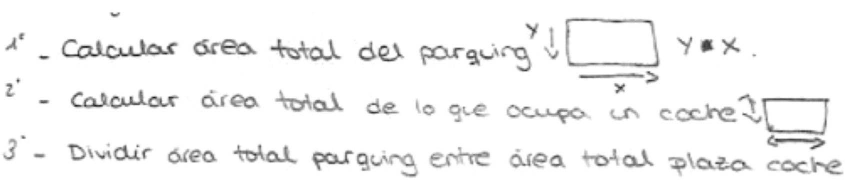


Fig. 2. Individual base unit solution for P4-Cars problem. "*1. Calculate the total area of the car park. 2- Calculate the total area occupied by a car. 3- Divide the total area of the car park by the area occupied by the car.*".

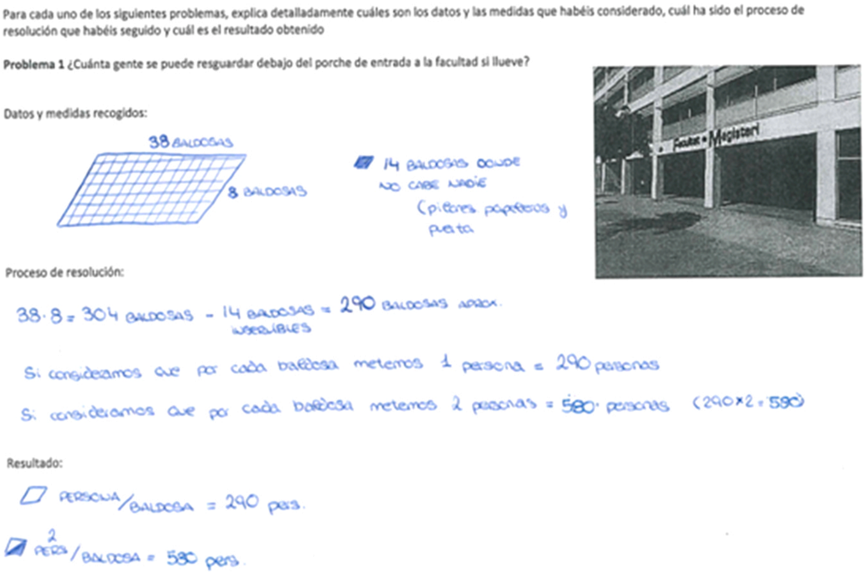


Fig. 3. Density-based solution for the P1-People problem carried out by a group in situ. "Data: 14 tiles where no one can stand. 38 × 8 = 304 tiles. Minus 14 unusable tiles = 290 tiles. Solution procedure: If we put one person on each tile, we have a total of 290 people. If we put two people on each tile, we have a total of 580 people.".

> "We would need to know the total size of the porch first. To do so we would have to estimate the total from the width and length."

In this proposal, the solver has not indicated what magnitude he is referring to by the "size" of the porch, nor what procedure would have to be developed to calculate it from the length and width. But first and foremost, it was categorized as incomplete because the elements (people) and their distribution on the surface are not mentioned. This proposal does not work as an action plan.

The *in-situ* group proposals categorized as *incomplete* were those characterized by the fact that although they presented the data of some measurements taken at the scene of the problem, they neither made their mathematical model clear nor developed the necessary methods of calculation.

6. Results

In the first section below we analyze how prospective teachers adapt their individual action plans when working collaboratively and experientially. In the second, we analyze the types of consensus that the participants reached to solve the problems in the sequence when working collaboratively and experientially.

6.1. Adaptive expertise in collaborative and experiential real-context estimation problem solving

Table 2 shows the absolute frequency and level of incidence in percentage terms of each individual (I) and collaborative, experiential (CE) proposal category with regard to each problem in the sequence. For each problem we have indicated the most frequent strategy in bold, in the cases of both the individual and the collaborative, experiential proposals.

Table 2
Relationships between problems and types of solution in 224 individual proposals and 62 group proposals, indicating the absolute frequency and, in brackets, the relative frequency in percentage terms for each problem in each phase of the activity.

	Incomplete	Counting	Linearization	Base unit	Density
P1-I	34 (15.2%)	1 (0.4%)	28 (12.5%)	**110 (49.1%)**	51 (22.8%)
P1-CE	0 (0%)	0 (0%)	1 (1.6%)	14 (22.6%)	**47 (75.8%)**
P2-I	34 (15.2%)	6 (2.7%)	**92 (41.1%)**	71 (31.7%)	21 (9.3%)
P2-CE	1 (1.6%)	0 (0%)	7 (11.3%)	**49 (79%)**	5 (8.1%)
P3-I	44 (19.6%)	2 (0.9%)	15 (6.7%)	67 (29.9%)	**96 (42.9%)**
P3-CE	3 (4.8%)	0 (0%)	2 (3.2%)	20 (32.3%)	**37 (59.7%)**
P4-I	28 (12.5%)	4 (1.8%)	31 (13.9%)	**160 (71.4%)**	1 (0.4%)
P4-CE	0 (%)	0 (0%)	4 (6.4%)	**58 (93.6%)**	0 (0%)
Total I	140 (15.6%)	13 (1.5%)	166 (18.5%)	**408 (45.5%)**	169 (18.9%)
Total CE	0 (%)	0 (0%)	4 (6.4%)	**58 (93.6%)**	0 (0%)

As can be seen in Table 2, problems set in a context containing elements that have a regular shape and an ordered arrangement promoted *linearization*, while a large element size encouraged *base unit* solutions, and a small element size and irregular arrangement promoted *density* solutions.

The results displayed in Table 2 contain some noteworthy data. First, we observe 13 individual solutions categorized as *counting*, although no *in-situ* group solutions were assigned to this category. We also observe a major behavioral change regarding the use of *linearization*. There are 166 individual proposals (18.5%) in the *linearization* category, the majority (92) dealing with *P2-Tiles* problem, but only 14 group and *in-situ* solutions (6%) fall into this same category. On the other hand, the *base unit* category was the most used in both individual and group solutions, and the *density* category shows a large increase when working in groups.

Based on the results shown in Table 2, here we analyze in detail the impact of collaborative and experiential work. Solutions categorized as *counting,* which amounted to 1.5% of the individual solutions were absent from *in-situ* group solutions. There were few action plans that resorted to counting all the elements, since this procedure is not efficient when there is a very high number of elements. This point was further confirmed when solutions were transferred to a real scenario to obtain a numerical estimate. The reason is that in the second part of the task, the action plan had to be put into practice, and it would have been very difficult to simulate the situations posed by the problems, as they were chosen to ensure that the number of elements to be estimated was very high.

As regards the *P1-People* problem there was a clear decrease in *base unit* usage between individual work (49%) and group work (22.5%). The type of solution that benefited was *density*, which increased from 22.8% to 75.8% in group solutions. A one-tailed direct test of comparison of proportions confirmed that this change was significant ($z = 7.8$ and $p < .001$). This change in the most frequent type of solution is related to changes in the approach to the *P1-People* problem when working in the scene of the problem. As shown in Fig. 3, the surface of the porch is divided up by large tiles, which can accommodate one or two people. In the first phase, prospective teachers had to develop an action plan envisioning the problem space. Although the faculty porch was familiar to the solvers, it was not easy for them to remember the appearance of the floor. So upon moving to the scene of the problem, the solvers found a division of the surface that made it easier to work with *density*, and a large number of groups decided to take advantage of the fact (Fig. 3). This division was used because the estimation could be calculated by counting the tiles and multiplying the total by the number of people that fit in each one, without the need to make any measurements of the space. However, in the individual action plans no solver mentions this floor pattern, and they worked on the problem as if the surface were smooth, so solutions based on the area occupied by one person appear more frequently (Fig. 2).

In the *P2-Tiles* problem, the use of *linearization* decreases from 41.1% of the total number of individual proposals to 11.3% of the total number of group proposals (although it is still the problem with the highest proportion of this type of solution), with a resulting increase in the *base unit* solutions in particular. In fact, while amounting to 31.7% of the total number of individual solutions this figure increases to 79% of the total number of group solutions. A one-tailed direct test of comparison of proportions confirmed that this change was significant ($z = 6.7$ and $p < .001$). Qualitative analysis of the solutions also confirms this. In the individual phase, the solvers had to take into account the rectangular space delimited by the faculty of education and the gym, and also the distribution of the tiles on that surface. It is natural to think that this distribution would have an orderly grid pattern, and that all tiles would be the same shape and size. Consequently, these aspects of the context were associated with linear patterns (number of tiles per row multiplied by the number of columns), as can be clearly observed in the action plan displayed in Fig. 1. Order and regularity are the reason why the *linearization* type of solution was associated with the *P2-Tiles* problem in the individual solutions. However, when the solvers began to work at the scene of the *P2-Tiles* problem, they found that the anticipated context features (order in the distribution of the elements, regular shape and size) did not respond exactly to the reality, which was more complex: there were irregularities in the distribution of the tiles, which in some areas changed direction and, therefore, messed up the linear model (this irregularity can be observed in Fig. 4). In addition, the tiles also varied in size (whole or half tiles). Consequently, the characteristics of the context were different to those envisaged in the individual action plans: there were areas with a disordered distribution of the elements, and these were sometimes whole tiles and sometimes half tiles. The grid model, typical of *linearization*, no longer seemed suitable when working *in situ*. The result was that the groups, when working *in situ*, adapted to the new context features by resorting to the *base unit* type of solution, which adjusts to disorder and irregularity.

However, we observe in Table 2 that in the case of the *P3-Grass* and *P4-Cars* problems the majority of individual action plans

Fig. 4. Detail of the paving at the scene of the *P2-Tiles* problem, with irregularities in the tile distribution.

coincided with the type of solution most used in the experiential and collaborative phase. In these two problems there is no observable element in the real situation that introduces changes in the contextual features; therefore, the groups were not obliged to adapt their individual action plans. As regards the *P3-Grass* problem, a reinforcement of adaptive expertise is observed during the collaborative and experiential phase, as there was an increase in density-based solutions as compared to individual action plans. Density-based solutions serve to overcome the difficulties in estimation caused by the irregularities in the shape and distribution of blades of grass. On the other hand, in the *P4-Cars* problem there was also an increase in the base-unit type of solution, and so the experiential and collaborative solution reinforced the idea that using the area occupied by a car, which has a fairly regular shape and a large size was the most convenient way to solve the problem.

Finally, we note that the percentage of incomplete proposals dropped from 15.6% in individual work to 2% in the groups. This change can be explained by what was mentioned previously: experiential and collaborative problem solving fosters adaptive expertise, and since the groups, working *in situ*, were better able to adapt their problem solving to the contextual features of the problem, this enabled them to overcome those difficulties in making an estimation that might have led to incomplete proposals. In the first phase, during the individual visualization of the context in the classroom, we observed that it was more challenging for solvers to develop the action plan best suited to each problem. The impact of collaborative and experiential work is most clearly seen in groups where all the solvers presented incomplete proposals for some problem when working on their own in the classroom. This was the case of Group 6 F, for example, with regard to *P1-People* problem. None of the four members of the group was able to develop a complete action plan, as shown in Fig. 5, where one of the solvers says: "It occurs to me to first measure how long and deep the porch is. Then by calculating how wide each person is on average, without stretching out their arms, I can calculate how many people fit."

It can be seen that the solver confused width and area, and then was unable to describe the calculations made to find the estimation. The other three solvers in Group 6 F made similar incomplete proposals. However, as can be observed in Fig. 6, Group 6 F was able to develop a density-based solution in the collaborative and experiential phase, adapting its model and procedures to the way the surface was divided up by large paving tiles.

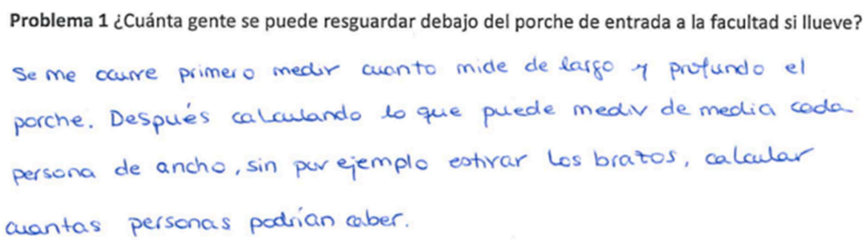
When analyzing the comparison of the failure rates for each sample (individual and collaborative solutions), we carried out a one-tailed direct test of comparison of proportions and we found that the null hypothesis could be rejected (there is no difference in the failure rate, with $z = 4.8$ and $p < .001$). A significant decrease in the proportion of incomplete proposals occurred for all problems, which suggests that collaborative and experiential work facilitates strategy management and the demonstration of adaptive expertise when solving real-context estimation problems.

6.2. Types of agreements reached during collaborative and experiential problem solving

Following the consensus rules established for workgroups by Stasson et al. (1991), we identified the following possible scenarios when contrasting the individual action plans of group members and their collaborative, experiential solutions: a) the group opted for a type of solution that none of its members had chosen individually (*total change*); b) the group used a type of solution that a minority of its members had chosen individually (*minority to majority*); c) the group used a type of solution that half of its members had chosen individually (*half and half*); d) the group used a type of solution that the majority of its members had chosen individually (*majority to minority*); and e) the group used the same type of solution as the one all of its members had chosen individually (*full consensus*). For each problem and overall, Table 3 shows the absolute frequency (and the relative frequency by problem) of each group as regards their choice of solution. The most frequent option(s) for each problem are indicated in bold for each problem.

As expected, when the four tasks were considered altogether, the solution type chosen by the groups coincided with the individual solutions of the majority of their members (the choice of solution in 33% of cases was *majority to minority*). Closer observation of each individual problem revealed this circumstance only changed for the *P1-People* and *P2-Tiles* problems, where a majority of the groups opted for a group solution that was different to the individual solutions of all its members. This finding concurs with the results of the comparative analysis shown in Table 2 and also with the results of the analysis of collaborative and experiential impact.

As regards the *P1-People* problem, we have already noted that in many of the groups all the members chose the *base unit* for their action plans – it was the majority choice in the individual phase. However, the influence of the collaborative and experiential work, as already discussed, prompted them to choose *density* instead for the group solution. Consequently, there was a majority of *total change* (34%) but given the effect of solving the problem by using the tiles and analyzing the groups' proposals in detail, we interpret the high proportion of total change as being due to contextual factors already specified in the previous section, which forced the group to agree


Problema 1 ¿Cuánta gente se puede resguardar debajo del porche de entrada a la facultad si llueve?

Se me ocurre primero medir cuanto mide de largo y profundo el porche. Después calculando lo que puede medir de media cada persona de ancho, sin por ejemplo estirar los brazos, calcular cuantas personas podrían caber.

Fig. 5. Incomplete proposal made by one of the members of group 6 F in the individual phase. "*It occurs to me to first measure how long and deep the porch is. Then by calculating how wide each person is on average, without stretching out their arms, I can calculate how many people fit.*".

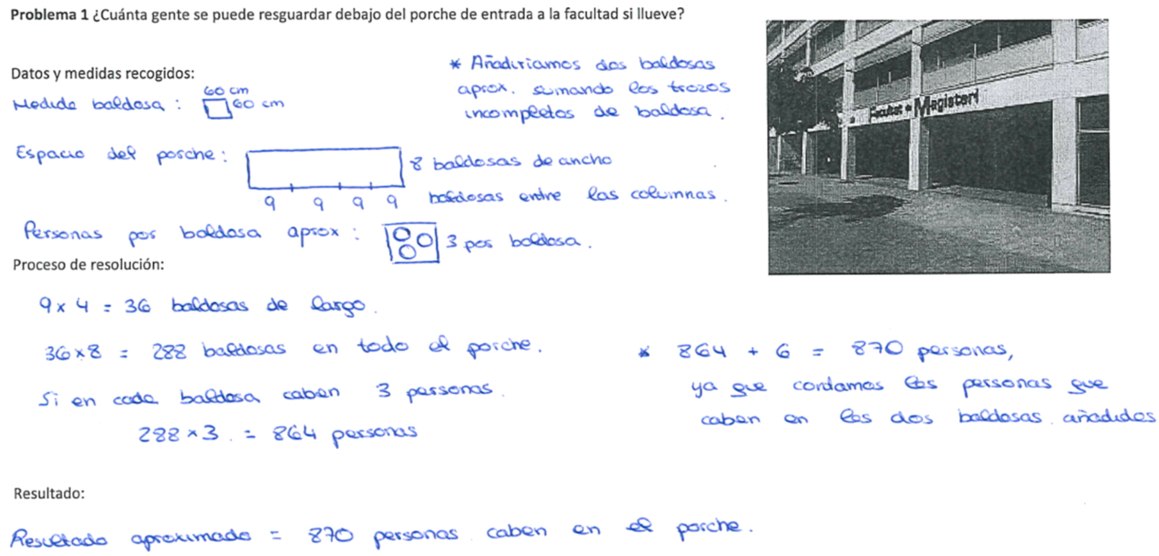


Fig. 6. Collaborative and experiential density-based solution proposed by group 6 F.

Table 3
Possible scenarios when sharing individual solutions among group members and choosing a group solution, for 62 groups.

	Total Change	Minority to majority	Half and half	Majority to minority	Full consensus
P1-People	**21 (34%)**	10 (16%)	7 (11%)	18 (29%)	6 (10%)
P2-Tiles	**19 (31%)**	18 (29%)	3 (5%)	15 (24%)	7 (11%)
P3-Grass	4 (6%)	16 (26%)	9 (15%)	**26 (42%)**	7(11%)
P4-Cars	1 (2%)	8 (13%)	6 (10%)	**23 (37%)**	**24 (39%)**
TOTAL	45(18%)	52(21%)	25(10%)	**82(33%)**	44(18%)

to a new solution.

We found similar behavior in the *P2-Tiles* problem: it was observed that in many groups all the members had chosen *linearization* individually, and what conditioned *total change* in the choice of group solution was the change in the perception of the characteristics of the context (from regular to irregular) when working collaborative and experientially.

In *P3-Grass* and *P4-Cars*, in the absence of *in situ* factors influencing a change, the most frequent scenario was *majority to minority*. Therefore, when there was no difference between the context visualized in the classroom and the real situation in which the experimental and collaborative work took place, the usual scenario was that the groups agreed on the type of solution that prevailed in their individual action plans. In the case of *P4-Cars*, the high proportion of total consensus among group members was due to the fact that the *base unit* strategy was by far the most commonly used type of solution in both the individual and group phases. In *P3-Grass*, however, the proportion of groups that opted for the type of solution chosen by the minority was also high (26%), and this is because both *base unit* and *density* were the types of solution with a high frequency of occurrence in the two phases. In fact, both types of solution were frequent in problems whose context was characterized by a disordered distribution of small elements. If we examine the proposals categorized as *base unit* and *density*, we find that inverse approaches: respectively, one type is based on the surface occupied by an element with respect to the total area, and the other is based on the number of elements occupying a surface with respect to the total area.

7. Discussion

The results of the study are discussed below with the goal of answering the two research questions.

R1- Do prospective teachers demonstrate adaptive expertise, switching their individual action plans to other solutions better suited to the setting, when they solve problems collaboratively and experientially?

The use of real-context estimation problem sequences in two-phase activities, in which an individual action plan is first proposed in the classroom, and then in the second phase they are re-solved experientially and collaboratively, offers opportunities for prospective teachers to demonstrate adaptive expertise. Indeed, by working *in situ*, making measurements and finding the solution collaboratively, the solvers are able to adapt their solutions appropriately to aspects of the context that had not been foreseen in the individual action plans, developing new models of the real context and more suitable strategies. Given that prospective teachers adapt their solutions to the corresponding real-world problem situations, it seems that sequences of real-context estimation problems, solved in two-phase activities where they are first tackled individually and then collaboratively and experientially, promote the development of flexible, non-stereotyped solutions, as is often the case when solving other types of problems *in situ* after tackling them in the classroom (Leikin, 2003). In fact, in this study we found that the proportion of incomplete proposals, where the solver was unable to develop a model or complete the strategies needed to reach an estimation, dropped significantly when working collaboratively and experientially, and this was due - especially in the *P1-People* and *P2-Tiles* tasks - to adaptive expertise, the ability to adopt new solutions better suited to the context of the problem. This is consistent with the findings of other studies that show that this type of sequence promotes inter-task flexibility (Ferrando & Segura, 2020). Previous studies have shown that prospective primary teachers have difficulties in successfully relating their solutions to the real setting when solving contextualized problems (Tirosh et al., 1991), but the results of this study suggest that experiential and collaborative approaches offer a possible way of overcoming these difficulties. These results highlight the need for prospective teachers to work on modeling activities outside the classroom and in contexts where tasks are prepared to establish the basic mechanisms of mathematical modeling activities. These results are in line with previous studies that highlight the importance of the opportunity to work on modeling activities using a collaborative, experiential approach (Egerbladh & Sjodin, 1986; Geiger et al., 2021).

R2. What types of consensus are reached among prospective teachers when they choose the type of solution collaboratively and experientially?

Prospective teachers may have difficulties in defining an action plan for a real-world open problem if they lack data about the problem setting and practical experience of it (Chapman, 2012). Working collaboratively and experientially facilitates the *in-situ* exchange of ideas (Szydlik et al., 2003) and leads, as we have seen, to a more appropriate adaptation of individual action plans to the context, thereby avoiding incomplete proposals. This exchange that fosters adaptive expertise can be observed in the way the groups reached a consensus about the solutions during the collaborative and experiential, second phase of the activity. When aspects emerged in the scene of the problem that were not envisaged in the individual in-class action plans (as in the cases of *P1-People* and *P2-Tiles*), most of the groups developed and agreed on a new type of solution during the collaborative, experiential phase, different from all their previous action plans. In other words, in collaborative and experiential work, adaptability prevails over prior ideas, even if those ideas are in the majority. However, if the scene of the problem does not elicit changes that alter the individual action plans (*P3-Grass* and *P4-Cars*), then the groups – while engaged in experiential and collaborative work – develop the type of solution that prevailed in the individual action plans. The results show that the prospective teachers who took part in the study were able to see that there were different approaches to solving the problems in the sequence when they compared their strategies. In fact, the two-phase design of the activity obliged them to listen and try to understand their peers' proposals and actively decide which proposal to choose during the experiential and collaborative phase.

8. Conclusions

Problem solving is an important competence for teachers because it is an essential part of specialized content knowledge in the MKT model (Ball et al., 2008; Chapman, 2013). This study points to the relevance for prospective teachers to solve problems experientially and collaboratively to improve their real-context problem solving and mathematical modeling skills, in line with Cai et al. (2014). Indeed, the results suggest that a collaborative, experiential approach may have a greater impact on the development of real-context problem-solving competence than individual problem solving in which the problem is simulated (or imagined) in the classroom.

Thus, it has been shown that when prospective teachers solve a sequence of real-context estimation problems experientially and collaboratively, they demonstrate adaptive expertise with respect to their individual solutions. We observed that the prospective teachers changed their strategies while solving problems in groups and *in situ*, when it had not been previously possible to anticipate all the contextual features that could influence the solving process. The results also show that the adaptive experience promoted by experiential and collaborative work leads to improved performance in solving sequences of real-context estimation problems.

Nevertheless, the design of this study had its limitations. It was not possible to identify which improvements in performance were due to collaborative work and which were due to experiential work. Therefore, further studies are needed to determine, separately, the individual effects of collaboration and experimental work on the development of problem-solving skills. A future research goal would be to compare adaptive expertise and improvements in performance in each case.

It was necessary to use the same problem sequence in both phases of the study in order to analyze prospective teachers' adaptive experience. However, this condition limited further analysis of the enhancing effect of collaborative and experiential work on individual work, because some of the improvements detected may have been due to repetition. A future study could use different problems in each phase, contrasting the results with this work, to better understand the advantages of collaborative and experiential real-world problem solving.

In addition, further research is called for to determine how problem-solving skills are transferred from pre-service teacher training to classroom practice. For example, asking prospective teachers, in groups, to share different points of view about the context of a problem, or to differentiate elements of their solutions, is an essential activity for their future teaching practice, as it is part of the so-called mathematical-task knowledge required for teaching (Chapman, 2013). There is a need to study whether future teachers understand the importance of discussing and comparing contexts and strategies so as to incorporate them into the design of future modeling activities, as well as to identify what difficulties they face when implementing such activities in the classroom.

Funding

Lluís Albarracín is Serra Húnter Fellow in Universitat Autònoma de Barcelona. Grant PID2021-126707NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". This research is also supported by the projects EDU2017-

84377-R (Ministerio de Economía, Industria y Competitividad, Spain), 2021 SGR 00159 (AGAUR, Generalitat de Catalunya) and UV-INV-AE-1557785 (Universitat de València).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

Albarracín, L., Ferrando, I., & Gorgorió, N. (2021). The role of context for characterising students' strategies when estimating large numbers of elements on a surface. *International Journal of Science and Mathematics Education, 19*(6), 1209–1227.

Albarracín, L., & Gorgorió, N. (2014). Devising a plan to solve Fermi problems involving large numbers. *Educational Studies in Mathematics, 86*(1), 79–96.

Albarracín, L., & Gorgorió, N. (2019). Using large number estimation problems in primary education classrooms to introduce mathematical modelling. *International Journal of Innovation in Science and Mathematics Education, 27*(2), 45–57.

Abassian, A., Safi, F., Bush, S., & Bostic, J. (2020). Five different perspectives on mathematical modelling in mathematics education. *Investigations in Mathematics Learning, 12*(1), 53–65.

Anthony, G., Hunter, J., & Hunter, R. (2015). Prospective teachers development of adaptive expertise. *Teaching and Teacher Education, 49*, 108–117.

Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Mathematics Enthusiast, 6*(3), 331–364.

Ärlebäck, J.B., & Doerr, H.M.. (2015). Moving beyond a single modelling activity. In G. A. Stillman, W. Blum, & M. S. Beimbegut (Eds.), Mathematical modelling in education and practice (pp. 293–303). New York: Springer.

Arbaugh, F., Lannin, J., Jones, D. L., & Park-Rogers, M. (2006). Examining instructional practices of CorePlus lessons: Implications for professional development. *Journal of Mathematics Teacher Education, 9*, 517–550.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM Mathematics Education, 45*, 797–810.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Blomhøj, M.. (2009). Different perspectives in research on the teaching and learning mathematical modelling. In M. Blomhøj & S. Carreira (Eds.), Mathematical applications and modelling in the teaching and learning of mathematics. Proceedings from topic study group 21 at the 11th international congress on mathematical education in monterrey, México. (pp. 1–17). Roskilde: Roskilde Universitet.

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In S. J. Cho (Ed.), The proceedings of the 12th international congress on mathematical education: Intellectual and attitudinal changes. (pp. 73–96). New York: Springer International Publishing.

Borromeo Ferri, R. (2018). Learning how to teach mathematical modelling in school and teacher education. Cham: Springer.

Buchholtz, N. (2017). How teachers can promote mathematising by means of mathematical city walks. In G. A. Stillman, W. Blum & G. Kaiser (Eds.), Mathematical modelling and applications - crossing and researching boundaries in mathematics education (pp. 49–58). Springer. https://doi.org/10.1007/978–3-319–62968-1_4.

Cai, J., Cirillo, M., Pelesko, J.A., Borromeo Ferri, R., Borba, M., Geiger, V., et al. (2014). Mathematical modelling in school education: Mathematical, cognitive, curricular, instructional, and teacher education perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), Proceedings of the 38th conference of the international group for the psychology of mathematics education and the 36th conference of the North American chapter of the psychology of mathematics education (pp. 145–172). Vancouver: PME.

Carlson, J.E. (1997). Fermi problems on gasoline consumption. The Physics Teacher, 35, 308–309. DOI: 10.1119/1.2344696.

Chapman, O. (1999). In-service teacher development in mathematical problem solving. *Journal of Mathematics Teacher Education, 2*, 121–142.

Chapman, O.. (2009). Self-study as a basis of prospective mathematics teachers' learning of problem solving for teaching. In S. Lerman & B. Davis (Eds.), Mathematical action & structures of noticing (pp. 163–173). Rotterdam: Sense Publishers.

Chapman, O. (2012). Prospective elementary school teachers' ways of making sense of mathematical problem posing. *PNA, 6*(4), 135–146.

Chapman, O. (2013). Mathematical-task knowledge for teaching. *Journal of Mathematics Teacher Education, 16*, 1–6.

Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT: International Journal on Math, Science and Technology Education, 3*(1), 19–36.

Czocher, J. A. (2016). Introducing modelling transition diagrams as a tool to connect mathematical modelling to mathematical thinking. *Mathematical Thinking and Learning, 18*(2), 77–106.

Czocher, J. A. (2018). How does validating activity contribute to the modelling process? *Educational Studies in Mathematics, 99*(2), 137–159.

Denzin, N. K. (2009). *The research act: A theoretical introduction to sociological methods* (3rd ed). Prentice Hall.

Doerr, H. M., & English, L. D. (2003). A modelling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*(2), 110–136.

Efthimiou, C. J., & Llewellyn, R. A. (2007). Cinema, Fermi problems and general education. *Physics Education, 42*(3), 253–261.

Egerbladh, T. & Sjodin, S. (1986). Joint effects of group composition, group norm, type of problem and group vs individual responding. Scandinavian Journal of Educational Research, 30(1) March 1986 pp 17 - 23.

Ferrando, I., & Albarracín, L. (2021). Students from grade 2 to grade 10 solving a Fermi problem: Analysis of emerging models. *Mathematics Education Research Journal, 33*(1), 61–78.

Ferrando, I., Segura, C., & Pla-Castells. (2020). Relations entre contexte, situation et schéma de résolution dans les problèmes d'estimation. *Canadian Journal of Science, Mathematics and Technology Education, 20*, 557–573.

Ferrando, I., & Segura, C. (2020). Fomento de la flexibilidad matemática a través de una secuencia de tareas de modelización. *Avances Délelőtt Investigación Eno¯sis Educación Matemática, 17*, 84–97.

Geiger, V., Galbraith, P., Niss, M., & Delzoppo, C. (2021). Developing a task design and implementation framework for fostering mathematical modelling competencies. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-021-10039-y

Hatano, G., & Inagaki, K. (1986). Two courses of expertise. In H. W. Stevenson, H. Azuma, & K. Hakuta (Eds.), Child development and education in Japan (pp. 262–272). W H Freeman/Times Books/ Henry Holt & Co.

Haberzettl, N., Klett, S., & Schukajlow, S.. (2018). Mathematik rund um die Schule—Modellieren mit Fermi-Aufgaben. In K. Eilerts & K. Skutella (Eds.), Neue Materialien für einen realitätsbezogenen Mathematikunterricht 5. Ein ISTRON-Band für die Grundschule (pp. 31–41). Wiesbaden: Springer Spectrum.

Heinze, A., Star, J., y Verschaffel, L.. (2009). Flexible and adaptive use of strategies and representations in mathematics education. ZDM - The International Journal on Mathematics Education, 41, 535–540.

Henze, J., & Fritzlar, T.. (2010). Primary school children's model building processes by the example of Fermi questions. In A. Ambrus & E. Vásárhelyi (Eds.), Problem solving in mathematics education. Proceedings of the 11th ProMath conference September 3–6, 2009 in Budapest (pp. 60–75). Budapest: Eötvös Loránd University.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM - The International Journal on Mathematics Education, 38*(3), 302–310.

Ko, P.Y., & Marton, F.. (2004). Variation and the secret of the virtuoso. En F. Marton & A. B. M. Tsui (Eds.), Classroom discourse and the space of learning (pp. 43–62). Mahwah, NJ: Erlbaum.

Landis, J., & Koch, G. (1977). The measurement of observer agreement for categorical data. *Biometrics, 33*, 159-74.

Leikin, R. (2003). Problem-solving preferences of mathematics teachers: Focusing on symmetry. *Journal of Mathematics Teacher Education, 6*(4), 297–329.

Lesh, R., & Harel, G. (2003). Problem solving, modelling, and local conceptual development. *Mathematical Thinking and Learning, 5*(2), 157–189.

Llinares, S., & Krainer, K.. (2006). Mathematics (student) teachers and teacher educators as learners. In Gutiérrez, A., Boero, P. (Eds.), Handbook of research on the psychology of mathematics education. past, present and future (pp. 429–459). Rotterdam: Sense Publishers.

Lott, J. W. (2007). Correcting the course of math education. *Principal Leadership, 7*(5), 27–31.

Marton, F., Runesson, U., & Tsui, A.B. M.. (2004). The space of learning. En F. Marton & A. B. M. Tsui (Eds.), Classroom discourse and the space of learning (pp. 3–40). Mahwah, NJ: Erlbaum.

Peter-Koop, A. (2009). Teaching and understanding mathematical modelling through fermi-problem. In B. Clarke, B. Grevholm & R. Millman (Eds.), Tasks in primary mathematics teacher education (pp. 131–146) Springer.

Pólya, G. (1957). *How to solve it: A new aspect of mathematical method*. Princeton University Press.

Robinson, A. W. (2008). Don't just stand there—teach Fermi problems! *Physics Education, 43*(1), 83–87.

Shoaf, M. M., Pollack, H., & Schneider, J. (2004). *Math trails*. Lexington: COMAP.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Son, J. W., & Lee, M. Y. (2021). Exploring the relationship between preservice teachers' conceptions of problem solving and their problem-solving performances. *International Journal of Science and Mathematics Education, 19*, 129–150.

Sriraman, B., & Knott, L. (2009). The mathematics of estimation: Possibilities for interdisciplinary pedagogy and social consciousness. *Interchange, 40*(2), 205–223.

Stacey, K. (1992). Mathematical problem solving in groups: Are two heads better than one. *Journal of Mathematical Behavior, 11*(3), 261–275.

Stasson, M. F., Kameda, T., Parks, C. D., Zimmerman, S. K., & Davis, J. H. (1991). Effects of assigned group consensus requirement on group problem solving and group members' learning. *Social Psychology Quarterly, 54*(1), 25–35.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488.

Szydlik, J. E., Szydlik, S. D., & Benson, S. R. (2003). Exploring changes in preservice elementary teachers' mathematics beliefs. *Journal of Mathematics Teacher Education, 6*(3), 253–279.

Tirosh, D., Tirosh, C., Graeber, A., & Wilson, J. (1991). Computer-based intervention to correct preservice teachers' misconceptions about the operation of division. *Journal of Computers in Mathematics and Science Teaching, 10*, 71–78.

Thompson, A.G.. (1985). Teachers' conceptions of mathematics and the teaching of problem solving. In E. A. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives (pp. 281–294). Erlbaum.

Vale, I., Barbosa, A., & Cabrita, I. (2019). Mathematics outside the classroom: Examples with preservice teachers. *Quaderni di Ricerca in Didacttica, 2*(3), 138–142.

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating,and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335–359.

Vorhölter, K., Kaiser, G. & Borromeo Ferri, R.. (2014) Modelling in mathematics classroom instruction: An innovative approach for transforming mathematics education. In Y. Li, E. A. Silver & S. Li (Eds.), Transforming mathematics instruction (pp. 21–36). Cham, Switzerland: Springer.

Widjaja, W. (2013). Building awareness of mathematical modelling in teacher education: A case study in Indonesia. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), Teaching mathematical modelling: Connecting to research and practice. Dordrecht: Springer.

Zawojewski, J.S., Lesh, R.A., & English, L.D.. (2003). A models and modelling perspective on the role of small group learning activities. In R. Lesh & H. M. Doerr (Eds.), Beyond constructivism: Models and modelling perspectives on mathematics problem solving, teaching, and learning (pp. 337 – 358). Mahwah, NJ: Lawrence Erlbaum Associates.