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THE EXTENDED 16TH HILBERT PROBLEM FOR A CLASS OF DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS

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ABSTRACT. This paper solves the extended 16th Hilbert problem for a family of discontinuous planar differential systems with two regions separated by the straight line $x = 0$. By using the first integrals, we prove that the maximum number of crossing limit cycles in the family of systems formed by a linear center and a class of Hamiltonian isochronous global centers with a polynomial first integral of degree $2n$ is 5.

1. INTRODUCTION

In this paper we are interesting is studying discontinuous piecewise differential systems defined by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}^-(\mathbf{x}) = (F_1^-(\mathbf{x}), F_2^-(\mathbf{x}))^T & \mathbf{x} \in \Sigma^- \\ \mathbf{F}^+(\mathbf{x}) = (F_1^+(\mathbf{x}), F_2^+(\mathbf{x}))^T & \mathbf{x} \in \Sigma^+. \end{cases} \tag{1}$$

with $\mathbf{x} = (x, y)$. Where Σ^- and Σ^+ are two regions in the plane defined by

$$\Sigma^- = \{(x, y) : x \le 0\}, \qquad \Sigma^+ = \{(x, y) : x \ge 0\},$$

and $\Sigma = \{(x, y) : x = 0\}$ is the separation line of the plane.

For this kind of systems the problem of analysing the existence and the maximum number of limit cycles remains in general open, and to find an upper bound for the maximum number of limit cycles in this class of systems it is known as the extended 16th Hilbert problem. The original 16th Hilbert problem consists in finding an upper bound for the maximum number of limit cycles that the class of polynomial differential systems of a given degree can exhibit, see [10, 11, 12].

The dynamics over the line of discontinuity $x = 0$ is defined following the Filippov's convention, see [8]. If $F_1^-(0, y)F_1^+(0, y) > 0$, then both vector fields have the same direction, the point $(0, y)$ is called a crossing point, and the cross region is defined as follows

$$\Sigma^c = \{(0, y) \in \Sigma \mid F_1^-(0, y)F_1^+(0, y) > 0\}.$$

Around 1920 Andronov, Vitt and Khaikin [1] started the study of the piecewise differential systems separated by a straight line and until nowadays such systems have deserved the attention of the researchers. Thus these differential systems are

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widely used to model processes appearing in mechanics, electronics, economy, etc ..., see for instance [3, 16, 19].

In the last years many authors worked on the extended 16th Hilbert's problem for distinct classes of discontinuous piecewise linear differential systems separated by a straight line, where they determine or bound how many limit cycles can appear in these classes of difFerential systems, see for instance [2, 4, 7, 9].

In what follows we deal with the following two classes of isochronous centers.

First the class of linear differential systems. The the following lemma provides a normal form for an arbitrary linear differential center.

Lemma 1. *Through a linear change of variables and a rescaling of the independent variable every linear differential center in $\mathbb{R}^2$ can be written*

$$\begin{aligned} \dot{x} &= -ax - \frac{a^2+\omega^2}{d}y + b, \\ \dot{y} &= dx + ay + c, \quad \textit{with} \;\; \omega > 0 \;\; d > 0. \end{aligned} \tag{2}$$

The first integral of this linear differential system is

$$H_1(x,y) = (ay+dx)^2 + 2d(cx-by) + \omega^2 y^2. \tag{3}$$

We consider the class of Hamiltonian isochronous global centers of the form

$$\dot{x} = -(\delta nxy^{n-1} + \delta^2 n y^{2n-1} + y), \qquad \dot{y} = \delta y^n + x, \tag{4}$$

with $n \in \mathbb{N}$ and $\delta \neq 0$. The first intgral of system (4) is given by

$$H(x,y) = \frac{1}{2}((\delta y^n + x)^2 + y^2).$$

For more details on the Hamiltonian system (4) see Theorem B of [17] and Theorem 2.3 of [18].

The main result of this work is to study the upper bounds of crossing limit cycles for discontinuous piecewise differential systems separated by the straight line $x = 0$, and formed by a linear center (2) and the class of Hamiltonian isochronous global center (4) after an arbitrary affine change of variables.

Theorem 2. *Consider discontinuous piecewise differential systems separated by the straight line $x = 0$ and formed by two differential systems, when these differential systems are linear centers* (2) *or the class of Hamiltonian isochronous global centers* (4) *after an arbitrary affine change of variables. The maximum number of limit cycles of these discontinuous piecewise differential systems is*

- *(a) no limit cycles if $n = 1$;*
- *(b) at most one if $n = 2$, and this maximum is reached, see Fig.* 1*;*
- *(c) at most two if $n = 3$, and this maximum is reached, see Fig.* 2*;*
- *(d) at most three if $n = 4$, and this maximum is reached, see Fig.* 3*;*
- *(e) at most four if $n = 5$, and this maximum is reached, see Fig.* 4*;*
- *(f) at most five if $n \geq 6$, and this maximum is reached, see Fig.* 5.

Theorem 2 is proved in section 3.

The phase portraits of the piecewise differential systems (2)-(4) could be studied using the techniques of [5, 6].

2. The class of Hamiltonian isochronous global center (4) after an arbitrary affine change of variables

We present the expression of the class of Hamiltonian isochronous global centers (4) and their first integrals after an arbitrary affine change of variables $(x, y) \to (a_1X + b_1Y + c_1, \alpha_1X + \beta_1Y + \gamma_1)$, with $b_1\alpha_1 - a_1\beta_1 \neq 0$. Thus, after this affine change of variables the differential system (4) becomes

$$
\begin{aligned}
\dot{X} = \; & \frac{-1}{(b_1\alpha_1 - a_1\beta_1)}\big(- b_1(c_1 + a_1x + b_1y + (x\alpha_1 + y\beta_1 + \gamma_1)^n\delta) \\
& - \frac{1}{(x\alpha_1 + y\beta_1 + \gamma_1)}\big(\beta_1\big((x\alpha_1 + y\beta_1 + \gamma_1)^2 + n(c_1 + a_1x + b_1y)(x\alpha_1 \\
& + y\beta_1 + \gamma_1)^n\delta + n(x\alpha_1 + y\beta_1 + \gamma_1)^{2n}\delta^2\big)\big)\big), \\
\dot{Y} = \; & \frac{-1}{(b_1\alpha_1 - a_1\beta_1)}\big(a_1(c_1 + a_1x + b_1y + (x\alpha_1 + y\beta_1 + \gamma_1)^n\delta) \\
& + \frac{1}{(x\alpha_1 + y\beta_1 + \gamma_1)}\big(\alpha_1\big((x\alpha_1 + y\beta_1 + \gamma_1)^2 + n(c_1 + a_1x + b_1y)(x\alpha_1 \\
& + y\beta_1 + \gamma_1)^n\delta + n(x\alpha_1 + y\beta_1 + \gamma_1)^{2n}\delta^2\big)\big)\big),
\end{aligned} \tag{5}
$$

with its first integral

$$H_2(x, y) = \frac{1}{2}((x\alpha_1 + y\beta_1 + \gamma_1)^2 + (c_1 + a_1x + b_1y + (x\alpha_1 + y\beta_1 + \gamma_1)^n\delta)^2).$$

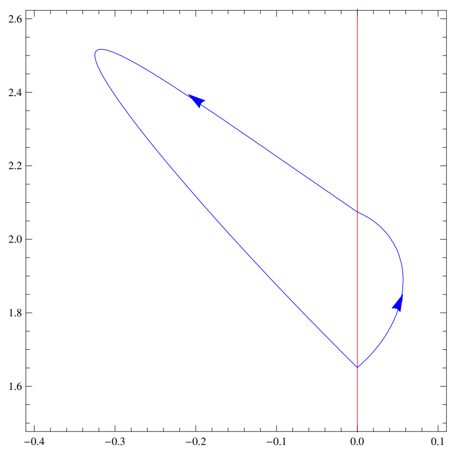

Figure 1. The unique crossing limit cycle of the discontinuous piecewise differential system (7)–(8)

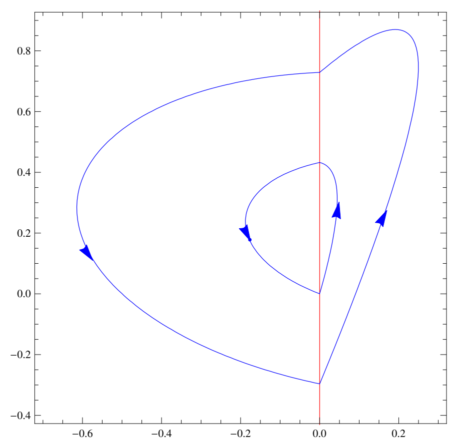

Figure 2. The two crossing limit cycles of the discontinuous piecewise differential system (10)–(11)

3. Proof of Theorem 2

The statement (a) of Theorem 2 has already been proved in Theorem 4 of [15] and in Theorem 3 of [13].

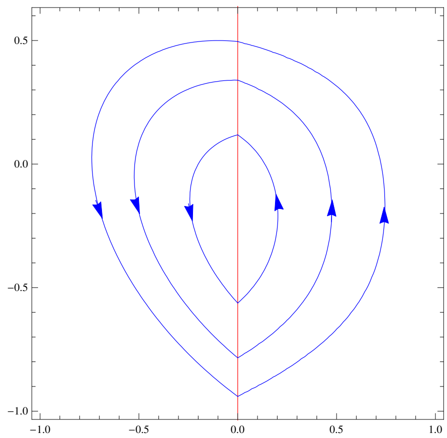

Figure 3. The three crossing limit cycles of the discontinuous piecewise differential system (13)–(14)

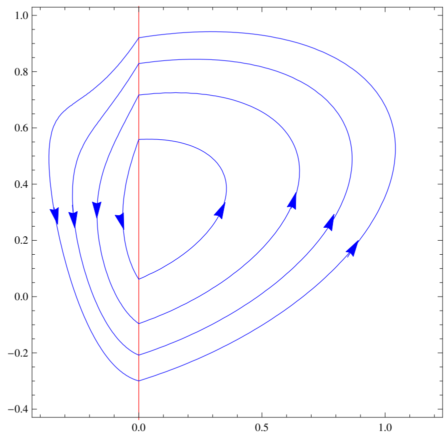

Figure 4. The four crossing limit cycles of the discontinuous piecewise differential system (16)–(17)

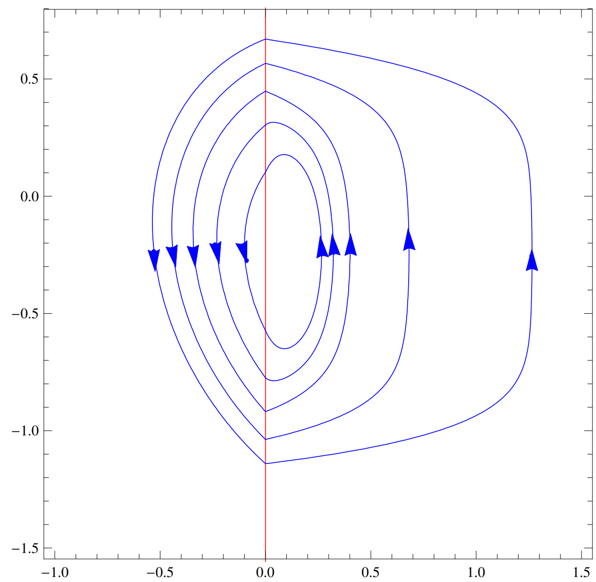

Figure 5. The five crossing limit cycles of the discontinuous piecewise differential systems (20)–(19)

Proof of statement (b) of Theorem 2. For $n = 2$ and in the half plane $R_2 = \{(x, y) : x < 0\}$ we consider the Hamiltonian isochronous global center (5) with its first integral $H_2(x, y)$, and in the half plane $R_1 = \{(x, y) : x > 0\}$ we consider the planar linear differential center (2) with its first integral $H_1(x, y)$.

Our goal is to prove that the discontinuous piecewise differential systems formed by system (5) and system (2) have at most one crossing limit cycle intersecting the line of discontinuity $x = 0$ in two different points $(0, y)$ and $(0, Y)$. We know that these two points must satisfy the system of equations

$$
\begin{aligned}
H_1(0,y) - H_1(0,Y) = &\ (y-Y)P_1(y,Y) = 0,\\
H_2(0,y) - H_2(0,Y) = &\ \frac{1}{2}(y-Y)Q_1(y,Y) = 0,
\end{aligned}
\tag{6}
$$

where $Q_1(y, Y)$ and $P_1(y, Y)$ are polynomials of degrees one and three, respectively, and their expressions are

$$
\begin{aligned}
P_1(y,Y) = &\ -2bd + a^2y + \omega^2 y + a^2 Y + \omega^2 Y,\\
Q_1(y,Y) = &\ b_1^2(y+Y) + 2b_1\big(c_1 + \delta\big(\gamma_1^2 + \beta_1^2\big(y^2 + yY + Y^2\big) + 2\beta_1\gamma_1(y\\
&+Y)\big)\big) + \beta_1(2\gamma_1 + \beta_1(y+Y))\big(2c_1\delta + \delta^2\big(2\gamma_1^2 + \beta_1^2\big(y^2+Y^2\big)\\
&+2\beta_1\gamma_1(y+Y)\big) + 1\big).
\end{aligned}
$$

From $P_1(y, Y) = 0$ we know that $y = f(Y)$. Substituting the expression of y in $Q_1(y, Y) = 0$, we obtain a quadratic equation in the variable Y, where this equation has at most two real solutions Y_1 and Y_2. Therefore system (6) has at most two real solutions. We can easily show that the two solutions verify the symmetry $(Y_1, f(Y_1)) = (f(Y_2), Y_2)$, which means that both solutions provide the same limit cycle for the discontinuous piecewise differential system (2)–(5). So statement (b) is proved if we provide an exemple of a discontinuous piecewise differential system (2)–(5) having one limit cycle for $n = 2$.

In the half plane R_1, we consider the linear differential center

$$
\dot{x} = 4 + x - 2.1473579210218228..y, \qquad \dot{y} = 2.5 + 7.916705377141101..x - y,
\tag{7}
$$

with its first integral

$$
H_1(x,y) = (7.91671..x - y)^2 + 15.8334..(2.5x - 4y) + 16y^2.
$$

In the half plane R_2, we consider the Hamiltonian isochronous global center

$$
\begin{aligned}
\dot{X} = &\ 0.0344828..(9((3x+0.4y-5.5)^2 + 5x - 9y - 6) - (0.4\big(2(3x\\
&+0.4y-5.5)^2 + (3x+0.4y-5.5)^2\big)\big)\frac{1}{(3x+0.4y-5.5)},\\
\dot{Y} = &\ 0.0344828..\big(9\big((3x+0.4y-5.5)^2 + 5x - 9y - 6\big) + \big(3\big(2(3x\\
&+0.4y-5.5)^2 + (3x+0.4y-5.5)^2)\big)\frac{1}{(3x+0.4y-5.5)},
\end{aligned}
\tag{8}
$$

with its first integral

$$
H_2(x,y) = \frac{1}{2}((3x+0.4y-5.5)^2 + ((3x+0.4y-5.5)^2 + 5x - 9y - 6)^2).
$$

The unique real solution of system (6) is $(1.6508469135171708.., 2.074661499255111..)$. So, the discontinuous piecewise differential system (7)–(8) has one limit cycle as in Fig. 1. $\square$

Proof of statement (c) of Theorem 2. For $n = 3$ in the half plane R_1 we consider the Hamiltonian isochronous global center (5) with its first integral $H_2(x, y)$, and in the half plane R_2 we consider the linear differential center (2) with its first integral $H_1(x, y)$. In order that the discontinuous piecewise differential system (2)–(5) has crossing limit cycle, it must intersect the line of discontinuit $x = 0$ in two different points $(0, y)$ and $(0, Y)$, and these two points must satisfy the system of equations

$$\begin{aligned} H_1(0,y) - H_1(0,Y) &= (y-Y)P_1(y,Y) = 0,\\ H_2(0,y) - H_2(0,Y) &= \frac{1}{2}(y-Y)Q_2(y,Y) = 0. \end{aligned} \tag{9}$$

where $Q_2(y, Y)$ is a polynomial of degree five given by

$$\begin{aligned} Q_2(y,Y) = \;& b_1^2(y+Y) + 2b_1\big(c_1 + \delta\big(\gamma_1^3 + \beta_1^3(y+Y)\big(y^2+Y^2\big) + 3\beta_1^2\gamma_1\big(y^2\\ & +yY+Y^2\big) + 3\beta_1\gamma_1^2(y+Y)\big)\big) + \beta_1\big(2c_1\delta\big(3\gamma_1^2 + \beta_1^2\big(y^2+yY+Y^2\big)\\ & +3\beta_1\gamma_1(y+Y)\big) + 2\gamma_1 + \delta^2(2\gamma_1 + \beta_1(y+Y))\big(\gamma_1^2 + \beta_1^2\big(y^2-yY\\ & +Y^2\big) + \beta_1\gamma_1(y+Y)\big)\big(3\gamma_1^2 + \beta_1^2\big(y^2+yY+Y^2\big) + 3\beta_1\gamma_1(y+Y)\big)\\ & +\beta_1(y+Y)\big). \end{aligned}$$

From $P_1(y, Y) = 0$ we obtain $y = f(Y)$. By Substituting the value of Y in $Q_2(y, Y) = 0$, we obtain a quartic equation in the variable y. This equation has at most four real solutions, symmetric in the sense stated in the proof of statment (b). Therefore system (9) has at most two real solutions (y, Y). Hence the discontinuous piecewise differential system (2) + (5) with $n = 3$ has at most two limit cycles.

Now we prove that our result is reached for $n = 3$, by giving an example with exactly two limit cycles. In the half plane R_1 we consider the Hamiltonian isochronous global center

$$\begin{aligned} \dot{X} = \;& 14.6449..\big(2\big(-(0.5x - 0.232929..y + 1.68341..)^3 + 4x - 2y + 2\big)\\ & +\frac{1}{(0.5x - 0.232929..y + 1.68341..)}\big(0.232929..\big(3(0.5x - 0.232929..y\\ & +1.68341..)^6 - 3(4x-2y+2)(0.5x - 0.232929..y + 1.68341..)^3\\ & +(0.5x - 0.232929..y + 1.68341..)^2\big)\big),\\ \dot{Y} = \;& 14.6449..\big(4\big(-(0.5x - 0.232929..y + 1.68341..)^3 + 4x - 2y + 2\big)\\ & +\frac{1}{(0.5x - 0.232929..y + 1.68341..)}\big(0.5\big(3(0.5x - 0.232929..y\\ & +1.68341..)^6 - 3(4x-2y+2)(0.5x - 0.232929..y + 1.68341..)^3\\ & +(0.5x - 0.232929..y + 1.68341..)^2\big)\big), \end{aligned} \tag{10}$$

with the first integral

$$\begin{aligned} H_2(x,y) = \;& \tfrac{1}{2}\big((0.5x - 0.232929..y + 1.68341..)^2 + \big(-(0.5x - 0.232929..y\\ & +1.68341..)^3 + 4x - 2y + 2\big)^2\big). \end{aligned}$$

In the half plane R_2, we consider the linear differential center

$$\dot{x} = 2 - x - \frac{37y}{4}, \qquad \dot{y} = -1 + 4x + y, \tag{11}$$

with the first integral

$$H_1(x, y) = 8(-x - 2y) + 36y^2 + (4x + y)^2.$$

The two real solutions of system (9) are $\Big(\dfrac{2}{37}(4-3\sqrt{10}), \dfrac{2}{37}\Big(4-3\sqrt{10}\Big)\Big)$ and $\Big(0, \dfrac{16}{37}\Big)$. Then the two crossing limit cycles of these systems are shown in Fig.2. $\square$

Proof of statement (d) of Theorem 2. For $n = 4$ and in the half plane R_1 we consider the Hamiltonian isochronous global center (5) with its first integral $H_2(x, y)$, and in the half plane R_2 we consider the linear differential center (2) with its first integral $H_1(x, y)$.

We have to prove that the piecewise differential systems formed by system (5) and system (2) have at most three crossing limit cycles, each one of them intersects the line of discontinuity $x = 0$ in two different points $(0, y)$ and $(0, Y)$. We know that these two points must satisfy the system of equations

$$
\begin{aligned}
H_1(0, y) - H_1(0, Y) &= (y - Y)P_1(y, Y) = 0,\\
H_2(0, y) - H_2(0, Y) &= \tfrac{1}{2}(y - Y)Q_3(y, Y) = 0.
\end{aligned} \tag{12}
$$

where $Q_3(y, Y)$ is a polynomial of degree seven given by

$$
\begin{aligned}
Q_3(y, Y) = \;& \tfrac{1}{2}\big(b_1^2(y + Y) + 2b_1\big(c_1 + \delta\big(\gamma_1^4 + 4\beta_1^3\gamma_1(y + Y)\big(y^2 + Y^2\big) + 6\beta_1^2\gamma_1^2\big(y^2\\
&+yY + Y^2\big) + \beta_1^4\big(y^4 + y^3Y + y^2Y^2 + yY^3 + Y^4\big) + 4\beta_1\gamma_1^3(y + Y)\big)\big)\\
&+\beta_1(2\gamma_1 + \beta_1(y + Y))\big(\delta\big(2\gamma_1^2 + \beta_1^2\big(y^2 + Y^2\big) + 2\beta_1\gamma_1(y + Y)\big)\big(2c_1\\
&+\delta\big(2\gamma_1^4 + \beta_1^4\big(y^4 + Y^4\big) + 4\beta_1^3\gamma_1\big(y^3 + Y^3\big) + 6\beta_1^2\gamma_1^2\big(y^2 + Y^2\big)\\
&+4\beta_1\gamma_1^3(y + Y)\big)\big) + 1\big)\big).
\end{aligned}
$$

From $P_1(y, Y) = 0$ we obtain $y = f(Y)$. By Substituting the value of Y in $Q_3(y, Y) = 0$, we obtain an equation of degree seven in the variable y, which has at most seven real solutions (y, Y), which are again symmetric. So these solutions provide at most three limit cycles of the discontinous piecewise differential systems (2)-(5) with $n = 4$.

Now we give an example of a discontinuous piecewise differential system (2)–(5) with $n = 4$ having three limit cycles. In the half plane R_1, we consider the Hamiltonian isochronous global center

$$
\begin{aligned}
\dot{X} = \;& 0.0526154..\big(0.2\big(- 0.00991602..(0.5x - 4.72646..y - 1)^4 - 4x\\
&-0.2y - 1.3326\big)\\
&+\frac{1}{(0.5x - 4.72646..y - 1)}\big(4.72646..\big(0.00039331..(0.5x - 4.72646..y\\
&-1)^8 - 0.0396641..(-4x - 0.2y - 1.3326..)(0.5x - 4.72646..y - 1)^4\\
&+(0.5x - 4.72646..y - 1)^2\big)\big)\big),\\
\dot{Y} = \;& 0.0526154..\big(4\big(- 0.00991602..(0.5x - 4.72646..y - 1)^4 - 4x - 0.2y\\
&-1.3326..\big) - \frac{1}{(0.5x - 4.72646..y - 1)}\big(0.5\big(0.00039331..(0.5x-\\
&4.72646..y - 1)^8 - 0.0396641..(-4x - 0.2y - 1.3326..)(0.5x\\
&-4.7264..6y - 1)^4 + (0.5x - 4.72646..y - 1)^2\big)\big)\big),
\end{aligned} \tag{13}
$$

with the first integral

$$
\begin{aligned}
H_2(x, y) = \;& \tfrac{1}{2}\big(\big(- 0.00991602..(0.5x - 4.72646..y - 1)^4 - 4x - 0.2y - 1.3326..\big)^2\\
&+(0.5x - 4.72646..y - 1)^2\big).
\end{aligned}
$$

In the half plane R_2, we consider the linear differential center

$$
\dot{x} = -2 - 3x - 9y, \qquad \dot{y} = -1 + 5x + 3y, \tag{14}
$$

with the first integral

$$
H_1(x, y) = 36y^2 + 10(-x + 2y) + (5x + 3y)^2.
$$

The three real solutions of system (12) are $(\frac{1}{45}(-10-\sqrt{1045}), \frac{1}{45}(-10-\sqrt{1045}))$, $(\frac{2}{45}(-5-4\sqrt{10}), \frac{2}{45}(-5+4\sqrt{10}))$ and $(\frac{1}{45}(-10-\sqrt{235}), \frac{1}{45}(-10+\sqrt{235}))$. Then the three crossing limit cycles of the discontinuous piecewise differential system (13)–(14) are shown in Fig. 3. □

Proof of statement (e) of Theorem 2. For $n=5$ and in the half plane R_2, we consider the Hamiltonian isochronous global center (5) with its first integral $H_2(x,y)$. In the half plane R_1 we consider the linear differential center (2) with its first integral $H_1(x,y)$. If the discontinuous piecewise differential system (2)–(5) has a crossing limit cycle intersecting the line of discontinuity $x=0$ in the points $(0,y)$ and $(0,Y)$, these points must satisfy the system of equations

$$
\begin{aligned}
&H_1(0,y)-H_1(0,Y)=(y-Y)P_1(y,Y),\\
&H_2(0,y)-H_2(0,Y)=\frac{1}{2}(y-Y)Q_4(y,Y)=0.
\end{aligned}
\tag{15}
$$

Where $Q_4(y,Y)$ is a polynomial of degree nine given by

$$
\begin{aligned}
Q_4(y,Y)=\ &\tfrac{1}{2}\big(b_1^2y+b_1^2Y+2\delta\big(b_1\gamma_1^5+10\beta_1^3\gamma_1^2\big(b_1(y+Y)\big(y^2+Y^2\big)+c_1\big(y^2\\
&+yY+Y^2\big)\big)+10\beta_1^2\gamma_1^3\big(b_1\big(y^2+yY+Y^2\big)+c_1(y+Y)\big)+\beta_1^5\big(b_1(y\\
&+Y)\big(y^2+yY+Y^2\big)\big(y^2-yY+Y^2\big)+c_1\big(y^4+y^3Y+y^2Y^2+yY^3\\
&+Y^4\big)\big)+5\beta_1^4\gamma_1\big(b_1\big(y^4+y^3Y+y^2Y^2+yY^3+Y^4\big)+c_1(y+Y)\big(y^2\\
&+Y^2\big)\big)+5\beta_1\gamma_1^4(b_1(y+Y)+c_1)\big)+2b_1c_1+2\beta_1\gamma_1+\beta_1\delta^2(2\gamma_1\\
&+\beta_1(y+Y))\big(\gamma_1^4+\beta_1^3\gamma_1(y+Y)\big(3y^2-4yY+3Y^2\big)+2\beta_1^2\gamma_1^2\big(2y^2\\
&-yY+2Y^2\big)+\beta_1^4\big(y^4-y^3Y+y^2Y^2-yY^3+Y^4\big)+2\beta_1\gamma_1^3(y+Y)\big)\\
&\big(5\gamma_1^4+5\beta_1^3\gamma_1(y+Y)\big(y^2+Y^2\big)+10\beta_1^2\gamma_1^2\big(y^2+yY+Y^2\big)+\beta_1^4\big(y^4\\
&+y^3Y+y^2Y^2+yY^3+Y^4\big)+10\beta_1\gamma_1^3(y+Y)\big)+\beta_1^2y+\beta_1^2Y\big).
\end{aligned}
$$

By Bezout Theorem (see for instance [20]), system (15) has at most nine real solutions (y,Y) with y different from Y, by symmetry the maximum number of crossing limit cycles of these systems is at most four.

We provide an example with exactly four limit cycles of the discontinuous piecewise differential system (2)–(5) for $n=5$. So in the half plane R_1 we consider the linear differential center

$$
\dot{x}=3+2x-\frac{29y}{3},\qquad \dot{y}=1+3x-2y,
\tag{16}
$$

with the first integral

$$
H_1(x,y)=6(x-3y)+(3x-2y)^2+25y^2.
$$

In the half plane R_2 we consider the Hamiltonian isochronous global center

$$
\begin{aligned}
\dot{X}=\ &1.93362..\big(1.03433..\big(-0.977806..(0.5x-1.)^5+3x-1.03433..y\\
&-0.656806..\big)\big),\\
\dot{Y}=\ &1.93362..\big(3\big(-0.977806(0.5x-1.)^5+3x-1.03433..y\\
&-0.656806..\big)+\frac{1}{(0.5x-1)}\big(0.5\big(-4.88903..(0.5x-1.)^5(3x\\
&-1.03433..y-0.656806..\big)+4.78052..(0.5x-1.)^{10}\\
&+(0.5x-1)^2\big)\big)\big),
\end{aligned}
\tag{17}
$$

with its first integral

$$H_2(x,y) = \frac{1}{2}((-0.977806..(0.5x-1.)^5+3x-1.03433..y-0.656806..)^2+(0.5x-1)^2).$$

The discontinuous piecewise differential system (16)–(17) has exactly four crossing limit cycles intersecting the straight line $x = 0$ in the following points: $\Big(\frac{1}{29}(9-\sqrt{313}), \frac{1}{29}(9+\sqrt{313})\Big)$, $\Big(\frac{1}{29}(9-\sqrt{226}), \frac{1}{29}(9+\sqrt{226})\Big)$, $\Big(\frac{1}{29}(9-\sqrt{134}), \frac{1}{29}(9+\sqrt{134})\Big)$, and $\Big(\frac{1}{29}(9-2\sqrt{13}), \frac{1}{29}(9+2\sqrt{13})\Big)$. These limit cycles are shown in Fig. 4. □

We shall need to use the following technical Proposition in the proof of statement (f) of Theorem 2.

Proposition 3. *Let $f_0, ..., f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. If $f_0, ..., f_n$ are linearly independent, then there exists $s_1, ..., s_n \in I$ and $\lambda_0, \dots, \lambda_n \in \mathbb{R}$ such that for every $j \in \{1, ..., n\}$ we have $\sum_{i=0}^{n} \lambda_i f_i(s_j) = 0$.*

For a proof of this result see for instance Proposition 1 of [14].

Proof of statement (f) of Theorem 2. For $n = 6$ we consider the linear differential center (2) with its first integral $H_1(x, y)$ in the half plane R_2, and the Hamiltonian isochronous global center (5) with its first integral $H_2(x, y)$ in the half plane R_1. In order that the discontinuous piecewise differential system (2)–(5) has a crossing limit cycle, it must intersect the discontinuous line $x = 0$ in two different points $(0, y)$ and $(0, Y)$. These points satisfy the equations

$$\begin{aligned} &H_1(0,y) - H_1(0,Y) = (y-Y)P_1(y,Y), \\ &H_2(0,y) - H_2(0,Y) = \frac{1}{2}(y-Y)Q_5(y,Y) = 0. \end{aligned} \tag{18}$$

Where $Q_5(y, Y)$ is a polynomial of degree eleven.

From $P_1(x, y) = 0$ we have $y = f(Y)$. Substituting the expression of y in $Q_5(y, Y)/(y - Y) = 0$, we obtain the polynomial

$$\begin{aligned} K(Y) = \quad & A_0 + A_1Y + A_2Y^2 + A_3Y^3 + A_4Y^4 + A_5Y^5 + A_6Y^6 + A_7Y^7 + A_8Y^8 \\ & +A_9Y^9 + A_{10}Y^{10}, \end{aligned}$$

of degree 10, where the A_i's for $i = 0, \dots, 10$, are given in the appendix.

Since the rank of the Jacobian matrix of the function $\mathcal{A} = (A_0, \dots, A_{10})$ with respect to its twelve parameters: $a, b, c, d, \alpha_1, \beta_1, \gamma_1, c_1, a_1, b_1, \delta, \omega$, is six of the eleven functions A_i, with $i = 0, \dots, 10$, which are linearly independent. according Proposition 3 it follows that the polynomial $K(Y)$ can have at most five real solutions.

We provide an example with exactly five limit cycles of the discontinuous piecewise differential system (2)-(5) for $n = 6$. So in the half plane R_2 we consider the linear differential center

$$\dot{x} = -x - \frac{17y}{4} - 1, \qquad \dot{y} = 4x + y - 2, \tag{19}$$

with the first integral

$$H_1(x,y) = (4x+y)^2 + 8(y-2x) + 16y^2.$$

In the half plane R_1, we consider the Hamiltonian isochronous global center

$$
(20)\quad \begin{aligned}
\dot{X} = &\frac{0.15486..}{(0.379849..+0.02x+1.61436..y)}\big(-(1.61436..((0.379849..+0.02x\\
&+1.61436..y)^2+2.96641..(-0.399469..+4x)(0.379849..+0.02x\\
&+1.61436..y)^6+1.46659..(0.379849..+0.02x+1.61436..y)^12))\big),\\
\dot{Y} = &\frac{0.15486..}{(0.379849..+0.02x+1.61436..y)}\big((4(-0.399469..+4x+0.494401..\\
&(0.379849..+0.02x+1.61436..y)^6)+(0.02((0.379849..+0.02x\\
&+1.61436..y)^2+2.96641..(-0.399469..+4x)(0.379849..+0.02x\\
&+1.61436..y)^6+1.46659..(0.379849..+0.02x+1.61436..y)^12))\big),
\end{aligned}
$$

with its first integral

$$
\begin{aligned}
H_2(x,y) = &\tfrac{1}{2}((0.379849..+0.02x+1.61436..y)^2+(-0.399469..+4x+0.494401..\\
&(0.379849..+0.02x+1.61436..y)^6)^2).
\end{aligned}
$$

The discontinuous piecewise differential system (19)–(20) has exactly five crossing limit cycles intersecting the straight line $x = 0$ in the following points: $\Big(\frac{1}{17}(-4-\sqrt{237}), \frac{1}{17}(-4+\sqrt{237}))\Big)$, $\Big(\frac{1}{17}(-4-\sqrt{186}), \frac{1}{17}(-4+\sqrt{186})\Big)$, $\Big(\frac{1}{17}(-4-3\sqrt{15}), \frac{1}{17}(-4+3\sqrt{15})\Big)$, $\Big(\frac{2}{17}(-2-\sqrt{21}), \frac{2}{17}(-2+\sqrt{21})\Big)$ and $\Big(\frac{1}{17}(-4-\sqrt{33}), \frac{1}{17}(-4+\sqrt{33})\Big)$. These limit cycles are shown in Fig. 5. This completes the proof of statement (f) of theorem 2 for $n = 6$.

In the general case for $n > 6$ we consider the linear differential center (2) with its first integral $H_1(x, y)$ in the half plane R_2, and the Hamiltonian isochronous global center (5) with $n > 6$ with its first integral $H_2(x, y)$, in the half plane R_1. If there exists a limit cycle of the discontinuous piecewise differential systems (2) –(5) it must intersect the discontinuity line $x = 0$ in two different points $(0, y)$ and $(0, Y)$. These two points must satisfy the system

$$
(21)\quad \begin{aligned}
H_1(0,y) - H_1(0,Y) = &\quad (y-Y)P_1(y,Y) = 0,\\
H_2(0,y) - H_2(0,Y) = &\quad (y-Y)Q_n(y,Y) = 0.
\end{aligned}
$$

where $Q_n(y, Y)$ is a polynomial of degree less than or equal $2n-1$. From $P_1(y, Y) = 0$ we get the expression of y. By replacing $y = f(Y)$ in the equation $Q_n(y, Y) = 0$, we obtain again a polynomial $D(Y)$ of degree at most $2n - 1$ in the variable Y. Assume that the degree of $D(Y)$ is $2n - 1$, it the degree is smaller we can use the same arguments for proving that the discontinupus piecewise differential system has at most five limit cycles. So we write

$$
D(Y) = C_0 + C_1Y + C_2Y^2 + \cdots + C_{2n-1}Y^{2n-1}.
$$

Let $M_{(2n)\times 12}$ be the Jacobian matrix of the function $\mathcal{C} = (C_0, \ldots, C_{2n-1})$ with respect to its twelve parameters.

We know that the rank of the matrix of order $(2n) \times 12$ for $n > 6$ is at most 12. Hence the maximum number of real roots of the polynomial $D(Y)$ is at most eleven, and due to the symmetry of its real roots we know that the maximum number of real solutions of system (21) is at most 5. Then the maximum number of crossing limit cycles of the discontinuous piecewise differential system (2)–(5) for $n > 6$ is at most 5. $\square$

4. The Appendix

Here we provide the values A_i, with $i = 0, \dots, 10$.

$$
\begin{aligned}
A_0 = {} & \frac{1}{(a^2+\omega^2)^{11}}(b_1c_1a^{22} + \beta_1\gamma_1 a^{22} + 11b_1c_1\omega^2a^{20} + bd\beta_1^2a^{20} + bb_1^2da^{20} + 11\omega^2 \\
& \beta_1 a^{20} + 55b_1c_1\omega^4a^{18} + 10bb_1^2d\omega^2a^{18} + 10bd\omega^2\beta_1^2a^{18} + 55\omega^4\beta_1\gamma_1a^{18} + 165b_1 \\
& c_1\omega^6a^{16} + 45bb_1^2d\omega^4a^{16} + 45bd\omega^4\beta_1^2a^{16} + 165\omega^6\beta_1\gamma_1a^{16} + 330b_1c_1\omega^8a^{14} \\
& +120bb_1^2d\omega^6a^{14} + 120bd\omega^6\beta_1^2a^{14} + 330\omega^8\beta_1\gamma_1a^{14} + 462b_1c_1\omega^{10}a^{12} + 210b \\
& b_1^2d\omega^8a^{12} + 210bd\omega^8\beta_1^2a^{12} + 462\omega^{10}\beta_1\gamma_1a^{12} + 462b_1c_1\omega^{12}a^{10} + 252bb_1^2d\omega^{10} \\
& a^{10} + 252bd\omega^{10}\beta_1^2a^{10} + 462\omega^{12}\beta_1\gamma_1a^{10} + 330b_1c_1\omega^{14}a^8 + 210bb_1^2d\omega^{12}a^8 \\
& +210bd\omega^{12}\beta_1^2a^8 + 330\omega^{14}\beta_1\gamma_1a^8 + 165b_1c_1\omega^{16}a^6 + 120bb_1^2d\omega^{14}a^6 + 120bd \\
& \omega^{14} + 165\omega^{16}\beta_1\gamma_1a^6 + 55b_1c_1\omega^{18}a^4 + 45bb_1^2d\omega^{16}a^4 + 45bd\omega^{16}\beta_1^2a^4 + 55\omega^{18} \\
& \beta_1\gamma_1a^4 + 11b_1c_1\omega^{20}a^2 + 10bb_1^2d\omega^{18}a^2 + 10bd\omega^{18}\beta_1^2a^2 + 11\omega^{20}\beta_1\gamma_1a^2 + b_1c_1 \\
& \omega^{22} + bb_1^2d\omega^{20} + bd\omega^{20}\beta_1^2 + 2\beta_1(512b^{11}d^{11}\beta_1^{11} + 3072b^{10}d^{10}(a^2+\omega^2)\gamma_1\beta_1^{10} \\
& +8448b^9d^9(a^2+\omega^2)^2\gamma_1^2\beta_1^9 + 14080b^8d^8(a^2+\omega^2)^3\gamma_1^3\beta_1^8 + 15840b^7d^7(a^2 \\
& +\omega^2)^4\gamma_1^4\beta_1^7 + 12672b^6d^6(a^2+\omega^2)^5\gamma_1^5\beta_1^6 + 7392b^5d^5(a^2+\omega^2)^6\gamma_1^6\beta_1^5 + 3168b^4 \\
& d^4(a^2+\omega^2)^7\gamma_1^7\beta_1^4 + 990b^3d^3(a^2+\omega^2)^8\gamma_1^8\beta_1^3 + 220b^2d^2(a^2+\omega^2)^9\gamma_1^9\beta_1^2 \\
& +33bd(a^2+\omega^2)^{10}\gamma_1^{10}\beta_1 + 3(a^2+\omega^2)^{11}\gamma_1^{11})\delta^2 + \omega^{22}\beta_1\gamma_1 + (a^2+\omega^2)^5(32b^5 \\
& d^5(2bb_1d + c_1(a^2+\omega^2))\beta_1^6 + 96b^4d^4(a^2+\omega^2)(2bb_1d+c_1(a^2+\omega^2))\gamma_1\beta_1^5 \\
& +120b^3d^3(a^2+\omega^2)^2(2bb_1d+c_1(a^2+\omega^2))\gamma_1^2\beta_1^4 + 80b^2d^2(a^2+\omega^2)^3(2bb_1d \\
& +c_1(a^2+\omega^2))\gamma_1^3\beta_1^3 + 30bd(a^2+\omega^2)^4(2bb_1d+c_1(a^2+\omega^2))\gamma_1^4\beta_1^2 + 6(a^2+\omega^2)^5 \\
& (2bb_1d+c_1(a^2+\omega^2))\gamma_1^5\beta_1 + b_1(a^2+\omega^2)^6\gamma_1^6)\delta),
\end{aligned}
$$

$$
\begin{aligned}
A_1 = {} & -\frac{1}{(a^2+\omega^2)^{10}}2bd\beta_1^2\delta(5\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)a^{18} + 5\gamma_1^2(9\gamma_1(22\beta_1\delta\gamma_1^6 \\
& +3b_1\gamma_1 + 4c_1\beta_1)\omega^2 + 2bd\beta_1(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1))a^{16} + 4\gamma_1(45\gamma_1^2(22\beta_1 \\
& \delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^4 + 20bd\beta_1\gamma_1(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^2 + 9b^2d^2\beta_1^2 \\
& (5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1)))a^{14} + 4(105\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^6 \\
& +70bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^4 + 63b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 \\
& +c_1))\omega^2 + 8b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1))a^{12} + 2(315\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1 \\
& \gamma_1 + 4c_1\beta_1)\omega^8 + 280bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^6 + 378b^2d^2\beta_1^2\gamma_1(5b_1 \\
& \gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^4 + 96b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^2 + 40b^4d^4 \\
& \beta_1^4(396\beta_1\delta\gamma_1^5 + b_1))a^{10} + 10(63\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^{10} + 70bd\beta_1\gamma_1^2 \\
& (99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^8 + 126b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^6 \\
& +48b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^4 + 40b^4d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1)\omega^2 \\
& +4752b^5d^5\beta_1^6\gamma_1^4\delta)a^8 + 20(21\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^{12} + 28bd\beta_1\gamma_1^2 \\
& (99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^{10} + 63b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^8 \\
& +32b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^6 + 40b^4d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1)\omega^4 \\
& +9504b^5d^5\beta_1^6\gamma_1^4\delta\omega^2 + 2464b^6d^6\beta_1^7\gamma_1^3\delta)a^6 + 4(45\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1) \\
& \omega^{14} + 70bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^{12} + 189b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1 \\
& (66\delta\gamma_1^6 + c_1))\omega^{10} + 120b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^8 + 200b^4d^4\beta_1^4(396 \\
& \beta_1\delta\gamma_1^5 + b_1)\omega^6 + 71280b^5d^5\beta_1^6\gamma_1^4\delta\omega^4 + 36960b^6d^6\beta_1^7\gamma_1^3\delta\omega^2 + 8448b^7d^7\beta_1^8\gamma_1^2\delta) \\
& a^4 + (45\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^{16} + 80bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1
\end{aligned}
$$

$\beta_1)\omega^{14} + 252b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^{12} + 192b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6$
$+6b_1\gamma_1 + c_1\beta_1)\omega^{10} + 400b^4d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1)\omega^8 + 190080b^5d^5\beta_1^6\gamma_1^4\delta\omega^6$
$+147840b^6d^6\beta_1^7\gamma_1^3\delta\omega^4 + 67584b^7d^7\beta_1^8\gamma_1^2\delta\omega^2 + 13824b^8d^8\beta_1^9\gamma_1\delta)a^2 + 2560b^9d^9$
$\beta_1^{10}\delta + 47520b^5d^5\omega^8\beta_1^6\gamma_1^4\delta + 49280b^6d^6\omega^6\beta_1^7\gamma_1^3\delta + 33792b^7d^7\omega^4\beta_1^8\gamma_1^2\delta$
$+13824b^8d^8\omega^2\beta_1^9\gamma_1\delta + 80b^4d^4\omega^{10}\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1) + 5\omega^{18}\gamma_1^3(22\beta_1\delta\gamma_1^6$
$+3b_1\gamma_1 + 4c_1\beta_1) + 10bd\omega^{16}\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1) + 32b^3d^3\omega^{12}\beta_1^3$
$(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1) + 36b^2d^2\omega^{14}\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))),$

$A_2 = \dfrac{1}{(a^2+\omega^2)^9}\beta_1^2\delta(5\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)a^{18} + 5\gamma_1^2(9\gamma_1(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1$
$+4c_1\beta_1)\omega^2 + 2bd\beta_1(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1))a^{16} + 4\gamma_1(45\gamma_1^2(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1$
$+4c_1\beta_1)\omega^4 + 20bd\beta_1\gamma_1(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^2 + 12b^2d^2\beta_1^2(5b_1\gamma_1 + 2\beta_1$
$(66\delta\gamma_1^6 + c_1)))a^{14} + 28(15\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^6 + 10bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6$
$+8b_1\gamma_1 + 6c_1\beta_1)\omega^4 + 12b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^2 + 2b^3d^3\beta_1^3(462\beta_1$
$\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1))a^{12} + 2(315\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^8 + 280bd\beta_1\gamma_1^2$
$(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^6 + 504b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^4$
$+168b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^2 + 88b^4d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1))a^{10}$
$+10(63\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^{10} + 70bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^8$
$+168b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^6 + 84b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1$
$\beta_1)\omega^4 + 88b^4d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1)\omega^2 + 12672b^5d^5\beta_1^6\gamma_1^4\delta)a^8 + 20(21\gamma_1^3(22\beta_1\delta\gamma_1^6$
$+3b_1\gamma_1 + 4c_1\beta_1)\omega^{12} + 28bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^{10} + 84b^2d^2\beta_1^2\gamma_1(5b_1$
$\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^8 + 56b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^6 + 88b^4d^4\beta_1^4$
$(396\beta_1\delta\gamma_1^5 + b_1)\omega^4 + 25344b^5d^5\beta_1^6\gamma_1^4\delta\omega^2 + 7744b^6d^6\beta_1^7\gamma_1^3\delta)a^6 + 4(45\gamma_1^3(22\beta_1\delta\gamma_1^6$
$+3b_1\gamma_1 + 4c_1\beta_1)\omega^{14} + 70bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1 + 6c_1\beta_1)\omega^{12} + 252b^2d^2\beta_1^2\gamma_1$
$(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^{10} + 210b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1)\omega^8 + 440b^4$
$d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1)\omega^6 + 190080b^5d^5\beta_1^6\gamma_1^4\delta\omega^4 + 116160b^6d^6\beta_1^7\gamma_1^3\delta\omega^2 + 30624b^7d^7$
$\beta_1^8\gamma_1^2\delta)a^4 + (45\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1)\omega^{16} + 80bd\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1$
$+6c_1\beta_1)\omega^{14} + 336b^2d^2\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6 + c_1))\omega^{12} + 336b^3d^3\beta_1^3(462\beta_1\delta\gamma_1^6$
$+6b_1\gamma_1 + c_1\beta_1)\omega^{10} + 880b^4d^4\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1)\omega^8 + 506880b^5d^5\beta_1^6\gamma_1^4\delta\omega^6 + 464640b^6d^6$
$\beta_1^7\gamma_1^3\delta\omega^4 + 244992b^7d^7\beta_1^8\gamma_1^2\delta\omega^2 + 56832b^8d^8\beta_1^9\gamma_1\delta)a^2 + 11776b^9d^9\beta_1^{10}\delta + 126720b^5d^5\omega^8$
$\beta_1^6\gamma_1^4\delta + 154880b^6d^6\omega^6\beta_1^7\gamma_1^3\delta + 122496b^7d^7\omega^4\beta_1^8\gamma_1^2\delta + 56832b^8d^8\omega^2\beta_1^9\gamma_1\delta + 176b^4d^4\omega^{10}$
$\beta_1^4(396\beta_1\delta\gamma_1^5 + b_1) + 5\omega^{18}\gamma_1^3(22\beta_1\delta\gamma_1^6 + 3b_1\gamma_1 + 4c_1\beta_1) + 10bd\omega^{16}\beta_1\gamma_1^2(99\beta_1\delta\gamma_1^6 + 8b_1\gamma_1$
$+6c_1\beta_1) + 56b^3d^3\omega^{12}\beta_1^3(462\beta_1\delta\gamma_1^6 + 6b_1\gamma_1 + c_1\beta_1) + 48b^2d^2\omega^{14}\beta_1^2\gamma_1(5b_1\gamma_1 + 2\beta_1(66\delta\gamma_1^6$
$+c_1))),$

$$A_3 = -\frac{1}{\left(a^2+\omega^2\right)^8}(4b\beta_1^4 d\delta(15a^{14}b_1\gamma_1^2 + 6a^{14}\beta_1 c_1\gamma_1 + 36a^{12}bb_1\beta_1\gamma_1 d + 6a^{12}b\beta_1^2 c_1 d$$
$$+105a^{12}b_1\gamma_1^2\omega^2 + 42a^{12}\beta_1 c_1\gamma_1\omega^2 + 26a^{10}b^2 b_1\beta_1^2 d^2 + 216a^{10}bb_1\beta_1\gamma_1 d\omega^2 + 36a^{10}b$$
$$\beta_1^2 c_1 d\omega^2 + 315a^{10}b_1\gamma_1^2\omega^4 + 126a^{10}\beta_1 c_1\gamma_1\omega^4 + 130a^8 b^2 b_1\beta_1^2 d^2\omega^2 + 540a^8 bb_1\beta_1\gamma_1 d\omega^4$$
$$+90a^8 b\beta_1^2 c_1 d\omega^4 + 525a^8 b_1\gamma_1^2\omega^6 + 210a^8\beta_1 c_1\gamma_1\omega^6 + 260a^6 b^2 b_1\beta_1^2 d^2\omega^4 + 720a^6 bb_1\beta_1$$
$$\gamma_1 d\omega^6 + 120a^6 b\beta_1^2 c_1 d\omega^6 + 525a^6 b_1\gamma_1^2\omega^8 + 210a^6\beta_1 c_1\gamma_1\omega^8 + 260a^4 b^2 b_1\beta_1^2 d^2\omega^6$$
$$+540a^4 bb_1\beta_1\gamma_1 d\omega^8 + 90a^4 b\beta_1^2 c_1 d\omega^8 + 315a^4 b_1\gamma_1^2\omega^{10} + 126a^4\beta_1 c_1\gamma_1\omega^{10} + 130a^2 b^2 b_1$$
$$\beta_1^2 d^2\omega^8 + 4\beta_1\delta(4368b^6\beta_1^6\gamma_1 d^6(a^2+\omega^2) + 8184b^5\beta_1^5\gamma_1^2 d^5(a^2+\omega^2)^2 + 8800b^4\beta_1^4\gamma_1^3 d^4$$
$$(a^2+\omega^2)^3 + 5940b^3\beta_1^3\gamma_1^4 d^3(a^2+\omega^2)^4 + 2574b^2\beta_1^2\gamma_1^5 d^2(a^2+\omega^2)^5 + 693b\beta_1\gamma_1^6 d(a^2$$
$$+\omega^2)^6 + 99\gamma_1^7(a^2+\omega^2)^7 + 2574 + 1024b^7\beta_1^7 d^7) + 216a^2 bb_1\beta_1\gamma_1 d\omega^{10} + 36a^2 b\beta_1^2 c_1 d$$
$$\omega^{10} + 105a^2 b_1\gamma_1^2\omega^{12} + 42a^2\beta_1 c_1\gamma_1\omega^{12} + 26b^2 b_1\beta_1^2 d^2\omega^{10} + 36bb_1\beta_1\gamma_1 d\omega^{12} + 6b\beta_1^2 c_1$$
$$d\omega^{12} + 15b_1\gamma_1^2\omega^{14} + 6\beta_1 c_1\gamma_1\omega^{14})),$$

$$A_4 = -\frac{1}{(a^2+\omega^2)^7}(\beta_1^4\delta(15a^{14}b_1\gamma_1^2 + 6a^{14}\beta_1 c_1\gamma_1 + 36a^{12}bb_1\beta_1\gamma_1 d + 6a^{12}b\beta_1^2 c_1 d + 105a^{12}b_1$$
$$\gamma_1^2\omega^2 + 42a^{12}\beta_1 c_1\gamma_1\omega^2 + 36a^{10}b^2 b_1\beta_1^2 d^2 + 216a^{10}bb_1\beta_1\gamma_1 d\omega^2 + 36a^{10}b\beta_1^2 c_1 d\omega^2$$
$$+315a^{10}b_1\gamma_1^2\omega^4 + 126a^{10}\beta_1 c_1\gamma_1\omega^4 + 180a^8 b^2 b_1\beta_1^2 d^2\omega^2 + 540a^8 bb_1\beta_1\gamma_1 d\omega^4 + 90a^8$$
$$b\beta_1^2 c_1 d\omega^4 + 525a^8 b_1\gamma_1^2\omega^6 + 210a^8\beta_1 c_1\gamma_1\omega^6 + 360a^6 b^2 b_1\beta_1^2 d^2\omega^4 + 720a^6 bb_1\beta_1\gamma_1 d$$
$$\omega^6 + 120a^6 b\beta_1^2 c_1 d\omega^6 + 525a^6 b_1\gamma_1^2\omega^8 + 210a^6\beta_1 c_1\gamma_1\omega^8 + 360a^4 b^2 b_1\beta_1^2 d^2\omega^6 + 540a^4$$
$$bb_1\beta_1\gamma_1 d\omega^8 + 90a^4 b\beta_1^2 c_1 d\omega^8 + 315a^4 b_1\gamma_1^2\omega^{10} + 126a^4\beta_1 c_1\gamma_1\omega^{10} + 180a^2 b^2 b_1\beta_1^2 d^2\omega^8$$
$$+4\beta_1\delta(14208b^6\beta_1^6\gamma_1 d^6(a^2+\omega^2) + 22704b^5\beta_1^5\gamma_1^2 d^5(a^2+\omega^2)^2 + 20240b^4\beta_1^4\gamma_1^3 d^4(a^2+$$
$$\omega^2)^3 + 10890b^3\beta_1^3\gamma_1^4 d^3(a^2+\omega^2)^4 + 3564b^2\beta_1^2\gamma_1^5 d^2(a^2+\omega^2)^5 + 693b\beta_1\gamma_1^6 d(a^2+\omega^2)^6$$
$$+99\gamma_1^7(a^2+\omega^2)^7 + 3824b^7\beta_1^7 d^7) + 216a^2 bb_1\beta_1\gamma_1 d\omega^{10} + 36a^2 b\beta_1^2 c_1 d\omega^{10} + 105a^2 b_1\gamma_1^2$$
$$\omega^{12} + 42a^2\beta_1 c_1\gamma_1\omega^{12} + 36b^2 b_1\beta_1^2 d^2\omega^{10} + 36bb_1\beta_1\gamma_1 d\omega^{12} + 6b\beta_1^2 c_1 d\omega^{12} + 15b_1\gamma_1^2\omega^{14}$$
$$+6\beta_1 c_1\gamma_1\omega^{14})),$$

$$A_5 = -\frac{1}{(a^2+\omega^2)^6}(2b\beta_1^6 d\delta(-(a^2+\omega^2)((a^2+\omega^2)(44\beta_1\gamma_1^2\delta(340b^2\beta_1^2\gamma_1 d^2(a^2+\omega^2) + 135b\beta_1$$
$$\gamma_1^2 d(a^2+\omega^2)^2 + 27\gamma_1^3(a^2+\omega^2)^3 + 480b^3\beta_1^3 d^3) + 3b_1(a^2+\omega^2)^3) + 15936b^4\beta_1^5\gamma_1 d^4\delta)$$
$$-5024b^5\beta_1^6 d^5\delta)),$$

$$A_6 = -\frac{1}{(a^2+\omega^2)^5}(\beta_1^6\delta((a^2+\omega^2)((a^2+\omega^2)(44\beta_1\gamma_1^2\delta(160b^2\beta_1^2\gamma_1 d^2(a^2+\omega^2 + 45b\beta_1\gamma_1^2 d(a^2$$
$$+\omega^2)^2 + 9\gamma_1^3(a^2+\omega^2)^3 + 300b^3\beta_1^3 d^3) + b_1(a^2+\omega^2)^3) + 12480b^4\beta_1^5\gamma_1 d^4\delta)$$
$$+4736b^5\beta_1^6 d^5\delta)),$$

$$A_7 = -\frac{1}{(a^2+\omega^2)^4}(-80b\beta_1^9 d\delta^2(42b^2\beta_1^2\gamma_1 d^2(a^2+\omega^2) + 33b\beta_1\gamma_1^2 d(a^2+\omega^2)^2 + 11\gamma_1^3(a^2$$
$$+\omega^2)^3 + 20b^3\beta_1^3 d^3)),$$

$$A_8 = -\frac{1}{(a^2+\omega^2)^3}(\beta_1^9\delta^2(60b^2\beta_1^2\gamma_1 d^2(a^2+\omega^2) + 33b\beta_1\gamma_1^2 d(a^2+\omega^2)^2 + 11\gamma_1^3(a^2+\omega^2)^3$$
$$+38b^3\beta_1^3 d^3),)),$$

$$A_9 = -\frac{1}{(a^2+\omega^2)^2}(-60b\beta_1^{11}d\delta^2(\gamma_1(a^2+\omega^2) + b\beta_1 d),$$

$$A_{10} = -\frac{1}{(a^2+\omega^2)^2}(-60b\beta_1^{11}d\delta^2(\gamma_1(a^2+\omega^2)+b\beta_1 d))).$$

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