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CHARACTERIZATION OF THE KUKLES POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING AN INVARIANT ALGEBRAIC CURVE

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Abstract. Let $f(x)$ and $g(x)$ be complex polynomials. We characterize all Kukles polynomial differential systems of the form

$$x' = y, \quad y' = -y^2 - f(x)y - g(x)$$

having an invariant algebraic curve. We show that expanding an invariant algebraic curve of these differential systems as a polynomial in the variable y, the first four higher coefficients of the polynomial defining the invariant algebraic curve determine completely these Kukles systems. In particular if the second and third higher coefficients of the polynomial defining the invariant algebraic curve satisfy a simple relation between them the invariant algebraic curve is of the form $(y + p(x))^n = 0$ for some polynomial $p(x)$ and $y + p(x) = 0$ is an invariant algebraic curve of the Kukles system for any complex polynomial $f(x)$.

1. Introduction and statement of the main results

Consider the Kukles system

$$x' = y, \quad y' = -y^2 - f(x)y - g(x), \tag{1}$$

where $f(x), g(x) \in \mathbb{C}[x]$, being $\mathbb{C}[x]$ the ring of polynomials in the variable x with coefficients in $\mathbb{C}$. Moreover we assume that f and g are not the zero polynomial. We also assume that system (1) has an invariant algebraic curve $F(x, y) = 0$, that we write as

$$F(x, y) = \sum_{j=0}^{n} a_j(x) y^{n-j}, \quad a_0(x) \neq 0, \ n \geq 2. \tag{2}$$

Without loss of generality we can assume that the coefficient of the highest degree of $a_0(x)$ in the variable x is 1, because we can always divide the invariant curve by such highest coefficient.

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We recall that a polynomial $F(x,y)=0$ is an *invariant algebraic curve* of system (1) if it satisfies

$$y\frac{\partial F}{\partial x}-(y^2+f(x)y+g(x))\frac{\partial F}{\partial y}=KF,$$

for some polynomial $K=K(x,y)$ called the *cofactor* of F.

The invariant algebraic curves are crucial elements in the study of the qualitative theory and integrability theory of the polynomial differential systems (see for instance [2, Chapter 8] and the references therein). In fact the existence of several invariant algebraic curves is a measure of the integrability of a polynomial differential system. The invariant algebraic curves are the main elements in the Darboux theory of integrability. This theory was started by Darboux [4], and developed later on for several authors as [1, 2, 3, 6, 7, 8, 9, 10, 11].

We introduce the main result of this paper.

Theorem 1. *Assume that the Kukles system* (1) *has an invariant algebraic curve $F(x,y)=0$ of the form* (2) *with $n\geq 2$. Then $K(x,y)=K_0(x)+yK_1(x)$, $a_0(x)=1$, $K_1(x)=-n$, $K_0(x)=a_1'(x)+a_1(x)-nf(x)$, and*

$$g(x)=\frac{-a_1(x)a_1'(x)+a_2'(x)-a_1^2(x)+2a_2(x)+a_1(x)f(x)}{n}. \tag{3}$$

Moreover the following hold:

(a) *If $(1-n)a_1(x)^2+2na_2(x)\neq 0$, then*

(a.1) *if $n=2$, write $a_2(x)-a_1^2(x)/4$ in irreducible polynomials as*

$$a_2(x)-a_1^2(x)/4=\alpha\prod_{j=0}^{r}(x-x_j)^{\beta_j}, \tag{4}$$

for some $\alpha\in\mathbb{C}$, $r\geq 1$ and positive integers β_j. Then there exists $Q(x)\in\mathbb{C}[x]$ such that

$$f(x)=\frac{a_1'(x)}{2}+a_1(x)+\frac{1}{4}Q(x)\sum_{j=0}^{r}\beta_j\prod_{k=0,k\neq j}^{r}(x-x_k),$$

and

$$a_1(x)=Q(x)\prod_{j=0}^{r}(x-x_j);$$

(a.2) *if $n\geq 3$, then*

$$f(x)=-\frac{A(x)}{(1-n)a_1(x)^2+2na_2(x)},$$

where

$$\begin{aligned}A(x)&=na_3'(x)+3na_3(x)+(n-1)a_1(x)^2a_1'(x)-(n-1)a_1(x)a_2'(x)\\&\quad+(n-1)a_1(x)^3-(3n-2)a_1(x)a_2(x)-na_1'(x)a_2(x),\end{aligned}$$

and $(1-n)a_1(x)^2 + 2na_2(x)$ *must divide* $A(x)$.

(b) *If* $(1-n)a_1(x)^2 + 2na_2(x) = 0$, *then*

$$F(x) = \Big(y + \frac{a_1(x)}{n}\Big)^n,$$

and $f(x)$ *is an arbitrary polynomial of* $\mathbb{C}[x]$.

Theorem 1 is proved in section 2. Note that it characterizes completely the possible invariant algebraic curves of system (1) when $(1-n)a_1(x)^2 + 2a_2(x)n = 0$. In this case, there is a unique possible invariant algebraic curve which is irreducible and it is an invariant algebraic curve of system (1) for any $f \in \mathbb{C}[x]$. In other words, the first two coefficients of the invariant algebraic curve determine completely the invariant algebraic curve for any system (1) independently of $f(x) \in \mathbb{C}[x]$ and $g(x) = -\frac{1}{n}(a_1(x) + a_1'(x)) - \frac{1}{n}f(x)$ for any $f \in \mathbb{C}[x]$.

As a concrete example of the application of Theorem 1 in the following proposition we characterize all the Kukles system (1) which has an invariant algebraic curve $F(x,y) = 0$ of the form (2) with $n = 2$ as in the statement (a.1) of Theorem 1 and being $a_i(x)$ for $i = 1, 2$ polynomials of degree two in the variable x.

Proposition 2. *Assume that the Kukles system* (1) *has an invariant algebraic curve* $F(x,y) = 0$ *of the form* (2) *with* $n = 2$, *with* $4a_2(x) - a_1(x)^2 \neq 0$ *and* $a_i(x) = \sum_{j=0}^{2} a_{ij}x^j$ *for* $i = 1, 2$.

(a) *If* $a_{12} \neq 0$, *then* $f(x) = a_{12}x^2 + (a_{11} + 2a_{12})x + a_{10} + a_{11}$ *and*

$$F(x,y) = y^2 + (a_{10} + a_{11}x + a_{12}x^2)y;$$

(b) *If* $a_{12} = 0$, *then we have*

(b.1) $f(x) = a_{11}x + \dfrac{1}{2}(2a_{10} + a_{11})$ *and*

$$F(x,y) = y^2 + (a_{10} + a_{11}x)y + \frac{a_{11}^2}{4}x^2 + \frac{a_{10}a_{11}}{2}x + a_{20};$$

(b.2) $a_{11} \neq 0$, $f(x) = a_{11}x + \dfrac{1}{4}(4a_{10} + 3a_{11})$ *and*

$$F(x,y) = y^2 + (a_{10} + a_{11}x)y + \frac{a_{11}^2}{4}x^2 + a_{21}x + \frac{4a_{10}a_{21} - a_{10}^2a_{11}}{4a_{11}};$$

(b.3) $f(x) = a_{10}$ *and* $F(y) = y^2 + a_{10}y + a_{20}$;

(b.4) $a_{11} \neq 0$, $f(x) = a_{11}x + a_{10} + a_{11}$ *and*

$$F(x,y) = y^2 + (a_{10} + a_{11}x)y + a_{22}x^2 + \frac{2a_{10}a_{22}}{a_{11}}x + \frac{a_{10}^2a_{22}}{a_{11}^2}.$$

The proof of Proposition 2 is given in section 3.

Similar results obtained for the generalized Liénard differential systems $x' = y$, $y' = -f(x)y - g(x)$ can be found in [5].

2. Proof of Theorem 1

We separate the proof of Theorem 1 into different lemmas.

Lemma 1. *Assume that the Kukles system* (1) *has an invariant algebraic curve* $F(x,y)=0$ *of the form* (2) *with* $n\geq 2$. *Then* $K(x,y)$, $a_0(x)$, $K_1(x)$, $K_0(x)$ *and* $g(x)$ *are as in the statement of Theorem 1.*

Proof. Let $F(x,y)=0$ be an invariant algebraic curve of system (1) as in (2). Then we have

$$\begin{aligned}
&\sum_{j=0}^{n} a_j'(x)y^{n+1-j}-\sum_{j=0}^{n}(n-j)a_j(x)y^{n-j+1}\\
&\quad -\sum_{j=0}^{n}(n-j)a_j(x)y^{n-j}f(x)-\sum_{j=0}^{n}(n-j)a_j(x)y^{n-j-1}g(x)\\
&\quad =\Big(\sum_{l=0}^{s}K_l(x)y^l\Big)\Big(\sum_{j=0}^{n}a_j(x)y^{n-j}\Big).
\end{aligned}\tag{5}$$

Note that it follows from (5) that $s\leq 1$. Therefore $K(x,y)=K_0(x)+K_1(x)y$.

Computing the coefficient of y^{n+1} in (5) we get

$$a_0'(x)-na_0(x)=K_1(x)a_0(x)\quad\text{that is}\quad a_0'(x)=(K_1(x)+n)a_0(x).$$

Since $K_1(x)$ must be a polynomial we must have that $a_0(x)$ is constant and so $a_0(x)=1$. Moreover $K_1(x)=-n$.

Computing the coefficient of y^n in (5) using that $a_0(x)=1$ and $K_1(x)=-n$ we get

$$a_1'(x)-(n-1)a_1(x)-nf(x)=K_1(x)a_1(x)+K_0(x)=-na_1(x)+K_0(x),$$

which yields $K_0(x)=a_1'(x)+a_1(x)-nf(x)$. Computing the coefficient of y^{n-1} in (5) we get

$$\begin{aligned}
&a_2'(x)-(n-2)a_2(x)-(n-1)f(x)a_1(x)-ng(x)\\
&=-na_2(x)+(a_1'(x)+a_1(x)-nf(x))a_1(x),
\end{aligned}$$

which provides $g(x)$ given in (3). Note that since f is a polynomial, so it is g. ∎

Lemma 2. *Assume that the Kukles system* (1) *has an invariant algebraic curve* $F(x,y)=0$ *of the form* (2) *with* $n=2$ *and* $4a_2(x)-a_1(x)^2\neq 0$. *Write* $a_2(x)-a_1^2(x)/4$ *into irreducible polynomials in* $\mathbb{C}[x]$ *as in* (4). *Then* $f(x)$ *and* $a_1(x)$ *are as in the statement (a.1) of Theorem 1.*

Proof. It follows from Lemma 1 that $K(x,y)=K_0(x)+K_1(x)y$ with $K_1(x)=-2$ and $K_0(x)$ and $g(x)$ as in the statement of Lemma 1 with $n=2$. Computing the independent term (coefficient of y^0 in (5)) we get

$$-g(x)a_1(x)=(a_1'(x)+a_1(x)-2f(x))a_2(x). \tag{6}$$

Introducing the value of $g(x)$ obtained in Lemma 1 in (12) we obtain

$$\begin{aligned}&a_1(x)^3+a_1(x)^2a_1'(x)-4a_1(x)a_2(x)-2a_1'(x)a_2(x)-a_1(x)a_2'(x)\\&\quad+(4a_2(x)-a_1(x)^2)f(x)=0.\end{aligned} \tag{7}$$

Since by hypothesis $a_2(x)\neq a_1^2(x)/4$, we have that

$$\begin{aligned}f(x)&=-\frac{a_1(x)^3+a_1(x)^2a_1'(x)-4a_1(x)a_2(x)-2a_1'(x)a_2(x)-a_1(x)a_2'(x)}{4a_2(x)-a_1(x)^2}\\&=\frac{a_1'(x)}{2}+a_1(x)+\frac{1}{4}a_1(x)\frac{(a_2(x)-a_1(x)^2/4)'}{a_2(x)-a_1^2(x)/4}.\end{aligned}$$

Taking into account that $f(x)$ must be a polynomial, we must have that expanding $a_2(x)-a_1^2(x)/4$ in irreducible factors in $\mathbb{C}[x]$ as in (4), then there exists $Q(x)\in\mathbb{C}[x]$ such that $f(x)$ and $a_1(x)$ are as in the statement (a.1) of Theorem 1. This concludes the proof of the lemma. □

Lemma 3. *Assume that the Kukles system* (1) *has an invariant algebraic curve* $F(x,y)=0$ *of the form* (2) *with* $n\geq 3$ *and* $(1-n)a_1(x)^2+2na_2(x)\neq 0$. *Then* $f(x)$ *is as in the statement (a.2) of Theorem 1.*

Proof. It follows from Lemma 1 that $K(x,y)=K_0(x)+K_1(x)y$ with $K_1(x)=-n$ and $K_0(x)$ and $g(x)$ as in the statement of Lemma 1. Computing the coefficient of y^{n-2} in (5) we get

$$\begin{aligned}&a_3'(x)-(n-3)a_3(x)-(n-2)f(x)a_2(x)-(n-1)g(x)a_1(x)\\&\quad=-na_3(x)+(a_1'(x)+a_1(x)-nf(x))a_2(x),\end{aligned}$$

and so

$$a_3'(x)+3a_3(x)+2f(x)a_2(x)-(n-1)g(x)a_1(x)=(a_1'(x)+a_1(x))a_2(x). \tag{8}$$

Substituting $g(x)$ given in Lemma 1 into (13) we obtain

$$f(x)((1-n)a_1(x)^2+2na_2(x)n)+A(x)=0, \tag{9}$$

with $A(x)$ as in statement (a.2) of Theorem 1. Taking into account that $f(x)$ is a polynomial, $(1-n)a_1(x)^2+2na_2(x)$ must divide $A(x)$. This completes the proof of the lemma. □

Lemma 4. *Assume that the Kukles system* (1) *has an invariant algebraic curve* $F(x,y)=0$ *of the form* (2) *with* $n\geq 2$ *and* $(1-n)a_1(x)^2+2na_2(x)n=0$. *Then* $F(x)$ *is as in statement (b) of Theorem 1 for any arbitrary polynomial* $f(x)\in\mathbb{C}[x]$.

Proof. We first study the case $n = 2$. In this case $4a_2(x) - a_1(x)^2 = 0$ and so $4a_2'(x) = 2a_1(x)a_1'(x)$ implying that (7) is automatically satisfied for any $f(x) \in \mathbb{C}[x]$. Hence,

$$F(x) = y^2 + a_1(x)y + a_2(x) = y^2 + a_1(x)y + \frac{a_1(x)^2}{4} = \Big(y + \frac{a_1(x)}{2}\Big)^2,$$

yielding the lemma for $n = 2$.

Now assume $n \geq 3$. Note that from Lemma 1 we have

$$g(x) = -\frac{a_1(x)}{n^2}(a_1(x) + a_1'(x)) + \frac{a_1(x)}{n}f(x). \tag{10}$$

We will first show by induction that

$$a_k(x) = \binom{n}{k}\frac{a_1(x)^k}{n^k}, \quad \text{for } k \geq 2. \tag{11}$$

Note that $a_2(x) = \binom{n}{2}a_1(x)^2/n^2$ by induction hypothesis.

Now we prove it for $a_3(x)$. It follows from (13) after introducing the value of $g(x)$ given in (10) that

$$a_3'(x) + 3a_3(x) - \frac{(n-1)(n-2)}{2n^2}a_1(x)^2(a_1(x) + a_1'(x)) = 0$$

and so

$$a_3(x) = \binom{n}{3}\frac{a_1(x)^3}{n^3}.$$

Assume that (11) holds for until $k - 1$ and we shall prove it for k.

Computing the coefficient of y^{n-k+1} in (5) we get

$$\begin{aligned} &a_k'(x) - (n-k)a_k(x) - (n-k+1)f(x)a_{k-1}(x) - (n-k+2)g(x)a_{k-2}(x) \\ &= -na_k(x) + (a_1'(x) + a_1(x) - nf(x))a_{k-1}(x), \end{aligned}$$

or equivalently, introducing the value of $g(x)$ given in (10),

$$\begin{aligned} &a_k'(x) + ka_k(x) + \Big((k-1)a_{k-1}(x) - \frac{n-k+2}{n}a_1(x)a_{k-2}(x)\Big)f(x) \\ &\quad + \Big(\frac{n-k+2}{n^2}a_1(x)a_{k-2}(x) - a_{k-1}(x)\Big)(a_1(x) + a_1'(x)) = 0 \end{aligned} \tag{12}$$

Note that by induction hypotheses we have

$$\begin{aligned} &(k-1)a_{k-1}(x) - \frac{n-k+2}{n}a_1(x)a_{k-2}(x) \\ &\quad = (k-1)\binom{n}{k-1}\frac{a_1(x)^{k-1}}{n^{k-1}} - (n-k+2)\binom{n}{k-2}\frac{a_1(x)^{k-1}}{n^{k-1}} = 0 \end{aligned}$$

and

$$\begin{aligned}\frac{n-k+2}{n^2}a_1(x)a_{k-2}(x)-a_{k-1}(x)&=\frac{k-1-n}{n}a_{k-1}(x)\\&=-\frac{n-k-1}{n}\binom{n}{k-1}\frac{a_1(x)^{k-1}}{n^{k-1}}=-k\binom{n}{k}\frac{a_1(x)^{k-1}}{n^k},\end{aligned}$$

and so (12) becomes

$$a_k'(x)+ka_k(x)-k\binom{n}{k}\frac{a_1(x)^{k-1}}{n^k}(a_1(x)+a_1'(x))=0,$$

which yields

$$a_k(x)=\binom{n}{k}\frac{a_1(x)^k}{n^k}.$$

So the induction hypothesis holds.

Computing the term of degree y^0 in (5) we get

$$\begin{aligned}&-g(x)a_{n-1}(x)-(a_1'(x)+a_1(x)-nf(x))a_n(x)\\&=\Big(\frac{a_1(x)}{n^2}(a_1'(x)+a_1(x))-\frac{a_1(x)}{n}f(x)\Big)\binom{n}{n-1}\frac{a_1(x)^{n-1}}{n^{n-1}}\\&\quad-(a_1'(x)+a_1(x)-nf(x))\binom{n}{n}\frac{a_1(x)^n}{n^n}\\&=\Big(\frac{1}{n}(a_1'(x)+a_1(x))-f(x)\Big)n\frac{a_1(x)^n}{n^n}-(a_1'(x)+a_1(x)-nf(x))\frac{a_1(x)^n}{n^n}=0,\end{aligned}$$

which is automatically satisfied.

Therefore for any $f(x)\in\mathbb{C}[x]$ and $g(x)$ given in (10) the unique possible invariant algebraic curve is

$$F(x)=\sum_{k=0}^{n}a_k(x)y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\frac{a_1(x)^k}{n^k}y^{n-k}=\Big(y+\frac{a_1(x)}{n}\Big)^n.$$

This completes the proof of the lemma. □

Proof of Theorem 1. The proof of Theorem 1 (a) follows from Lemmas 1, 2 and 3. The proof of Theorem 1 (b) follows from Lemma 4. □

3. Proof of Proposition 2

It follows from (7) that the function $f(x)$ has degree at most 2 in the variable x. We write it as $f(x)=b_0+b_1x+b_2x^2$. Introducing it into (5) with $K_1=-2$, $K_0=a_1'(x)+a_1(x)-2f(x)$ and $g(x)=(-a_1a_1'(x)+a_2'(x)-a_1^2(x)+2a_2(x)+a_1(x)f(x))/2$ we get that equation (5) multiplied by 2 becomes

$$
\begin{aligned}
&a_{10}^3 + a_{10}^2 a_{11} - 4a_{10}a_{20} - 2a_{11}a_{20} - a_{10}a_{21} - a_{10}^2 b_0 + 4a_{20}b_0 \\
&\quad + (3a_{10}^2 a_{11} + 2a_{10}a_{11}^2 + 2a_{10}^2 a_{12} - 4a_{11}a_{20} - 4a_{12}a_{20} - 4a_{10}a_{21} \\
&\quad - 3a_{11}a_{21} - 2a_{10}a_{22} - 2a_{10}a_{11}b_0 + 4a_{21}b_0 - a_{10}^2 b_1 + 4a_{20}b_1)x \\
&\quad + (3a_{10}a_{11}^2 + a_{11}^3 + 3a_{10}^2 a_{12} + 6a_{10}a_{11}a_{12} - 4a_{12}a_{20} - 4a_{11}a_{21} \\
&\quad - 5a_{12}a_{21} - 4a_{10}a_{22} - 4a_{11}a_{22} - a_{11}^2 b_0 - 2a_{10}a_{12}b_0 + 4a_{22}b_0 \\
&\quad - 2a_{10}a_{11}b_1 + 4a_{21}b_1 - a_{10}^2 b_2 + 4a_{20}b_2)x^2 + (a_{11}^3 + 6a_{10}a_{11}a_{12} \\
&\quad + 4a_{11}^2 a_{12} + 4a_{10}a_{12}^2 - 4a_{12}a_{21} - 4a_{11}a_{22} - 6a_{12}a_{22} - 2a_{11}a_{12}b_0 \\
&\quad - a_{11}^2 b_1 - 2a_{10}a_{12}b_1 + 4a_{22}b_1 - 2a_{10}a_{11}b_2 + 4a_{21}b_2)x^3 + (3a_{11}^2 a_{12} \\
&\quad + 3a_{10}a_{12}^2 + 5a_{11}a_{12}^2 - 4a_{12}a_{22} - a_{12}^2 b_0 - 2a_{11}a_{12}b_1 - a_{11}^2 b_2 \\
&\quad - 2a_{10}a_{12}b_2 + 4a_{22}b_2)x^4 + a_{12}(3a_{11}a_{12} + 2a_{12}^2 - a_{12}^2 b_1 - 2a_{11}b_2)x^5 \\
&\quad + a_{12}^2(a_{12} - b_2)x^6 = 0.
\end{aligned} \tag{13}
$$

We consider two cases: either $a_{12} \neq 0$ and $b_2 = a_{12}$, or $a_{12} = 0$.

In the first case, imposing $b_2 = a_{12}$ and $a_{12} \neq 0$ in (13), we obtain readily the solution in statement (a) of the proposition.

In the second case, imposing $b_2 = 0$ in (13) and using that $4a_2(x) - a_1(x)^2 \neq 0$ we get that $b_2 = 0$, $b_1 = a_{11}$ and (13) becomes

$$
\begin{aligned}
&(a_{10}^3 + a_{10}^2 a_{11} - 4a_{10}a_{20} - 2a_{11}a_{20} - a_{10}a_{21} - a_{10}^2 b_0 + 4a_{20}b_0) \\
&\quad + (2a_{10}^2 a_{11} + 2a_{10}a_{11}^2 - 4a_{10}a_{21} - 3a_{11}a_{21} - 2a_{10}a_{22} - 2a_{10}a_{11}b_0 \\
&\quad + 4a_{21}b_0)x + (a_{11}^2 - 4a_{22})(a_{10} + a_{11} - b_0)x^2 = 0.
\end{aligned} \tag{14}
$$

We have two possibilities: either $a_{22} = a_{11}^2/4$, or $b_0 = a_{10} + a_{11}$.

If $a_{22} = a_{11}^2/4$ then (14) becomes

$$
\begin{aligned}
&(a_{10}^3 + a_{10}^2 a_{11} - 4a_{10}a_{20} - 2a_{11}a_{20} - a_{10}a_{21} - a_{10}^2 b_0 + 4a_{20}b_0) \\
&\quad + \frac{1}{2}(a_{10}a_{11} - 2a_{21})(4a_{10} + 3a_{11} - 4b_0)x = 0,
\end{aligned}
$$

and we have two possible solutions: either $a_{21} = a_{10}a_{11}/2$, or $b_0 = (4a_{10} + 3a_{11})/4$. In the first case we get the solution as in statement (b.1) in the proposition. In the second case we get the condition $a_{10}^2 a_{11} + 4a_{11}a_{20} - 4a_{10}a_{21} = 0$. Taking into account that $f(x)$ is not the zero polynomial, we obtain the solutions (b.2) and (b.3) as in the statement of the proposition.

If $b_0 = a_{10} + a_{11}$, then (14) becomes

$$
(2a_{11}a_{20} - a_{10}a_{21}) + (a_{11}a_{21} - 2a_{10}a_{22})x = 0.
$$

Taking into account that $f(x)$ is not the zero polynomial, we get the two solutions (b.4) and (b.3) as in the statement of the proposition. This concludes the proof of the proposition.

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