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ON A CLASS OF GLOBAL CENTERS OF LINEAR SYSTEMS WITH QUINTIC HOMOGENEOUS NONLINEARITIES

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ABSTRACT. One of the classical and difficult problems in the qualitative theory of differential systems in the plane is the characterization of their centers. In this paper we characterize the linear and nilpotent global centers of polynomial differential systems with quintic homogeneous terms, with the symmetry $(x, y, t) \to (-x, y, -t)$ and without infinite singular points.

1. Introduction and statement of the main results

When all the orbits of a planar differential system in a punctured neighborhood of a singular point p are periodic we say that p is a *center*. If the orbits of a planar differential system in a punctured neighborhood of p spiral to p when $t \to \pm\infty$ then p is a *focus*. If the origin is either a focus or a center we say that it is a *monodromic singular point*. The clasical center-focus problem started with Poincaré [8] and Dulac [3] and in the present day many questions remain open about this problem.

It is known that if a real planar analytic system has a center, then after an affine change of variables and a change of scale of the time variable, it can be written in one of the following three ways:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called *linear type center*, which has a pair of purely imaginary eigenvalues,

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y)$$

called *nilpotent center*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

called *degenerated center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

We recall that a *global center* for a vector field on the plane is a singular point p having $\mathbb{R}^2$ filled of periodic orbits with the exception of the singular

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point. The easiest global center is the linear center $\dot{x} = -y$, $\dot{y} = x$. It is known (see [10, 1]) that quadratic polynomial differential systems have no global centers. The global degenerated centers of homogeneous or quasi-homogeneous polynomial differential systems were characterized in [2] and [7], respectively. However, the characterization of the global centers in the cases that the center is nilpotent or of linear-type has been done only when $P(x, y)$ and $Q(x, y)$ are cubic homogeneous polynomials (see [6, 5]).

In this paper we give a classification of linear and nilpotent type global centers for systems that have a linear part at the origin with purely imaginary eigenvalues and homogeneous quintic nonlinearities that satisfy the symmetry $(x, y, t) \to (-x, y, -t)$ and without singular points at infinity. We note that due to the symmetries the origin is a center.

We characterize the global centers of the two families of systems.

$$\begin{aligned} \dot{x} &= -y + a_2x^4y + a_4x^2y^3 + a_6y^5, \\ \dot{y} &= \ x + b_1x^5 + b_3x^3y^2 + b_5xy^4, \end{aligned} \tag{1}$$

(with a linear type center at the origin), and

$$\begin{aligned} \dot{x} &= y + a_2x^4y + a_4x^2y^3 + a_6y^5, \\ \dot{y} &= b_1x^5 + b_3x^3y^2 + b_5xy^4, \end{aligned} \tag{2}$$

(with a nilpotent center). Let

$$\begin{aligned} \Delta = \ & \frac{1}{a_6^4}\Big(b_1(4a_4^3 - 27a_6^2b_1) - 2a_6(a_2 - b_3)\big(2(a_2 - b_3)^2 + 9b_1(a_4 - b_5)\big) \\ & + a_4^2\big((a_2 - b_3)^2 - 12b_1b_5\big) - 2a_4b_5\big((a_2 - b_3)^2 - 6b_1b_5\big) \\ & + b_5^2\big((a_2 - b_3)^2 - 4b_1b_5\big)\Big). \end{aligned} \tag{3}$$

The first main theorem of the paper is the following.

Proposition 1. *Let* Δ *be as in* (3) *and consider system*

$$\begin{aligned} \dot{x} &= a_2x^4y + a_4x^2y^3 + a_6y^5, \\ \dot{y} &= b_1x^5 + b_3x^3y^2 + b_5xy^4. \end{aligned} \tag{4}$$

(A) *If* $a_6 < 0$ *and* $b_1 > 0$ *then system* (4) *has no singular points at infinity if and only if one of the following five sets of conditions holds:*
  - (A.1) $b_3 > a_2$, *and* $b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
  - (A.2) $b_3 \geq a_2$, $b_5 < a_4$, $\Delta < 0$;
  - (A.3) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$;
  - (A.4) $b_3 \leq a_2$, $b_5 > a_4$, $\Delta < 0$;
  - (A.5) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$.

(B) *If* $a_6 > 0$ *and* $b_1 < 0$ *then system* (4) *has no singular points at infinity if and only if one of the following five sets of conditions holds:*
  - (B.1) $b_3 \leq a_2$, $b_5 > a_4$;

(B.2) $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(B.3) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(B.4) $b_3 \geq a_2$, $b_5 < a_4$, $\Delta < 0$;
(B.5) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$.

The proof of Proposition 1 is given in section 3.

In order to state the following results we introduce the notation

$$\begin{aligned} L &= (a_4+b_3)^2 - 4(a_2+b_1)(a_6+b_5), \\ M &= 2(a_6+b_5)(a_2b_5 - a_6b_1) + (a_4+b_3)N, \\ N &= a_6b_3 - a_4b_5, \\ R &= 2b_5(a_6b_1 - a_2b_5) - b_3N, \\ S &= b_3^2 - 4b_1b_5. \end{aligned} \tag{5}$$

Proposition 2. *System* (1) *has a linear type center at the origin and no more finite singular points if and only if either* $a_6 < 0$, $b_1 > 0$ *and* $L < 0$, *or* $a_6 < 0$, $b_1 > 0$, $L > 0$, *and one of the following sets of conditions hold:*

(1) $M \pm N\sqrt{L} < 0$;
(2) $M + N\sqrt{L} < 0$, $M - N\sqrt{L} > 0$, $a_4 + b_3 - \sqrt{L} > 0$, $a_6 + b_5 > 0$;
(3) $M + N\sqrt{L} < 0$, $M - N\sqrt{L} > 0$, $a_4 + b_3 - \sqrt{L} < 0$, $a_6 + b_5 < 0$;
(4) $M + N\sqrt{L} > 0$, $M - N\sqrt{L} < 0$, $a_4 + b_3 + \sqrt{L} > 0$, $a_6 + b_5 > 0$;
(5) $M + N\sqrt{L} > 0$, $M - N\sqrt{L} < 0$, $a_4 + b_3 + \sqrt{L} < 0$, $a_6 + b_5 < 0$;
(6) $M \pm N\sqrt{L} > 0$, $a_4 + b_3 \pm \sqrt{L} < 0$, $a_6 + b_5 < 0$;
(7) $M \pm N\sqrt{L} > 0$, $a_4 + b_3 \pm \sqrt{L} > 0$, $a_6 + b_5 > 0$.

The proof of Proposition 2 is given in section 4.

A polynomial differential system can be extended in a unique analytic way to infinity using the Poincaré compactification, for more details on the Poincaré compactification see Chapter 5 of [4].

Theorem 3. *Consider* Δ *given in* (3) *and* L, M, N *given in* (5). *Systems* (1) *have a global center at the origin and do not have infinite singular points if and only if* $a_6 < 0$, $b_1 > 0$ *and one of the following sets of conditions hold:*

(I) $L < 0$, *and*
  (I.1) $b_3 > a_2$, $b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
  (I.2) $b_3 \geq a_2$, $b_5 < a_4$, $\Delta < 0$;
  (I.3) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$;
  (I.4) $b_3 \leq a_2$, $b_5 > a_4$, $\Delta < 0$;
  (I.5) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$.

(II) $L > 0$, $M \pm N\sqrt{L} < 0$, *and*
  (II.1) $b_3 > a_2$, $b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;

(II.2) $b_3 \geq a_2,\ b_5 < a_4,\ \Delta < 0$;
(II.3) $b_3 < a_2\ \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4,\ \Delta < 0$;
(II.4) $b_3 \leq a_2,\ b_5 > a_4,\ \Delta < 0$;
(II.5) $b_3 > a_2,\ a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3},\ \Delta < 0$.

(III) $L > 0,\ M+N\sqrt{L} < 0,\ M-N\sqrt{L} > 0,\ a_4+b_3-\sqrt{L} > 0,\ a_6+b_5 > 0$, *and*
(III.1) $b_3 > a_2,\ b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
(III.2) $b_3 \geq a_2,\ b_5 < a_4,\ \Delta < 0$;
(III.3) $b_3 < a_2,\ \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4,\ \Delta < 0$;
(III.4) $b_3 \leq a_2,\ b_5 > a_4,\ \Delta < 0$;
(III.5) $b_3 > a_2,\ a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3},\ \Delta < 0$.

(IV) $L > 0,\ M+N\sqrt{L} < 0,\ M-N\sqrt{L} > 0,\ a_4+b_3-\sqrt{L} < 0,\ a_6+b_5 < 0$, *and*
(IV.1) $b_3 > a_2,\ b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
(IV.2) $b_3 \geq a_2,\ b_5 < a_4,\ \Delta < 0$;
(IV.3) $b_3 < a_2,\ \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4,\ \Delta < 0$;
(IV.4) $b_3 \leq a_2,\ b_5 > a_4,\ \Delta < 0$;
(IV.5) $b_3 > a_2,\ a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3},\ \Delta < 0$.

(V) $L > 0,\ M+N\sqrt{L} > 0,\ M-N\sqrt{L} < 0,\ a_4+b_3+\sqrt{L} > 0,\ a_6+b_5 > 0$, *and*
(V.1) $b_3 > a_2,\ b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
(V.2) $b_3 \geq a_2,\ b_5 < a_4,\ \Delta < 0$;
(V.3) $b_3 < a_2,\ \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4,\ \Delta < 0$;
(V.4) $b_3 \leq a_2,\ b_5 > a_4,\ \Delta < 0$;
(V.5) $b_3 > a_2,\ a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3},\ \Delta < 0$.

(VI) $L > 0,\ M+N\sqrt{L} > 0,\ M-N\sqrt{L} < 0,\ a_4+b_3+\sqrt{L} < 0,\ a_6+b_5 < 0$, *and*
(VI.1) $b_3 > a_2,\ b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
(VI.2) $b_3 \geq a_2,\ b_5 < a_4,\ \Delta < 0$;
(VI.3) $b_3 < a_2,\ \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4,\ \Delta < 0$;
(VI.4) $b_3 \leq a_2,\ b_5 > a_4,\ \Delta < 0$;
(VI.5) $b_3 > a_2,\ a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3},\ \Delta < 0$.

(VII) $L > 0,\ M \pm N\sqrt{L} > 0,\ a_4 + b_3 \pm \sqrt{L} < 0,\ a_6 + b_5 < 0$, *and*
(VII.1) $b_3 > a_2,\ b_5 > \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
(VII.2) $b_3 \geq a_2,\ b_5 < a_4,\ \Delta < 0$;
(VII.3) $b_3 < a_2,\ \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4,\ \Delta < 0$;
(VII.4) $b_3 \leq a_2,\ b_5 > a_4,\ \Delta < 0$;
(VII.5) $b_3 > a_2,\ a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3},\ \Delta < 0$.

(VIII) $L>0$, $M \pm N\sqrt{L}>0$, $a_4+b_3 \pm \sqrt{L}>0$, $a_6+b_5>0$, *and*
(VIII.1) $b_3>a_2$, $b_5>\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$;
(VIII.2) $b_3 \geq a_2$, $b_5<a_4$, $\Delta<0$;
(VIII.3) $b_3<a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}<b_5<a_4$, $\Delta<0$;
(VIII.4) $b_3 \leq a_2$, $b_5>a_4$, $\Delta<0$;
(VIII.5) $b_3>a_2$, $a_4<b_5<\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta<0$.

The proof of Theorem 3 is given in section 5.

Remark 1. *All the conditions given in Theorem 3 are not empty except perhaps for the conditions (V.2)-(V.5) and (VI.2)-(VI.5) for which we cannot find examples. The following values of the parameters satisfy the set of the remaining conditions.*

(I.1) $a_6=-1$, $a_4=-1$, $a_2=-\frac{13}{4}$, $b_1=\frac{1}{2}$, $b_3=-2$, $b_5=0$;
(I.2) $a_6=-1$, $a_4=-68$, $a_2=-\frac{1367}{16}$, $b_1=-\frac{171}{2}$, $b_3=\frac{13}{4}$, $b_5=-\frac{283}{4}$;
(I.3) $a_6=-1$, $a_4=1$, $a_2=-1$, $b_1=\frac{3}{8}$, $b_3=0$, $b_5=0$;
(I.4) $a_6=-1$, $a_4=-1/2$, $a_2=-1$, $b_1=\frac{7}{32}$, $b_3=-1$, $b_5=0$;
(I.5) $a_6=-1$, $a_4=-68$, $a_2=-\frac{346709}{4096}$, $b_1=\frac{3}{512}$, $b_3=-\frac{173311}{2048}$, $b_5=-\frac{8679}{128}$.

(II.1) $a_6=-1$, $a_4=-16$, $a_2=0$, $b_1=\frac{5}{4}$, $b_3=1$, $b_5=\frac{7}{64}$;
(II.2) $a_6=-1$, $a_4=-\frac{123}{8}$, $a_2=0$, $b_1=4096$, $b_3=-2$, $b_5=-16$;
(II.3) $a_6=-1$, $a_4=8$, $a_2=-131072$, $b_1=10923$, $b_3=-512$, $b_5=3$;
(II.4) $a_6=-1$, $a_4=1$, $a_2=-\frac{15}{16}$, $b_1=\frac{1}{1024}$, $b_3=-1$, $b_5=\frac{69}{32}$;
(II.5) $a_6=-1$, $a_4=-\frac{57}{32}$, $a_2=-512$, $b_1=\frac{8179}{16}$, $b_3=-1$, $b_5=-1$.

(III.1) $a_6=-1$, $a_4=-\frac{495}{512}$, $a_2=0$, $b_1=\frac{5}{1024}$, $b_3=1$, $b_5=\frac{131}{128}$;
(III.2) $a_6=-1$, $a_4=4$, $a_2=\frac{427}{128}$, $b_1=\frac{165}{128}$, $b_3=\frac{105}{32}$, $b_5=2$;
(III.3) $a_6=-1$, $a_4=\frac{11}{8}$, $a_2=\frac{1}{1024}$, $b_1=\frac{3}{8192}$, $b_3=1$, $b_5=\frac{8388613}{8388608}$;
(III.4) $a_6=-1$, $a_4=\frac{1}{32}$, $a_2=1$, $b_1=\frac{3}{512}$, $b_3=1$, $b_5=\frac{257}{256}$;
(III.5) $a_6=-1$, $a_4=4$, $a_2=\frac{427}{128}$, $b_1=\frac{165}{128}$, $b_3=\frac{105}{32}$, $b_5=2$.

(IV.1) $a_6=-1$, $a_4=-32$, $a_2=-8$, $b_1=\frac{3}{512}$, $b_3=-1$, $b_5=-4$;
(IV.2) $a_6=-1$, $a_4=4$, $a_2=\frac{397}{64}$, $b_1=\frac{105}{16}$, $b_3=\frac{393}{64}$, $b_5=\frac{1}{2}$;
(IV.3) $a_6=-1$, $a_4=-16$, $a_2=1$, $b_1=7013$, $b_3=1$, $b_5=-43$;
(IV.4) $a_6=-1$, $a_4=-16$, $a_2=1$, $b_1=\frac{1}{65536}$, $b_3=1$, $b_5=-\frac{509}{32}$;
(IV.5) $a_6=-1$, $a_4=-13988$, $a_2=-32768$, $b_1=536870912$, $b_3=1$, $b_5=-4$.

(V.1) $a_6=-1$, $a_4=-37$, $a_2=-2$, $b_1=\frac{7}{256}$, $b_3=1$, $b_5=2$.

(VI.1) $a_6=-1$, $a_4=-16$, $a_2=-\frac{511}{8}$, $b_1=\frac{1}{8192}$, $b_3=1$, $b_5=\frac{1}{8}$.

(VII.1) $a_6 = -1,\ a_4 = -22,\ a_2 = -8,\ b_1 = \frac{3}{512},\ b_3 = -1,\ b_5 = -4;$
(VII.2) $a_6 = -1,\ a_4 = -\frac{163833}{32768},\ a_2 = -1,\ b_1 = \frac{7}{2048},\ b_3 = -\frac{65}{64},\ b_5 = -5;$
(VII.3) $a_6 = -1,\ a_4 = -\frac{4093}{4096},\ a_2 = -\frac{91}{512},\ b_1 = \frac{435}{8192},\ b_3 = 0,\ b_5 = -1;$
(VII.4) $a_6 = -\frac{107308641}{4},\ a_4 = -\frac{288142404537758509}{8796093022208},\ a_2 = -\frac{41943039}{4194304},$
$b_1 = \frac{72101605851328245}{83076749736557242056487941267521536},\ b_3 = -10,\ b_5 = -16384;$
(VII.5) $a_6 = -\frac{2409}{2048},\ a_4 = -\frac{2829}{512},\ a_2 = -\frac{600161531}{268435456},\ b_1 = 1,\ b_3 = 0,\ b_5 = -5.$

(VIII.1) $a_6 = -1,\ a_4 = -\frac{7}{8},\ a_2 = -\frac{87}{65536},\ b_1 = \frac{3}{2048},\ b_3 = 1,\ b_5 = \frac{53}{2};$
(VIII.2) $a_6 = -\frac{27}{128},\ a_4 = \frac{263}{128},\ a_2 = \frac{67543}{16384},\ b_1 = 1,\ b_3 = 4,\ b_5 = 2;$
(VIII.3) $a_6 = -\frac{11}{128},\ a_4 = \frac{69}{32},\ a_2 = 4,\ b_1 = \frac{479}{512},\ b_3 = 4,\ b_5 = 2;$
(VIII.4) $a_6 = -\frac{205}{128},\ a_4 = 2,\ a_2 = \frac{1355}{128},\ b_1 = 1,\ b_3 = \frac{169}{16},\ b_5 = 5;$
(VIII.5) $a_6 = -\frac{37}{128},\ a_4 = \frac{1}{2},\ a_2 = 2,\ b_1 = \frac{217}{1024},\ b_3 = \frac{515}{256},\ b_5 = 1.$

Proposition 4. *System* (2) *has a nilpotent center at the origin and no more finite singular points if and only if either* $a_6 > 0$, $b_1 < 0$ *and* $S < 0$ *or* $a_6 > 0$, $b_1 < 0$, $S > 0$, *and one of the following sets of conditions hold:*

(1) $R \pm N\sqrt{S} < 0$;
(2) $R + N\sqrt{S} < 0$, $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} < 0$, $b_5 < 0$;
(3) $R + N\sqrt{S} < 0$, $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} > 0$, $b_5 > 0$;
(4) $R + N\sqrt{S} > 0$, $R - N\sqrt{S} < 0$, $b_3 - \sqrt{S} < 0$, $b_5 < 0$;
(5) $R + N\sqrt{S} > 0$, $R - N\sqrt{S} > 0$, $b_3 \pm \sqrt{S} < 0$, $b_5 < 0$.

The proof of Proposition 4 is given in section 6.

Theorem 5. *Consider* Δ *given in* (3) *and* N, S, R *given in* (5). *Systems* (2) *have a global center at the origin and do not have infinite singular points if and only if* $a_6 > 0$, $b_1 < 0$ *and one of the following sets of conditions hold:*

(i) $S < 0$, *and*
  (i.1) $b_3 \le a_2$, $b_5 > a_4$;
  (i.2) $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
  (i.3) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
  (i.4) $b_3 \ge a_2$, $b_5 < a_4$, $\Delta < 0$;
  (i.5) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$.

(ii) $S > 0$, $R \pm N\sqrt{S} < 0$, *and*
  (ii.1) $b_3 \le a_2$, $b_5 > a_4$;
  (ii.2) $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
  (ii.3) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
  (ii.4) $b_3 \ge a_2$, $b_5 < a_4$, $\Delta < 0$;
  (ii.5) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$.

(iii) $S > 0$, $R + N\sqrt{S} < 0$, $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} < 0$, $b_5 < 0$, *and*

(iii.1) $b_3 \le a_2$, $b_5 > a_4$;
(iii.2) $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(iii.3) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(iii.4) $b_3 \ge a_2$, $b_5 < a_4$, $\Delta < 0$;
(iii.5) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$.

(iv) $S > 0$, $R + N\sqrt{S} < 0$, $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} > 0$, $b_5 > 0$, *and*
(iv.1) $b_3 \le a_2$, $b_5 > a_4$;
(iv.2) $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(iv.3) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(iv.4) $b_3 \ge a_2$, $b_5 < a_4$, $\Delta < 0$;
(iv.5) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$.

(v) $S > 0$, $R + N\sqrt{S} > 0$, $R - N\sqrt{S} < 0$, $b_3 - \sqrt{S} < 0$, $b_5 < 0$, *and*
(v.1) $b_3 \le a_2$, $b_5 > a_4$.

(vi) $S > 0$, $R + N\sqrt{S} > 0$, $R - N\sqrt{S} > 0$, $b_3 \pm \sqrt{S} < 0$, $b_5 < 0$, *and*
(vi.1) $b_3 \le a_2$, $b_5 > a_4$;
(vi.2) $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(vi.3) $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3}$, $\Delta < 0$;
(vi.4) $b_3 \ge a_2$, $b_5 < a_4$, $\Delta < 0$;
(vi.5) $b_3 < a_2$, $\frac{(a_2-b_3)a_4+a_6b_1}{a_2-b_3} < b_5 < a_4$, $\Delta < 0$.

The proof of Theorem 5 is given in section 7.

Remark 2. *The conditions given in Theorem 5 are not empty. For instance, the following values of the parameters satisfy the set of conditions for each item*

(i.1) $a_6 = 1$, $a_4 = 1$, $a_2 = 1$, $b_1 = -1$, $b_3 = 0$, $b_5 = -1$;
(i.2) $a_6 = \frac{9}{4}$, $a_4 = -3$, $a_2 = -1$, $b_1 = -1$, $b_3 = 0$, $b_5 = -1$;
(i.3) $a_6 = 1$, $a_4 = -2$, $a_2 = 0$, $b_1 = -1$, $b_3 = 0$, $b_5 = -1$;
(i.4) $a_6 = \frac{1}{8}$, $a_4 = -\frac{29}{32}$, $a_2 = 1$, $b_1 = -1$, $b_3 = 0$, $b_5 = -1$;
(i.5) $a_6 = 1$, $a_4 = 0$, $a_2 = 0$, $b_1 = -1$, $b_3 = 0$, $b_5 = -1$.

(ii.1) $a_6 = 1$, $a_4 = -\frac{11}{4}$, $a_2 = -1$, $b_1 = -1$, $b_3 = -4$, $b_5 = -\frac{7}{2}$;
(ii.2) $a_6 = 64$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{1}{8}$, $b_3 = -\frac{249}{256}$, $b_5 = -\frac{8189}{8192}$;
(ii.3) $a_6 = 1$, $a_4 = -1$, $a_2 = -1$, $b_1 = -7200$, $b_3 = -128$, $b_5 = -\frac{1}{2}$;
(ii.4) $a_6 = 2048$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{32765}{2048}$, $b_3 = -128$, $b_5 = -256$;
(ii.5) $a_6 = 4096$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{3}{256}$, $b_3 = -\frac{27}{32}$, $b_5 = -8$.

(iii.1) $a_6 = 1$, $a_4 = -\frac{11}{8}$, $a_2 = -1$, $b_1 = -1$, $b_3 = -4$, $b_5 = -\frac{7}{2}$;
(iii.2) $a_6 = 18$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{1}{8}$, $b_3 = -\frac{249}{256}$, $b_5 = -\frac{8189}{8192}$;
(iii.3) $a_6 = 1$, $a_4 = -1$, $a_2 = -1$, $b_1 = -16$, $b_3 = -32$, $b_5 = -\frac{1}{2}$;

(iii.4) $a_6 = 194561$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{1}{4}$, $b_3 = -128$, $b_5 = -256$;
(iii.5) $a_6 = 1873$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{3}{256}$, $b_3 = -\frac{27}{32}$, $b_5 = -8$.

(iv.1) $a_6 = \frac{1}{2}$, $a_4 = 2$, $a_2 = -1$, $b_1 = -1$, $b_3 = -4$, $b_5 = 1$;
(iv.2) $a_6 = \frac{5}{2}$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{3}{8}$, $b_3 = -\frac{249}{256}$, $b_5 = \frac{5}{1024}$;
(iv.3) $a_6 = 1$, $a_4 = -1$, $a_2 = -1$, $b_1 = -64$, $b_3 = -128$, $b_5 = \frac{1}{4}$;
(iv.4) $a_6 = \frac{129}{4096}$, $a_4 = 1$, $a_2 = -1$, $b_1 = -2048$, $b_3 = -64$, $b_5 = \frac{1}{128}$;
(iv.5) $a_6 = 1$, $a_4 = 1$, $a_2 = -1$, $b_1 = -1$, $b_3 = -\frac{27}{32}$, $b_5 = \frac{7}{64}$.

(v.1) $a_6 = \frac{1}{16}$, $a_4 = \frac{3}{4}$, $a_2 = 1$, $b_1 = -1$, $b_3 = -1$, $b_5 = -\frac{1}{8}$.

(vi.1) $a_6 = 1$, $a_4 = 0$, $a_2 = -1$, $b_1 = -1$, $b_3 = -4$, $b_5 = -\frac{7}{2}$;
(vi.2) $a_6 = \frac{1}{8}$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{1}{8}$, $b_3 = -\frac{249}{256}$, $b_5 = -\frac{8189}{8192}$;
(vi.3) $a_6 = \frac{1}{512}$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{63237}{65536}$, $b_3 = -\frac{125}{64}$, $b_5 = -\frac{253}{256}$;
(vi.4) $a_6 = 4054$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{15747}{1048576}$, $b_3 = -\frac{251}{128}$, $b_5 = -64$;
(vi.5) $a_6 = 57$, $a_4 = -1$, $a_2 = -1$, $b_1 = -\frac{3}{256}$, $b_3 = -\frac{27}{32}$, $b_5 = -8$.

2. Preliminary Results

Routh-Hurwitz criterion (see [9] pg. 167). For the algebraic equation

$$A_3x^3 + A_2x^2 + A_1x + A_0 = 0, \tag{6}$$

with real coefficients and $A_3 > 0$, the number of roots with positive real part is equal to the number of sign alterations in the sequence A_3, A_2, $A_2(A_1A_2 - A_0A_3)$, A_0. Moreover, all the roots of (6) have negative real part if and only if all the expressions A_3, A_2, $A_1A_2 - A_0A_3$, $A_0(A_1A_2 - A_0A_3)$ are positive.

Lemma 6. *Consider the polynomial*

$$P(Y) = Y^3 + \frac{a_4 - b_5}{a_6}Y^2 + \frac{a_2 - b_3}{a_6}Y - \frac{b_1}{a_6}, \tag{7}$$

where $a_6 \neq 0$, and let Δ be its discriminant. Then $P(Y)$ has

i) *three roots with negative real part if and only if one of the following two sets of conditions holds*
   1) $a_6 > 0$, $b_1 < 0$, $b_3 < a_2$, *and* $b_5 < \frac{(a_2-b_3)a_4-b_1}{a_2-b_3}$;
   2) $a_6 < 0$, $b_1 > 0$, $b_3 > a_2$, *and* $b_5 > \frac{(a_2-b_3)a_4-b_1}{a_2-b_3}$.

ii) *two complex roots with positive real part and one negative real root if and only if $\Delta < 0$ and one of the following eight sets of conditions holds*
   1) $a_6 > 0$, $b_1 < 0$, $b_3 \leq a_2$, $b_5 > a_4$;
   2) $a_6 < 0$, $b_1 > 0$, $b_3 \geq a_2$, $b_5 < a_4$;
   3) $a_6 > 0$, $b_1 < 0$, $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4-b_1}{a_2-b_3}$;

4) $a_6 < 0$, $b_1 > 0$, $b_3 < a_2$, $\frac{(a_2-b_3)a_4-b_1}{a_2-b_3} < b_5 < a_4$;
5) $a_6 > 0$, $b_1 < 0$, $b_3 \geq a_2$, $b_5 < a_4$;
6) $a_6 < 0$, $b_1 > 0$, $b_3 \leq a_2$, $b_5 > a_4$;
7) $a_6 > 0$, $b_1 < 0$, $b_3 < a_2$, $\frac{(a_2-b_3)a_4-b_1}{a_2-b_3} < b_5 < a_4$;
8) $a_6 < 0$, $b_1 > 0$, $b_3 > a_2$, $a_4 < b_5 < \frac{(a_2-b_3)a_4-b_1}{a_2-b_3}$.

The proof of the previous lemma is follows by using the Routh-Hurwitz criterion.

3. Proof of Proposition 1

The infinite singular points of system (4) in the Poincaré disc are the real solutions of the expression

$$-b_1x^6 + (a_2 - b_3)x^4y^2 + (a_4 - b_5)x^2y^4 + a_6y^6 = 0.$$

Doing the change of variables $x = \sqrt{X}$, $y = \sqrt{Y}$ in the previous expression we have

$$-b_1X^3 + (a_2 - b_3)X^2Y + (a_4 - b_5)XY^2 + a_6Y^3 = 0.$$

Since this expression is homogeneous, we can consider $X = 1$ and solve it in the variable Y. Doing so, we obtain the equation

$$a_6Y^3 + (a_4 - b_5)Y^2 + (a_2 - b_3)Y - b_1 = 0.$$

Note that $a_6 = 0$ must be different from zero, otherwise system (4) would have singular points at infinity, so we can consider the polynomial

$$P(Y) = Y^3 + \frac{a_4 - b_5}{a_6}Y^2 + \frac{a_2 - b_3}{a_6}Y - \frac{b_1}{a_6}.$$

Note that $P(Y)$ has degree three in the variable Y and so it always has a real solution. In order to guarantee that system (4) does not have any singular points at infinity and due to the change of variables, we require that solutions Y of $P(Y)$ are either complex or negative. Thus, Lemma 6 give us the conditions on the parameters so that the polynomial $P(Y)$ has either complex roots or negative roots.

4. Proof of Proposition 2

The finite singular points of system (1) are $(0, 0)$, $(0, a_6^{-1/4})$, $(-b_1^{-1/4}, 0)$, and the eight solutions of the algebraic system

$$-1 + a_2x^4 + a_4x^2y^2 + a_6y^4 = 0 \quad \text{and} \quad 1 + b_1x^4 + b_3x^2y^2 + b_5y^4 = 0,$$

that is, $(\pm x_+, \pm y_+)$, $(\pm x_-, \pm y_-)$, where

$$x_\pm = \sqrt{\frac{\mp\sqrt{2}(a_6 + b_5)}{\sqrt{M + N\sqrt{L}}}}, \quad y_\pm = \sqrt{\frac{\pm(a_4 + b_3 + \sqrt{L})}{\sqrt{M + N\sqrt{L}}}},$$

and $(\pm\bar{x}_+, \pm\bar{y}_+)$, $(\pm\bar{x}_-, \pm\bar{y}_-)$, where

$$\bar{x}_\pm = \sqrt{\frac{\mp\sqrt{2}(a_6 + b_5)}{\sqrt{M - N\sqrt{L}}}}, \quad \bar{y}_\pm = \sqrt{\frac{\pm(a_4 + b_3 - \sqrt{L})}{\sqrt{M - N\sqrt{L}}}},$$

with L, M and N introduced in (5).

Note that if $a_6 < 0$ and $b_1 > 0$ then the points $(0, a_6^{-1/4})$, $(-b_1^{-1/4}, 0)$ do not exist. On the other hand, if $L < 0$ then the other singular points do not exist either. If $L > 0$ then in order that all the singular points different from the origin do not exist, we must have (besides the conditions $a_6 < 0$ and $b_1 > 0$) the sets of conditions (1) to (7) given in the statement of the theorem.

5. Proof of theorem 3

The proof of this theorem follows from Proposition 2 which gives us the conditions for system (1) to have the origin as the unique finite singular point and from Proposition 1 item (A) which gives us the conditions for the non-existence of singular points at infinity.

6. Proof of Proposition 4

The origin is a nilpotent singular point of system (2) so we can apply Theorem 3.5 of [4]. Doing so, we obtain that $y = f(x) = 0$ and then $B(x, f(x)) = b_1 x^5 + \dots$ and $G(x) \equiv 0$. Taking into account the symmetry of system (2) we conclude that the origin of system (2) is a center if $b_1 < 0$.

Other finite singular points of system (2) are $(0, -(-a_6)^{-1/4})$, $(0, (-a_6)^{-1/4})$, and the eight solutions of the algebraic system

$$1 + a_2 x^4 + a_4 x^2 y^2 + a_6 y^4 = 0 \quad \text{and} \quad b_1 x^4 + b_3 x^2 y^2 + b_5 y^4 = 0,$$

that is, $(\pm x_+, \pm y_+)$, $(\pm x_-, \pm y_-)$, where

$$x_\pm = \sqrt{\frac{\mp\sqrt{2}\, b_5}{\sqrt{R + N\sqrt{S}}}}, \quad y_\pm = \sqrt{\frac{\pm(b_3 - \sqrt{S})}{\sqrt{2}\sqrt{R + N\sqrt{S}}}}, \tag{8}$$

and $(\pm\bar{x}_+, \pm\bar{y}_+)$, $(\pm\bar{x}_-, \pm\bar{y}_-)$, where

$$\bar{x}_\pm = \sqrt{\frac{\mp\sqrt{2}\, b_5}{\sqrt{R - N\sqrt{S}}}}, \quad \bar{y}_\pm = \sqrt{\frac{\pm(b_3 + \sqrt{S})}{\sqrt{2}\sqrt{R - N\sqrt{S}}}}, \tag{9}$$

with R, S and N introduced in (5).

Note that if $a_6 > 0$ then the points $(0, -(-a_6)^{-1/4})$, $(0, (-a_6)^{-1/4})$, do not exist. On the other hand, if $S < 0$ then the other eight singular points do not exist either. So, the first set of conditions in order that system (2) has a nilpotent center and no more finite singular points is $a_6 > 0$, $b_1 < 0$ and $S < 0$.

If $a_6 > 0$, $b_1 < 0$ and $S > 0$ then in order that the singular points given by (8) and (9) do not exist we must have that either $R + N\sqrt{S} < 0$ and $R - N\sqrt{S} < 0$ (this give us condition (1) of the theorem), or all other possible combinations so that the points do not exist. The first one is $R + N\sqrt{S} < 0$ (with this condition the points $(\pm x_+, \pm y_+)$, $(\pm x_-, \pm y_-)$ do not exist the points), and either $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} < 0$, $b_5 < 0$ (this give us the set of conditions (2) of the theorem), or $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} > 0$ and $b_5 > 0$ (this give us the set of conditions (3) of the theorem). Also, we have two set of conditions corresponding to $R - N\sqrt{S} < 0$ (with this condition the points $(\pm\bar{x}_+, \pm\bar{y}_+)$, $(\pm\bar{x}_-, \pm\bar{y}_-)$ do not exist), and either $R + N\sqrt{S} > 0$, $b_3 - \sqrt{S} < 0$, $b_5 < 0$ (this give us the set of conditions (4) of the theorem), or $R + N\sqrt{S} > 0$, $b_3 - \sqrt{S} > 0$ and $b_5 > 0$. However, these last two inequalities cannot hold simultaneously, and so we do not need to consider this set of conditions. Finally, we have the cases $R + N\sqrt{S} > 0$, $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} < 0$, $b_3 - \sqrt{S} < 0$, $b_5 < 0$ (this give us the set of conditions (5) of the theorem), or $R + N\sqrt{S} > 0$, $R - N\sqrt{S} > 0$, $b_3 + \sqrt{S} > 0$, $b_3 - \sqrt{S} > 0$, $b_5 > 0$. Again these last two inequalities cannot be satisfied simultaneously, and so we do not consider this set of conditions.

7. Proof of theorem 5

The proof of this theorem follows from Proposition 4 which gives us the conditions for system (2) to have the origin as the unique finite singular point and from Proposition 1 item (B) which gives us the conditions for the non-existence of singular points at infinity.

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