

Article

# Integrability Properties of the Slepyan–Palmov Model Arising in the Slepyan–Palmov Medium

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**Abstract:** This study investigates the Slepyan–Palmov (SP) model, which describes plane longitudinal waves propagating within a medium comprising a carrier medium and nonlinear oscillators. The primary objective is to analyze the integrability properties of this model. The research entails two key aspects. Firstly, the study explores the group invariant solution by utilizing reductions in symmetry subalgebras based on the optimal system. Secondly, the conservation laws are studied using the homotopy operator, which offers advantages over the conventional multiplier approach, especially when arbitrary functions are absent from both the equation and characteristics. This method proves advantageous in handling complex multipliers and yields significant outcomes.

**Keywords:** Lie symmetry method; the Slepyan–Palmov model; optimal system; longitudinal wave; conservation laws; homotopy operator

**MSC:** 70S10; 70G65; 35B06; 37K06; 17B81; 35R03



**Citation:** Usman, M.; Hussain, A.; Zaman, F.D.; Ibeas, A.; Almalki, Y.

Integrability Properties of the Slepyan–Palmov Model Arising in the Slepyan–Palmov Medium.

*Mathematics* **2023**, *11*, 4545. <https://doi.org/10.3390/math11214545>

Academic Editor: Alain Rapaport

Received: 8 October 2023

Revised: 27 October 2023

Accepted: 2 November 2023

Published: 3 November 2023



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## 1. Introduction

The Cauchy stress principle is a fundamental assumption in classical continuum mechanics which establishes a key equivalence between the action of all internal forces applied to an elementary area and the action of their resultant force applied at the center of the area. This principle forms the backbone of classical continuum mechanics and plays a crucial role in understanding the behavior of continuous materials under various loads and deformations. However, as engineering and scientific investigations have advanced, it has become evident that the simple stress principle embodied by Cauchy’s hypothesis might not fully capture the complexity of certain materials and deformation scenarios. In more general cases, the action of an arbitrary system of forces on a deformable solid cannot be solely represented by stresses alone. In addition to stresses, moment stresses emerge, leading to the formation of asymmetric tensors. These moment stresses significantly influence the mechanical response of materials, especially in situations where classical continuum mechanics fall short of describing real-world phenomena accurately.

To adequately account for these complex behaviors and to broaden the understanding of material responses, the introduction of additional degrees of freedom in the medium becomes necessary. This involves considering a physically infinitesimal volume (over which the properties of the medium are averaged) not as a simple material point, but as a more intricate object with new degrees of freedom. This recognition of additional degrees of freedom allows for the inclusion of a microstructure within the material, such as

graininess, fibrousness, or cellular structures present in real-world materials. By embracing this expanded perspective, the range of properties that can be modeled within a continuous medium expands considerably. The presence of internal microstructure and the consideration of additional degrees of freedom enable a more comprehensive representation of the mechanical response of materials, particularly in cases where classical continuum mechanics might be limited in its accuracy. Both the classical continuum model and the generalized continuum model are extensively utilized in modern deformable solid mechanics [1–3].

In addition to the classical continuum model and the generalized continuum model, there are other models used in deformable solid mechanics. One example is the Cosserat continuum [4], which considers internal rotational degrees of freedom. This theory was originally published in Russian and it has been subject to comments and discussions [5]. Gradient models, like the Leroux continuum [6], Jeremillo model, and Tupin model [7], also exist. These models incorporate gradients of certain fields to handle materials with spatial variations in properties, proving useful in describing complex materials.

Furthermore, there are models that account for media with oscillatory degrees of freedom. These models address the presence of oscillations or vibrations within the material. The works of Slepyan [8] in one-dimensional systems and Palmov [9] in three-dimensional systems have significantly contributed to this area of research. These diverse models enrich our understanding of material behaviors and offer valuable insights into various physical phenomena in deformable solids.

The SP model combines a linearly elastic carrier medium with non-interacting oscillators (elastic or viscoelastic) suspended at each point. It postulates that the dynamic behavior of the carrier model is described by the Lamé equations and the oscillators fixed within it have continuously distributed eigenfrequencies.

The following equations govern the dynamics of the SP model

$$\begin{aligned}
 (\rho + \mu) \operatorname{grad} \operatorname{div} \mathbf{v} + \mu \Delta \mathbf{v} - \rho \ddot{\mathbf{v}} - \int_0^\infty m(q) \ddot{\mathbf{w}}_q \, dq + \mathbf{K} + \mathbf{Q} &= 0, \\
 m(q) \ddot{\mathbf{w}}_q + c(q) \left[ 1 + \tilde{R} \left( \frac{\partial}{\partial \kappa} \right) \right] (\mathbf{w}_q - \mathbf{v}) &= \mathbf{Q}_q.
 \end{aligned}
 \tag{1}$$

where

- $\rho$  represents the mass density of the carrier medium.
- $\rho$  and  $\mu$  are the Lamé elastic moduli characterizing the carrier medium’s elasticity.
- $\mathbf{v}$  denotes the displacement vector of points within the carrier medium.
- $\mathbf{K}$  stands for the intensity of the external body force acting on the medium.
- $\mathbf{w}_q$  represents the absolute displacement vector of the oscillator mass with respect to its equilibrium position.
- $\mathbf{Q}_q$  denotes the external force applied to the mass of the oscillator.
- The quantity  $m(q) \, dq$  corresponds to the mass of all oscillators with eigenfrequencies lying within the interval  $(q, q + dq)$  multiplied by a unit volume.
- $m = \int_0^\infty m(q) \, dq$  is the total mass density of all oscillators fixed to the carrier medium.
- $c(q) = q^2 m(q)$  represents the static stiffness of the oscillator suspension.

The quantity  $\tilde{R}(\partial/\partial\kappa)$  characterizes the energy dissipation in the oscillator suspension. An interesting characteristic of this model is that even with the low damping of oscillators, the spatial attention of vibrations in the medium is finite [10]. As a result, the model finds effective application in calculating the vibrations of aircraft, rockets, and space technology objects, as well as submarines. It proves to be a valuable tool in analyzing and predicting the behavior of these structures under various vibrational conditions.

In the context of an isolated environment with  $\mathbf{K} = 0$  and  $\mathbf{Q}_q = 0$ , we focus on a one-dimensional version of the system (1). To account for nonlinearity attributed to the carrier

medium, we introduce nonlinear terms into the system. The resulting one-dimensional system is expressed as

$$\begin{aligned}
 (\varrho + 2\mu) \frac{\partial^2 v_y(y, \kappa)}{\partial y^2} + \frac{\partial}{\partial y} \left[ F \left( \frac{\partial v_y}{\partial y} \right)^2 \right] - \rho \frac{\partial^2 v_y}{\partial \kappa^2} - \int_0^\infty m(q) \frac{\partial^2 w_{qy}(y, \kappa)}{\partial \kappa^2} dq = 0, \\
 m(q) \frac{\partial^2 w_{qy}}{\partial \kappa^2} + c(q) \left[ 1 + \tilde{R} \left( \frac{\partial}{\partial \kappa} \right) \right] (w_{qy} - v_y) = 0,
 \end{aligned}
 \tag{2}$$

where  $F$  is the coefficient characterizing the nonlinearity of the carrier medium.

Under the assumption that the absolute displacement of the oscillator does not depend on its eigenfrequency, we can express it as ( $w_{qy} = w_y$ )

$$\int_0^\infty m(q) \frac{\partial^2 w_{qy}}{\partial \kappa^2} dq = \frac{\partial^2 w_y(y, \kappa)}{\partial \kappa^2} \int_0^\infty m(q) dq.$$

The system (2) is simplified to a single equation representing the longitudinal displacement of the carrier medium, denoted as  $v_y$  given by

$$\begin{aligned}
 \frac{\partial^2 v_y}{\partial \kappa^2} - c_1^2 \frac{\partial^2 v_y}{\partial y^2} - \frac{c_1^2}{q^2} \frac{\partial^4 v_y}{\partial y^2 \partial \kappa^2} + \frac{1}{\alpha^2 q^2} \frac{\partial^4 v_y}{\partial \kappa^4} + \tilde{R} \frac{\partial^3 v_y}{\partial \kappa^3} - \tilde{R} c_1^2 \frac{\partial^3 v_y}{\partial y^2 \partial \kappa} \\
 - \frac{c_2^2}{q^2} \frac{\partial^3}{\partial y \partial \kappa^2} \left[ \left( \frac{\partial v_y}{\partial y} \right)^2 \right] - c_2^2 \frac{\partial}{\partial y} \left[ \left( \frac{\partial v_y}{\partial y} \right)^2 \right] - \tilde{R} c_2^2 \frac{\partial^2}{\partial y \partial \kappa} \left[ \left( \frac{\partial v_y}{\partial y} \right)^2 \right] = 0,
 \end{aligned}
 \tag{3}$$

with

$$\tilde{\rho} = \rho + \int_0^\infty m(q) dq, \quad c_1^2 = \frac{\varrho + 2\mu}{\tilde{\rho}}, \quad c_2^2 = \frac{F}{\tilde{\rho}}, \quad \alpha^2 = \frac{\tilde{\rho}}{\rho}.$$

We define the dimensionless parameters as

$$\theta = \frac{v_x}{v_0}, \quad x = \frac{\alpha q}{c_1} y, \quad t = \alpha q \kappa.$$

By introducing  $v_0$  as the maximum displacement, within which the deformation of the carrier medium remains elastic, Equation (3) is modified to take the following form

$$\begin{aligned}
 \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \theta}{\partial t^2} - \alpha^2 \frac{\partial^2 \theta}{\partial x^2} \right) + \tilde{R} \alpha q \frac{\partial}{\partial t} \left( \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x^2} \right) \\
 - \tilde{N} \frac{\partial}{\partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] - \tilde{N} \alpha^2 \frac{\partial^3}{\partial t^2 \partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] - \tilde{R} \tilde{N} \alpha q \frac{\partial^2}{\partial t \partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] = 0,
 \end{aligned}$$

where  $\tilde{N} = c_2^2 \alpha q v_0 / c_1^3$

$$\begin{aligned}
 \text{or } \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \theta}{\partial t^2} - \alpha^2 \frac{\partial^2 \theta}{\partial x^2} \right) + R \frac{\partial}{\partial t} \left( \frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x^2} \right) \\
 - N_1 \frac{\partial}{\partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] - R N_1 \frac{\partial^2}{\partial t \partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] - \beta \frac{\partial^3}{\partial t^2 \partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] = 0,
 \end{aligned}$$

where  $\beta / N_1 = \alpha^2 > 1$ .

Considering the nonlinearity of the medium in the absence of dissipation ( $R = 0, N_1 = 0$ ), Equation (3) can be expressed as follows

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \theta}{\partial t^2} - \alpha^2 \frac{\partial^2 \theta}{\partial x^2} \right) - \beta \frac{\partial^3}{\partial t^2 \partial x} \left[ \left( \frac{\partial \theta}{\partial x} \right)^2 \right] = 0.$$

In this study, we conduct a Lie symmetry analysis of the SP model [11] in the given mathematical form

$$\theta_{tt} - \theta_{xx} + (\theta_{tt} - \alpha^2\theta_{xx})_{xx} - \beta((\theta_x)^2)_{tx} = 0. \tag{4}$$

Lie theorists like Ovsyannikov [12], Ibragimov [13], Bluman [14], Olver [15], Hydon [16], Stephani [17], and others [18,19] have played a vital role in connecting group structures to differential equations. This link has been crucial in discovering invariant solutions for nonlinear partial differential equations and gaining deeper insight into their behavior. A significant area of study has been the group analysis of the SP model, where Lie theory techniques have led to the identification of invariant solutions and conservation laws, highlighting essential underlying symmetries and dynamic properties. This study provides group invariant solutions that remain unchanged under a specific Lie symmetry generator, adding novelty to our research as such solutions are not present in the existing literature. Additionally, the conservation laws outlined in this study are reported for the first time, contributing to the enrichment of the SP model’s applications from a divergence perspective. Our study is specifically centered on invariant solutions and nonlocal conservation laws, which represents a limitation in terms of the methods we have employed. References [20–22] represent valuable contributions to this field, shaping our understanding of the SP model and its applications in various scientific and engineering domains.

The article is organized as follows: Section 1 presents a detailed explanation of the SP model. In Section 2, the Lie group method and one-dimensional subalgebras for the SP model are discussed. Section 3 is devoted to exploring the invariant solutions of the SP model. Moving on to Section 4, the article examines the derivation of conservation laws using the homotopy operator. Section 5 focuses on the physical interpretation of the solutions. Lastly, in Section 6, the article concludes by highlighting potential future research directions.

## 2. Lie Group Method

In this section, we investigate the Lie symmetries and optimal system of Equation (4). We consider a one-parameter Lie group of transformations to identify the inherent symmetries in the equation

$$\begin{aligned} \tilde{x} &\rightarrow x + \zeta\varphi_1(x, t, \theta) + O(\zeta^2), \\ \tilde{t} &\rightarrow t + \zeta\varphi_2(x, t, \theta) + O(\zeta^2), \\ \tilde{\theta} &\rightarrow \theta + \zeta\varrho(x, t, \theta) + O(\zeta^2), \end{aligned} \tag{5}$$

where  $\zeta$  is the group parameter. The vector field associated with the above transformations is

$$\mathcal{E} = \varphi_1(x, t, \theta)\frac{\partial}{\partial x} + \varphi_2(x, t, \theta)\frac{\partial}{\partial t} + \varrho(x, t, \theta)\frac{\partial}{\partial \theta}. \tag{6}$$

The coefficient functions  $\varphi_1$ ,  $\varphi_2$ , and  $\varrho$  are to be found, and the operator  $\mathcal{E}$  fulfills the Lie symmetry condition [23]

$$\mathcal{E}^{[4]}(\theta_{tt} - \theta_{xx} + (\theta_{tt} - \alpha^2\theta_{xx})_{xx} - \beta((\theta_x)^2)_{tx})|_{(4)} = 0, \tag{7}$$

where  $\mathcal{E}^{[4]}$  is the fourth extension of  $\mathcal{E}$ .

This leads to the four-dimensional symmetry algebra for Equation (4) given by (Table 1):

$$\mathcal{E}_1 = \frac{\partial}{\partial t}, \quad \mathcal{E}_2 = \frac{\partial}{\partial \theta}, \quad \mathcal{E}_3 = \frac{\partial}{\partial x}, \quad \mathcal{E}_4 = t\frac{\partial}{\partial \theta}. \tag{8}$$

The adjoint representation is given by (Table 2):

$$Ad(\exp(\epsilon\mathcal{E}_i).\mathcal{E}_j) = \mathcal{E}_j - \epsilon[\mathcal{E}_i, \mathcal{E}_j] + \frac{\epsilon^2}{2!}[\mathcal{E}_i, [\mathcal{E}_i, \mathcal{E}_j]] - \dots \tag{9}$$

**Table 1.** Commutator table.

$[\mathcal{E}_i, \mathcal{E}_j]$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_1$	0	0	0	$\mathcal{E}_2$
$\mathcal{E}_2$	0	0	0	0
$\mathcal{E}_3$	0	0	0	0
$\mathcal{E}_4$	$-\mathcal{E}_2$	0	0	0

**Table 2.** Adjoint table.

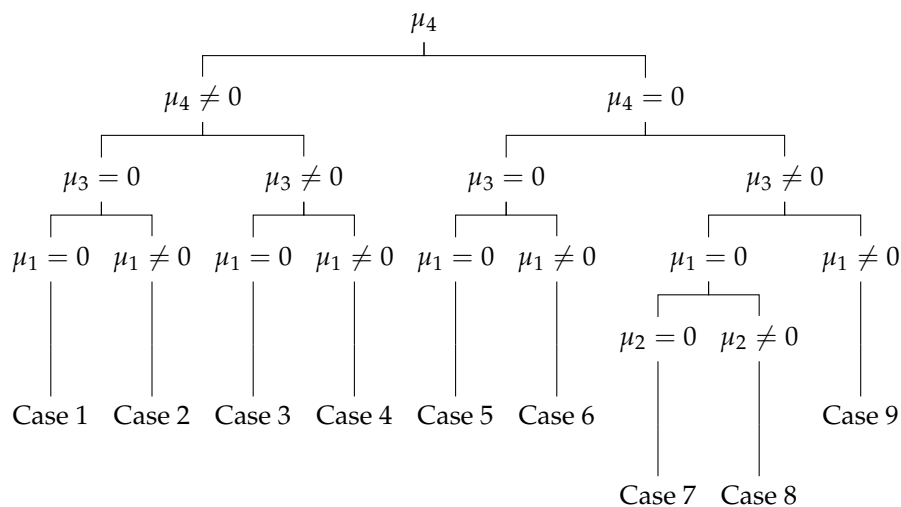
$Ad(e^e)$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_1$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4 - \epsilon\mathcal{E}_2$
$\mathcal{E}_2$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_3$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_4$	$\mathcal{E}_1 + \epsilon\mathcal{E}_2$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$

**Proposition 1.** Let  $\mathcal{L}^4$  be the Lie algebra of the SP model (4). The optimal system of one-dimensional subalgebras is then generated by the generators listed below

$$\begin{aligned}
 \mathcal{J}_1 &= \langle \mathcal{E}_4 \rangle, \\
 \mathcal{J}_2 &= \langle \mathcal{E}_1 + c\mathcal{E}_4 \rangle, \quad c \neq 0, \\
 \mathcal{J}_3 &= \langle \mathcal{E}_3 + c\mathcal{E}_4 \rangle, \quad c \neq 0, \\
 \mathcal{J}_4 &= \langle \mathcal{E}_1 + c\mathcal{E}_3 + d\mathcal{E}_4 \rangle, \quad c, d \neq 0, \\
 \mathcal{J}_5 &= \langle \mathcal{E}_2 \rangle, \\
 \mathcal{J}_6 &= \langle \mathcal{E}_1 \rangle, \\
 \mathcal{J}_7 &= \langle \mathcal{E}_3 \rangle, \\
 \mathcal{J}_8 &= \langle \mathcal{E}_2 + c\mathcal{E}_3 \rangle, \quad c \neq 0, \\
 \mathcal{J}_9 &= \langle \mathcal{E}_1 + c\mathcal{E}_3 \rangle, \quad c \neq 0.
 \end{aligned}
 \tag{10}$$

**Proof.** Take any element  $\mathcal{E} \in \mathcal{L}^4$ . We have,

$$\mathcal{E} = \mu_1\mathcal{E}_1 + \mu_2\mathcal{E}_2 + \mu_3\mathcal{E}_3 + \mu_4\mathcal{E}_4
 \tag{11}$$



**Case 1:**  $\mu_4 \neq 0, \mu_3 = 0, \mu_1 = 0$ . Then we have,

$$\mathcal{E} = \mu_2\mathcal{E}_2 + \mu_4\mathcal{E}_4
 \tag{12}$$

Under the adjoint action on  $\mathcal{E}$ , we have

$$\mathcal{E}' = Ad(e^\epsilon \mathcal{E}_1)\mathcal{E} = \mu_4 \mathcal{E}_4 \tag{13}$$

Hence, we obtain

$$\mathcal{J}_1 = \mathcal{E}_4, \tag{14}$$

**Case 2:**  $\mu_4 \neq 0, \mu_3 = 0, \mu_1 \neq 0$ . Then we have,

$$\mathcal{E} = \mu_1 \mathcal{E}_1 + \mu_2 \mathcal{E}_2 + \mu_4 \mathcal{E}_4 \tag{15}$$

$$\mathcal{E}' = Ad(e^\epsilon \mathcal{E}_4)\mathcal{E} = \mu_1 \mathcal{E}_1 + \mu_4 \mathcal{E}_4 \tag{16}$$

Hence, we obtain

$$\mathcal{J}_2 = \mathcal{E}_1 + c\mathcal{E}_4, c \neq 0, \tag{17}$$

**Case 3:**  $\mu_4 \neq 0, \mu_3 \neq 0, \mu_1 = 0$ . Then we have,

$$\mathcal{E} = \mu_2 \mathcal{E}_2 + \mu_3 \mathcal{E}_3 + \mu_4 \mathcal{E}_4 \tag{18}$$

$$\mathcal{E}' = Ad(e^\epsilon \mathcal{E}_1)\mathcal{E} = \mu_3 \mathcal{E}_3 + \mu_4 \mathcal{E}_4 \tag{19}$$

Hence, we obtain

$$\mathcal{J}_3 = \mathcal{E}_3 + c\mathcal{E}_4, c \neq 0, \tag{20}$$

**Case 4:**  $\mu_4 \neq 0, \mu_3 \neq 0, \mu_1 \neq 0$ . Then we have,

$$\mathcal{E} = \mu_1 \mathcal{E}_1 + \mu_2 \mathcal{E}_2 + \mu_3 \mathcal{E}_3 + \mu_4 \mathcal{E}_4 \tag{21}$$

$$\mathcal{E}' = Ad(e^\epsilon \mathcal{E}_1)\mathcal{E} = \mu_1 \mathcal{E}_1 + \mu_3 \mathcal{E}_3 + \mu_4 \mathcal{E}_4 \tag{22}$$

So, we obtain

$$\mathcal{J}_4 = \mathcal{E}_1 + c\mathcal{E}_3 + d\mathcal{E}_4, c, d \neq 0, \tag{23}$$

**Case 5:**  $\mu_4 = 0, \mu_3 = 0, \mu_1 = 0$ . Then we have,

$$\mathcal{E} = \mu_2 \mathcal{E}_2 \tag{24}$$

So, we obtain

$$\mathcal{J}_5 = \mathcal{E}_2, \tag{25}$$

**Case 6:**  $\mu_4 = 0, \mu_3 = 0, \mu_1 \neq 0$ . Then we have,

$$\mathcal{E} = \mu_1 \mathcal{E}_1 + \mu_2 \mathcal{E}_2 \tag{26}$$

$$\mathcal{E}' = Ad(e^\epsilon \mathcal{E}_4)\mathcal{E} = \mu_1 \mathcal{E}_1 \tag{27}$$

So, we obtain

$$\mathcal{J}_6 = \mathcal{E}_1, \tag{28}$$

**Case 7:**  $\mu_4 = 0, \mu_3 \neq 0, \mu_1 = 0, \mu_2 = 0$ . Then we have,

$$\mathcal{E} = \mu_3 \mathcal{E}_3 \tag{29}$$

So, we obtain

$$\mathcal{J}_7 = \mathcal{E}_3, \tag{30}$$

**Case 8:**  $\mu_4 = 0, \mu_3 \neq 0, \mu_1 = 0, \mu_2 \neq 0$ . Then we have,

$$\mathcal{E} = \mu_2 \mathcal{E}_2 + \mu_3 \mathcal{E}_3 \tag{31}$$

So, we obtain

$$\mathcal{J}_8 = \mathcal{E}_2 + c\mathcal{E}_3, \quad c \neq 0, \tag{32}$$

**Case 9:**  $\mu_4 = 0, \mu_3 \neq 0, \mu_1 \neq 0$ . Then we have,

$$\mathcal{E} = \mu_1\mathcal{E}_1 + \mu_2\mathcal{E}_2 + \mu_3\mathcal{E}_3 \tag{33}$$

$$\mathcal{E}' = Ad(e^\epsilon \mathcal{E}_4)\mathcal{E} = \mu_1\mathcal{E}_1 + \mu_3\mathcal{E}_3 \tag{34}$$

So, we obtain

$$\mathcal{J}_9 = \mathcal{E}_1 + c\mathcal{E}_3, \quad c \neq 0. \tag{35}$$

□

### 3. Similarity Reductions and Invariant Solutions

**Vector field  $\mathcal{J}_6 = \langle \mathcal{E}_1 \rangle$ .**

The characteristic equation associated with the vector field  $\mathcal{E}_1 = \frac{\partial}{\partial t}$  is written as

$$\frac{dx}{0} = \frac{dt}{1} = \frac{d\theta}{0},$$

and provides a transformation  $\theta(x, t) = k(r), r = x$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$-a^2k^{(iv)} - k'' = 0, \tag{36}$$

which gives,

$$k(r) = c_1 + c_2r + c_3 \sin\left(\frac{r}{\alpha}\right) + c_4 \cos\left(\frac{r}{\alpha}\right).$$

Hence, the solution of (4) in original variables becomes,

$$\theta(x, t) = c_1 + c_2x + c_3 \sin\left(\frac{x}{\alpha}\right) + c_4 \cos\left(\frac{x}{\alpha}\right). \tag{37}$$

**Vector field  $\mathcal{J}_7 = \langle \mathcal{E}_3 \rangle$ .**

The characteristic equation associated with the vector field  $\mathcal{E}_3 = \frac{\partial}{\partial x}$  is written as

$$\frac{dx}{1} = \frac{dt}{0} = \frac{d\theta}{0},$$

and provides a transformation  $\theta(x, t) = k(r), r = x$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$k'' = 0, \tag{38}$$

which gives,

$$k(r) = c_1r + c_2.$$

Hence, the solution of (4) in original variables becomes,

$$\theta(x, t) = c_1t + c_2. \tag{39}$$

**Vector field  $\mathcal{J}_2 = \langle \mathcal{E}_1 + c\mathcal{E}_4 \rangle$ .**

The characteristic equation associated with the vector field  $\mathcal{E}_1 + c\mathcal{E}_4 = \frac{\partial}{\partial t} + ct\frac{\partial}{\partial \theta}$  is written as

$$\frac{dx}{0} = \frac{dt}{1} = \frac{d\theta}{ct},$$

and provides a transformation  $\theta(x, t) = \frac{t^2c}{2} + k(r)$ ,  $r = x$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$-\alpha^2 k^{(iv)} - k'' + c = 0, \tag{40}$$

which gives,

$$k(r) = -\alpha^2 c_1 \cos\left(\frac{r}{\alpha}\right) - \alpha^2 c_2 \sin\left(\frac{r}{\alpha}\right) + \frac{r^2c}{2} + c_3r + c_4.$$

Hence, the solution of (4) in original variables becomes,

$$\theta(x, t) = -\alpha^2 c_1 \cos\left(\frac{x}{\alpha}\right) - \alpha^2 c_2 \sin\left(\frac{x}{\alpha}\right) + \frac{(x^2 + t^2)c}{2} + c_3x + c_4. \tag{41}$$

**Vector field**  $\mathcal{J}_8 = \langle \mathcal{E}_2 + c\mathcal{E}_3 \rangle$ .

The characteristic equation associated with the vector field  $\mathcal{E}_2 + c\mathcal{E}_3 = \frac{\partial}{\partial \theta} + c \frac{\partial}{\partial x}$  is written as

$$\frac{dx}{c} = \frac{dt}{0} = \frac{d\theta}{1},$$

and provides a transformation  $\theta(x, t) = \frac{x}{c} + k(r)$ ,  $r = t$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$k'' = 0, \tag{42}$$

which gives,

$$k(r) = c_1r + c_2.$$

Hence, the solution of (4) in original variables becomes,

$$\theta(x, t) = c_1t + c_2 + \frac{x}{c}. \tag{43}$$

**Vector field**  $\mathcal{J}_3 = \langle \mathcal{E}_3 + c\mathcal{E}_4 \rangle$ .

The characteristic equation associated with the vector field  $\mathcal{E}_3 + c\mathcal{E}_4 = \frac{\partial}{\partial x} + ct \frac{\partial}{\partial \theta}$  is written as

$$\frac{dx}{1} = \frac{dt}{0} = \frac{d\theta}{ct},$$

and provides a transformation  $\theta(x, t) = cxt + k(r)$ ,  $r = t$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$k'' = 0, \tag{44}$$

which gives,

$$k(r) = c_1r + c_2.$$

Hence, the solution of (4) in original variables becomes,

$$\theta(x, t) = c_1t + c_2 + cxt. \tag{45}$$

**Vector field**  $\mathcal{J}_9 = \langle \mathcal{E}_1 + c\mathcal{E}_3 \rangle$ .

The characteristic equation associated with the vector field  $\mathcal{E}_1 + c\mathcal{E}_3 = \frac{\partial}{\partial t} + c \frac{\partial}{\partial x}$  is written as

$$\frac{dx}{c} = \frac{dt}{1} = \frac{d\theta}{0},$$

and provides a transformation  $\theta(x, t) = k(r)$ ,  $r = t - \frac{x}{c}$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$(2c\beta k' + c^2 - \alpha^2)k^{(iv)} + c(c^3 + 6\beta k''' - c)k'' = 0. \tag{46}$$



We suggest solving the above equation numerically.

**Vector field**  $\mathcal{J}_4 = \langle \mathcal{E}_1 + c\mathcal{E}_3 + l\mathcal{E}_4 \rangle$ .

The characteristic equation associated with the vector field  $\mathcal{E}_1 + c\mathcal{E}_3 + l\mathcal{E}_4 = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + lt\frac{\partial}{\partial \theta}$  is written as

$$\frac{dx}{c} = \frac{dt}{1} = \frac{d\theta}{lt},$$

and provides a transformation  $\theta(x, t) = \frac{2clxt - lx^2}{2c^2} + k(r), r = t - \frac{x}{c}$ . With the application of this transformation, we acquire the simplified version of Equation (4) presented as follows,

$$(-2cl\beta r + 2c\beta k' + c^2 - \alpha^2)k^{(iv)} + (-6\beta(l - k'')k''' + ((c^2 - 1)k'' + k)c)c = 0. \tag{47}$$

We suggest solving the above equation numerically.

#### 4. Local Conservation Laws via Homotopy Operator

When dealing with complex multipliers and/or equations in the context of inverting divergence operators, an effective approach involves the utilization of homotopy operators derived from differential geometry. These operators help simplify the task of determining fluxes, reducing it to a more manageable problem of integration in single-variable calculus. In this section, we introduce the initial series of such formulas, as outlined in reference [19]. It is crucial to emphasize that when dealing with PDE systems and/or multipliers that are not excessively intricate but involve arbitrary constitutive functions, the direct method of flux computation is typically employed.

Let us consider a system of partial differential equations of order  $k$ , denoted as  $\Delta(x, \theta^{(k)}) = 0$ , where  $x = (x^1, x^2, \dots, x^n)$  represents the independent variables and  $\theta = (\theta^1, \theta^2, \dots, \theta^m)$  represents the dependent variables. The system can be expressed as a collection of individual equations as follows

$$\Delta^{(\sigma)} = \Delta^{(\sigma)}(x, \theta, \partial\theta, \dots, \partial\theta^k) = 0, \quad \sigma = 1, 2, \dots, N. \tag{48}$$

Now, let us define a local conservation law for the system (48). This law is characterized by a divergence expression

$$D_i \Omega^i = D_1 \Omega^1 + D_2 \Omega^2 + \dots + D_n \Omega^n, \tag{49}$$

which holds true for all solutions of the system (48). Here, the operators  $D_i$  represent the total derivatives with respect to the variable  $x_i$ , while the quantities  $D_i, i = 1, 2, \dots, n$ , correspond to the fluxes associated with the conservation laws. The Euler operator is defined as follows

$$E_\theta = \frac{\partial}{\partial \theta} - D_i \frac{\partial}{\partial \theta_i} + \dots + (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial \theta_{i_1 \dots i_s}} + \dots \tag{50}$$

Now, let us define the  $n$ -dimensional higher-order Euler operator concerning a function  $\theta(x^1, x^2, \dots, x^n)$

$$E_\theta^{(s_1, s_2, \dots, s_n)} = \sum_{k_1=s_1}^\infty \dots \sum_{k_n=s_n}^\infty \binom{k_1}{s_1} \dots \binom{k_n}{s_n} D_1^{k_1-s_1} \dots D_n^{k_n-s_n} \frac{\partial}{\partial \theta^{(k_1+k_2+\dots+k_n)}}, \tag{51}$$

where  $\theta^{(k_1+k_2+\dots+k_n)} = \frac{\partial^{(k_1+k_2+\dots+k_n)} \theta}{\partial^{k_1} x^1 \dots \partial^{k_n} x^n}$ . It is worth noting that when  $s_1 = 0$  and  $s_n = 0$ , we obtain the original Euler operator  $E_\theta^{(0,0)} = E_\theta$  as defined in (50).

The  $n$ -dimensional homotopy operator [24] is introduced for an expression  $g[\theta] = g(x, \theta, \partial\theta, \dots)$ , where  $\theta = (\theta^1(x), \theta^2(x), \dots, \theta^m(x))$  and  $x = (x^1, x^2, \dots, x^n)$ . This

operator is defined through its  $n$  components, each corresponding to a specific independent variable  $x^i$ ,  $i = 1, 2, \dots, n$ , and can be expressed as

$$\mathcal{H}^{(x^i)} = \int_0^1 \sum_{j=1}^m I_j^{(x^i)}(g[\hat{\theta}]) \Big|_{\hat{\theta}=\Lambda\theta} \frac{d\Lambda}{\Lambda}, \tag{52}$$

where  $I_j^{(x^i)}(g[\hat{\theta}])$  is determined using the expression

$$I_j^{(x^i)}(g[\hat{\theta}]) = \sum_{s_1=0}^{\infty} \dots \sum_{s_n=0}^{\infty} \left( \frac{1 + s_i}{1 + s_1 + \dots + s_n} \right) \times D_1^{s_1} \dots D_n^{s_n} (\hat{\theta}^j E_{\hat{\theta}^j}^{(s_1, \dots, s_n)}(g[\hat{\theta}])), \tag{53}$$

where  $j = 1, \dots, m$ . In essence, the  $n$ -dimensional homotopy operator enables us to deal with expressions  $g[\theta]$  involving multiple variables  $\theta$  and  $x$  and it provides a systematic way of handling each independent variable  $x^i$  individually. The main theorem is presented below.

**Proposition 2.** Let  $g[\theta]$  be a divergence expression given by [24]

$$g[\theta] = \text{div } \Omega = D_1 \Omega^1[\theta] + \dots + \dots, D_n \Omega^n[\theta], \tag{54}$$

and assume that  $g[0] = 0$ . Then the fluxes  $\Omega^i$  can be expressed as

$$\Omega^i = \mathcal{H}^{(x^i)}(g[\theta]), \quad i = 1, \dots, n, \tag{55}$$

up to the corresponding fluxes of a trivial conservation law, provided that the integrals (53) converge.

By utilizing the direct (multiplier) method to solve Equation (4), we have derived the first-order multipliers as follows

$$\begin{aligned} \Psi(x, t, \theta, \theta_x) = & -\alpha^2(C_3t + C_7) \cos\left(\frac{x}{\alpha}\right) - \alpha^2(C_4t + C_8) \sin\left(\frac{x}{\alpha}\right) + \frac{1}{6}C_1t^3 \\ & + \frac{1}{2}C_2t^2 + \frac{1}{6}(3C_1x^2 + 6C_5x + 6C_6)t + \frac{1}{2}C_2x^2 + C_9x + C_{10}. \end{aligned} \tag{56}$$

In the above equation,  $\Psi(x, t, \theta, \theta_x)$  represents the first-order multipliers obtained from the direct (multiplier) method applied to Equation (4). The constants  $C_1$  to  $C_{10}$  are coefficients determined during the solution process. Further details and the complete derivation process can be found in the reference [25,26].

- Using the characteristic  $\Psi_1 = \frac{1}{6}t^3 + \frac{1}{2}x^2t$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_1^t = -\frac{1}{3}\beta\theta_{txx}\theta_x x^2t - \beta x^2t\theta_{txx} + \beta t^2\theta_{xx}\theta_x + \beta x^2\theta_x\theta_{xx} - \frac{1}{3}\beta t^2\theta_{tx}\theta_{xx} \\ \quad - \beta x^2t\theta_{tx}\theta_{xx} - \frac{1}{2}\theta x^2 + \frac{1}{6}t^3\theta_t + \frac{1}{2}x^2t\theta_t + \frac{1}{2}t^2\theta_{xx} + \frac{1}{2}x^2\theta_{xx} - \frac{1}{6}t^3\theta_{txx} \\ \quad - \frac{1}{2}x^2t\theta_{txx}, \\ \Omega_1^x = \theta_{xt} - \alpha^2t\theta_x - \beta t\theta_x^2 - \frac{1}{2}x^2t\theta_x - \frac{1}{6}\alpha^2t^3\theta_{xxx} - t\theta_x - \frac{1}{6}t^3\theta_x \\ \quad + \alpha^2xt\theta_{xx} - \frac{1}{2}\alpha^2x^2t\theta_{xxx}. \end{cases} \tag{57}$$

- Using the characteristic  $\Psi_2 = \frac{1}{2}t^2 + \frac{1}{2}x^2$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_2^t = -\beta t^2\theta_x\theta_{txx} - \beta x^2\theta_x\theta_{txx} + 2\beta t\theta_x\theta_{xxx} - \beta t^2\theta_{tx}\theta_{xx} - \beta x^2\theta_{tx}\theta_{xx} - t\theta \\ \quad + \frac{1}{2}t^2\theta_t + \frac{1}{2}x^2\theta_t + t\theta_{xx} - \frac{1}{2}t^2\theta_{txx} - \frac{1}{2}x^2\theta_{txx}, \\ \Omega_2^x = \alpha^2x\theta_{xx} - \frac{1}{2}\alpha^2t^2\theta_{xxx} - \frac{1}{2}\alpha^2x^2\theta_{xxx} - \frac{1}{2}t^2\theta_x - \frac{1}{2}x^2\theta_x - \beta\theta_x^2 \\ \quad - x\theta - \theta_x - \alpha^2\theta_x. \end{cases} \tag{58}$$

- Using the characteristic  $\Psi_3 = -\alpha^2 t \cos(\frac{x}{\alpha})$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_3^t &= \alpha^2 \cos(\frac{x}{\alpha})(2\beta t \theta_{tx} \theta_{xx} + 2\beta t \theta_x \theta_{txx} - 2\beta \theta_x \theta_{xx} - t\theta_t + t\theta_{txx} + \theta - \theta_{xx}), \\ \Omega_3^x &= \alpha^3 t(\alpha \theta_{xx} \cos(\frac{x}{\alpha}) + \theta_{xx} \sin(\frac{x}{\alpha})). \end{cases} \quad (59)$$

- Using the characteristic  $\Psi_4 = -\alpha^2 t \sin(\frac{x}{\alpha})$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_4^t &= \alpha^2 \sin(\frac{x}{\alpha})(2\beta t \theta_{tx} \theta_{xx} + 2\beta t \theta_x \theta_{txx} - 2\beta \theta_x \theta_{xx} - t\theta_t + t\theta_{txx} + \theta - \theta_{xx}), \\ \Omega_4^x &= -\alpha^3 t(-\alpha \theta_{xxx} \sin(\frac{x}{\alpha}) + \theta_{xx} \cos(\frac{x}{\alpha})). \end{cases} \quad (60)$$

- Using the characteristic  $\Psi_5 = xt$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_5^t &= -2\beta xt \theta_{tx} \theta_{xx} - 2\beta xt \theta_{txx} \theta_x + 2\beta x \theta_x \theta_{xx} + xt \theta_t - xt \theta_{txx} - x\theta + x\theta_{xx}, \\ \Omega_5^x &= -\alpha^2 xt \theta_{xxx} + \alpha^2 t \theta_{xx} - xt \theta_x + t\theta. \end{cases} \quad (61)$$

- Using the characteristic  $\Psi_6 = t$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_6^t &= -2\beta t \theta_{tx} \theta_{xx} - 2\beta t \theta_{txx} \theta_x + 2\beta \theta_x \theta_{xx} + t\theta_t - t\theta_{txx} - \theta + \theta_{xx}, \\ \Omega_6^x &= -\alpha^2 t \theta_{xxx} - t\theta_x. \end{cases} \quad (62)$$

- Using the characteristic  $\Psi_7 = -\alpha^2 \cos(\frac{x}{\alpha})$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_7^t &= \alpha^2 \cos(\frac{x}{\alpha})(2\beta \theta_{tx} \theta_{xx} + 2\beta \theta_x \theta_{txx} - \theta_t + \theta_{txx}), \\ \Omega_7^x &= \alpha^3(\alpha \cos(\frac{x}{\alpha}) \theta_{xxx} + \theta_{xx} \sin(\frac{x}{\alpha})). \end{cases} \quad (63)$$

- Using the characteristic  $\Psi_8 = -\alpha^2 \sin(\frac{x}{\alpha})$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_8^t &= \alpha^2 \sin(\frac{x}{\alpha})(2\beta \theta_{tx} \theta_{xx} + 2\beta \theta_x \theta_{txx} - \theta_t + \theta_{txx}), \\ \Omega_8^x &= -\alpha^3(-\alpha \sin(\frac{x}{\alpha}) \theta_{xxx} + \theta_{xx} \cos(\frac{x}{\alpha})). \end{cases} \quad (64)$$

- Using the characteristic  $\Psi_9 = x$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_9^t &= -2\beta x \theta_{tx} \theta_{xx} - 2\beta x \theta_x \theta_{txx} + x\theta_t - x\theta_{txx}, \\ \Omega_9^x &= -\alpha^2 x \theta_{xxx} + \alpha^2 \theta_{xx} - x\theta_x + \theta. \end{cases} \quad (65)$$

- Using the characteristic  $\Psi_{10} = 1$  in Equation (55) and following the integral Formula (52), we obtain

$$\begin{cases} \Omega_{10}^t &= -2\beta \theta_{tx} \theta_{xx} - 2\beta \theta_x \theta_{txx} + \theta_t - \theta_{txx}, \\ \Omega_{10}^x &= -\alpha^2 \theta_{xxx} - \theta_x. \end{cases} \quad (66)$$

### 5. Wave Nature of the Obtained Solutions

The graphical interpretation of a solution is of paramount importance as it offers a visual representation of complex mathematical relationships. It provides intuitive insights into the behavior, trends, and critical points of the solution. Graphs aid in understanding the sensitivity to varying parameters, validating results, and facilitating communication with a broader audience. They serve as valuable tools for exploration, model selection, and

enhancing our understanding of intricate systems and phenomena. Figures 1 and 2 show the behavior of nonlinear longitudinal waves in an SP medium using the SP model.

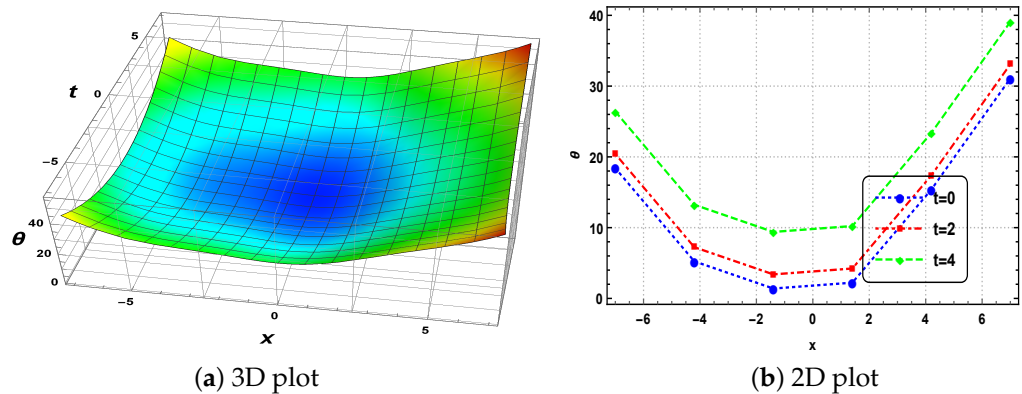


Figure 1. Wave behavior of the SP model (4) with  $c_1 = c_2 = c_3 = c_4 = 0$ .

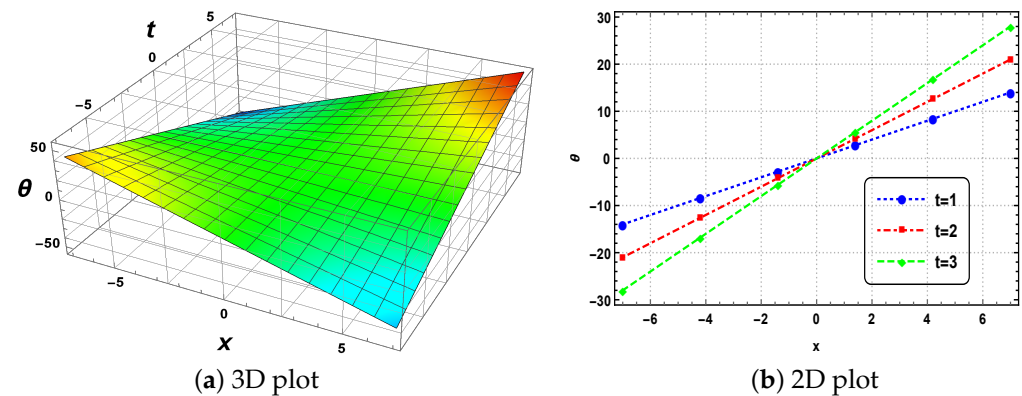


Figure 2. Wave behavior of the SP model (4) with  $c_1 = c_2 = c_3 = c_4 = 0$ .

### 6. Concluding Remarks

In this study, we successfully applied the Lie symmetry method to analyze the integrability properties of the SP model. By exploring the group invariant solutions via the reduction in symmetry subalgebras based on the optimal system, we gained valuable insights into the behavior of the system. Additionally, we listed the conservation laws using the homotopy operator, which proved advantageous over the traditional multiplier approach due to the absence of arbitrary functions in both the equation and characteristics. In the existing literature, Erofeev et al. [11] attempted to explore linear and non-linear plane longitudinal waves in the SP medium. However, their work did not present any soliton solutions or invariant ones. Our study focuses on invariant solutions, emphasizing solutions that remain unchanged under specific Lie symmetry transformations. This unique aspect adds novelty to our results. Furthermore, to illustrate the behavior of the solution at specific points in the model, we introduced nonlocal conservation laws reported for the first time. These findings contribute to the enhanced exploration of plane longitudinal waves in the SP medium via the application of the SP model. These findings shed light on the behavior of nonlinear longitudinal waves in the Slepyan–Palmov medium and the impact of various parameters on wave characteristics. These results motivate us to continue using the Lie symmetry method and the homotopy operator in tackling mathematical physics problems to further our understanding and contribute to this field of research.

**Author Contributions:** Conceptualization, A.H., M.U. and F.D.Z.; methodology, M.U., A.H. and F.D.Z.; software, A.H. and M.U.; validation, A.I., F.D.Z. and Y.A.; formal analysis, Y.A. and A.I.; investigation, F.D.Z. and Y.A.; resources, M.U. and A.H.; data curation, A.I.; writing—original draft preparation, M.U. and A.H.; writing—review and editing, M.U., A.H. and F.D.Z.; visualization, A.I.

and Y.A.; supervision, F.D.Z.; project administration, A.I. and Y.A.; funding acquisition, A.I. All authors have read and agreed to the published version of the manuscript.

**Funding:** The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a large group Research Project under the grant number RGP2/366/44.

**Data Availability Statement:** All data generated or analyzed during this study are included in this published article.

**Conflicts of Interest:** The authors declare no conflict of interest.

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