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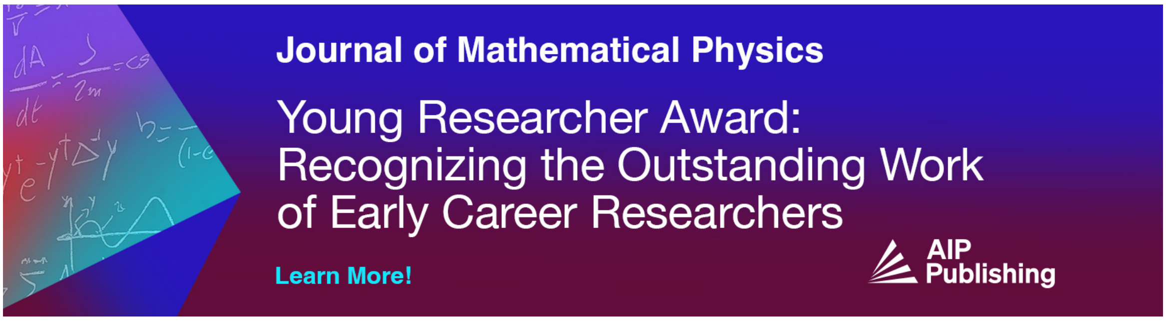

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ABSTRACT

An equilibrium point of a differential system in $\mathbb{R}^3$ such that the eigenvalues of the Jacobian matrix of the system at the equilibrium are 0 and $\pm\,\omega i$ with $\omega > 0$ is called a zero-Hopf equilibrium point. First, we prove that the Chua's circuit can have three zero-Hopf equilibria varying its three parameters. Later, we show that from the zero-Hopf equilibrium point localized at the origin of coordinates can bifurcate one periodic orbit. Moreover, we provide an analytic estimation of the expression of this periodic orbit and we have determined the kind of the stability of the periodic orbit in function of the parameters of the perturbation. The tool used for proving these results is the averaging theory of second order.

I. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In 1993, Chua *et al.*[1] proposed to replace the piecewise linear function of his famous circuit by a cubic one. The system they designed then became a relaxation oscillator with a cubic nonlinear characteristic elaborated from a circuit comprising a harmonic oscillator for which the operation was based on a field-effect transistor, coupled to a relaxation–oscillator composed of a tunnel diode. The modeling of the circuit used a capacity preventing from abrupt voltage drops and allowing to describe the fast motion of this oscillator. This gave rise to Eq. (1), which constitutes a singularly perturbed system and is since considered the paradigm of complex dynamics.[2]

The Chua's circuit is analyzed using the Kirchhoff's laws. Then, the dynamics of the Chua's circuit is modeled by means of the following system of three nonlinear ordinary differential equations in the variables $x(t)$, $y(t)$, and $z(t)$:

$$\begin{aligned} \dot{x} &= a(y - cx - x^3), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -by, \end{aligned} \tag{1}$$

where a, b, and c are real parameters and the dot indicates derivative with respect to the time t. For more details on the Chua's circuit, see Refs. 3 and 4.

Note that this differential system is invariant under the symmetry $(x,y,z) \to (-x,-y,-z)$. Then, if $(x(t),y(t),z(t))$ is a solution of the differential system (1), then $(-x(t),-y(t),-z(t))$ is another solution of this differential system, eventually both solutions coincide.

The study of the periodic orbits of a differential system is one of the main objectives of the qualitative theory of the differential systems. In general, this is not an easy task, mainly if we want to study the periodic orbits analytically.

A way of finding periodic orbits is through a Hopf bifurcation, which in $\mathbb{R}^3$ takes place when a periodic orbit bifurcates from an equilibrium point whose linear part has eigenvalues $\lambda \neq 0$ and $\pm\omega i$ with $\omega > 0$, and moving the parameters of the differential system this equilibrium changes its kind of stability (for more details, see Ref. 5 and an application in Ref. 6). While there is a well-developed theory for studying the Hopf bifurcation when $\lambda \neq 0$, this is not the case when $\lambda = 0$.

Here, we shall use the averaging theory of second order for studying the periodic orbits that bifurcate from the zero-Hopf equilibria of the differential system (1). We note that this technique applied for studying the zero-Hopf bifurcations of the Chua's system can be used for studying arbitrary zero-Hopf bifurcations of other differential systems. Sometimes the averaging theory of first order for studying zero-Hopf bifurcations is sufficient (see, for instance, Ref. 7), but here this is not the case.

A *zero-Hopf equilibrium* p of a differential system in $\mathbb{R}^3$ is an equilibrium point with eigenvalues 0 and $\pm\omega i$ with $\omega > 0$. A *zero-Hopf bifurcation* takes place when one or several periodic orbits bifurcate from the equilibrium p when the parameters of the system move.

Easy computations show that the equilibria of the differential system (1) are

$$E_1 = (0,0,0), \qquad E_2 = (\sqrt{-c}, 0, -\sqrt{-c}), \qquad E_3 = (-\sqrt{-c}, 0, \sqrt{-c}).$$

Of course, the last two equilibria exist only if the parameter $c < 0$.

Proposition 1. The following statements hold.

(a) *The point E_1 is a zero-Hopf equilibrium if and only if $b = 0$, $c = -1/a$, and $a < -1$.*
(b) *The points E_2 and E_3 are zero-Hopf equilibria if and only if $b = 0$, $c = 1/(2a)$, and $a < -1$.*

Proposition 1 is proved in Sec. III.

Theorem 2. *The equilibrium point E_1 of the Chua's circuit (1) has a zero-Hopf bifurcation when $b = 0$, $c = -1/a$, and $a = -1 - a_1^2 < -1$ with $a_1 > 0$ and it is perturbed as follows:*

$$b = b_2\varepsilon^2, \qquad c = -\frac{1}{a} + c_2\varepsilon^2, \tag{2}$$

where $b_2 c_2 \neq 0$ and $b_2 - a_1^2 c_2(1 + a_1^2) > 0$. Then, for $\varepsilon \neq 0$ sufficiently small the periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ equal to

$$\varepsilon\left(r_0 \cos(a_1 t), r_0\left(a_1 + \frac{1}{a_1}\right)\sin(a_1 t) + \frac{1}{a_1}\cos(a_1 t), \mathcal{O}(1)\right) + \mathcal{O}(\varepsilon^2) \tag{3}$$

bifurcates from E_1 (see Fig. 1). Moreover, this periodic solution is asymptotically unstable if $b_2 > 0$. It has a local stable and a local unstable manifold each one formed by two topological cylinders if $b_2 < 0$.

Theorem 2 is proved in Sec. III.

We note that in the expressions of b and c given in (2), there are no terms in $b_1\varepsilon$ and $c_1\varepsilon$, which is due to the fact that these terms do not contribute to the existence of periodic orbits when we compute such periodic orbits for this differential system using the averaging theory. Of course, we see that doing all the computations of the averaging theory of first and second order. But we have omitted them in (2) because otherwise the expressions of the Proof of Theorem 2 become longer without providing any new information.

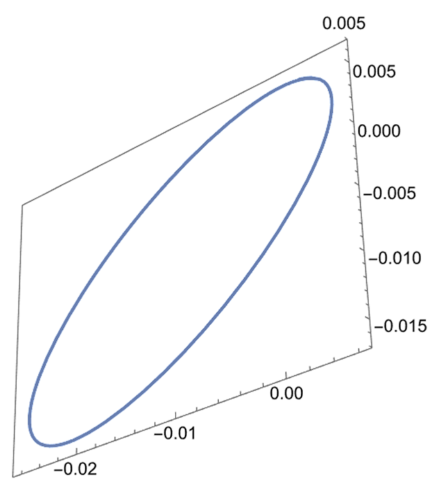

FIG. 1. The periodic orbit of Theorem 2 for the values $\varepsilon = 0.005$, $a = -2$, $a_1 = 1$, $b = 0$, $b_2 = 3$, $c = 0.5$, and $c_2 = 1$.

The averaging theory does not provide information about the possible orbits bifurcating from the zero-Hopf equilibria E_2 and E_3. This is due to the fact the first averaged function $f_1(r,u) = (f_{11}(r,u), f_{12}(r,u))$ at these equilibria is

$$f_{11}(r,u) = \frac{\pi r(2Aa_1^2c_1 + 3\sqrt{2}u)}{A^{3/2}a_1^3},$$
$$f_{12}(r,u) = \frac{2\pi b_1 u}{a_1^3},$$

where $A = 1/\sqrt{a_1^2+1}$. The unique solution of the system $f_1(r,u) = (0,0)$ is $(r,u) = (0,0)$, but this solution only provides an equilibrium point, instead of a periodic solution. Then, since the averaging theory of first order does not provide information, we must do the averaged function $f_1(r,u)$ of first order identically zero and compute the averaged function of second order. But in this case, we cannot do identically zero the averaged function of first order because we cannot vanish the coefficient $3\sqrt{2}\pi/(A^{3/2}a_1^3)$ of the term ru in the function $f_{11}(r,u)$. For more details on the averaging theory, see Sec. II.

II. THE AVERAGING THEORY OF FIRST AND SECOND ORDER

In this section, we present the averaging theory of second order for finding periodic orbits that we need for proving our main theorem. A proof of the following result can be found in Ref. 8 or 9, and in Theorem 11.6 of Ref. 10, it is studied the kind of stability of the periodic solutions obtained using the averaging theory. For more details on the general theory of averaging theory, see the book 11, and for another application to the zero-Hopf bifurcation, see Ref. 12.

Theorem 3. *Let*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon) \tag{4}$$

be a non-autonomous differential system such that D is an open subset of $\mathbb{R}^n$, where the functions F_1, $F_2 : \mathbb{R}\times D \to \mathbb{R}^n$, and $R : \mathbb{R}\times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous and periodic of period T in the variable t. Assume that the next conditions are satisfied.

(i) *The functions $F_1(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R, and D_xF_1 are locally Lipschitz with respect to x, and the function R is C^1 with respect to ε. The functions f_1, $f_2 : D \to \mathbb{R}^n$ are defined as*

$$f_1(z) = \int_0^T F_1(s,z)ds,$$
$$f_2(z) = \int_0^T \left[D_zF_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)\right]ds. \tag{5}$$

(ii) *For a bounded and open set $V \subset D$ and for every $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$, there is $a \in V$ satisfying $f_1(a) + \varepsilon f_2(a) = 0$. Moreover the Brouwer degree of the function $f_1 + \varepsilon f_2$ at a is not zero.*

Then, for $|\varepsilon| > 0$ sufficiently small, there is a periodic solution $x(t,\varepsilon)$ of period T of the differential system (4) verifying that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$. Furthermore the eigenvalues of the Jacobian matrix $D(f_1(a) + \varepsilon f_2(a))$ provides the kind of stability of the Poincaré map associated with the periodic solution $x(t,\varepsilon)$.

Assume that a is a fixed point of a function f. When the Jacobian of the function f at a (if it is defined) does not vanish, then the Brouwer degree of function f at a is non-zero; for a proof, see Ref. 13.

When the first averaged function f_1 is not identically zero, for ε sufficiently small the zeros of $f_1 + \varepsilon f_2$ are essentially the zeros of f_1. Then, we say that Theorem 3 provides the *averaging theory of first order*.

When the first averaged function f_1 is identically zero and the second averaged function f_2 is not identically zero, the zeros of the function $f_1 + \varepsilon f_2$ are essentially the zeros of f_2, and then we say that Theorem 3 provides the *averaging theory of second order*.

III. PROOF OF THE RESULTS

Proof of Proposition 1. The Jacobian matrix of the vector field associated to system (1) is

$$M = \begin{pmatrix} -a(c+3x^2) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix}.$$

In order that the eigenvalues of the matrix M at some equilibrium E_k for $k = 1,2,3$ be 0 and $\pm\omega i$ with $\omega > 0$, the characteristic polynomial of M must be $-\lambda(\lambda^2 + \omega^2)$. Imposing this fact for every equilibrium E_k for $k = 1,2,3$, we obtain the results stated in the statement of the proposition. □

Proof of Theorem 2. We consider system (1) for the values of the parameters given in (2), so the equilibrium E_1 is zero-Hopf. Now we shall do several changes of variables until to write the differential system into the normal form (4) for applying the averaging theory of second order, and in this way to study using this theory the possible periodic orbits bifurcate from the equilibrium E_1.

We shall write the matrix of the linear part of the differential system (1) at the equilibrium E_1 in its real Jordan normal form

$$\begin{pmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For doing this, we do the change of variables, which pass the variables (x, y, z) to the variables (X, Y, Z), through

$$(x, y, z) = \left(X - \left(\frac{1}{a_1^2} + 1\right)Z,\ \frac{a_1 Y + X}{a_1^2 + 1} - \frac{Z}{a_1^2},\ Z\right). \tag{6}$$

Then, system (1) in the new variables (X, Y, Z) becomes

$$\begin{aligned}
\dot{X} = &\ -a_1 Y + \frac{\left(a_1^2+1\right)\left(a_1^2(X-Z)-Z\right)^3}{a_1^6} \\
&+ \varepsilon^2\left(-\frac{b_2(a_1 Y + X)}{a_1^2} + \left(a_1^2+1\right)c_2 X + \frac{\left(a_1^2+1\right)Z\left(b_2 - a_1^2\left(a_1^2+1\right)c_2\right)}{a_1^4}\right) + \mathcal{O}(\varepsilon^3), \\
\dot{Y} = &\ a_1 X + \frac{3\left(a_1^2+1\right)^2 a_1^2 XZ\left(a_1^2 X - \left(a_1^2+1\right)Z\right) + \left(a_1^2+1\right)^4 Z^3 - \left(a_1^2+1\right)a_1^6 X^3}{a_1^7} \\
&+ \varepsilon^2 \frac{\left(a_1^2+1\right)c_2\left(a_1^2(Z-X)+Z\right)}{a_1^3} + \mathcal{O}(\varepsilon^3), \\
\dot{Z} = &\ \varepsilon^2\left(\frac{b_2 Z}{a_1^2} - \frac{b_2(a_1 Y + X)}{a_1^2+1}\right) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{7}$$

Now, we change the coordinates (X, Y, Z) to the variables (r, θ, u) taking $X = \varepsilon r \sin\theta$, $Y = \varepsilon r\cos\theta$ and $Z = \varepsilon u$. Finally, taking θ as the new independent variable the differential system (7) writes in the new variables (r, θ) as

$$\begin{aligned}
r' &= \varepsilon^2 F_{21} + \mathcal{O}(\varepsilon^3), \\
u' &= \varepsilon^2 F_{22} + \mathcal{O}(\varepsilon^3),
\end{aligned} \tag{8}$$

where the prime denotes derivative with respect to the variable θ, and

$$\begin{aligned}
F_{21} = &\ \frac{1}{a_1^8}\Big(\left(a_1^2+1\right)a_1^7 r^3\cos^4\theta - \left(a_1^2+1\right)a_1^5 r^2\cos^3\theta\left(3\left(a_1^2+1\right)u + a_1 r\sin\theta\right) \\
&- a_1\cos\theta\Big(a_1^2\left(a_1^2+1\right)u\left(a_1^2\left(a_1^2+1\right)c_2 - b_2\right) + \left(a_1^2+1\right)^4 u^3 \\
&+ a_1 r\sin\theta\left(3\left(a_1^2+1\right)^3 u^2 + a_1^4\left(a_1^2 c_2 + b_2 + c_2\right)\right)\Big) \\
&+ \left(a_1^2+1\right)^2 u\sin\theta\left(a_1^4 c_2 + \left(a_1^2+1\right)^2 u^2\right) \\
&+ a_1^3 r\cos^2\theta\left(a_1^6 c_2 + a_1^4 c_2 - a_1^2 b_2 + 3\left(a_1^2+1\right)^2 a_1 r u\sin\theta + 3\left(a_1^2+1\right)^3 u^2\right)\Big), \\
F_{22} = &\ \frac{b_2}{a_1^3}\left(u - \frac{a_1^2 r(a_1\sin\theta + \cos\theta)}{a_1^2+1}\right).
\end{aligned}$$

Differential system (8) is written into the normal form (4) for applying the averaging theory, where using the notation of Sec. II, we have $t = \theta$, $x = (r, u)$, $T = 2\pi$, $n = 2$, $F_1 = (F_{11}, F_{12}) = (0, 0)$, and $F_2 = (F_{21}, F_{22})$. Since all the assumptions of Theorem 3 of Sec. II are satisfied, we can apply it to the differential system (8). Then, the first averaged function $f_1(r, u)$ defined in (5) is identically zero, and the second averaged function $f_2(r, u) = (f_{21}(r, u), f_{22}(r, u))$ is

$$\begin{aligned}
f_{21}(r, u) = &\ \frac{1}{a_1^8}\left(\frac{3}{4}\pi\left(a_1^2+1\right)a_1^7 r^3 + \pi r\left(a_1^9 c_2 + a_1^7 c_2 - a_1^5 b_2 + 3\left(a_1^3 + a_1\right)^3 u^2\right)\right), \\
f_{22}(r, u) = &\ \frac{2\pi b_2 u}{a_1^3}.
\end{aligned}$$

The unique zero of the second averaged function $f_2(r,u)$, which going back through the changes of variables, is not associated with an equilibrium point of system (2) and, which has $r \ge 0$, is

$$(r,u) = \left(2\sqrt{\frac{b_2 - a_1^2 c_2(1+a_1^2)}{3a_1^2(1+a_1^2)}}, 0\right) = (r_0, 0),$$

which is real because by assumptions $b_2 - a_1^2 c_2(1+a_1^2) > 0$. Since the Jacobian of the function $f_2(r,u)$ at that zero is $4\pi^2 b_2(b_2 - a_1^2 c_2 (1+a_1^2))/a_1^6 \ne 0$ by assumptions, it follows from Theorem 3 that the differential system (8) has a periodic solution $(r(\theta,\varepsilon), u(\theta,\varepsilon))$ such that

$$(r(0,\varepsilon), u(0,\varepsilon)) = (r_0, 0) + \mathcal{O}(\varepsilon).$$

Since the eigenvalues of the Jacobian matrix $D(f_{21}, f_{22})(r_0, 0)$ are

$$\frac{2\pi(b_2 - a_1^2 c_2(1+a_1^2))}{a_1^3} > 0 \quad \text{and} \quad \frac{2\pi b_2}{a_1^3}.$$

From Theorem 3, the corresponding periodic solution $(r(\theta,\varepsilon), u(\theta,\varepsilon))$ is asymptotically unstable if $b_2 > 0$. If $b_2 < 0$ the fixed point of the Poincaré map associated with the periodic orbit $(r(\theta,\varepsilon), u(\theta,\varepsilon))$ is a saddle, so this periodic orbit has a local unstable manifold and a local stable manifold each one formed by two topological cylinders.

Going back through the changes of variables the periodic solution $(r(\theta,\varepsilon), u(\theta,\varepsilon))$ of system (8) becomes the periodic solution $(r(t,\varepsilon), \theta(t,\varepsilon), u(t,\varepsilon))$ of the differential system $(\dot{r}, \dot{\theta}, \dot{u})$, where as usual the dot denotes derivative with respect to the time t. We did not write explicitly the differential $(\dot{r}, \dot{\theta}, \dot{u})$, which is in the middle of the differential systems (7) and (8). The periodic solution $(r(t,\varepsilon), \theta(t,\varepsilon), u(t,\varepsilon))$ satisfies

$$(r(0,\varepsilon), \theta(0,\varepsilon), u(0,\varepsilon)) = (r_0, a_1 t, 0) + \mathcal{O}(\varepsilon).$$

This periodic solution becomes the periodic solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ of system (7) such that

$$(X(0,\varepsilon), Y(0,\varepsilon), Z(0,\varepsilon)) = \varepsilon(r_0 \cos(a_1 t), r_0 \sin(a_1 t), 0) + \mathcal{O}(\varepsilon^2).$$

Finally, going back to the coordinates (x,y,z) undoing the change (6), we obtain the periodic solution of system (1) described in the statement of the theorem. □

IV. CONCLUSIONS

We have proved that from the zero-Hopf equilibrium point localized at the origin of coordinates of the Chua's circuit can bifurcate one periodic orbit. Moreover, we have provided an analytic estimation of the expression of this periodic orbit, and additionally we have determined its kind of the stability in function of the perturbation of the parameters $b = 0$, $c = -1/a$, and $a < -1$, for which the zero-Hopf equilibrium at the origin of coordinates exists. The tool used for proving these results has been the averaging theory of second order.

Unfortunately, the averaging theory does not provide information about the possible periodic orbits bifurcating from the other two zero-Hopf equilibria that the Chua's circuit can have.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jean-Marc Ginoux: Investigation (equal); Methodology (equal). **Jaume Llibre**: Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal).

DATA AVAILABILITY

Our paper has no data.

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