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Formal Relationship

Dynamics of the coefficient of variation of the age at death distribution

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Demographic Research: Volume 49, Article 38
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Abstract

BACKGROUND

Indicators of lifespan inequality, such as the life table entropy or variance of age at death, provide a measure of inequality in the timing of death. A range of indicators of relative and absolute inequality exist, and their evolution over time and sensitivity to changes in age-specific mortality have been studied. However, the coefficient of variation, a relative indicator defined as the standard deviation divided by the mean of the age at death distribution, has yet to be studied, and the existence and form of the threshold age has not been determined.

RESULTS

As with other lifespan inequality indicators, changes in the coefficient of variation can be written as a weighted sum of changes in age-specific mortality rates, and a unique threshold age exists. The threshold for the coefficient of variation occurs later than that of its absolute counterpart, the standard deviation, a result verified for other pairs of relative and absolute indicators. Empirical applications show differing trajectories over time of the threshold age for different countries and different educational groups.

CONCLUSION

Change over time of the coefficient of variation can be expressed in a similar way to other indicators and provides a way to study the sensitivity of the indicator to changes in mortality. Although empirical applications show a similar trajectory for the threshold age

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of the coefficient of variation as for other indicators, the differences between them can be substantial.

CONTRIBUTION
We formally determine the threshold age for the coefficient of variation and contextualize the sensitivities and threshold ages of lifespan inequality indicators.

1. Introduction

Life expectancy measures the average length of life in a stationary population and serves as an indicator of health and mortality (Preston, Heuveline, and Guillot 2001). Life expectancy alone provides an incomplete picture of the mortality profile of a population since the average age at death may hide important heterogeneity in the overall age pattern of mortality. Indicators of lifespan inequality measure the variability of age at death and can be used to monitor inequalities in mortality, complementing life expectancy. While historically life expectancy received more attention from demographers, governments, and international organizations as an indicator of the health and mortality status of a population, more recently lifespan inequality has gained recognition as a key demographic indicator (van Raalte, Sasson, and Martikainen 2018). Demographers have proposed many different indicators, including the variance, standard deviation, coefficient of variation, years of life lost, Gini coefficient, interquartile range, and Theil index, which are highly correlated (van Raalte and Caswell 2013; Colchero et al. 2016).

Populations around the world have experienced both large rises in life expectancy and decreases in lifespan inequality since the mid-20th century (Smits and Monden 2009; Permanyer and Scholl 2019). Studies of mortality data across a range of countries and time periods have found higher levels of life expectancy to be correlated with lower levels of lifespan inequality (Shkolnikov 2003; Shkolnikov et al. 2011; Vaupel, Zhang, and van Raalte 2011), and, when considering the dynamics of change over time, increases in life expectancy have been shown to be correlated with decreases in lifespan inequality (Aburto et al. 2020). However, the parallel rise of life expectancy and decrease of lifespan inequality is not guaranteed. In contexts such as eastern Europe, the United States, and some Latin American countries, the negative association between changes in life expectancy and changes in lifespan inequality is weakened or even reversed (Aburto and van Raalte 2018; Aburto et al. 2020; van Raalte, Sasson, and Martikainen 2018; Aburto and Beltrán-Sánchez 2019; García and Aburto 2019). Reductions in mortality at any age will increase life expectancy, and the amount of increase depends on the potential years of life saved at each age (Vaupel 1986). On the other hand, whether a reduction in mortality will increase or decrease lifespan inequality depends on the age at which the reduction happens since the dynamics of change in lifespan inequality are not identical to those of life expectancy (Aburto et al. 2020). Thus, an increase in life expectancy can be accompanied by an increase in lifespan inequality. For several indicators of lifespan inequality, there exists a 'threshold age' above which improvements in mortality will increase lifespan inequality since extending lifespans past that age will increase the overall variation in length of life.

The threshold age reveals the ages at which a reduction in mortality will simultaneously increase life expectancy and decrease lifespan inequality. Zhang and Vaupel (2009) argue that the threshold age separates 'early' from 'late' deaths. Hence, the threshold age can be interpreted as providing a definition of what constitutes 'premature' mortality. The existence and form of the threshold age has been mathematically proven for life disparity (Zhang and Vaupel 2009), the life table entropy (Aburto et al. 2019), the variance (Gillespie, Trotter, and Tuljapurkar 2014), and the Gini coefficient (Aburto et al. 2022), but both the existence and form remain to be determined for the coefficient of variation.

The coefficient of variation is a standard statistical indicator of dispersion for strictly positive random variables and is defined as the standard deviation divided by the mean. For example, the coefficient of variation of the distribution of incomes in a population is used to operationalize economic inequality (Atkinson 1970; De Maio 2007), though more generally it can be applied as an indicator of social and health inequalities in human populations (Allison 1978; Gakidou, Murray, and Frenk 2000) or inequalities in any biological property in nonhuman populations (Bendel et al. 1989). When applied to the age at death distribution, the coefficient of variation is an indicator of lifespan inequality (Wrycza, Missov, and Baudisch 2015; Aburto et al. 2018, 2020). It is the relative counterpart to the standard deviation of the age at death distribution (Edwards and Tuljapurkar 2005; Tuljapurkar 2010; Alvarez, Aburto, and Canudas-Romo 2020), which is a commonly used indicator of absolute lifespan inequality.

2. Relationship

The coefficient of variation of the age at death distribution, denoted c_v, is defined as

$$c_v(t) = \frac{\sigma(t)}{e_0(t)} = \frac{\sqrt{\int_0^\infty d(x,t)(x - e_0(t))^2 dx}}{e_0(t)}, \tag{1}$$

where t is time period, x is age, $d(x,t)$ is the probability density function of death at age x and time t, $e_0(t) = \int_0^\infty \ell(x,t)dx$ is life expectancy at birth at time t, $\ell(x,t)$ is the probability of survival to age x at time t, and $\sigma(t) = \sqrt{\int_0^\infty d(x,t)(x - e_0(t))^2 dx}$ is the standard deviation of the age at death distribution at time t.

Let $\dot{c_v}$ be the derivative of $c_v(t)$ with respect to time (we will use a single dot to denote the derivative or partial derivative with respect to time, and we omit t to simplify notation). We will prove that the time derivative of c_v can be written as the weighted sum of $\rho(x) = -\dot{\mu}(x)/\mu(x)$, the age-specific rates of mortality improvements:

$$\dot{c_v} = \int_0^\infty \rho(x) \frac{\mu(x)}{\sigma e_0} \int_x^\infty \ell(y) \left(y - e_0 - \frac{\sigma^2}{e_0} \right) dy dx. \tag{2}$$

Furthermore, we will show that this form of the derivative implies that there exists a unique threshold age below which mortality improvements reduce the coefficient of variation and above which improvements increase it. This corresponds to the age x at which

$$\bar{a}_x = e_0 + \frac{\sigma^2}{e_0}, \tag{3}$$

where

$$\bar{a}_x = \frac{\int_x^\infty \ell(y) y dy}{\int_x^\infty \ell(y) dy}$$

is the average age of the population older than x in the stationary population defined by the survivorship schedule of $\ell(x)$. The quantity $e_0 + \frac{\sigma^2}{e_0}$ has demographic meaning: It is the average lifespan of the living in a stationary population (Wrigley-Field and Feehan 2021), which corresponds to the average age at death of the length-biased sample of only those alive at any given moment in a stationary population, and is equal to twice the average age of the whole stationary population.

3. Proof

We will first show that the derivative of the coefficient of variation with respect to time can be written as in equation (2). We will start by giving a form for the derivative of the variance, which can then be used to determine the derivative of the standard deviation. Let $X \sim d(x)$ be the random variable denoting time until death. The variance of the age at death distribution is

$$\sigma^2 = \int_0^\infty d(x)(x - e_0)^2 dx = E[X^2] - E[X]^2,$$

where $E[X]$ and $E[X^2]$ represent the first and second moments of the age at death distribution, respectively. Taking the derivative with respect to time,

$$\dot{\sigma^2} = \frac{d}{dt}E[X^2] - \frac{d}{dt}E[X]^2 = \frac{d}{dt}\int_0^\infty d(x)x^2 dx - 2e_0\dot{e_0}. \tag{4}$$

Through integration by parts (details shown in Appendix A-1), the second moment can be rewritten as

$$E[X^2] = 2\int_0^\infty x\ell(x)dx.$$

Hence the derivative of the second moment is

$$\frac{d}{dt}E[X^2] = \frac{d}{dt}2\int_0^\infty x\ell(x)dx.$$

Assuming that $\ell(x,t)$ and $\frac{\partial}{\partial t}\ell(x,t)$ are continuous in (x,t), we can bring the derivative inside the integral, and we have that

$$\begin{aligned} \frac{d}{dt}E[X^2] &= 2\int_0^\infty x\dot{\ell}(x)dx \\ &= -2\int_0^\infty x\ell(x)\int_0^x \dot{\mu}(y)dydx \\ &= 2\int_0^\infty \rho(x)\mu(x)\int_x^\infty y\ell(y)dydx, \end{aligned} \tag{5}$$

the last step coming from changing the order of integration and renaming the variables of integration. Furthermore, Vaupel and Canudas-Romo (2003) show that

$$\dot{e}_0 = \int_0^\infty \rho(x)\mu(x)\ell(x)e(x)dx,$$

where $e(x) = \frac{1}{\ell(x)} \int_x^\infty \ell(y)dy$ is the remaining life expectancy at age x, so this becomes

$$\dot{e}_0 = \int_0^\infty \rho(x)\mu(x) \int_x^\infty \ell(y)dydx. \tag{6}$$

Replacing equations (5) and (6) into equation (4) for $\dot{\sigma^2}$, we have

$$\begin{aligned} \dot{\sigma^2} &= 2\int_0^\infty \rho(x)\mu(x) \int_x^\infty y\ell(y)dydx - 2e_0 \int_0^\infty \rho(x)\mu(x) \int_x^\infty \ell(y)dydx \\ &= 2\int_0^\infty \rho(x)\mu(x) \int_x^\infty \ell(y)(y - e_0)dydx, \end{aligned} \tag{7}$$

which is equivalent to the result given by Gillespie, Trotter, and Tuljapurkar (2014) but is justified assuming continuous change over time.

Since $\sigma = \left(\sigma^2\right)^{\frac{1}{2}}$, by the chain rule

$$\dot{\sigma} = \frac{d}{dt}\sigma = \frac{1}{2}\left(\sigma^2\right)^{-\frac{1}{2}} \frac{d}{dt}\sigma^2 = \frac{\dot{\sigma^2}}{2\sigma}.$$

Then, $c_v = \frac{\sigma}{e_0}$, so by the quotient rule,

$$\begin{aligned} \dot{c_v} &= \frac{1}{e_0^2}\left(\dot{\sigma}e_0 - \sigma\dot{e_0}\right) \\ &= \frac{1}{e_0^2}\left(\frac{\dot{\sigma^2}e_0}{2\sigma} - \sigma\dot{e_0}\right) \\ &= \frac{1}{2\sigma e_0}\left(\dot{\sigma^2} - \frac{2\sigma^2\dot{e_0}}{e_0}\right). \end{aligned}$$

Now, replacing $\dot{e_0}$ and $\dot{\sigma^2}$ with the forms given in equations (6) and (7), respectively, we have that

$$\begin{aligned} \dot{c_v} &= \frac{1}{\sigma e_0}\left(\int_0^\infty \rho(x)\mu(x) \int_x^\infty \ell(y)(y - e_0)dydx - \frac{\sigma^2}{e_0}\int_0^\infty \rho(x)\mu(x) \int_x^\infty \ell(y)dydx\right) \\ &= \frac{1}{\sigma e_0}\int_0^\infty \rho(x)\mu(x) \int_x^\infty \ell(y)\left(y - e_0 - \frac{\sigma^2}{e_0}\right)dydx, \end{aligned}$$

which proves equation (2) and shows that the derivative of c_v with respect to time is a weighted sum of the rates of change of age-specific mortality.

Now we will prove the existence and uniqueness of the threshold age. Let

$$g(x) = \int_x^\infty \ell(y)\left(y - e_0 - \frac{\sigma^2}{e_0}\right)dy,$$

which corresponds to the part of the weights in equation (2) that can change sign (the quantities $\mu(x)$, e_0, and σ are always positive). Then, differentiating with respect to x we have that

$$g'(x) = -\ell(x)\left(x - e_0 - \frac{\sigma^2}{e_0}\right)$$

and

$$g''(x) = -\ell'(x)\left(x - e_0 - \frac{\sigma^2}{e_0}\right) - \ell(x).$$

Since $x - e_0 - \frac{\sigma^2}{e_0} < 0$ for $x \in [0, e_0 + \frac{\sigma^2}{e_0})$, $g(x)$ is an increasing function on that interval. Furthermore, $g(x)$ reaches a critical point at $x = e_0 + \frac{\sigma^2}{e_0}$, and the second derivative at this point is equal to $-\ell(x)$, which is always negative, so this point is a local maximum. We also have that $g(e_0 + \frac{\sigma^2}{e_0}) > 0$ unless $\ell(y)$ is identically equal to 0 on $[e_0 + \frac{\sigma^2}{e_0}, \infty)$, and $g(x)$ decreases as $x \to \infty$ but remains non negative since for $x > e_0 + \frac{\sigma^2}{e_0}$, $g(x) \geq 0$. Hence, if $g(0) < 0$ the intermediate value theorem guarantees that there exists a unique age at which $g(x) = 0$ in the interval $(0, e_0 + \frac{\sigma^2}{e_0})$, which corresponds to the threshold age at which mortality improvements stop reducing the coefficient of variation and translate to increases in lifespan inequality.

To show that $g(0) < 0$, we evaluate the integral in $g(x)$, and we have that

$$\begin{aligned} g(x) &= \int_x^\infty \ell(y)y\,dy - \left(e_0 + \frac{\sigma^2}{e_0}\right)\int_x^\infty \ell(y)dy \\ &= \ell(x)e(x)\bar{a}_x - \ell(x)e(x)\left(e_0 + \frac{\sigma^2}{e_0}\right). \end{aligned} \tag{8}$$

Since $\ell(x)$ and $e(x)$ are always non negative, $g(0) < 0$ if

$$\bar{a}_0 < e_0 + \frac{\sigma^2}{e_0},$$

which is always true: The average age in a stationary population is always less than the average age at death of the length-biased sample of those alive (Wrigley-Field and Feehan 2021), so we have proven the existence and uniqueness of the threshold. The form for the threshold age follows directly from equation (8) since it implies that $g(x) = 0$ at the age x such that

$$\bar{a}_x = e_0 + \frac{\sigma^2}{e_0},$$

which corresponds to the threshold age and proves equation (3).

4. Related results

The existence and form of the threshold age for several other indicators of lifespan inequality have been mathematically proven and are summarized in Table 1. Gillespie, Trotter, and Tuljapurkar (2014) analyze the sensitivity of the variance to changes in age-specific mortality and determine that the threshold age is the age x at which

$$\int_x^\infty \ell(y)(y - e_0)dy = 0.$$

Evaluating the integral as we did for the coefficient of variation, an equivalent form for the threshold age of the variance is when

$$\bar{a}_x = e_0.$$

Furthermore, assuming $\sigma^2 > 0$,

$$\int_x^\infty \ell(y)(y - e_0)dy > \int_x^\infty \ell(y)\left(y - e_0 - \frac{\sigma^2}{e_0}\right)dy, \tag{9}$$

so the threshold age for the coefficient of variation will be at an older age than that of the variance. To see this, note that both sides of equation (9), when considered as functions of x, are increasing on the intervals $(0, e_0)$ for the left-hand side and $(0, e_0 + \frac{\sigma^2}{e_0})$ on the right-hand side. Since the left-hand side is always greater than the right-hand side, it will achieve the value of 0 before the right-hand side does.

Table 1: Lifespan inequality indicators and their threshold ages

Indicator	Notation	Definition	Threshold age
Life disparity	$e^\dagger$	$\int_0^\infty d(x)e(x)dx$	$H(x) + \bar{H}(x) = 1$
Life table entropy	$\bar{H}$	$\frac{e^\dagger}{e_0}$	$H(x) + \bar{H}(x) = 1 + \bar{H}$
Variance	σ^2	$\int_0^\infty d(x)(x - e_0)^2 dx$	$\bar{a}_x = e_0$
Coefficient of variation	c_v	$\frac{\sqrt{\sigma^2}}{e_0}$	$\bar{a}_x = e_0 + \frac{\sigma^2}{e_0}$
Absolute Gini	G^{abs}	$e_0 - \int_0^\infty \ell^2(x)dx$	$2\ell(x)D(x) = 1$
Gini coefficient	G	$1 - \frac{\int_0^\infty \ell^2(x)dx}{\int_0^\infty \ell(x)dx}$	$2\ell(x)D(x) = D$

Notes: $H(x)$ is the cumulative hazard at age $x : H(x) = \int_0^x \mu(y)dy$. $\bar{H}(x)$ is the life table entropy after age x: $\bar{H}(x) = \int_x^\infty d(y)e(y)dy/e(x)$. $\bar{a}_x$ is the average age in the stationary population after age x: $\bar{a}_x = \frac{\int_x^\infty \ell(y)y\,dy}{\int_x^\infty \ell(y)dy}$. D is Drewnowski's index: $D = 1 - G$. $D(x)$ is Drewnowski's index after age x: $D(x) = \frac{1}{\ell(x)} \frac{\int_x^\infty \ell^2(y)dy}{\int_x^\infty \ell(y)dy}$.

Life disparity $e^\dagger$ is defined as

$$e^\dagger = \int_0^\infty d(x)e(x)dx$$

and is an absolute indicator of lifespan inequality (Vaupel and Canudas-Romo 2003). Its relative counterpart is the life table entropy, defined as (Mitra 1978; Vaupel 1986; Goldman and Lord 1986; Vaupel and Canudas-Romo 2003)

$$\bar{H} = \frac{e^\dagger}{e_0},$$

which can also be shown to be the survivor weighted average of the cumulative hazard rates:

$$\bar{H} = \frac{\int_0^\infty \ell(x)H(x)dx}{\int_0^\infty \ell(x)dx}, \tag{10}$$

where $H(x) = \int_0^x \mu(y)dy$ is the cumulative hazard at age x. Leser (1955) and Keyfitz (1977) derive the life table entropy as the elasticity of life expectancy with respect to a constant proportional change in mortality and give its form as equation (10). Zhang and Vaupel (2009) find that the threshold age for $e^\dagger$ occurs at the age x when

$$H(x) + \bar{H}(x) = 1,$$

where

$$\bar{H}(x) = \frac{\int_x^\infty d(y)e(y)dy}{e(x)}$$

is the life table entropy after age x. For the life table entropy, Aburto et al. (2019) determine that the threshold age occurs at the age x such that

$$H(x) + \bar{H}(x) = 1 + \bar{H}.$$

Furthermore, this implies that the threshold age for the life table entropy will always be greater than that of $e^\dagger$. Assuming that $\bar{H} > 0$, we have that for any value of x,

$$H(x) + \bar{H}(x) - 1 > H(x) + \bar{H}(x) - 1 - \bar{H}.$$

Both sides of equation (4) are strictly increasing in x (Aburto et al. 2019), so the left-hand side will reach 0 before the right-hand side does.

The Gini coefficient of lifespans is defined as

$$G = \frac{1}{2e_0} \int_0^\infty \int_0^\infty d(x)d(y)|x - y|dxdy,$$

and Hanada (1983) shows that this can be rewritten as

$$G = 1 - \frac{\int_0^\infty \ell^2(x)dx}{\int_0^\infty \ell(x)dx}.$$

Aburto et al. (2022) determine the threshold age for Drewnowski's index, an indicator of lifespan equality defined as $D = 1 - G$, to be the age at which

$$2\ell(x)D(x) = D,$$

where

$$D(x) = \frac{1}{\ell(x)} \frac{\int_x^\infty \ell^2(y)dy}{\int_x^\infty \ell(y)dy}$$

is Drewnowski's index after age x. Furthermore, they prove that the threshold age for Drewnowski's index and the Gini coefficient are the same. Both the Gini coefficient and Drewnowski's index are relative measures, and the corresponding absolute measure is the absolute Gini coefficient (Shkolnikov 2003; Vigezzi et al. 2022), which can be written as

$$G^{abs} = e_0 - \int_0^\infty \ell^2(x)dx.$$

The results of Aburto et al. (2022) imply that the threshold age of G^{abs} occurs when

$$2\ell(x)D(x) = 1,$$

which implies that the threshold age for the Gini coefficient will always be greater than that of the absolute Gini coefficient (proof in Appendix A-2).

Hence, for the three pairs of absolute and relative indicators (here we use the term pair to refer to an absolute indicator and the relative indicator obtained by dividing it by life expectancy), the threshold age for the absolute indicator occurs at a younger age than the corresponding relative indicator. Furthermore, Gillespie, Trotter, and Tuljapurkar (2014) find that in some populations there may be no threshold age, implying that any improvement in mortality would increase the variance. This can also occur with life disparity (Zhang and Vaupel 2009) and the absolute Gini coefficient, but it cannot with the relative measures the life table entropy, the Gini coefficient, and the coefficient of variation. This means that in certain populations, a reduction in mortality at any age can provoke an increase in absolute inequality (as measured by the standard deviation, life disparity, or the absolute Gini coefficient), but the same is not true for relative inequality (as measured by the coefficient of variation, the life table entropy, or the Gini coefficient). There will always exist some ages at which mortality improvements will reduce relative inequality.

5. Applications

The code and data used to produce the results in this section are available in the the public repository at the following link: https://osf.io/q2uhe.

5.1 Cross-country comparison

From equation (2) we can calculate the relative derivative of the coefficient of variation:

$$\frac{\dot{c_v}}{c_v} = \dot{c_v} \cdot \frac{e_0}{\sigma} = \int_0^{\infty} \rho(x) \frac{\mu(x)}{\sigma^2} \int_x^{\infty} \ell(y) \left(y - e_0 - \frac{\sigma^2}{e_0} \right) dy dx.$$

From this we can define the age-specific weights of mortality change:

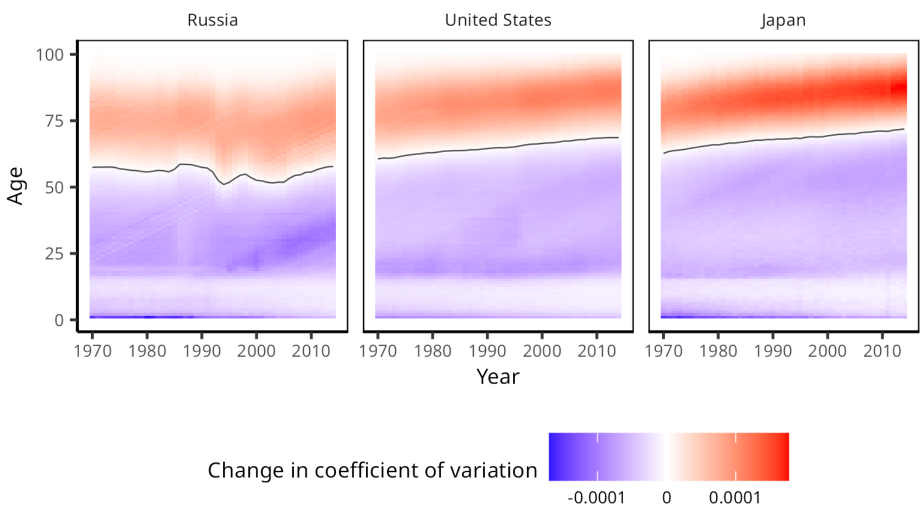
$$W_{cv}(x) = \frac{\mu(x)}{\sigma^2} \int_x^{\infty} \ell(y) \left(y - e_0 - \frac{\sigma^2}{e_0} \right) dy,$$

which represent the relative sensitivity of the coefficient of variation to a proportional change in mortality at age x. For example, the relative change provoked by a 1% reduction in mortality constant in the age interval x to $x + 1$ is

$$0.01 \cdot \int_x^{x+1} W_{cv}(y) dy.$$

To demonstrate these sensitivities we used period life tables from the Human Mortality Database for Russia, the United States, and Japan for the years 1970 to 2014 (Human Mortality Database 2023). We chose these three countries since they represent a low-mortality longevity leader (Japan), a country that has experienced severe stagnations and declines in life expectancy (Russia), and an example of a country with overall high life expectancy but not a leader (United States). We chose period rates (opposed to cohort rates) because they provide a snapshot of inequality and can be used to monitor year-to-year changes in the health and mortality conditions of a population (van Raalte, Sasson, and Martikainen 2018), but our results could be used in any analysis of mortality data, including cohort data and data for nonhuman populations. Figure 1 shows the sensitivity of the coefficient of variation to a 1% mortality reduction for ages 1 to 100 and for the years 1970 to 2014. These years allow us to see a comparison of how the sensitivity of the coefficient of variation evolves over both steady mortality improvements and extreme crises, as Russia experienced fluctuating life expectancy starting in the 1990s after the breakup of the Soviet Union. We show only ages 1 to 100 since age 0 represents for all countries and all years by far the greatest magnitude of weights and distorts the scale of the graph. We do not show ages above 100 because the sensitivity to the change in the coefficient of variation is extremely close to 0.

Figure 1: Sensitivity of the coefficient of variation to a 1% reduction in age-specific mortality



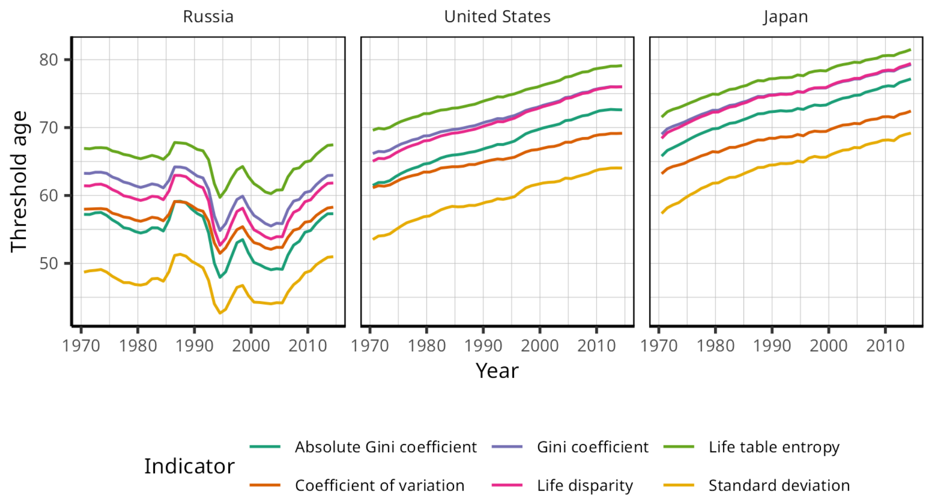
Darker shades of blue indicate the ages at which mortality improvements would most reduce lifespan inequality, and darker shades of red show the ages that would most increase it. In all three countries, we see that improvements in mortality in infancy and early childhood would result in the largest decreases in the coefficient of variation, but after very early childhood the potential decreases diminish. Then, between ages 15 and 45, reductions in mortality rates would produce substantial decreases in lifespan inequality, though this is much more pronounced for Russia and somewhat more for the United States.

The white band in the middle of the graph for all three countries represents the age at which the weights are very close to zero, and the threshold age is located at the black line in the middle of this white band. We see a steady postponement of the threshold age in the United States and Japan, indicating that over time mortality improvements at ever older ages can contribute to reducing lifespan inequality. Russia presents a different picture: The threshold age fluctuates throughout the time periods, and at the end of the series it remains at roughly the same level as the beginning.

Figure 1 also shows that the sensitivity of the coefficient of variation is driven by age, period, and cohort effects. A notable age effect is the small sensitivity in childhood between the ages 5 and 15. Period effects can be seen in Russia and the United States in the 1990s as the sensitivity increases overall for Russia during this period, reflecting the overall health crisis. For the United States, a slight decrease is visible for adult ages after the mid-1990s. Diagonal patterns in the graph show cohort effects, which are visible for all three countries. Increased sensitivity across a cohort indicates that efforts to reduce mortality rates in a specific cohort would cause greater reductions in lifespan inequality over time than only targeting specific ages. Nepomuceno et al. (2022) propose a new indicator in an effort to take into account the cohort drivers of lifespan inequality.

In order to compare our result of the threshold age for the coefficient of variation to the threshold age of other indicators, we calculated the thresholds for all six indicators in Table 1 again for Russian, US, and Japanese males between 1970 and 2014. Figure 2 shows that the threshold age for the standard deviation is the lowest across all countries and time periods, and the threshold for the life table entropy is the highest. However, the order of the threshold ages can change across countries and time periods. For example, for Russia during the whole time period, the threshold age for the absolute Gini coefficient is lower than that of the coefficient of variation, but this is not true for the United States and Japan. This also shows that the threshold age of an absolute indicator is not necessarily less than that of a relative indicator. The threshold age of an absolute indicator is always less than that of the relative indicator obtained by dividing the absolute indicator by life expectancy, but we have proven no general relationship between arbitrary pairs of indicators. Furthermore, Figure 2 shows that while the threshold ages seem to follow the same trend, the difference between them can be substantial. For all years and countries the difference between the threshold ages of the life table entropy and the standard deviation was greater than 12 years and reached over 18 years for some periods in Russia.

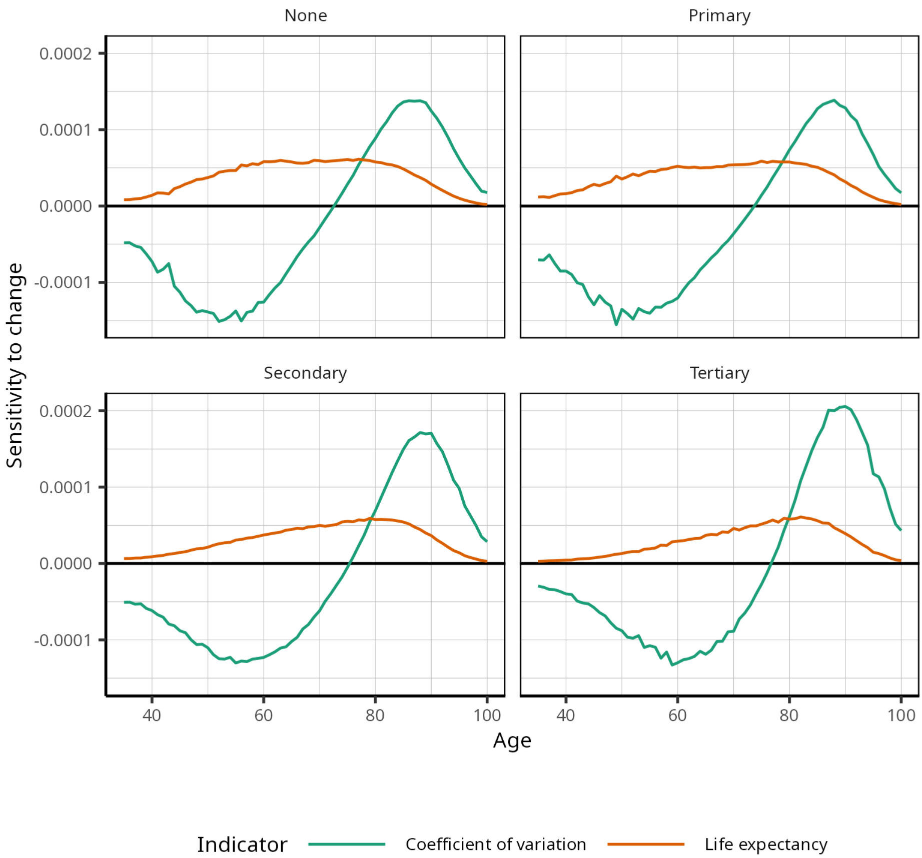
Figure 2: Threshold ages of lifespan inequality indicators



5.2 The threshold age for educational groups

We also illustrate our findings by examining lifespan inequality by educational level. There is a growing interest in understanding differences in mortality patterns over time among subgroups in a larger population, especially socioeconomic groups. A number of studies calculate life expectancy and lifespan inequality by educational level (van Raalte, Sasson, and Martikainen 2018; Permanyer et al. 2018; Sasson 2016; Gómez-Ugarte and García-Guerrero 2023) and find that the negative relationship between life expectancy and lifespan inequality is reversed in certain educational groups. Our results can help better understand these dynamics. We used life tables by education for Spain for the time period 2013–2015 taken from Permanyer et al. (2018). In that paper, individuals were classified in four educational attainment categories: "no education," "primary education," "secondary education," and "tertiary education." Figure 3 shows the sensitivity of life expectancy (Vaupel and Canudas-Romo 2003) and the coefficient of variation to a 1% reduction in age-specific mortality for Spanish males by educational level (we show results for Spanish females in Figure A-1 in Appendix A-3). In the case of the coefficient of variation, the minimum of the curve shows at which age a mortality reduction would most reduce lifespan inequality, and the intersection with the x-axis corresponds to threshold age. The maximum of the curve for life expectancy shows where mortality improvements would most increase life expectancy. Overall, we can observe that the threshold age increases progressively with increasing levels of education. For the two lower educational groups both the life expectancy and the coefficient of variation are more sensitive to change at younger ages compared to the two higher educational groups. This means that in lower educational groups there is greater potential to simultaneously increase life expectancy and reduce lifespan variation. However, the lower threshold age means that there is a larger range of ages in which a reduction in mortality will simultaneously increase both life expectancy and lifespan inequality.

Figure 3: Sensitivity of life expectancy and the coefficient of variation to a 1% reduction in mortality among Spanish males by educational level



6. Conclusion

The coefficient of variation of the age at death distribution is a relative lifespan inequality indicator equal to the average distance between all individual lifespans and life expectancy, divided by life expectancy itself. In this article we have shown that the rate of change over time of the coefficient of variation can be written as a weighted sum of rates of change of age-specific mortality times weights that determine the sensitivity of the coefficient of variation to the changes in mortality at each age. Furthermore, this result implies a unique threshold age that determines at which ages a reduction in mortality will also reduce lifespan inequality. The form for the derivative in equation (2) and the threshold in equation (3) can be used to better understand the dynamics over time of the coefficient of variation since they show the effect of mortality changes at each age on the indicator. When applied to data for the present, they can be used to determine at which ages reductions in mortality will most reduce inequality, which can guide public health policy.

The coefficient of variation complements the standard deviation of age at death, which is an absolute indicator. The rate of change over time of the standard deviation can also be written as a weighted sum of rates of change of the age-specific force of mortality in a similar fashion to the result for the coefficient of variation. When the threshold age exists for the standard deviation, it is unique and always less than or equal to the threshold age for the coefficient of variation. The property that the threshold age of an absolute indicator is always less than or equal to that of the corresponding relative indicator is verified for two other pairs of lifespan inequality indicators.

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Appendices

1. The moments of the age at death distribution

Since $d(x,t) = \mu(x,t)\ell(x,t)$, $\ell'(x) = -d(x,t)$, so we have (by integration by parts and application of L'Hôpital's rule)

$$\begin{aligned}\int_0^\infty x d(x,t)dx &= -\int_0^\infty x\ell'(x)dx \\ &= -\lim_{t\to\infty} \frac{x^2}{2}\ell(x)\bigg|_0^t + \int_0^\infty \ell(x,t)dx \\ &= \int_0^\infty \ell(x,t)dx.\end{aligned}$$

Similarly, for the second moment (again by integration by parts and L'Hôpital's rule),

$$\begin{aligned}\int_0^\infty x^2 d(x,t)dx &= -\int_0^\infty x^2\ell'(x)dx \\ &= -\lim_{t\to\infty} \frac{x^3}{3}\ell(x)\bigg|_0^t + 2\int_0^\infty x\ell(x,t)dx \\ &= 2\int_0^\infty x\ell(x,t)dx.\end{aligned}$$

2. The threshold age of the absolute Gini coefficient

Following Aburto et al. (2022), Drewnowski's index can be written as

$$D = \frac{\int_0^\infty \ell^2(x)dx}{\int_0^\infty \ell(x)dx} = \frac{\vartheta}{e_0} = 1 - G,$$

where $\vartheta = \int_0^\infty \ell^2(x)dx$ is life expectancy if mortality rates were doubled at all ages. Then and the absolute Gini coefficient can then be written as

$$G^{abs} = e_0 G = e_0(1 - D) = e_0 - \vartheta.$$

The derivative of ϑ with respect to time was shown to be

$$\dot{\vartheta} = \int_0^\infty 2\rho(x)e(x)d(x)\ell(x)D(x)dx,$$

so

$$\begin{aligned}\dot{G}^{abs} &= \dot{e_0} - \dot{\vartheta} \\ &= \int_0^\infty \rho(x)e(x)d(x)dx - \int_0^\infty 2\rho(x)e(x)d(x)\ell(x)D(x)dx \\ &= \int_0^\infty \rho(x)e(x)d(x)\big(1 - 2\ell(x)D(x)\big)dx.\end{aligned}$$

Since, as a function of x, $2\ell(x)D(x) - D$ is strictly decreasing, an immediate consequence is that $1 - 2\ell(x)D(x)$ is strictly increasing. Furthermore,

$$\lim_{x\to\infty} \ell(x)D(x) = 0,$$

so

$$\lim_{x\to\infty} 1 - 2\ell(x)D(x) = 1 > 0,$$

which implies that if $1 - 2\ell(0)D(0) = 1 - 2D < 0$, there exists a unique threshold age greater than 0 for the absolute Gini coefficient, which occurs when

$$2\ell(x)D(x) = 1.$$

However, this is not always the case as high lifespan inequality will cause $1 - 2D$ to be positive. Because the threshold age for both the Gini coefficient and Drewnowski's index occurs when

$$2\ell(x)D(x) = D,$$

$2\ell(x)D(x)$ is strictly decreasing, and $0 < D \leq 1$, the threshold age for G^{abs} must necessarily occur at less than or equal to the threshold age for G.

3. Results for Spanish females

Figure A-1: Sensitivity of life expectancy and the coefficient of variation to a 1% reduction in mortality among Spanish females by educational level