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# POLYNOMIAL DIFFERENTIAL SYSTEMS WITH HYPERBOLIC LIMIT CYCLES

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

ABSTRACT. Given an algebraic curve of degree $n$ we provide polynomial differential systems of degree greater or equal than $n$ which admit the ovals components of the curve as hyperbolic limit cycles.

## 1. Introduction and statement of the main results

The second part of the 16th Hilbert problem aims to obtain the maximum number of limit cycles of the polynomial differential equation

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where the dot means derivative with respect to the independent variable $t$ and $P, Q$ are polynomials. There is an extensive literature on the existence, number and stability of limit cycles for the differential equation (1) (see for instance [3, 4, 6, 7, 13, 16] and the references therein). It is a very hard problem to know the existence of limit cycles for a given polynomial differential equation and it is even harder to know its exact analytical expression and this has been done for very few and specific cases. The aim of this paper is to provide a contribution in this direction by determining the number of limit cycles and their expression for certain polynomial differential systems (1). Guided by [1, 5, 9, 10, 11, 12, 14] we will give polynomial differential systems where we will provide the number and explicit form of the limit cycles by just choosing the components of the system in a clever way.

Before stating the main result of the paper we introduce some preliminary definitions. Let $\mathbb{R}[x, y]$ be the ring of polynomials with real coefficients. Given $U \in \mathbb{R}[x, y]$ the algebaric curve $U = 0$ is called

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*invariant* of the polynomial differential equation (1) if for some polynomial $K \in \mathbb{R}[x,y]$ called *the cofactor* of the algebraic curve, we have

$$P(x,y)\frac{\partial U}{\partial x} + Q(x,y)\frac{\partial U}{\partial y} = KU.$$

It is clear that $U = 0$ is formed by trajectories of the polynomial differential equation (1).

The curve $\Omega = \{(x,y) \in \mathbb{R}^2 : U(x,y) = 0\}$ is a *non-singular curve* of the polynomial differential equation (1) if the equilibrium points of the system, that is, the points $(x,y) \in \mathbb{R}^2$ such that $P(x,y) = Q(x,y) = 0$ are not contained in $\Omega$.

A *limit cycle* $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$ is a $T$-periodic solution isolated in the set of all periodic solutions of the system. A limit cycle $\Gamma$ is called *hyperbolic* if

$$\int_0^T \operatorname{div}(\Gamma)\, dt \neq 0,$$

see for instance [15].

Take $P \in \mathbb{R}[y]$, $Q \in \mathbb{R}[x]$ of degrees $n_1 \geq 0$, $n_2 \geq 0$, respectively, and set $\Phi \in \mathbb{R}[x,y]$ of degree $n$. Consider the curve

$$R(x,y) = \alpha x + \beta y + \int Q(x)\, dx - \int P(y)\, dy,$$

with $\alpha, \beta \in \mathbb{R}$ and the function

$$w = w(x,y) = \int Q(x)\, dx - \int P(y)\, dy.$$

**Theorem 1.** *Let $U = 0$ be a non-singular algebraic curve of degree $m$ and $\Phi$ a polynomial function of degree $n$, chosen so that the curve*

$$R(x,y) + \Phi(w(x,y)) = 0$$

*lies outside all oval components of $U = 0$. If $Q(x)\beta + P(y)\alpha \neq 0$ then the polynomial differential equation*

$$\begin{aligned} \dot{x} &= P(y)U + \big(R(x,y) + \Phi(w(x,y))\big)U_y, \\ \dot{y} &= Q(x)U - \big(R(x,y) + \Phi(w(x,y))\big)U_x, \end{aligned} \tag{2}$$

*has all the oval components of $U = 0$ as hyperbolic limit cycles.*

The proof of Theorem 1 is given in section 2. When $\Phi$ is constant we obtain the same result that was proved in [5, Theorem 2.1] and when $P$ and $Q$ are of degree zero then we obtain the same result as in [2].

The last part of section 2 is devoted to providing an example satisfying all the conditions of Theorem 1.

## 2. Proof of Theorem 1

By assumptions the curve of $U = 0$ which is a non-singular curve of system (2) and the curve

$$R(x, y) + \Phi(w(x, y)) = 0$$

lies outside all oval components of $\Gamma$. To show that all the oval components of $U = 0$ are hyperbolic limit cycles of the polynomial differential equation (2) we will show that $U = 0$ is an invariant algebraic curve of the polynomial differential equation (2) and that if $U^*$ is an oval of the curve $U = 0$, corresponding to the periodic solution $(x(t), y(t))$ with period $T$, then

$$\int_0^T \operatorname{div}(U^*)\, dt \neq 0$$

Clearly, $U = 0$ is an invariant algebraic curve with cofactor $K = P(y)U_x + Q(x)U_y$ because

$$\begin{aligned}\frac{dU}{dt} &= U_x\big(P(y)U + \big(R(x,y) + \Phi(w(x,y))\big)U_y\big) + U_y\big(Q(x)U \\ &\quad - \big(R(x,y) + \Phi(w(x,y))\big)U_y\big) \\ &= (P(y)U_x + Q(x)U_y)U.\end{aligned}$$

It was proved in [8] that for an algebraic invariant curve $U = 0$ with cofactor $K$ we have

$$\int_0^T \operatorname{div}(U^*)\, dt = \int_0^T K(x(t), y(t))\, dt.$$

We claim that

$$\begin{aligned}\int_0^T K(x(t), y(t))\, dt &= -\int_{U^*=0} \frac{P(y)U_x}{\big(R(x,y) + \Phi(w(x,y))\big)U_x}\, dy \\ &\quad + \int_{U^*=0} \frac{Q(x)U_y}{\big(R(x,y) + \Phi(w(x,y))\big)U_y}\, dx \\ &= -\int_{U^*=0} \frac{P(y)}{R(x,y) + \Phi(w(x,y))}\, dy \\ &\quad + \int_{U^*=0} \frac{Q(x)}{R(x,y) + \Phi(w(x,y))}\, dx\end{aligned}$$

Now we prove the claim. By the line integral we have

$$\int_{U^*=0}\left(\frac{Q(x)}{R(x,y)+\Phi(w(x,y))},-\frac{P(y)}{R(x,y)+\Phi(w(x,y))}\right)d\gamma$$
$$=\int_0^T\left(\frac{Q(x)}{R(x,y)+\Phi(w(x,y))}\dot{x}(t)-\frac{P(y)}{R(x,y)+\Phi(w(x,y))}\dot{y}(t)\right)dt$$
$$=\int_0^T\frac{Q(x)}{R(x,y)+\Phi(w(x,y))}\big(P(y)U+(R(x,y)+\Phi(w(x,y)))U_y\big)\,dt$$
$$-\int_0^T\frac{P(y)}{R(x,y)+\Phi(w(x,y))}\big(Q(x)U-(R(x,y)+\Phi(w(x,y)))U_x\big)\,dt$$
$$=\int_0^T\big(Q(x)U_y+P(y)U_x\big)\,dt=\int_0^T K(x(t),y(t))\,dt=\int_0^T\operatorname{div}(U)\,dt.$$

By applying Green's formula we have

$$\int_0^T\operatorname{div}(U)\,dt=\int\int_{\mathrm{Int}(U^*=0)}\left(\frac{\partial}{\partial y}\left(\frac{Q(x)}{R(x,y)+\Phi(w(x,y))}\right)\right)dxdy$$
$$+\int\int_{\mathrm{Int}(U^*=0)}\left(\frac{\partial}{\partial x}\left(\frac{P(y)}{R(x,y)+\Phi(w(x,y))}\right)\right)dxdy$$
$$=-\int\int_{\mathrm{Int}(U^*=0)}\frac{Q(x)\left(\frac{\partial R}{\partial y}+\frac{\partial\Phi}{\partial w}\frac{\partial w}{\partial y}\right)}{\big(R(x,y)+\Phi(w(x,y))\big)^2}\,dxdy$$
$$-\int\int_{\mathrm{Int}(U^*=0)}\frac{P(y)\left(\frac{\partial R}{\partial x}+\frac{\partial\Phi}{\partial w}\frac{\partial w}{\partial x}\right)}{\big(R(x,y)+\Phi(w(x,y))\big)^2}\,dxdy,$$

where $\mathrm{Int}(U^*=0)$ denotes the interior of the bounded region limited by $U^*=0$. Using that

$$\frac{\partial R}{\partial x}=\alpha+Q(x),\quad \frac{\partial R}{\partial y}=\beta-P(y),\quad \frac{\partial w}{\partial x}=Q(x),\quad \frac{\partial w}{\partial y}=-P(y),$$

we get

$$\int_0^T\operatorname{div}(U^*)\,dt=-\int\int_{\mathrm{Int}(U^*=0)}\frac{Q(x)\beta+P(y)\alpha}{\big(R(x,y)+\Phi(w(x,y))\big)^2}\,dxdy.$$

Since by assumption $Q(x)\beta+P(y)\alpha\neq 0$ we get that $\int_0^T\operatorname{div}(\Gamma)\,dt\neq 0$. This concludes the proof of Theorem 1.

Now we provide a polynomial differential system satisfying all the assumptions of Theorem 1. Let $\alpha=1$, $\beta=\varepsilon>0$, $P(y)=y^2$, $Q(x)=$

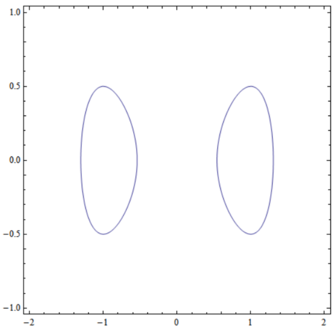


**Figure 1.** The two ovals of the algebraic curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$.

$x^2$ and $\Phi(w) = -w$. Here $\varepsilon$ is sufficiently small. Then system (3) becomes

$$
\begin{aligned}
\dot{x} &= y^2(2x^4 - 4x^2 + 4y^2 + 1) + 8y(x + \varepsilon y), \\
\dot{y} &= x^2(2x^4 - 4x^2 + 4y^2 + 1) - 8(x^3 - x)(x + \varepsilon y).
\end{aligned} \tag{3}
$$

This system has the invariant algebraic curve $U = 2x^4-4x^2+4y^2+1 = 0$ with cofactor $K = 8xy(x-y+x^2y)$. Since $Q(x)\beta+P(y)\alpha = y^2+\varepsilon x^2 \neq 0$ and the straight line $R(x, y)+\Phi(w(x, y)) = x+\varepsilon y = 0$ does not intersect the two ovals of $U = 0$ (see Figure 1), these two ovals are hyperbolic limit cycles of system (3).

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

*Email address*: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

*Email address*: cvalls@math.ist.utl.pt