EXISTENCE OF A CYLINDER FOLIATED BY PERIODIC ORBITS IN THE GENERALIZED CHAZY DIFFERENTIAL EQUATION

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ABSTRACT. The generalized Chazy differential equation

$$\dddot{x} + |x|^q \ddot{x} + \frac{k|x|^q}{x}\dot{x}^2 = 0$$

has its regularity varying with q. Indeed, it is discontinuous on the straight line $x = 0$ for $q = 1$, but when q is an even positive integer it is polynomial, and when $q > 1$ is an odd integer it is continuous but not differentiable on the straight line $x = 0$.

In 1999, it was detected numerically the existence of periodic solutions in the generalized Chazy differential equation for $q = 2$ and $k = q + 1$. In this paper, we develop a strategy to prove analytically the existence of such periodic solutions. More specifically, we consider $k = q + 1$ and $q = 1, 2, 3$, which are representatives of the different classes of regularity. In these cases, we prove that the generalized Chazy differential system has an invariant cylinder foliated by periodic orbits.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In 1997, Feix et al. [2] introduced the following general third order ordinary differential equation

(1) $$\dddot{x} + x^{3q+1} f(a, b) = 0, \quad \text{where } a = \frac{\dot{x}}{x^{q+1}} \text{ and } b = \frac{\ddot{x}}{x^{2q+1}},$$

which are invariant under the time translation and the rescaling symmetries. The Chazy differential equation [1]

$$\dddot{x} + x^q \ddot{x} + kx^{q-1}\dot{x}^2 = 0,$$

introduced in 1911, is a particular case of the differential equations (1) when $f(a, b) = ka^2 + b$, $q = 1$, and $k = -3/2$.

In general, numerical computations on the actual Chazy equation do not find any periodic solution except for $q = 2$ and $k = 3$, where periodic solutions were detected for a big number of initial conditions (for more details see [4]). Also numerically, periodic solutions are observed for q even and $k = q + 1$ in the so-called *generalized Chazy differential equation*

(2) $$\dddot{x} + |x|^q \ddot{x} + \frac{k|x|^q}{x}\dot{x}^2 = 0,$$

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introduced in 1999 by Géronimi, Feix and Leach [4].

Usually, to prove analytically the existence of periodic orbits in a given differential equation is not an easy problem. The difficult of the problem increases significantly when the dimension or the order is greater than two. The objective of this paper is double, first to provide an analytic proof of the existence of the periodic orbits detected only numerically in the generalized Chazy differential equation, and second to show that the generalized Chazy differential equation has, actually, an invariant cylinder foliated by periodic orbits.

The generalized Chazy differential equation can be written as the following first order differential system in $\mathbb{R}^3$

$$(3) \qquad X : \begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -|x|^q z + \dfrac{k|x|^q}{x} y^2. \end{cases}$$

When $k = q + 1$ system (3) has the first integral

$$(4) \qquad H(x, y, z) = xz - \frac{y^2}{2} + |x|^q xy.$$

Notice that the regularity of the generalized Chazy differential system (3) changes with q. Indeed, for $q = 1$ the generalized Chazy differential system (3) is discontinuous on the straight line $x = 0$, whereas for q a positive even integer it is polynomial, and for $q > 1$ an odd integer it is continuous but not differentiable on the straight line $x = 0$. In this paper, we will focus our analysis in the cases $q = 1, 2, 3$, which are representatives of the above different classes of regularity.

Our main result is the following.

Theorem 1. *For $q = 1, 2, 3$ and $k = q + 1$ the generalized Chazy differential system (3) has an invariant cylinder foliated by periodic orbits.*

The proof of Theorem 1 follows by showing the existence of a fixed point of a suitable Poincaré return map in each negative energy level of the first integral H. Due to the nonlinear nature of the generalized Chazy differential system (3), it is not possible to integrate the flow for obtaining an explicit expression of a Poincaré return map, thus tools from the qualitative theory must be employed.

In what follows, we provide the idea of the proof. First, by applying a rescaling in the variables and in the time, we reduce the analysis to the energy level $H = -1$, which is a topological cylinder. Second, by taking advantage of a symmetry of the system in this level, we show the existence of a Poincaré return map, for which we can ensure the existence of a fixed point via the construction of a convenient trapping region. This implies the existence of a periodic solution in each negative energy level, which varies smoothly with the energy, producing then a topological cylinder foliated by periodic orbits.

Theorem 1 is proven in Section 2. Its proof is divided into three subsections, which cover the cases $q = 1$ (discontinuous), $q = 2$ (polynomial) and $q = 3$ (continuous), respectively.

It is worthy mentioning that the strategy develop here for proving the existence of invariant cylinders foliated by periodic orbits for $q = 1, 2, 3$ can also be used for other values of q integer. A general approach for any value of q is currently under investigation.

2. Proof of Theorem 1

Consider the first integral (4). By solving $H(x, y, z) = -\omega^2$, with $\omega > 0$, in the variable y, we obtain a family of invariant surfaces $S_{\omega,q} := S_{\omega,q}^- \cup S_{\omega,q}^+$, where

$$S_{\omega,q}^\pm = \{(x, y, z) \in \mathbb{R}^3 : y = f_{\omega,q}^\pm(x, z) \text{ and } xz + \omega^2 \geq 0\},$$

with

$$f_{\omega,q}^\pm(x, z) = x|x|^q \pm \sqrt{x^{2(1+q)} + 2xz + 2\omega^2}.$$

Notice that, for each ω, $S_{\omega,q}^-$ and $S_{\omega,q}^+$ intersect the plane $y = 0$ on the hyperbola

$$\Sigma_{\omega,q} = \{(x, y, z) : xz + \omega^2 = 0\}$$

and, therefore, $S_{\omega,q}$ is a topological cylinder. We shall prove that, fixed $q \in \{1, 2, 3\}$, the invariant surface $S_{\omega,q}$ contains a periodic orbit $\gamma_{\omega,q}$ for each ω.

In order to do that, we will define a Poincaré return map on the branches of the hyperbola $\Sigma_{\omega,q}$, namely,

$$\Sigma_\omega^- = \{(x, y, z) : xz + \omega^2 = 0, \ x < 0\}, \quad \Sigma_\omega^+ = \{(x, y, z) : xz + \omega^2 = 0, \ x > 0\}.$$

Denote by $X_{\omega,q}^-$ and $X_{\omega,q}^+$ the reduced systems of (3) on $S_{\omega,q}^-$ and $S_{\omega,q}^+$, respectively, which are given by

$$X_{\omega,q}^\pm : \begin{cases} \dot{x} = x|x|^q \pm \sqrt{x^{2(1+q)} + 2xz + 2\omega^2}, \\ \dot{z} = \dfrac{|x|^q}{x}\left(-xz - (q+1)\left(x|x|^q \pm \sqrt{x^{2(1+q)} + 2xz + 2\omega^2}\right)^2\right), \end{cases}$$

for $xz + \omega^2 \geq 0$. Let $\phi_{\omega,q}^\pm(t, \cdot)$ denote the flow of the vector field $X_{\omega,q}^\pm$.

In order to conclude the existence of a periodic orbit contained in $S_{\omega,q}$, it is sufficient to show the existence of a point $p_{0,q} \in \Sigma_{\omega,q}^-$ and $t_{0,q}, t_{1,q} > 0$ such that

$$(5) \qquad p_{1,q} = \phi_{\omega,q}^-(t_{0,q}, p_{0,q}) \in \Sigma_\omega^+ \text{ and } \phi_{\omega,q}^+(t_{1,q}, p_{1,q}) = p_{0,q},$$

in this case the periodic orbit is given by $\gamma_{\omega,q} = \gamma_{\omega,q}^+ \cup \gamma_{\omega,q}^-$, where

$$\gamma_{\omega,q}^- = \{(x, y, z) : (x, z) = \phi_{\omega,q}^-(t, p_{0,q}), \ y = f_{\omega,q}^-(\phi_{\omega,q}^-(t, p_{0,q})), \text{ and } 0 \leq t \leq t_{0,q}\} \text{ and}$$

$$\gamma_{\omega,q}^+ = \{(x, y, z) : (x, z) = \phi_{\omega,q}^+(t, p_{1,q}), \ y = f_{\omega,q}^+(\phi_{\omega,q}^+(t, p_{1,q})), \text{ and } 0 \leq t \leq t_{1,q}\}.$$

Now, in order to simplify the reduced systems $X_{\omega,q}^{\pm}$, we apply the following change of variables and time-rescaling

$$(x, z) = (\omega^{\frac{1}{1+q}} u, \omega^{\frac{1+2q}{1+q}} v) \text{ and } t = \omega^{-\frac{q}{1+q}} \tau.$$

Thus, the vector fields $X_{\omega,q}^{\pm}$ become

$$(6) \qquad \widetilde{X}_q^{\pm} : \begin{cases} u' = u|u|^q \pm \sqrt{u^{2(1+q)} + 2uv + 2}, \\ v' = \dfrac{|u|^q}{u} \left(-uv - (q+1) \left(u|u|^q \pm \sqrt{u^{2(1+q)} + 2uv + 2} \right)^2 \right), \end{cases}$$

for $uv + 1 \geq 0$, and the curves Σ_{ω}^{+} and Σ_{ω}^{-} become, respectively,

$$\widetilde{\Sigma}^{+} = \{(u, v) : uv + 1 = 0, \, u < 0\} \text{ and } \widetilde{\Sigma}^{-} = \{(u, v) : uv + 1 = 0, \, u > 0\}.$$

Notice that, if $\phi_q^{\pm}(t, \cdot)$ denote the flow of the vector fields $\widetilde{X}_q^{\pm}$, then

$$\phi_{q,\omega}^{\pm}(t, p) = \Omega \, \phi_q^{\pm}(\omega^{\frac{q}{1+q}} t, \Omega^{-1} p) \text{ where } \Omega = \begin{pmatrix} \omega^{\frac{1}{1+q}} & 0 \\ 0 & \omega^{\frac{1+2q}{1+q}} \end{pmatrix}.$$

In the sequel, we shall prove the existence of a periodic orbit by showing, for $q = 1, 2, 3$, the existence of a point $p_q^* = (u_q^*, -1/u_q^*) \in \widetilde{\Sigma}^{-}$ and a $t_q^* > 0$ such that

$$\phi_q^{-}(t_q^*, p_q^*) = -p_q^* = (-u_q^*, 1/u_q^*) \in \widetilde{\Sigma}^{+}.$$

Indeed the vector fields $\widetilde{X}_q^{+}$ and $\widetilde{X}_q^{-}$ satisfy $\widetilde{X}_q^{+}(u, v) = -\widetilde{X}_q^{-}(-u, -v)$. This means that $\phi_q^{+}(t, p) = -\phi_q^{-}(t, -p)$ and, therefore, $\phi_q^{+}(t_q^*, -p_q^*) = -\phi_q^{-}(t_q^*, p_q^*) = p_q^*$. Hence, relationship (5) will hold by taking

$$p_{0,q} = \Omega \, p_q^* \text{ and } t_{0,q} = t_{1,q} = \omega^{\frac{1+q}{q}} t_q^*.$$

2.1. **Proof of Theorem 1 for the discontinuous case $q = 1$.** In this case, it follows from (6) that

$$\widetilde{X}_1^{\pm} = \begin{cases} \widetilde{X}_{1,L}^{\pm} & \text{if } u < 0, \\ \widetilde{X}_{1,R}^{\pm} & \text{if } u > 0, \end{cases}$$

for $uv + 1 \geq 0$, where

$$\widetilde{X}_{1,L}^{\pm} : \begin{cases} u' = -u^2 \pm \sqrt{u^4 + 2uv + 2}, \\ v' = uv + 2 \left(u|u| \pm \sqrt{u^4 + 2uv + 2} \right)^2, \end{cases}$$

and

$$\widetilde{X}_{1,R}^{\pm} : \begin{cases} u' = u^2| \pm \sqrt{u^4 + 2uv + 2}, \\ v' = -uv - 2 \left(u|u| \pm \sqrt{u^4 + 2uv + 2} \right)^2. \end{cases}$$

See Figure 1 for an idea of the phase space of the vector fields $\widetilde{X}_1^{-}$ (in the left) and $\widetilde{X}_1^{+}$ (in the right). Here, the Filippov's convention [3] is assumed for the trajectories of $\widetilde{X}_1^{\pm}$.
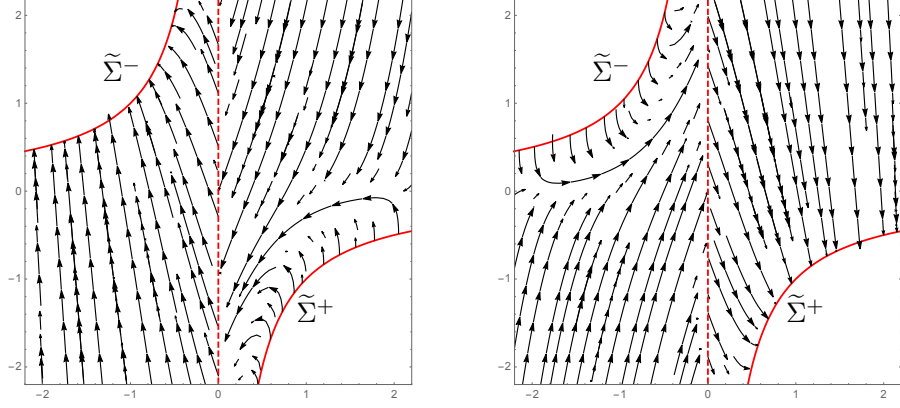
FIGURE 1. Phase space of the vector fields (left) $\widetilde{X}_1^-$ and (right) $\widetilde{X}_1^+$ defined for $uv \geq -1$. On the dashed line both vector fields (left) $\widetilde{X}_1^-$ and (right) $\widetilde{X}_1^+$ are discontinuous.

Consider the sections Σ_D^1 and Σ_I^1 on the hyperbola $u\,v + 1 = 0$, given by

$$\Sigma_D^1 = \{(u,v) : v = -1/u : u_{i,1}^D \leq u \leq u_{f,1}^D\} \quad \text{and}$$
$$\Sigma_I^1 = \{(u,v) : v = -1/u : u_{i,1}^I \leq u \leq u_{f,1}^I\},$$

where

$$u_{i,1}^D = 1/2^{3/4}, \quad u_{f,1}^D = 2, \quad u_{i,1}^I = -\frac{1+\sqrt{2}}{2^{3/4}} \quad \text{and} \quad u_{f,1}^I = -1/2^{3/4}.$$

We are going to show that the flow of $\widetilde{X}_1^-$ induces a map

(7) $$P_1 : \Sigma_D^1 \to \Sigma_I^1.$$

To do that, we will construct a compact connected trapping region K^1 such that $\Sigma_D^1 \cup \Sigma_I^1 \subset \partial K^1$ and $\widetilde{X}_1^-$ points inward everywhere on ∂K^1 except at Σ_I^1 (see Figure 2). Since $\widetilde{X}_1^-$ does not have singularities in K^1 and both vector fields $\widetilde{X}_{1,L}^-$ and $\widetilde{X}_{1,R}^-$ are transversal on $u = 0$ and point to the left, applying the Poincaré-Bendixson Theorem first in the region $K^1 \cap \{u \geq 0\}$ and then in the region $K^1 \cap \{u \leq 0\}$, we conclude that:

(C1) for each point $p \in \Sigma_D^1$, there exists $t(p) > 0$ such that $P_1(p) := \phi_1^-(t(p), p) \in \Sigma_I^1$.

Let K^1 (see Figure 2) be the compact region delimited by

$$\partial K^1 = \Sigma_D^1 \cup \Sigma_U^1 \cup R_1 \cup \cdots \cup R_5,$$

where
$$R_1 = \{(u,0), 0 < u \le 5/2\},$$
$$R_2 = \{(u,v) : v = 2u - 5/2 : u_{f,1}^D \le u \le 5/2\},$$
$$R_3 = \{(u,v) : v = 2\sqrt{2}|u| - 2^{7/4} : 0 \le u \le u_{i,1}^D\},$$
$$R_4 = \{(u,v) : v = 2\sqrt{2}|u| - 2^{7/4} : u_{i,1}^I \le u < 0\},$$
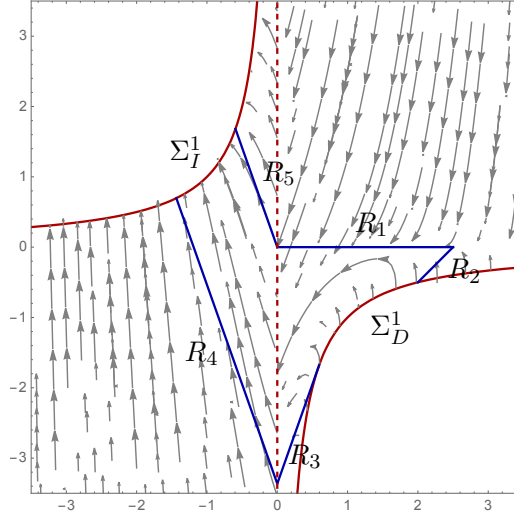$$R_5 = \{(u,v) : v = 2\sqrt{2}|u| : u_{f,1}^I \le u \le 0\}.$$

FIGURE 2. Phase space of the vector field $\widetilde{X}_1^-$ defined for $uv \ge -1$ and the trapping region K^1.

In what follows, we are going to analyze the behavior of the vector field $\widetilde{X}_1^-$ on each component of the boundary of K^1.

Behavior of $\widetilde{X}_1^-$ on Σ_D^1 and Σ_I^1. The hyperbola $g(u,v) = uv + 1 = 0$ on Σ_D^1 satisfies

$$\langle \nabla g(u,v), \widetilde{X}_1^-(u,v) \rangle|_{(u,v) \in \Sigma_D^1} = u,$$

which does not vanish on Σ_D^1. Note that the flow of the vector field $\widetilde{X}_1^-$ points inwards K^1 on Σ_D^1, because for instance $\widetilde{X}_1^-(1, -1) = (0, 1)$.

The curve $g(u,v) = uv + 1 = 0$ on Σ_I^1 satisfies

$$\langle \nabla g(u,v), \widetilde{X}_1^-(u,v) \rangle|_{(u,v) \in \Sigma_I^1} = u(1 + 8u^4),$$

which does not vanish on Σ_I^1. Note that the flow of the vector field $\widetilde{X}_1^-$ points outwards K^1 on Σ_I^1, because for instance $\widetilde{X}_1^-(-1, 1) = (-2, 7)$.

Behavior of $\widetilde{X}_1^-$ on R_1. The curve $g(u,v) = v = 0$ satisfies

$$\langle \nabla g(u,v), \widetilde{X}_1^-(u,v) \rangle|_{g(u,v)=0} = -4(1 + u^4 - u^2\sqrt{2 + u^4}).$$

We will show that this derivative does not vanish for $u > 0$. We proceed by contradiction. Assume that $1 + u^4 - u^2\sqrt{2 + u^4} = 0$. Then,

$$\sqrt{2 + u^4} = \frac{1 + u^4}{u^2} \quad \Leftrightarrow \quad 2 + u^4 = \left(\frac{1 + u^4}{u^2}\right)^2 = \frac{1 + u^8 + 2u^4}{u^4} \quad \Leftrightarrow \quad -\frac{1}{u^4} = 0,$$

which is not possible. This shows that the flow along R_1 points always inwards K^1, because for instance $\widetilde{X}_1^-(1,0) = (1 + \sqrt{3}, -4(2 + \sqrt{3}))$.

Behavior of $\widetilde{X}_1^-$ on R_2. The curve $g(u, v) = v - u + 5/2 = 0$ satisfies

$$\langle \nabla g(u, v), \widetilde{X}_1^-(u, v)\rangle|_{g(u,v)=0} = -4 + \frac{25}{2}u - 6u^2 - 4u^4 + (4u^2 + 1)\sqrt{2 - 5u + 2u^2 + u^4}.$$

We will show that this derivative does not vanish on R_2 that is, for $u_{f,1}^D \leq u \leq 5/2$. We proceed by contradiction. Assume that it is zero at some u. Then, proceeding analogously to R_1 to get rid of the square root, we obtain

$$\left(-4 + \frac{25}{2}u - 6u^2 - 4u^4\right)^2 = (1 + 4u^2)^2(2 - 5u + 2u^2 + u^4),$$

or equivalently

$$p_0(u) := -14 + 95u - \frac{745}{4}u^2 + 110u^3 - 19u^4 + 20u^5 - 8u^6 = 0.$$

Using the Sturm procedure we conclude that the polynomial $p_0(u)$ does not vanish on $u_{f,1}^D < u < 5/2$ which is a contradiction (see the appendix). Hence, the flow of the vector field $\widetilde{X}_1^-$ points inwards K^1 on R_2.

Behavior of $\widetilde{X}_1^-$ on R_3 and R_4. On the curve $g(u, v) = v - 2\sqrt{2}|u| - 2^{7/4} = 0$ we have: for $u > 0$

$$\langle \nabla g(u, v), \widetilde{X}_1^-(u, v)\rangle|_{g(u,v)=0} = p_1(u) := -4 + 10 \cdot 2^{3/4}u - 12\sqrt{2}u^2 - 4u^4$$
$$+ (4u^2 + 2\sqrt{2})\sqrt{2 - 4 \cdot 2^{3/4}u + 4\sqrt{2}u^2 + u^4},$$

and for $u < 0$,

$$\langle \nabla g(u, v), \widetilde{X}_1^-(u, v)\rangle|_{g(u,v)=0} = p_2(u) := 4 - 10 \cdot 2^{3/4}u - 12\sqrt{2}u^2 + 4u^4$$
$$+ (4u^2 - 2\sqrt{2})\sqrt{2 - 4 \cdot 2^{3/4}u + 4\sqrt{2}u^2 + u^4}.$$

Of course on $u = 0$ the above derivative is zero but since it is a quadratic tangency it is enough to show that $p_1(u)$ does not vanish on $0 < u \leq u_{i,1}^D$ and that the $p_2(u)$ does not vanish on $u_{i,1}^I \leq u < 0$.

We proceed by contradiction. Assume first that $p_1(u)$ has a zero. Then, proceeding as in R_1 to get rid of the square root, we have that $p_1(u) = 0$ implies

$$p_3(u) = 12 \cdot 2^{3/4} - 58\sqrt{2}u + 88 \cdot 2^{1/4}u^2 - 38u^3 + 4 \cdot 2^{3/4}u^4 - 4\sqrt{2}u^5 = 0.$$

Using the Sturm procedure we have that the polynomial $p_3(u)$ has a unique simple real zero on $u > 0$ (see the appendix). Moreover, we have that

$$p_1(0) = 0, \quad p_1'(0) = 6 \cdot 2^{3/4}, \quad p_1(u_{i,1}^D) = 1.$$

These computations show that if $p_1(u)$ has a zero in $(0, u_{i,1}^D)$ then it has another (or a multiple) zero on that interval. This would produce another zero of $p_3(u)$ in $(0, \infty)$, which is a contradiction. Hence the flow of the vector field $\widetilde{X}_1^-$ points inwards K^1 on R_3.

On the other hand assume that $p_2(u)$ has a zero. Then, proceeding as in R_1 to get rid of the square root, we have that $p_2(u) = 0$ implies

$$p_4(u) = 12 \cdot 2^{3/4} - 42\sqrt{2}u - 88 \cdot 2^{1/4}u^2 - 38u^3 + 4 \cdot 2^{3/4}u^4 + 4\sqrt{2}u^5 = 0.$$

Using the Sturm procedure we conclude that the polynomial $p_4(u)$ has a unique simple real zero on $u < 0$ (see the appendix). Moreover, we have that

$$p_2(0) = 0, \quad p_2'(0) = -6 \cdot 2^{3/4}, \quad p_2(u_{i,1}^I) = 38.5802...$$

These computations show that if $p_2(u)$ has a zero in $(u_{i,1}^I, 0)$, then either it is multiple or $p_2(u)$ has at least two negative zeros in contradiction with uniqueness of negative zeros provided by Sturm procedure. Hence the flow of the vector field $\widetilde{X}_1^-$ points inwards K^1 on R_4.

Behavior of $\widetilde{X}^-$ on R_5. Note that on the curve $g(u,v) = v - 2\sqrt{2}|u| = 0$ we have

$$\langle \nabla g(u,v), \widetilde{X}_1^-(u,v) \rangle|_{g(u,v)=0} = p_5(u) := 4 - 12\sqrt{2}u^2 + 4u^4 + (4u^2 - 2\sqrt{2})\sqrt{2 - 4\sqrt{2}u^2 + u^4}.$$

Of course on $u = 0$ the above derivative is zero but since it is a quadratic tangency it is sufficient to show that this derivative does not vanish on $u_{f,1}^I < u < 0$. We proceed by contradiction. Assume that it is zero. Then, proceeding as in the region R_1 to get rid of the square root, we have that $p_5(u) = 0$ implies

$$\frac{2u^2(4\sqrt{2} - 19u^2 + 2\sqrt{2}u^4)}{(\sqrt{2} - 2u^2)^2} = 0.$$

Note that the denominator does not vanish on $u_{f,1}^I < u < 0$, because the unique negative real solution of $\sqrt{2} - 2u^2 = 0$, which is $-1/2^{1/4}$, is outside the mentioned interval. On the other hand the unique two negative real solutions of the numerator are

$$u_1 = -\frac{1}{2}\left(\frac{19\sqrt{2} - 3\sqrt{66}}{2}\right)^{1/2} \quad \text{and} \quad u_2 = -\frac{1}{2}\left(\frac{19\sqrt{2} + 3\sqrt{66}}{2}\right)^{1/2}.$$

We observe that u_1 belongs to the interval $u_{f,1}^I < u < 0$ while u_2 does not. However, u_1 is not a solution of $p_5(u) = 0$ because

$$p_5(u_1) = -\frac{3}{8}\left(-78 + 14\sqrt{33} + 3\sqrt{22(19 - 3\sqrt{33})} - 5\sqrt{6(19 - 3\sqrt{33})}\right) \neq 0.$$

This contradiction shows that the flow of the vector field $\widetilde{X}_1^-$ points inwards K^1 on R_5.

Existence of p_1^*. Consider the map $h_1 \colon \Sigma_D^1 \to \mathbb{R}$ defined by $h_1(p) = \pi(P_1(p) + p)$, where $\pi \colon \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate (see (C1)). Note that by the continuous dependence of the flow of $\widetilde{X}_1^-$ with respect to the initial conditions the map h_1 is continuous. Moreover, the image of the point $(u_{f,1}^D, -1/u_{f,1}^D)$ by P_1 (which is the point w_4 in Figure 6) is inside Σ_I^1, and its symmetric is below its image because $u_{f,1}^D + u_{i,1}^I > 0$. Hence $h_1(u_{f,1}^D, -1/u_{f,1}^D) > 0$. On the other hand, the image of the point $(u_{i,1}^D, -1/u_{i,1}^D)$ by P_1 (which is the point w_3 in Figure 6) is inside Σ_I^1, and its symmetric is above its image because $u_{i,1}^D + u_{f,1}^I < 0$. Therefore $h_1(u_{i,1}^D, -1/u_{i,1}^D) < 0$. Thus, by continuity, there exists $p_1^* \in \Sigma_D^1$ such that $h_1(p_1^*) = 0$, i.e, $P_1(p_1^*) = -p_1^*$ and so $\phi_1^-(t_1^*, p_1^*) = -p_1^*$ as we wanted to prove.
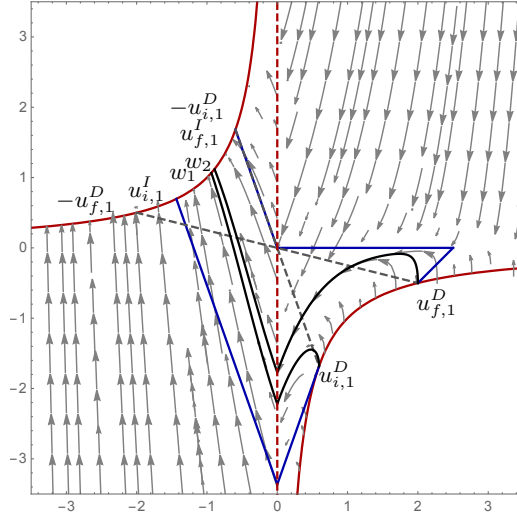

FIGURE 3. Idea of the obtention of the fixed point: $w_1 = P(u_{i,1}^D)$ and $w_2 = P(u_{f,1}^D)$.

2.2. **Proof of Theorem 1 for the smooth case $q = 2$.** In this case, it follows from (6) that

$$\widetilde{X}_2^\pm : \begin{cases} u' = u^3 \pm \sqrt{u^6 + 2uv + 2}, \\ v' = u\left(-uv - 3\left(u^3 \pm \sqrt{u^6 + 2uv + 2}\right)^2\right), \end{cases}$$

for $uv + 1 \geq 0$. See Figure 4 for the phase space of the vector field $\widetilde{X}_2^-$ (in the left) and $\widetilde{X}_2^+$ (in the right).

Consider the arcs Σ_D^2 and Σ_I^2 on the hyperbola $uv + 1 = 0$, given by

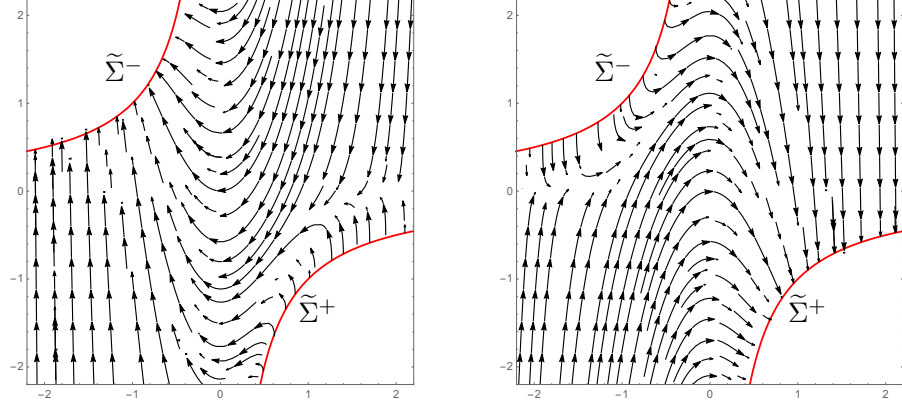$$\Sigma_I^2 = \{(u, v) : v = -1/u : u_{i,2}^I \leq u \leq u_{f,2}^I\},$$

FIGURE 4. Phase space of the vector fields (left) $\widetilde{X}_2^-$ and (right) $\widetilde{X}_2^+$ defined for $uv \geq -1$.

where

$$u_{i,2}^D = 2^{-1/6}3^{-1/3}, \quad u_{f,2}^D = 2, \quad u_{i,2}^I = -2^{5/6}/3^{1/3} \quad \text{and} \quad u_{f,2}^I = -2^{1/6}/3^{1/3}.$$

As for the discontinuous case $q = 1$ we are going to show that in this case the flow of $\widetilde{X}_2^-$ induces a map

$$(8) \qquad\qquad\qquad\qquad P_2 : \Sigma_D^2 \to \Sigma_I^2,$$

by constructing a trapping region K^2 such that $\Sigma_D^2 \cup \Sigma_I^2 \subset \partial K^2$ and $\widetilde{X}_2^-$ will point inwards K^2 everywhere on ∂K^2 except at Σ_I^2, see Figure 5. Since $\widetilde{X}_2^-$ does not have singularities in K^2, from the Poincaré-Bendixson Theorem, we conclude that:

(C2) for each point $p \in \Sigma_D^2$, there is $t(p) > 0$ such that $P_2(p) := \phi_2^-(t(p), p) \in \Sigma_I^2$.

Let K^2 be the compact region delimited by

$$\partial K^2 = \Sigma_D^2 \cup \Sigma_U^2 \cup S_1 \cup \cdots \cup S_5,$$

where

$$S_1 = \{(u,0), 0 < u \leq 9/4\},$$
$$S_2 = \{(u,v) : v = 2u - 9/2 : u_{f,2}^D \leq u \leq 9/4\},$$
$$S_3 = \{(u,v) : v = 3u^2/\sqrt{2} - 3^{4/3}/2^{5/6} : 0 \leq u \leq u_{i,2}^D\},$$
$$S_4 = \{(u,v) : v = 3u^2/\sqrt{2} - 3^{4/3}/2^{5/6} : u_{i,2}^I \leq u < 0\},$$
$$S_5 = \{(u,v) : v = 3u^2/\sqrt{2} : u_{f,2}^I \leq u \leq 0\}.$$

In what follows, we are going to analyze the behavior of the vector field $\widetilde{X}_2^-$ on each component of the boundary of K^2.

Behavior of $\widetilde{X}_2^-$ on Σ_D^2 and Σ_I^2. The hyperbola $g(u,v) = uv + 1 = 0$ satisfies

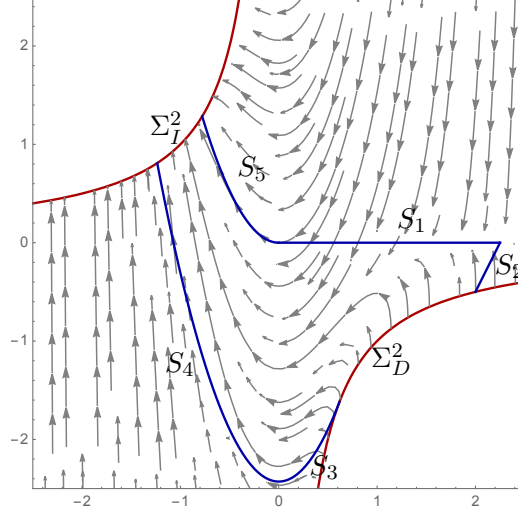$$\langle \nabla g(u,v), \widetilde{X}_2^- \rangle|_{g(u,v)=0} = u^2,$$

FIGURE 5. Phase space of the vector field $\widetilde{X}_2^-$ defined for $uv \geq -1$ and the trapping region K^2.

which does not vanish on $\Sigma_D^2 \cup \Sigma_I^2$. Note that the flow of the vector field $\widetilde{X}_2^-$ points inwards K^2 on $\Sigma_D^2 \cup \Sigma_I^2$, because, for instance, $\widetilde{X}_2^-(1, -1) = (0, 1)$ and $\widetilde{X}_2^-(-1, 1) = (-2, 11)$.

Behavior of $\widetilde{X}_2^-$ on S_1. The curve $g(u, v) = v = 0$ satisfies

$$\langle \nabla g(u, v), \widetilde{X}_2^- \rangle|_{g(u,v)=0} = 6u(1 + u^6 - u^3\sqrt{2 + u^6}).$$

We will show that this derivative does not vanish on $u > 0$. We proceed by contradiction. Assume that $6u(1 + u^6 - u^3\sqrt{2 + u^6}) = 0$. Then, since $u > 0$ we have $1 + u^6 - u^3\sqrt{2 + u^6} = 0$, and so

$$\sqrt{2 + u^6} = \frac{1 + u^6}{u^3} \quad \Leftrightarrow \quad 2 + u^6 = \left(\frac{1 + u^6}{u^3}\right)^2 = \frac{1 + u^{12} + 2u^6}{u^6} \quad \Leftrightarrow \quad -\frac{1}{u^6} = 0,$$

which is not possible. This shows that the flow of the vector field $\widetilde{X}_2^-$ points inwards K^2 on S_1, because for instance $\widetilde{X}_2^-(1, 0) = (-1 - \sqrt{2}, -3(1 + \sqrt{3}))$.

Behavior of $\widetilde{X}_2^-$ on S_2. The curve $g(u, v) = v - 2u + 9/2 = 0$ satisfies

$$\langle \nabla g(u, v), \widetilde{X}_2^- \rangle|_{g(u,v)=0} = -6u + \frac{63}{2}u^2 - 16u^3 - 6u^7 + (2 + 6u^4)\sqrt{u^6 + (u - 2)(4u - 1)}.$$

We will show that this derivative does not vanish on S_2, i.e, for $u_{f,2}^D \leq u \leq 9/4$. We proceed by contradiction. Assume that it is zero at some u. Then, proceeding as in S_1 to get rid of the square root, we obtain

$$\left(6u - \frac{63}{2}u^2 + 16u^3 + 6u^7\right)^2 = (2 + 6u^4)^2(u^6 + (-2 + u)(-1 + 4u)),$$

and so $u^2 p_6(u) = 0$, where

$$p_6(u) := 32 - 144u - 80u^2 + 1512u^3 - 4545u^4 + 3168u^5 - 624u^6 + 216u^9 - 96u^{10}.$$

Using the Sturm procedure we obtain that the polynomial $p_6(u)$ does not vanish on $u_{f,2}^D < u < 9/4$ which is a contradiction (see the appendix). Hence, the flow of the vector field $\widetilde{X}_2^-$ points inwards K^2 on S_2.

Behavior of $\widetilde{X}_2^-$ on S_3 and S_4. Note that on $g(u,v) = v - 3u^2/\sqrt{2} - 3^{4/3}/2^{5/6} = 0$ we have

$$\langle \nabla g(u,v), \widetilde{X}_2^- \rangle|_{g(u,v)=0} = -u\left(6 - \frac{21 \cdot 3^{1/3}u}{2^{5/6}} + \frac{27u^3}{\sqrt{2}} + 6u^6 + (3\sqrt{2} + 6u^3)\right.$$
$$\left. \times \sqrt{2 - 3 \cdot 2^{1/6}3^{1/3}u + 3\sqrt{2}u^3 + u^6}\right).$$

Of course on $u = 0$ the above derivative is zero but since it is a quadratic tangency it is enough to show that this derivative does not vanish neither on $0 < u \leq u_{i,2}^D$, nor on $u_{i,2}^I \leq u < 0$. We proceed by contradiction. Assume that

$$p_7(u) := 6 - \frac{21 \cdot 3^{1/3}u}{2^{5/6}} + \frac{27u^3}{\sqrt{2}} + 6u^6 + (3\sqrt{2} + 6u^3)\sqrt{2 - 3 \cdot 2^{1/6}3^{1/3}u + 3\sqrt{2}u^3 + u^6}$$

has a zero. Then, proceeding as in S_1 to get rid of the square root, we have that $p_7(u) = 0$ implies

$$p_8(u) := 32 \cdot 2^{1/6}3^{1/3} - 49 \cdot 2^{1/3}3^{2/3}u - 16\sqrt{2}u^2 + 78 \cdot 2^{2/3}3^{1/3}u^3 - 58u^5$$
$$+ 8 \cdot 2^{1/6}3^{1/3}u^6 - 8\sqrt{2}u^8 = 0.$$

Using the Sturm procedure we show that the polynomial $p_8(u)$ has a unique simple real zero on $u > 0$, and a unique simple real zero on $u < 0$ (see the appendix). Moreover we have that

$$p_7(0) = 0,\ p_7'(0) = 0,\ p_7''(0) = 12 \cdot 2^{1/6}3^{1/3},\ p_7(u_{i,2}^D) = 0.617715,\ p_7(u_{i,2}^I) = 31.7094.$$

These computations show that if $p_7(u)$ has a zero in $(0, u_{i,2}^D)$, then it has another (or a multiple) zero on that interval. This would produce another zero of $p_8(u)$ in $(0, \infty)$, which is a contradiction. On the other hand, if $p_7(u)$ has a zero in $(u_{i,2}^I, 0)$ then it has another (or a multiple) zero on that interval and this would produce another zero of $p_8(u)$ in $(-\infty, 0)$, which is again a contradiction. In short we have proved that the flow of the vector field $\widetilde{X}_2^-$ points inwards K^2 on S_3 and S_4.

Behavior of $\widetilde{X}_2^-$ on S_5. The curve $g(u,v) = v - 3u^2/\sqrt{2} = 0$ satisfies

$$\langle \nabla g(u,v), \widetilde{X}_2^- \rangle|_{g(u,v)=0} = p_9(u) := -\frac{3}{2}u\left(4 + 9\sqrt{2}u^3 + 4u^6 + (2\sqrt{2} + 4u^3)\sqrt{2 + 3\sqrt{2}u^3 + u^6}\right).$$

Of course on $u = 0$ the above derivative is zero but since it is a quadratic tangency it is sufficient to show that this derivative does not vanish on $u_{f,2}^I < u < 0$. Proceeding as in S_1 to get rid of the square root, we have that $p_9 = 0$ implies

$$\sqrt{2 + 3\sqrt{2}u^3 + u^6} = \frac{4 + 9\sqrt{2}u^3 + 4u^6}{2\sqrt{2} + 4u^3} \quad \Rightarrow \quad 2 + 3\sqrt{2}u^3 + u^6 = \left(\frac{4 + 9\sqrt{2}u^3 + 4u^6}{2\sqrt{2} + 4u^3} \right)^2,$$

and so

$$-\frac{u^3(8\sqrt{2} + 29u^3 + 4\sqrt{2}u^6)}{2(\sqrt{2} + 2u^3)^2} = 0.$$

Note that the denominator does not vanish on $u_{f,2}^I < u < 0$, because the unique negative real solution of $\sqrt{2} + 2u^3 = 0$, which is $-1/2^{1/6}$ is outside the mentioned interval. On the other hand the unique two real solutions of the numerator are

$$u_1 = -\frac{1}{2} \left(\frac{29\sqrt{2} - 3\sqrt{130}}{2} \right)^{1/3} \quad \text{and} \quad u_2 = -\frac{1}{2} \left(\frac{29\sqrt{2} + 3\sqrt{130}}{2} \right)^{1/3}.$$

We observe that u_1 belongs to the interval $u_{f,2}^I < u < 0$ while u_2 does not. However, u_1 is not a solution of $p_9(u) = 0$ because

$$p_9(u_1) = -\frac{1}{2} \left(\frac{29\sqrt{2} - 3\sqrt{130}}{2} \right)^{1/3} \neq 0.$$

This contradiction shows that the flow $\widetilde{X}_2^-$ point inwards K^2 on S_5.

Existence of p_2^*. Consider the map $h_2 \colon \Sigma_D^2 \to \mathbb{R}$ defined by $h_2(p) = \pi(P_2(p) + p)$, where $\pi \colon \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate (see (C2)). Note that by the continuous dependence of the flow of $\widetilde{X}_2^-$ with respect to the initial conditions the map h_2 is continuous. Moreover, the image of the point $(u_{f,2}^D, -1/u_{f,2}^D)$ by P_2 (which is the point w_4 in Figure 6) is inside Σ_I^2, and its symmetric is below its image because $u_{f,2}^D + u_{i,2}^I > 0$. Hence $h_2(u_{f,2}^D, -1/u_{f,2}^D) > 0$. On the other hand, the image of the point $(u_{i,2}^D, -1/u_{i,2}^D)$ by P_2 (which is the point w_3 in Figure 6) is inside Σ_I^2, and its symmetric is above its image because $u_{i,2}^D + u_{f,2}^I < 0$. Therefore $h_2(u_{i,2}^D, -1/u_{i,2}^D) < 0$. Thus, by continuity, there exists $p_2^* \in \Sigma_D^2$ such that $h_2(p_2^*) = 0$, i.e, $P_2(p_2^*) = -p_2^*$ and so $\phi_2^-(t_2^*, p_2^*) = -p_2^*$ as we wanted to prove.

2.3. **Proof of Theorem 1 for the continuous case $q = 3$.** In this case, it follows from (6) that

$$\widetilde{X}_2^\pm : \begin{cases} u' = u|u|^3 \pm \sqrt{u^8 + 2uv + 2}, \\ v' = \dfrac{|u|^3}{u} \left(-uv - 4 \left(u|u|^3 \pm \sqrt{u^8 + 2uv + 2} \right)^2 \right). \end{cases}$$
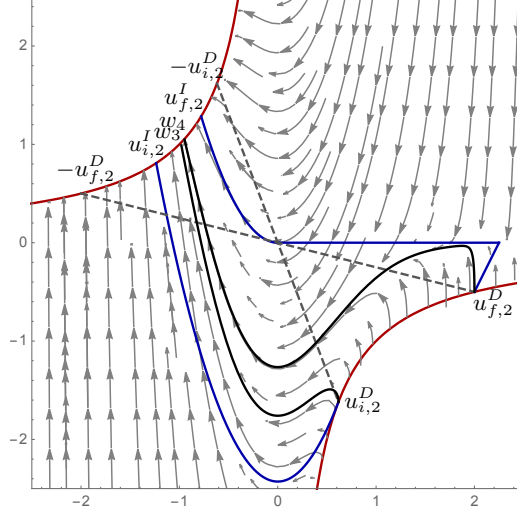
FIGURE 6. Idea of the obtention of the fixed point: $w_3 = P_2(u_{i,2}^D)$ and $w_4 = P_2(u_{f,2}^D)$.

for $uv + 1 \geq 0$. See Figure 7 for the phase spaces of the vector field $\widetilde{X}_3^-$ (in the left) and $\widetilde{X}_3^+$ (in the right), defined for $uv \leq -1$.
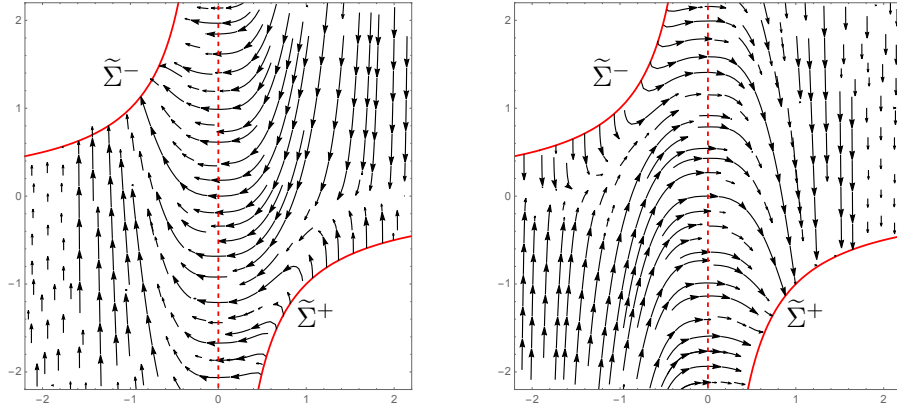

FIGURE 7. Phase space of the vector fields (left) $\widetilde{X}_3^-$ and (right) $\widetilde{X}_3^+$ defined for $uv \geq -1$. On the dashed line both vector fields (left) $\widetilde{X}_3^-$ and (right) $\widetilde{X}_3^+$ are discontinuous.

Consider the sections Σ_D^3 and Σ_I^3 on the hyperbola $uv + 1 = 0$, given by

$$\Sigma_D^3 = \{(u,v) : v = -1/u : u_{i,3}^D \leq u \leq u_{f,3}^D\} \quad \text{and}$$
$$\Sigma_I^3 = \{(u,v) : v = -1/u : u_{i,3}^I \leq u \leq u_{f,3}^I\},$$

where

$$u_{i,3}^D = \frac{1}{2^{5/8}}, \quad u_{f,3}^D = 2, \quad u_{i,3}^I = \frac{N_{i,3}^I}{D_{i,3}^I} \quad \text{and} \quad u_{f,3}^I = -\frac{3^{1/4}}{2^{5/8}}$$

being

$$D_{i,3}^I = 2^{13/8}(1+\sqrt{2})^{1/3}\sqrt{1+\sqrt{2}-(1+\sqrt{2})^{1/3}},$$
$$N_{i,3}^I = -2-\sqrt{2}+\sqrt{2}(1+\sqrt{2})^{1/3}-\Big(-6-4\sqrt{2}-2(1+\sqrt{2})^{2/3}+4(1+\sqrt{2})^{4/3}$$
$$+4(2+\sqrt{2})\sqrt{1+\sqrt{2}-(1+\sqrt{2})^{1/3}}\Big)^{1/2}.$$

As for the discontinuous case $q = 1$ we are going to show that in this case the flow of $\widetilde{X}_3^-$ induces a map

$$(9) \qquad\qquad\qquad P_3 : \Sigma_D^2 \to \Sigma_I^3,$$

by constructing a trapping region K^3 such that $\Sigma_D^3 \cup \Sigma_I^3 \subset \partial K^3$ and $\widetilde{X}_3^-$ will point inwards K^3 everywhere on ∂K^3 except at Σ_I^3, see Figure 8. Since $\widetilde{X}_3^-$ does not have singularities in K^3, from the Poincaré-Bendixson Theorem,

(C3) for each point $p \in \Sigma_D^3$, there is $t(p) > 0$ such that $P_3(p) := \phi_3^-(t(p), p) \in \Sigma_I^3$.

Let K^3 be the compact region delimited by

$$\partial K^3 = \Sigma_D^3 \cup \Sigma_U^3 \cup T_1 \cup \cdots \cup T_5,$$

where

$$T_1 = \{(u,0), 0 < u \le 13/6\},$$
$$T_2 = \{(u,v) : v = 3u - 13/2 : u_{f,3}^D \le u \le 13/6\},$$
$$T_3 = \{(u,v) : v = 4\sqrt{2}|u|^3/3 - 2^{21/8}/3 : 0 \le u \le u_{i,3}^D\},$$
$$T_4 = \{(u,v) : v = 4\sqrt{2}|u|^3/3 - 2^{21/8}/3 : u_{i,3}^I \le u \le 0\},$$
$$T_5 = \{(u,v) : v = 4\sqrt{2}|u|^3/3 : u_{f,3}^I \le u \le 0\}.$$

In what follows, we are going to analyze the behavior of the vector field $\widetilde{X}_3^-$ on each component of the boundary of K^3.

Behavior of $\widetilde{X}_3^-$ on Σ_D^3 and Σ_I^3. The hyperbola $g(u,v) = uv + 1 = 0$ satisfies

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{(u,v)\in\Sigma_D^3} = u^3,$$

which does not vanish on Σ_D^3. Note that the flow of the vector field $\widetilde{X}_3^-$ points inwards K^3 on Σ_D^3, because for instance $\widetilde{X}_1^-(1,-1) = (0,1)$.

The curve $g(u,v) = uv + 1 = 0$ on Σ_I^3 satisfies

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{(u,v)\in\Sigma_I^3} = u^3(1+16u^8),$$

which does not vanish on Σ_I^3. Note that the flow of the vector field $\widetilde{X}_3^-$ points outwards K^3 on Σ_I^3, because for instance $\widetilde{X}_3^-(-1,1) = (-2,15)$.
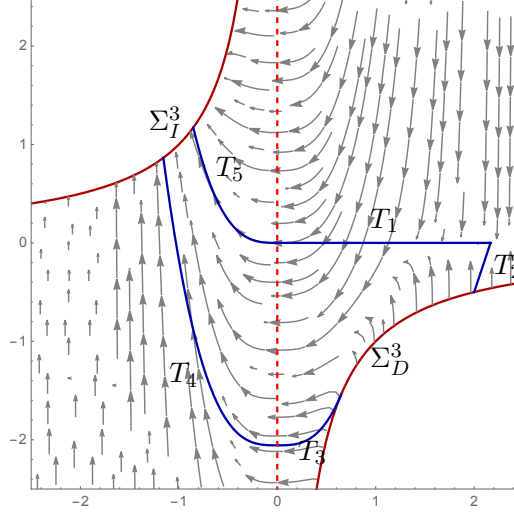
FIGURE 8. Phase space of the vector field $\widetilde{X}_3^-$ defined for $uv \geq -1$ and the trapping region K^3.

Behavior of $\widetilde{X}_3^-$ on T_1. The curve $g(u,v) = v = 0$ on $u > 0$ satisfies

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{g(u,v)=0} = -8u^2\big((1+u^8) - u^4\sqrt{2+u^8}\big).$$

We will show that this derivative does not vanish on $u > 0$. We proceed by contradiction. Assume that $6u(1 + u^6 - u^3\sqrt{2+u^6}) = 0$. Then, since $u > 0$ we have $(1+u^8) - u^4\sqrt{2+u^8} = 0$, and so

$$\sqrt{2+u^8} = \frac{1+u^8}{u^4} \;\Leftrightarrow\; 2+u^8 = \left(\frac{1+u^8}{u^4}\right)^2 = \frac{1+u^{16}+2u^8}{u^8} \;\Leftrightarrow\; \frac{1+4u^8+2u^{16}}{u^8} = 0,$$

which is not possible. This shows that the flow of the vector field $\widetilde{X}_3^-$ points inwards K^3 on T_1, because for instance $\widetilde{X}_2^-(1,0) = (1 - \sqrt{3}, -16 + 8\sqrt{3})$.

Behavior of $\widetilde{X}_3^-$ on T_2. The curve $g(u,v) = v - 3u + 13/6 = 0$ satisfies

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{g(u,v)=0} = -8u^2 + \frac{117}{2}u^3 - 30u^4 - 8u^{10} + (8u^6+3)\sqrt{2 + u(-13 + 6u + u^7)}.$$

We will show that this derivative does not vanish on T_2, i.e. for $u_{f,3}^D \leq u \leq 13/6$. We proceed by contradiction. Assume that it is zero at some u. Then, proceeding as in T_1 to get rid of the square root, we obtain

$$p_{10}(u) := 72 - 468u + 216u^2 - 256u^4 + 3744u^5 - 15225u^6 + 11544u^7 - 2412u^8 + 416u^{13} - 192u^{14}.$$

Using the Sturm procedure we prove that the polynomial $p_{10}(u)$ does not vanish on $u_{f,3}^D < u < 13/6$ which is a contradiction (see the appendix). Hence, the flow of the vector field $\widetilde{X}_3^-$ points inwards K^3 on T_2.

Behavior of $\widetilde{X}_3^-$ on T_3 and T_4. On the curve $g(u,v) = v - 4\sqrt{2}|u|^3/3 - 2^{21/8}/3 = 0$ for $u > 0$ we have

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{g(u,v)=0} = p_{11}(u) := -u^2 \Big(8 - 3 \cdot 2^{21/8} u + 16\sqrt{2}u^4 + 8u^8$$

$$- 4(\sqrt{2} + 2u^4)\sqrt{2 - \frac{2^{29/8}}{3}u + \frac{8\sqrt{2}}{3}u^4 + u^8} \Big),$$

and for $u < 0$,

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{g(u,v)=0} = p_{12}(u) := u^2 \Big(8 - 3 \cdot 2^{21/8} u - 16\sqrt{2}u^4 + 8u^8$$

$$- 4(\sqrt{2} - 2u^4)\sqrt{2 - \frac{2^{29/8}}{3}u - \frac{8\sqrt{2}}{3}u^4 + u^8} \Big).$$

Of course on $u = 0$ the above derivative is zero but since it is a quadratic tangency it is enough to show that $p_{11}(u)$ does not vanish on $0 < u \leq u_{i,3}^D$, and that the $p_{12}(u)$ does not vanish on $u_{i,3}^I \leq u < 0$.

We proceed by contradiction. Assume first that $p_{11}(u)$ has a zero. Then, proceeding as in R_1 to get rid of the square root, we have that $p_{11}(u) = 0$ implies

$$p_{13}(u) = 20 \cdot 2^{5/8} - 54 \cdot 2^{1/4}u - 8\sqrt{2}u^3 + 80 \cdot 2^{1/8}u^4 - 26u^7 + 4 \cdot 2^{5/8}u^8 - 4\sqrt{2}u^{11} = 0.$$

Using the Sturm procedure we obtain that the polynomial $p_{13}(u)$ has a unique simple real zero on $u > 0$ (see the appendix). Moreover, we have that

$$p_{11}(0) = 0, \quad p_{11}'(0) = 0, \quad p_{11}''(0) = 0, \quad p_{11}'''(0) = 40 \cdot 2^{5/8}, \quad p_1(u_{i,3}^D) = 0.420448...$$

These computations show that if $p_{11}(u)$ has a zero in $(0, u_{i,3}^D)$, then it has another (or a multiple) zero on that interval. This would produce another zero of $p_{13}(u)$ in $(0, \infty)$, which is a contradiction. Hence the flow of the vector field $\widetilde{X}_3^-$ points inwards K^3 on T_3.

On the other hand assume that $p_{12}(u)$ has a zero. Then, proceeding as in T_1 to get rid of the square root, we have that $p_{12}(u) = 0$ implies

$$p_{14}(u) = 20 \cdot 2^{5/8} - 54 \cdot 2^{1/4}u + 8\sqrt{2}u^3 - 80 \cdot 2^{1/8}u^4 - 26u^7 + 4 \cdot 2^{5/8}u^8 + 4\sqrt{2}u^{11} = 0.$$

Using the Sturm procedure we have that the polynomial $p_{14}(u)$ has a unique simple real zero on $u < 0$ (see the appendix). Moreover, we have that

$$p_{12}(0) = 0, \quad p_{12}'(0) = 0, \quad p_{12}''(0) = 0, \quad p_{12}'''(0) = -40 \cdot 2^{5/8}, \quad p_2(u_{i,3}^I) = 40.3062...$$

These computations show that if $p_{12}(u)$ has a zero in $(u_{i,3}^I, 0)$, then either it is multiple or $p_{12}(u)$ has at least two negative zeros in contradiction with uniqueness of negative zeros provided by Sturm procedure. So the flow of the vector field $\widetilde{X}_3^-$ points inwards K^3 on T_4.

Behavior of $\widetilde{X}_3^-$ on T_5. On the curve $g(u,v) = v - 4\sqrt{2}|u|^3/3 = 0$ we have

$$\langle \nabla g(u,v), \widetilde{X}_3^- \rangle|_{g(u,v)=0} = 4u^2 \left(2\left(u^8 - 2\sqrt{2}u^4 + 1\right) + \left(2u^4 - \sqrt{2}\right)\sqrt{u^8 - \frac{8\sqrt{2}u^4}{3} + 2} \right).$$

Of course on $u = 0$ the above derivative is zero but since it is a quadratic tangency it is to show that this derivative does not vanish on $u_{f,3}^I < u < 0$. Proceeding as in T_1 to get rid of the square root, we have that $p_{15}(u) := 6 - 12\sqrt{2}u^4 + 6u^8(\sqrt{6} - 2\sqrt{3}u^4)\sqrt{6 - 8\sqrt{2}u^4 + 3u^8} = 0$ implies

$$\frac{2u^4(4\sqrt{2} - 13u^4 + 2\sqrt{2}u^8)}{3(\sqrt{2} - 2u^4)^2} = 0.$$

Note that the denominator does not vanish on $u_{f,3}^I < u < 0$, because the unique negative real solution of $\sqrt{2} - 2u^4 = 0$, which is $-1/2^{1/8}$ is outside the mentioned interval. On the other hand the unique two real solutions of the numerator are

$$u_1 = -\frac{1}{2^{3/4}}\left(13\sqrt{2} - \sqrt{210}\right)^{1/4} \quad \text{and} \quad u_2 = -\frac{1}{2^{3/4}}\left(13\sqrt{2} + \sqrt{210}\right)^{1/4}.$$

We observe that u_1 belongs to the interval $u_{f,3}^I < u < 0$ while u_2 does not. However, u_1 is not a solution of $p_{15}(u) = 0$ because

$$p_{15}(u_1) = -\frac{1}{2^{3/4}}\left(77\sqrt{5} - 37\sqrt{21}\right) \neq 0.$$

This contradiction shows that the flow $\widetilde{X}_3^-$ point inwards K^3 on T_5.

Existence of p_3^*. Consider the map $h_3 \colon \Sigma_D^3 \to \mathbb{R}$ defined by $h_3(p) = \pi(P_3(p) + p)$, where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate (see (C3)). Note that by the continuous dependence of the flow of $\widetilde{X}_3^-$ with respect to the initial conditions the map h_3 is continuous. Moreover, the image of the point $(u_{f,3}^D, -1/u_{f,3}^D)$ by P_3 (which is the point w_6 in Figure 9) is inside Σ_I^3, and its symmetric is below its image because $u_{f,3}^D + u_{i,3}^I > 0$. Hence $h_3(u_{f,3}^D, -1/u_{f,3}^D) > 0$. On the other hand, the image of the point $(u_{i,3}^D, -1/u_{i,3}^D)$ by P_3 (which is the point w_5 in Figure 9) is inside Σ_I^3, and its symmetric is above its image because $u_{i,3}^D + u_{f,3}^I < 0$. Therefore $h_2(u_{i,3}^D, -1/u_{i,3}^D) < 0$. Thus, by continuity, there exists $p_3^* \in \Sigma_D^2$ such that $h_3(p_3^*) = 0$, i.e, $P_3(p_3^*) = -p_3^*$ and so $\phi_3^-(t_3^*, p_3^*) = -p_3^*$ as we wanted to prove.

<center>APPENDIX: STURM PROCEDURE</center>

Let $p(u)$ be a square-free polynomial of degree d, and consider the so-called *Sturm sequence* $q_i(u)$, for $i = 0, \ldots, \ell$, given by: $q_0(u) = p_0(u)$, $q_1(u) = p_0'(u)$, and for $i = 2, \ldots, \ell$, $-q_i(u)$ is the polynomial reminder of the division of q_{i-2} by q_{i-1}, where ℓ is the first index for which q_ℓ is constant. Sturm Theorem [5] provides that the number of real roots of $p(x)$ in the half-open interval $(a, b]$ is $V(a) - V(b)$, where $V(x)$ is the number of sign variation of the sequence $q_0(x), q_1(x) \ldots, q_\ell(x)$. Notice that when computing
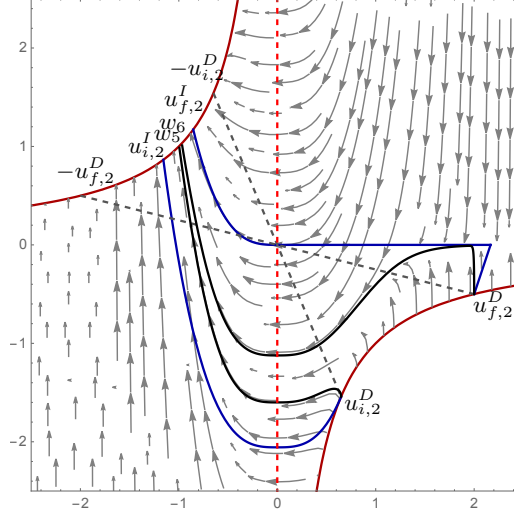
FIGURE 9. Idea of the obtention of the fixed point: $w_5 = P_3(u_{i,3}^D)$ and $w_6 = P_3(u_{f,3}^D)$.

$V(x)$, x can take the value ∞ (resp. $-\infty$), in these cases $V(\infty)$ (resp. $V(-\infty)$) denotes the number of sign variation of the leading terms of the sequence $q_0(x), q_1(x) \ldots, q_\ell(x)$ (resp. $q_0(-x), q_1(-x) \ldots, q_\ell(-x)$).

2.4. **Sturm Procedure for $q = 1$.** First, note that the polynomial

$$p_0(u) = -14 + 95u - \frac{745}{4}u^2 + 110u^3 - 19u^4 + 20u^5 - 8u^6$$

and its derivative

$$p_0'(u) = -95 - \frac{745}{2}u + 330u^2 - 76u^3 + 100u^4 - 48u^5$$

have no common zeroes because its resultant with respect to u is $-1.52127.. \cdot 10^{18}$. In particular, $p(x)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 6$, we see $V(2) = V(5/2) = 2$, which implies that $p_0(u)$ has no real zero in the interval $(2, 5/2)$.

Now, notice that the polynomial

$$p_3(u) = 12 \cdot 2^{3/4} - 58\sqrt{2}u + 88 \cdot 2^{1/4}u^2 - 38u^3 + 4 \cdot 2^{3/4}u^4 - 4\sqrt{2}u^5$$

has no common zeroes with its derivative because the resultant between $p_3(u)$ and $p_3'(u)$ with respect to the variable u is $-5637568724992\sqrt{2}$ which is not zero. In particular, $p_3(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 5$, we see that $V(0) = 3$ and $V(\infty) = 2$. Therefore it follows from the Sturm process that $p_3(u)$ has a unique positive real zero.

Finally, the polynomial

$$p_4(u) = 12 \cdot 2^{3/4} - 42\sqrt{2}u - 88 \cdot 2^{1/4}u^2 - 38u^3 + 4 \cdot 2^{3/4}u^4 + 4\sqrt{2}u^5$$

has no common zeroes with its derivative because the resultant between $p_4(u)$ and $p'_4(u)$ with respect to the variable u is $-88414837800960\sqrt{2}$ which is not zero. In particular, $p_4(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 5$, we see that $V(-\infty) = 4$ and $V(0) = 3$. Therefore, it follows from the Sturm process that $p_4(u)$ has a unique negative real zero.

2.5. Sturm Procedure for $q = 2$.

First, note that the polynomial

$$p_6(u) = 32 - 144u - 80u^2 + 1512u^3 - 4545u^4 + 3168u^5 - 624u^6 + 216u^9 - 96u^{10}$$

has no common zeroes with its derivative because the resultant between $p_6(u)$ and $p'_6(u)$ with respect to u is $-4.41356.. \cdot 10^{57}$. In particular, $p_6(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 10$, we see $V(2) = V(9/4) = 3$, which implies that $p_6(u)$ has no real zero in the interval $(2, 9/4)$.

Now, notice that the polynomial

$$p_8(u) = 32 \cdot 2^{1/6}3^{1/3} - 49 \cdot 2^{1/3}3^{2/3}u - 16\sqrt{2}u^2 + 78 \cdot 2^{2/3}3^{1/3}u^3 - 58u^5 + 8 \cdot 2^{1/6}3^{1/3}u^6 - 8\sqrt{2}u^8$$

has no common zeroes with its derivative because the resultant between $p_8(u)$ and $p'_8(u)$ with respect to the variable u is

$$966930076692226595132899328 00 \cdot 2^{1/6}3^{1/3}$$

which is not zero. In particular, $p_8(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 10$, we see that $V(-\infty) = 5$, $V(0) = 4$, and $V(\infty) = 3$. Therefore, it follows from the Sturm process that $p_8(u)$ has a unique positive real zero and a unique negative real zero.

2.6. Sturm Procedure for $q = 3$.

First, note that the polynomial

$$p_{10}(u) := 72 - 468u + 216u^2 - 256u^4 + 3744u^5 - 15225u^6 + 11544u^7 - 2412u^8 + 416u^{13} - 192u^{14}.$$

has no common zeroes with its derivative because the resultant between $p_{10}(u)$ and $p'_{10}(u)$ with respect to u is $-2.74875.. \cdot 10^{97}$. In particular, $p_{10}(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 10$, we see $V(2) = V(13/6) = 3$, which implies that $p_{10}(u)$ has no real zero in the interval $(2, 13/6)$.

Now, notice that the polynomial

$$p_{13}(u) = 20 \cdot 2^{5/8} - 54 \cdot 2^{1/4}u - 8\sqrt{2}u^3 + 80 \cdot 2^{1/8}u^4 - 26u^7 + 4 \cdot 2^{5/8}u^8 - 4\sqrt{2}u^{11}$$

has no common zeroes with its derivative because the resultant between $p_{13}(u)$ and $p'_{13}(u)$ with respect to the variable u is $-5.12026.. \cdot 10^{35}$ which is not zero. In particular,

$p_{13}(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 11$, we see that $V(0) = 5$ and $V(\infty) = 4$. Therefore it follows from the Sturm process that $p_{13}(u)$ has a unique positive real zero.

Finally, the polynomial

$$p_{14}(u) = 20 \cdot 2^{5/8} - 54 \cdot 2^{1/4}u + 8\sqrt{2}u^3 - 80 \cdot 2^{1/8}u^4 - 26u^7 + 4 \cdot 2^{5/8}u^8 + 4\sqrt{2}u^{11}$$

has no common zeroes with its derivative because the resultant between $p_{14}(u)$ and $p'_{14}(u)$ with respect to the variable u is $-2.29037.. \cdot 10^{37}$ which is not zero. In particular, $p_{14}(u)$ is a square-free polynomial and the Sturm procedure can be applied directly. By computing the Sturm sequence $q_i(u)$ for $i = 0, \ldots, 11$, we see that $V(-\infty) = 7$ and $V(0) = 6$. Therefore, it follows from the Sturm process that $p_{14}(u)$ has a unique negative real zero.

Acknowledgments

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