ON FAMILIES OF BOWEN-SERIES-LIKE MAPS FOR SURFACE GROUPS

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ABSTRACT. We review some recent results on a class of maps, called Bowen-Series-like, obtained from a class of group presentations for surface groups. These maps are piecewise homeomorphisms of the circle with finitely many discontinuities. The topological entropy of each map in the class and its relationship with the growth function of the group presentation is discussed, as well as the computation of these invariants.

This paper is dedicated to Alain Chenciner for his 80th birthday.

1. INTRODUCTION

This paper is a review of recent results [2] on a particular relationship between two classical theories: geometric groups and dynamical systems. The relationship we consider has a clear origin in two papers published in 1979 by Bowen [8] and Bowen-Series [9]. On one side, the groups are the well known Fuchsian groups given by some specific action A on the hyperbolic plane $\mathbb{H}^2$. The idea is to associate, to such an action, a dynamical system given as a "piecewise Möbius" map $\Phi_A\colon S^1 \longrightarrow S^1$, where $S^1 = \partial\mathbb{H}^2$. The Bowen-Series idea has been revisited in [17] for surface groups $G = \pi_1(\Sigma)$, where Σ is a closed compact surface of negative Euler characteristic. In this approach, G is given by a finite presentation $P = \langle X|R \rangle$, where X is a symmetric generating set and R is a set of relations. We will consider *geometric* presentations, meaning that the Cayley two complex $\mathrm{Cay}^2(G, P)$ is planar. This implies, in particular, that the Cayley graph $\mathrm{Cay}^1(G, P)$ is a planar graph. Observe that the classical presentations of surface groups are geometric in this sense [21]. The basic idea is to associate a dynamical system to such a group presentation. In this case the dynamics is given by a piecewise homeomorphism of the circle $\Phi_P\colon S^1 \longrightarrow S^1$. In this approach, G is a Gromov hyperbolic group [16] and its (Gromov) boundary is $\partial G = S^1$.

The maps Φ_A and Φ_P constructed respectively in [8, 9] and [17] are particular realizations of this idea and are different on several aspects (when they can be compared, i.e. for the classical presentations). The Bowen-Series maps Φ_A are "piecewise Möbius", so

in particular they are piecewise analytic, while the maps Φ_P are only piecewise homeomorphisms. The maps are also different from a dynamical point of view, since they are not conjugate or even semi-conjugate. But they share several properties:

(a) The maps are Markov and expanding.
(b) Each map and the group G are orbit equivalent. For this statement the group G is viewed as acting on its Gromov boundary $\partial G = S^1$ by homeomorphisms and is considered as a discrete subgroup of $\mathrm{Homeo}(S^1)$.
(c) For the particular map Φ_P defined in [17], the topological entropy $h_{top}(\Phi_P)$ of the map and the volume entropy $h_{vol}(G, P)$ of the group presentation are equal.

The orbit equivalence is a classical property for two dynamical systems acting on the same space. What was new in [9] was to compare orbits of different actions on the same space, one for the group and one for the map. The orbit equivalence means that if two points belong to the same orbit under the group action then they belong to the same orbit under the map and conversely. In our context this property is satisfied in the strong sense, namely up to finitely many orbits.

Property (c) is a comparison between two kinds of exponential growth rates. The *topological entropy* is a classical conjugacy invariant of a dynamical system [1, 7]. It has been introduced for continuous maps of compact metric spaces, and extended to piecewise monotone and discontinuous maps [20]. The topological entropy of a map f will be denoted by $h_{top}(f)$.

The *volume entropy* of a group presentation, denoted $h_{vol}(G, P)$, is an invariant of the presentation (and not of the group) and is defined as

$$(1) \qquad h_{\mathrm{vol}}(G, P) = \lim_{m \to \infty} \frac{1}{m} \log(\sigma_m)$$

where σ_m is the number of vertices in the Cayley graph $\mathrm{Cay}^1(G, P)$ at distance m from the identity element [12, 13, 15].

Computing the topological entropy for general maps on a metric space is difficult. The same is true in group theory: the computation of the volume entropy from a presentation is rarely possible. In dynamics, being *Markov* is a strong property for a map. In principle, this property allows the computation of the topological entropy when there is an explicit way to obtain a Markov partition. For group presentations the same difficulty exists. Knowing that the group is Gromov hyperbolic is a strong property. Cannon introduced the notion of *cone types* in [11]. This notion has been extended to all hyperbolic groups [14] and, in principle, it allows a computation of the growth function σ_m. In practice, finding the cone types from a presentation is not feasible in general.

In this paper we review some recent results obtained in [2] about a wide class of piecewise homeomorphisms of the circle, called the "Bowen-Series-like family", for any geometric presentation P of a surface group. We define a family of maps $\Phi_\Theta \colon S^1 \longrightarrow S^1$, where Θ is a parameter in a cube of dimension $2N$ that is the cardinality of the generating set X of the presentation P. The map Φ_P defined in [17] is a particular member of this family. The parameter Θ belongs to a product $J_1 \times \ldots \times J_{2N}$ of $2N$ intervals of S^1, each interval is defined from the geometry of the Cayley graph. For each parameter Θ, the circle admits a partition into $2N$ intervals $S^1 = \cup_{j=1}^{2N} I_j$ and Φ_Θ restricted to each I_j is a homeomorphism onto its image [17, 18].

The maps in the family Φ_Θ are dynamically different (in particular they are not pairwise conjugate neither semi-conjugate). The map can be Markov or non Markov depending on the parameter. Among the many questions about the family Φ_Θ, the following two are central:

- How the topological entropy varies with the parameter Θ?
- How the orbit equivalence property varies with the parameter Θ?

The first question is answered completely by the main theorem obtained in [2], which is described in Section 3. The result is that the topological entropy is the same for all parameters and is explicitly computable using the Milnor-Thurston kneading theory [19].

For the second question, some works in progress indicate that, for all parameters, the group G and each map Φ_Θ are orbit equivalent. The orbit equivalence result can only be stated at this moment for a subclass of presentations and an open subset of parameters. This partial answer is possible by combining several results from [2] and the main Theorem in [18]. Indeed, we obtained that one of the main assumptions in [18], called the *eventual coincidence* condition (EC), is satisfied for an open set of parameters in the family Φ_Θ. This property says that, for each discontinuity point θ_j, the left and right orbits coincide after some iterate k_j. In addition, for $i < k_j$, the i-th iterates from the left and the right are defined by the combinatorics of the presentation. The class of maps satisfying a condition (EC) is interesting by itself. The family Φ_Θ has a rich dynamics that is not yet studied, for instance for parameters not in the open set mentioned above.

2. Definition of the Bowen-Series-like family

We start this section by gathering some properties of geometric presentations for hyperbolic surface groups (c.f. [17]). Then we give the precise definition of the Bowen-Series-like family Φ_Θ.

Recall that a presentation $P = \langle X|R \rangle$ of a surface group is called *geometric* if the Cayley 2-complex $\mathrm{Cay}^2(G, P)$ is planar. The next result (Lemma 2.1 of [17]) states some elementary consequences of this planarity property.

Lemma 2.1. *Let G be a co-compact hyperbolic surface group and let $P = \langle X|R \rangle = \langle g_1^{\pm 1}, \ldots, g_N^{\pm 1}|R_1, \ldots, R_k \rangle$ be a geometric presentation of G. The following conditions hold.*

(a) *The set $\{g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_N^{\pm 1}\}$ admits a cyclic ordering that is preserved by the group action.*

(b) *There exists a planar fundamental domain Δ_P.*

(c) *Each generator appears exactly twice (with $+$ or $-$ exponent) in the set R of relations.*

Let us fix some notations. The elements of the generating set $X := \{g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_N^{\pm 1}\}$ are denoted as $x_1, x_2, \ldots, x_{2N}$ in such a way that $x_{i\pm 1}$ are the elements adjacent to x_i with respect to the cyclic ordering from Lemma 2.1(a) (the indices are defined modulo $2N$). We also adopt the convention that x_i is on the *left* of x_{i+1} (see Example 2.2 below). This convention defines a clockwise orientation of the plane $\mathrm{Cay}^2(G, P)$.
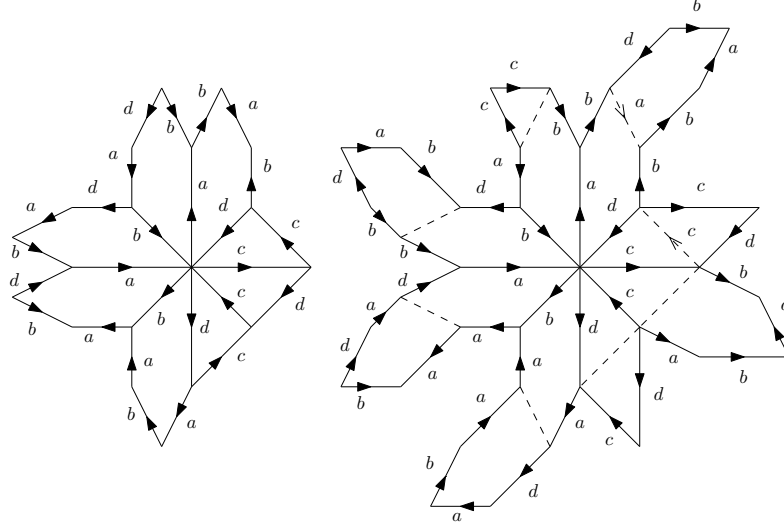
FIGURE 1. Left: the cells adjacent to the base vertex Id and the cyclic ordering associated to the presentation P of Example 2.2. Right: attaching cells to reach the closest vertices with two minimal geodesic representations.

Example 2.2. Take the presentation $P = \langle X \mid R \rangle = \langle a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1} \mid abab^{-1}d, c^2d \rangle$. We fix the identity element Id of the group as the base vertex of the Cayley 2-complex $\mathrm{Cay}^2(G, P)$. If this 2-complex is planar, then the boundary of each cell adjacent to Id is a closed path from Id to itself that, read in the clockwise direction, is (a cyclic shift of) a relation in $R \cup R^{-1}$. The presentation P is geometric if and only if it is possible to set up $2N = 8$ cells adjacent to Id in such a way that the 8 edges departing from Id are in bijection with the generating set $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$. For this presentation this is possible: see Figure 1 (left). The cyclic ordering stated in Proposition 2.1(a) is then $(x_1, x_2, \dots, x_8) = (a, d^{-1}, c, c^{-1}, d, b, a^{-1}, b^{-1})$ in the clockwise direction. The Cayley 2-complex $\mathrm{Cay}^2(G, P)$ is a tessellation of the plane obtained by replicating *ad infinitum* the cells drawn in Figure 1 (left). For instance, in Figure 1 (right) we have represented the Cayley graph up to the shortest elements of the group having an ambiguity in their representation as *minimal* words in the alphabet X. Indeed, $abd^{-1}b$ and $d^{-1}b^2a$ are two equivalent minimal words that represent the same vertex (group element) in the Cayley graph.

Such pairs of geodesic segments connecting two vertices are called *bigons*. The set of bigons $\{\gamma, \gamma'\}$ starting at the identity such that the words γ and γ' start respectively by two generators x and y is denoted by $B(x, y)$. One result in [17] states that $B(x, y) \neq \emptyset$ if and only if the two generators x, y are adjacent with respect to the cyclic ordering given by Lemma 2.1(a). In addition, if x, y are adjacent then there exists a unique bigon $\beta(x, y) \in B(x, y)$ of minimal length. For instance, in Example 2.2 we obtain that $\beta(a, d^{-1}) = \{abd^{-1}b, d^{-1}b^2a\}$.

A *geodesic ray* is a copy of $\mathbb{R}^+$ in the Cayley graph $\mathrm{Cay}^1(G, P)$ such that each subsegment joining two vertices is geodesic. A point $z \in \partial G$ (see [16]) is the limit of possibly

many geodesic rays starting at the identity, all remaining at a uniform bounded distance from each other. Let $\zeta \in \partial G$ be a point on the boundary and let $[\zeta]$ be a particular geodesic ray expression of ζ starting at the identity, i.e. an infinite word in the alphabet X. We denote by $[\zeta]_m$ the initial word of length m of the infinite word $[\zeta]$. The *cylinder sets* of length 1 are defined as

$$\mathscr{C}_x = \{\zeta \in \partial G : \text{ there exists } [\zeta] \text{ with } [\zeta]_1 = x\}.$$

The cylinder sets of length 1 have been characterized in terms of the geometry of the Cayley graph in [6].

The above notion of cylinder set can be naturally extended to all lengths. In symbolic dynamics a cylinder set, say of one letter, is the set of infinite words starting with that letter. Here the infinite word is replaced by the notion of geodesic ray, and the set of infinite words is replaced by the boundary of the hyperbolic space. The main difference, if the space is not a tree, is that many geodesic rays might define the same point on the boundary. Therefore, cylinder sets intersect in general. This notion of cylinder sets for hyperbolic spaces is sometimes called "shadow" in the geometry literature [10].

The following result is a direct consequence of Section 3 of [17].

Proposition 2.3. *Let $P = \langle X|R \rangle$ be a geometric presentation of a hyperbolic surface group G. Then, for any $x \in X$, $\mathscr{C}_x$ is connected and $\mathscr{C}_x \cap \mathscr{C}_y \neq \emptyset$ if and only if x and y are adjacent generators for the cyclic ordering given by Lemma 2.1(a). In this case, $\mathscr{C}_x \cap \mathscr{C}_y$ is an interval.*

We are now ready to define the Bowen-Series-like family.

Definition 1. *Let Σ be a closed, compact surface with negative Euler characteristic and let P be a geometric presentation of $G := \pi_1(\Sigma)$. We denote the elements of the generating set X as $x_1, x_2, \ldots, x_{2N}$, where the indices are defined modulo $2N$ (with $1, 2, \ldots, 2N$ as representatives of the classes modulo $2N$) in such a way that x_j is adjacent to $x_{j\pm1}$. From Proposition 2.3 there are $2N$ disjoint intervals*

$$J_j := \mathscr{C}_{x_{j-1}} \cap \mathscr{C}_{x_j} \subset S^1.$$

For each $\Theta := (\theta_1, \theta_2, \ldots, \theta_{2N}) \in J_1 \times J_2 \times \ldots \times J_{2N}$ we consider the intervals $I_j := [\theta_j, \theta_{j+1}) \subset S^1$ and the map

$$(2) \qquad \Phi_\Theta : S^1 \longrightarrow S^1 \text{ such that } \Phi_\Theta(z) = x_j^{-1}(z) \text{ if } z \in I_j.$$

The notation $x_j^{-1}(z)$ in (2) means that the map is the restriction, to the interval $I_j \subset \partial G$, of the action by the group element $x_j^{-1} \in G$ on ∂G. It is well known [14] that this action is by homeomorphisms. The map Φ_Θ in (2) is thus a piecewise homeomorphism of $S^1 = \partial G$. Such a map is called a Bowen-Series-like map. Each point θ_i is called a cutting point, and Θ is called a cutting parameter.

The definition of the interval I_j once the cutting points θ_j and θ_{j+1} are chosen is illustrated by Figure 2 for the particular presentation of Example 2.2. Note, for instance, that the infinite geodesic ray converging to θ_a admits two different infinite expressions in the alphabet $\{a^{\pm1}, b^{\pm1}, c^{\pm1}, d^{\pm1}\}$, one starting by b^{-1} and the other one starting by a. In other words, θ_a belongs to the intersection of cylinders $\mathscr{C}_{b^{-1}} \cap \mathscr{C}_a$.
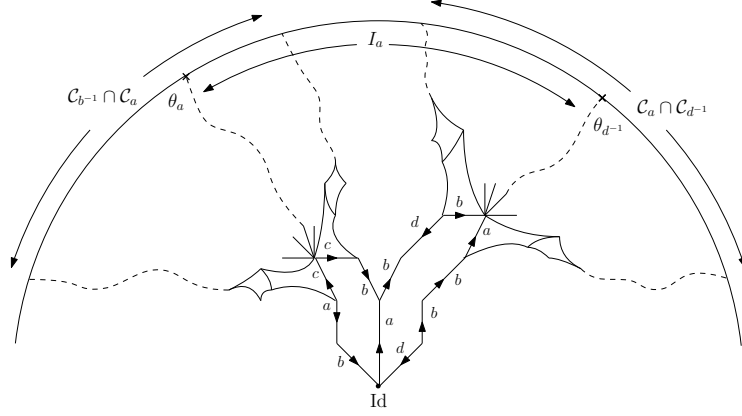
FIGURE 2. The interval I_a for the presentation P of Example 2.2.

The definitions imply that, for each $j \in \{1, 2, \ldots, 2N\}$, $I_j \subset \mathscr{C}_{x_j}$ and the map $\Phi_\Theta\big|_{I_j}$ is a homeomorphism onto its image. At the cutting points the map is not continuous. From the definition of the cylinder sets $\mathscr{C}_{x_j}$, if $z \in I_j$ then there exists a geodesic ray, starting at Id in the Cayley graph, denoted by $[z]$ and converging to $z \in \partial G$, such that $[z] = x_j A$, where A is an infinite word in the alphabet X. The map applied to the point z thus gives:

$$\text{if } z \in I_j \text{ then } [z] = x_j A \text{ and } [\Phi_\Theta(z)] = A.$$

In other words, Φ_Θ is a standard "shift map" for the particular coding of the boundary ∂G obtained from the intervals I_j. Different choices of the cutting parameter Θ in $J_1 \times J_2 \times \ldots \times J_{2N}$ define different maps and different codings of ∂G.

3. STATEMENT AND CONSEQUENCES OF THE MAIN THEOREM

With the definitions and the standing notation introduced in Section 2, we are ready to announce the main result of this paper.

Theorem A. *Let Σ be a closed, compact surface with negative Euler characteristic and let P be any geometric presentation of $G := \pi_1(\Sigma)$. Then, for any parameter $\Theta \in J_1 \times J_2 \times \ldots \times J_{2N}$, the Bowen-Series-like map Φ_Θ satisfies:*

(a) *$h_{top}(\Phi_\Theta) = h_{vol}(G, P) = \log(\lambda)$, where $1/\lambda$ is the smallest root in $(0, 1)$ of an integer polynomial $Q_P(t)$ that can be explicitly computed from P.*

(b) *Φ_Θ is topologically conjugate to a piecewise affine map $\widetilde{\Phi_\Theta}$ whose slope is constant for all intervals I_j and is equal to $\pm\lambda$.*

Theorem A is an "a priori" surprising result since the dynamics of two different maps in the family are quite different. In particular they are not pairwise topologically conjugate or even semi-conjugate. In addition, the orbit equivalence property discussed in Section 1 implies that for many presentations and parameters, the map Φ_Θ and the group G are orbit equivalent (which implies that, for those parameters and those presentations, any two maps are orbit equivalent to each other). The orbit equivalence property is much weaker than conjugacy and, in principle, does not preserve the topological entropy. Theorem A is thus a remarkable entropy stability property.

From a geometric group theory point of view, Theorem A implies also a surprising property. The map Φ_Θ is defined by the action of the generators of G restricted to the partition intervals of the circle, which, as the Gromov boundary of G, is just a topological space. The boundary, here $\partial G = S^1$, is metrizable and admits many classes of metrics. The conjugacy of Theorem A(b) implies that there exists $g_\Theta \in \mathrm{Homeo}(S^1)$ such that $\widetilde{\Phi}_\Theta = g_\Theta \circ \Phi_\Theta \circ g_\Theta^{-1}$ is piecewise affine with constant slope $\pm\lambda$, where $\log(\lambda)$ is the volume entropy of the presentation P. The map $\widetilde{\Phi}_\Theta$ is defined by a partition of $S^1 = \bigcup_{j=1}^{2N} \widetilde{I}_j$ with $\widetilde{I}_j = g_\Theta(I_j)$. The piecewise affine property makes sense if each interval $\widetilde{I}_j$ (and, thus, $S^1 = \partial G$) admits a well defined metric $\widetilde{\rho}$ for which the map is piecewise affine. Recall that G is a discrete subgroup of $\mathrm{Homeo}(S^1)$ and by conjugating G (i.e. each generator) via $g_\Theta \in \mathrm{Homeo}(S^1)$ we obtain a presentation $\widetilde{P}$ which is combinatorially the same as P and, in particular, has the same volume entropy. For that presentation, each generator $\widetilde{x}_j$ acts on some interval (the $\widetilde{I}_j$) as an affine map of slope $\pm\lambda$ with respect to the metric $\widetilde{\rho}$. The metric $\widetilde{\rho}$ is thus very special and, in particular, it reflects the asymptotic growth of the presentation. The existence of such particular metrics on the boundary of a hyperbolic group seems to be a new phenomenon. In some cases this metric satisfies more surprising properties for all group elements (see for instance Corollary 4 in [18]). The existence of metrics of that type for general hyperbolic groups is a new question.

The proof of Theorem A has several steps. The first one shows that the topological entropy is constant for all parameters. This is obtained as a consequence of two inequalities comparing the growth functions for the group presentation and for the dynamics. The growth function for the presentation P is σ_m in (1). For the dynamics, we count the number X_m of different *itineraries* of length m for each map Φ_Θ. More precisely, an interval of the form $I_{j_0, j_1, \ldots, j_{m-1}} := I_{j_0} \cap \Phi_\Theta^{-1}(I_{j_1}) \cap \ldots \cap \Phi_\Theta^{-(m-1)}(I_{j_{m-1}})$, which is the subinterval of points of I_{j_0} whose orbit of length m visits $I_{j_1}, I_{j_2}, \ldots, I_{j_{m-1}}$, is called an *itinerary interval of length m*. The function X_m is then defined as the number of nonempty itinerary intervals of level m. The topological entropy $h_{top}(\Phi_\Theta)$, in this context of discontinuous monotone maps, is obtained from [20] as

$$h_{top}(\Phi_\Theta) = \lim_{m \to \infty} \frac{1}{m} \log(X_m).$$

The entropy equality of Theorem A(a) is a direct consequence of the following result (Proposition 7.5 of [2]).

Proposition 3.1. *For any cutting parameter $\Theta \in J_1 \times \ldots \times J_{2N}$, the following inequalities are satisfied: $\sigma_m \leq X_m \leq m\sigma_m$.*

The next step is the proof of a property, called the *eventual coincidence* in [18]. Recall that each cutting point θ_j is a point of discontinuity and thus admits two different orbits, one from the left and one from the right: $\Phi_\Theta^n(\theta_j^\pm)$. The eventual coincidence property states that

(EC)   For all $j \in \{1, 2, \ldots, 2N\}$, there exists $k_j \geq 2$ such that $\Phi_\Theta^{k_j}(\theta_j^+) = \Phi_\Theta^{k_j}(\theta_j^-)$.

We obtain, in Lemma 5.1 of [2], that for an open set of parameters Θ the maps Φ_Θ satisfy the condition (EC), where each integer k_j is the length of the minimal bigon

Relations in P	Volume entropy	$Q_P(t)$
$[abd^{-1}b, d^{-1}b^2a]$	$\log(5.863240079)$	$t^{10} - 3t^9 - 14t^8 - 13t^7 - 17t^6$ $-12t^5 - 17t^4 - 13t^3 - 14t^2 - 3t + 1$
$[aba^{-1}b^{-1}cdc^{-1}d^{-1}]$	$\log(6.979835779)$	$t^4 - 6t^3 - 6t^2 - 6t + 1$
$[acd^{-1}, b^2e, adce]$	$\log(6.995719375)$	$t^8 - 4t^7 - 19t^6 - 10t^5$ $-24t^4 - 10t^3 - 19t^2 - 4t + 1$
$[aef^{-1}d^2, b^{-1}c^2gf, gabe]$	Non geometric	$-$
$[abac^{-1}f, dih^2, bdgi,$ $ece, gf^{-1}j^2]$	$\log(17.9527833)$	$t^{20} - 13t^{19} - 80t^{18} - 149t^{17}$ $-187t^{16} - 196t^{15} - 252t^{14} - 348t^{13}$ $-370t^{12} - 426t^{11} - 312t^{10} - 426t^9$ $-370t^8 - 348t^7 - 252t^6 - 196t^5$ $-187t^4 - 149t^3 - 80t^2 - 13t + 1$

TABLE 1. Some outputs of the algorithm.

$\beta(x_{j-1}, x_j)$. In addition, the two orbits $\Phi_\Theta^m(\theta_j^+)$ and $\Phi_\Theta^m(\theta_j^-)$ for $m < k_j$ can be obtained from the combinatorics of the minimal bigon $\beta(x_{j-1}, x_j)$.

Another observation is that we can choose the cutting parameter Θ^0 in such a way the forward orbit of each cutting point θ_j^0 never visits a cutting point. This remark, together with (EC), are used in the last step of the proof, consisting of a crucial use of the Milnor-Thurston theory of kneading invariants [19, 4, 5] for the particular map Φ_{Θ^0}.

The Milnor-Thurston invariant is, a priori, a formal power series. From the properties above we obtain that this power series is in fact a polynomial with integer coefficients $Q_P(t)$ which depend only on the combinatorics of the presentation P. The theory says that $h_{top}(\Phi_{\Theta^0}) = \log(\lambda)$, where $\frac{1}{\lambda}$ is the smallest root in $(0, 1)$ of $Q_P(t)$. Since $h_{top}(\Phi_\Theta) = h_{top}(\Phi_{\Theta^0})$ for all Θ, this concludes the proof of the statement (a) in Theorem A. Statement (b) is also a direct consequence of the Milnor-Thurston theory, since all Φ_Θ are transitive.

Theorem A has a strong algorithmic consequence. Indeed, since the polynomial invariant of the presentation $Q_P(t)$ does not depend on the cutting parameter Θ, it is possible to make an astute choice of a particular parameter for which the Milnor-Thurston invariants can be easily computed using elementary algebraic operations.

At the end, all the steps of the computation can be organized in an algorithm that uses only the information encoded in the relations of the presentation P and gives as output the polynomial $Q_P(t)$ and the corresponding volume entropy, provided that P is geometric (otherwise, the procedure reports that P is not geometric). The program has been written in Mapple and Maxima languages and is freely available to the scientific community upon request to the authors.

The algorithm works on any geometric presentation, in contrast to the techniques in [11, 3], that allow this computation only for very specific cases. Compared to [17] the computations are considerably simpler, since no explicit Markov partitions are necessary.

Going back to the geometric presentation P of Example 2.2, the obtained tiling of the plane is not regular and, in particular, the Cayley graph is not bipartite. It is worth noticing that, in this case, the classic technique of *cone types* [11] fails in computing the volume entropy $h_{\text{vol}}(G, P)$, since it is difficult to estimate the growth rate of the number σ_m of vertices of the graph at distance m from Id.

Table 1 shows some examples of execution of the program. The first entry of the table corresponds to the presentation of Example 2.2. The second entry is the classical presentation for the orientable surface of genus 2. Note that the corresponding $Q_P(t)$ is the well known growth polynomial for the minimal geometric presentations of surface groups [3, 12].

This combinatorial approach for computing the polynomial $Q_P(t)$ has possibly a purely geometric or algebraic group theory interpretation, which is not visible from our dynamical point of view.

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