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THE DISCONTINUOUS MATCHING OF TWO GLOBALLY ASYMPTOTICALLY STABLE CROSSING PIECEWISE SMOOTH SYSTEMS IN THE PLANE DO NOT PRODUCE IN GENERAL A PIECEWISE DIFFERENTIAL SYSTEM GLOBALLY ASYMPTOTICALLY STABLE

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Abstract. A differential system in the plane $\mathbb{R}^2$ is globally asymptotically stable if it has an equilibrium point p and all the other orbits of the system tend to p in forward time. In other words if the basin of attraction of p is $\mathbb{R}^2$. The problem of determining the basin of attraction of an equilibrium point is one of the main problems in the qualitative theory of differential equations. We prove that planar crossing piecewise smooth systems with two zones formed by two globally asymptotically stable differential systems sharing the same equilibrium point localized in the separation line are not necessarily globally asymptotically stable.

1. Introduction and statement of the main result

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n \geq 2$, be a C^1 vector field. As usual, we will identify the vector field F with the ordinary differential equation

$$\dot{x} = \frac{dx}{dt} = F(x), \quad x = (x_1, \ldots, x_n), \tag{1}$$

where the dot denotes the derivative with respect to the independent variable t, called here the time.

Assume that $x^* \in \mathbb{R}^n$ is the only equilibrium point of system (1). Assume further that x^* is *locally asymptotically stable*, i.e. there exists an open neighborhood U of system x^* such that the orbits of (1) starting from U tend to x^* in forward time. The *basin of attraction* of x^* is the largest open set whose elements satisfy the above condition. Of course, the neighborhood U is contained in the basin of attraction of x^*. The equilibrium point x^* or the vector field F is *globally asymptotically stable* if its basin of attraction is the whole $\mathbb{R}^n$.

The problem of determining the basin of attraction of an equilibrium point of a vector field is of great importance for applications of the stability theory of ordinary differential equations. The globally asymptotically stable problem is closely related to the Markus-Yamabe Conjecture [9].

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Conjecture 1 (Markus-Yamabe)**.** *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 vector field having the origin as its unique equilibrium point. If the eigenvalues of the Jacobian matrix of F at $x \in \mathbb{R}^n$ have negative real parts at every point $x \in \mathbb{R}^n$, then F is globally asymptotically stable.*

The Markus-Yamabe Conjecture was proved to be true in dimension two independently by Feßler [5], Glutsyuk [7] and Gutiérrez [8]. However, if n is greater than or equal to three, the conjecture has been proven to be false. See the articles Bernat and Llibre [2] and Cima, van den Essen, Gasull, Hubbers and Mañosas [4].

Here we are interested in the study of the globally asymptotically stable piecewise differential systems in the plane. Of course, these piecewise systems are not necessarily under the assumptions of the Markus-Yamabe Conjecture.

Discontinuous piecewise smooth differential systems with two zones in the plane separated by the line $\mathcal{H}(x,y) = 0$ are

$$(\dot{x}, \dot{y}) = \begin{cases} F^+(x,y), & \text{if } \mathcal{H}(x,y) \geq 0, \\ F^-(x,y), & \text{if } \mathcal{H}(x,y) \leq 0, \end{cases} \tag{2}$$

where $F^{\pm}$ are smooth vector fields defined in the plane. The function $\mathcal{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is smooth having the zero as a regular value and the set $\Sigma = \mathcal{H}^{-1}(\{0\})$, called the *separation line*, divides the plane into two unbounded components (zones) Σ^+ and Σ^- where $\mathcal{H}$ is positive and negative, respectively. Thus $\mathbb{R}^2 = \Sigma^+ \cup \Sigma \cup \Sigma^-$.

Usually, the points on Σ are classified as crossing, sliding, escaping or tangency points, for more details see [10]. Recall that a point $(x_0, y_0) \in \Sigma = \mathcal{H}^{-1}(\{0\})$ is a *crossing* point if

$$T(x_0, y_0) = \big(F^-(x_0, y_0) \cdot \nabla\mathcal{H}(x_0, y_0)\big)\, \big(F^+(x_0, y_0) \cdot \nabla\mathcal{H}(x_0, y_0)\big) > 0. \tag{3}$$

What somehow gives simplicity to crossing points on Σ is that the orbit of the piecewise system (2) through such points are the concatenation of the orbits of the vector fields $F^+|_{\Sigma^+}$ and $F^-|_{\Sigma^-}$ through these points.

Since the seminal work of Andronov et al. [1] a lot of articles were published about questions like the existence, number, stability and distribution of limit cycles mainly for the case where $F^{\pm}$ are linear.

Nevertheless, to the best of our knowledge, there are almost no articles studying the problem of the global asymptotic stability of an equilibrium point in this class of piecewise systems (2). The landmark of a such study is the article [6] in which the authors analyzed the following case:

- H1. $F^{\pm}$ are Hurwitz homogeneous linear vector fields (the real parts of all its eigenvalues are negative);
- H2. The separation boundary Σ is a straight line that contains the unique equilibrium point of the piecewise system at the origin;
- H3. The vector fields $F^{\pm}$ are continuous on $\Sigma \setminus \{(0,0)\}$.

With such hypotheses it was proved in [6] that the unique equilibrium point is globally asymptotically stable. Similar result was obtained in [3] weakening the assumption of the continuity on $\Sigma \setminus \{(0,0)\}$ in [6] for crossing, that is, the hypothesis H3 was changed by:

H3'. The points on $\Sigma \setminus \{(0,0)\}$ are crossing points.

In this article we study piecewise smooth systems in the plane satisfying the hypotheses H2, H3' and

H1'. $F^{\pm}$ are globally asymptotically stable.

The following question is natural: *Are piecewise smooth systems with two zones in the plane satisfying the hypotheses H1', H2 and H3' globally asymptotically stable*? In the light of the articles [3] and [6] if we look for a negative answer to this question then at least one of the vector fields $F^{\pm}$ must be nonlinear. The main result of this article is the following.

Theorem 2. *Consider piecewise smooth differential systems in the plane separated by a straight line and formed by two differential systems satisfying the hypotheses H1', H2 and H3'. Then there exist piecewise polynomial differential systems in this class which are not globally asymptotically stable.*

The proof of Theorem 2 is presented in Section 2.

2. Proof of Theorem 2

Lemma 3. *Consider the piecewise linear differential system of the form* (2) *defined by*

$$(\dot{x},\dot{y}) = \begin{cases} A^{+}(x,y) &= \left(-\dfrac{5}{8}x + \dfrac{11}{20}y, -x + \dfrac{5}{8}y\right), \qquad \text{if } \widetilde{\mathcal{H}}(x,y) \geq 0, \\[2ex] A^{-}(x,y) &= \left(-\dfrac{13}{16}x + \dfrac{31}{40}y, -x + \dfrac{13}{16}y\right), \quad \text{if } \widetilde{\mathcal{H}}(x,y) \leq 0, \end{cases} \tag{4}$$

where $\widetilde{\mathcal{H}}(x,y) = y - x^2$ *and the separation boundary is given by* $\widetilde{\Sigma} = \widetilde{\mathcal{H}}^{-1}(0)$. *The following statements hold.*

(a) *The homogeneous linear vector fields* A^+ *and* A^- *define global centers on* $\mathbb{R}^2$ *and the unique equilibrium point of both systems is* $(0,0) \in \widetilde{\Sigma}$.
(b) *The points* $(x,y) \in \widetilde{\Sigma} \setminus \{(0,0)\}$ *are crossing points.*
(c) *There exists a hyperbolic repelling crossing limit cycle in the phase portrait of* (4).

Proof. The proof of statement (a) is immediate since $\mathrm{tr} A^{\pm} = 0$ and $\det A^{\pm} > 0$. The linear differential systems in (4) are the Hamiltonian systems

$$A^{\pm}(x,y) = \left(\frac{\partial H^{\pm}}{\partial y}(x,y), -\frac{\partial H^{\pm}}{\partial x}(x,y)\right),$$

with the Hamiltonians

$$H^+(x,y) = \frac{x^2}{2} - \frac{5}{8}xy + \frac{11}{40}y^2, \qquad H^-(x,y) = \frac{x^2}{2} - \frac{13}{16}xy + \frac{31}{80}y^2. \tag{5}$$

Now we prove statement (b). As the gradient of the function $\widetilde{\mathcal{H}}(x,y) = y - x^2$ is given by $\nabla\widetilde{\mathcal{H}}(x,y) = (-2x, 1)$, from (3) we obtain

$$T(x,x^2) = \frac{x^2}{3200}\left(44x^2 - 75x + 40\right)\left(124x^2 - 195x + 80\right) > 0,$$

for all $x \neq 0$. This implies that the points $(x,y) \in \widetilde{\Sigma} \setminus \{(0,0)\}$ are crossing points, proving statement (b).

Consider the points $(-1,1) \in \widetilde{\Sigma}$ and $(2,4) \in \widetilde{\Sigma}$. It immediately follows that

$$H^+(-1,1) = H^+(2,4) = \frac{7}{5}, \qquad H^-(-1,1) = H^-(2,4) = \frac{17}{10}.$$

Thus the points $(-1,1) \in \widetilde{\Sigma}$ and $(2,4) \in \widetilde{\Sigma}$ belong to a possible crossing periodic orbit of the piecewise linear differential system (4). If we draw the arc of $H^+(x,y) = H^+(-1,1)$ inside the region $\widetilde{\mathcal{H}} \geq 0$ and the arc of $H^-(x,y) = H^-(-1,1)$ inside the region $\widetilde{\mathcal{H}}(x,y) \leq 0$, we obtain the crossing limit cycle of the piecewise differential system (4), see Figure 1.

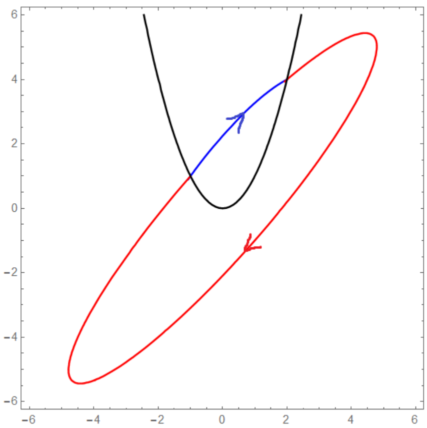

FIGURE 1. The limit cycle of the piecewise differential system (4).

Given $(x_0, x_0^2) \in \widetilde{\Sigma}$, $x_0 < 0$, consider the intersections of the level sets ${H^{\pm}}^{-1}(\{H^{\pm}(x_0, x_0^2)\})$ with $\widetilde{\Sigma}$. Denote these points by $(x_+, x_+^2) \in \widetilde{\Sigma}$ and $(x_-, x_-^2) \in \widetilde{\Sigma}$ with $x_+ > 0$ and $x_- > 0$. By construction, the points x_+ and x_- satisfy the following system

$$H^+(x_+, x_+^2) = H^+(x_0, x_0^2), \quad H^-(x_-, x_-^2) = H^-(x_0, x_0^2), \quad x_0 < 0,\ x_+ > 0,\ x_- > 0. \tag{6}$$

System (6) defines $x_+ = x_+(x_0)$ and $x_- = x_-(x_0)$. So we can define the separation function

$$\Delta(x_0) = x_-(x_0) - x_+(x_0), \qquad x_0 < 0. \tag{7}$$

The expression of the separation function Δ is too long to write here. Of course, by the above analysis, $\Delta(-1) = 0$. Using algebraic manipulations from a software system such as Mathematica, we get

$$\Delta'(-1) = 0.13196 > 0$$

with five decimals. Therefore the crossing periodic orbit of the piecewise linear differential system (4) passing through the point $(x_0, x_0^2) = (-1, 1) \in \widetilde{\Sigma}$ is a hyperbolic repelling crossing limit cycle. Statement (c) is proved. $\square$

Consider the following perturbation of system the piecewise (4)

$$(8) \qquad (\dot{x}, \dot{y}) = \begin{cases} B^+(x,y) &= \left(-(1+\varepsilon)\frac{5}{8}x + \frac{11}{20}y, -x + \frac{5}{8}y\right), \qquad \widetilde{\mathcal{H}}(x,y) \geq 0, \\ B^-(x,y) &= \left(-(1+\varepsilon)\frac{13}{16}x + \frac{31}{40}y, -x + \frac{13}{16}y\right), \quad \widetilde{\mathcal{H}}(x,y) \leq 0, \end{cases}$$

where $\widetilde{\mathcal{H}}(x,y) = y - x^2$, $\widetilde{\Sigma} = \widetilde{\mathcal{H}}^{-1}(0)$, and $\varepsilon > 0$ is small enough.

Since a hyperbolic limit cycle and crossing points on $\widetilde{\Sigma}$ are preserved by slightly perturbations of the piecewise linear system (4) the following lemma has an immediate proof.

Lemma 4. *Consider the piecewise linear differential system* (8). *The following statements hold.*

(i) *The homogeneous linear vector fields B^+ and B^- define stable focus on $\mathbb{R}^2$ and the unique equilibrium point of both systems is $(0,0) \in \widetilde{\Sigma}$.*
(ii) *The points $(x,y) \in \widetilde{\Sigma} \setminus \{(0,0)\}$ are crossing points.*
(iii) *There exists a hyperbolic repelling crossing limit cycle in the phase portrait of* (8).

In the next step we study the action of the diffeomorphism

$$(9) \qquad S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad S(x,y) = \left(x, y - x^2\right),$$

on the whole piecewise linear system (8). By simplicity, we maintain the notation x and y for the variables in the target plane.

Since $S(x, x^2) = (x, 0)$ for all $x \in \mathbb{R}$, it follows that the separation line $\widetilde{\Sigma}$ is mapped onto the straight line $\Sigma = \mathcal{H}^{-1}(0)$ where $\mathcal{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\mathcal{H}(x,y) = y$.

The diffeomorphism S transforms the vector fields $B^{\pm}$ into the vector fields

$$(10) \qquad F^{\pm} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F^{\pm}(x,y) = DS(S^{-1}(x,y))B^{\pm}(S^{-1}(x,y)).$$

As the vector fields $B^{\pm}$ are globally asymptotically stable then the vector fields $F^{\pm}$ are also globally asymptotically stable on $\mathbb{R}^2$.

The piecewise polynomial differential system (4) under the diffeomorphism S becomes the piecewise differential system (2) with $\mathcal{H}(x,y)=y$, $\Sigma=\mathcal{H}^{-1}(0)$ and

$$\begin{aligned} F^+(x,y) &= \left(-(1+\varepsilon)\frac{5}{8}x+\frac{11}{20}y+\frac{11}{20}x^2, -x+\frac{5}{8}y+\left(\varepsilon+\frac{3}{2}\right)\frac{5}{4}x^2-\frac{11}{10}xy-\frac{11}{10}x^3\right), \\ F^-(x,y) &= \left(-(1+\varepsilon)\frac{13}{16}x+\frac{31}{40}y+\frac{31}{40}x^2, -x+\frac{13}{16}y+\left(\varepsilon+\frac{3}{2}\right)\frac{13}{8}x^2-\frac{31}{20}xy-\frac{31}{20}x^3\right), \end{aligned} \tag{11}$$

where $\varepsilon>0$ is small parameter.

From Lemma 4 and the definition of the diffeomorphism (9), in order to finish the proof of Theorem 2 it is enough to prove that the points on $\Sigma\setminus\{(0,0)\}$ are crossing points of (11). From (3) we obtain

$$T_\varepsilon(x,0)=x^2 f_\varepsilon(x)\, g_\varepsilon(x), \tag{12}$$

where

$$f_\varepsilon(x)=\frac{11}{10}x^2-\left(\varepsilon+\frac{3}{2}\right)\frac{5}{4}x+1, \quad g_\varepsilon=\frac{31}{20}x^2-\left(\varepsilon+\frac{3}{2}\right)\frac{13}{8}x+1, \quad x\in\mathbb{R}.$$

As $f_0(x)>0$ and $g_0(x)>0$ for all $x\in\mathbb{R}$, it follows that $T_\varepsilon(x,0)>0$ for all $x\neq 0$, if $\varepsilon>0$ is small enough. In short, Theorem 2 is proved.

Conflict of interest

The authors declare that they have no conflict of interest.

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