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PHASE PORTRAITS OF TWO CLASSES OF LIÉNARD EQUATIONS

JIE LI[1] AND JAUME LLIBRE[2]

Abstract. In this paper we classify the phase portraits in the Poincaré disc of the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with either $f(x) = 0$ and $g(x)$ is a quadratic or cubic polynomial, or $f(x)$ is a quadratic or cubic polynomial and $g(x) = 0$.

1. Introduction and main results

Consider the well-known Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where g is the restoring force and f denotes the friction coefficient being f and g continuous functions and $g(0) = 0$, see for more details the reference [3]. Here the dot denotes derivative with respect to the time t.

Much attention has been paid to the Liénard equation with $f(x) \neq 0$ and $g(x) \neq 0$, thus when this paper is being written in MathSciNet appears 903 papers when you look in Anywhere for "Liénard equation" and "Liénard system". Thus two recent papers on Liénard equations are [4, 7].

In this paper we consider the Liénard equation with either $f(x) = 0$ and $g(x) = \sum_{\ell=0}^{m} b_\ell x^\ell$, or $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = 0$, where $m, n \geq 2$ are integers and a_k, b_ℓ are real with $a_n, b_m \neq 0$. Then the particular Liénard equations that we will study are $\ddot{x}+g(x) = 0$ and $\ddot{x}+f(x)\dot{x} = 0$.

These Liénard equations can be written as the following two differential systems of first order

$$\dot{x} = y, \qquad \dot{y} = -g(x), \tag{1}$$

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$$\dot{x} = y, \qquad \dot{y} = -f(x)y. \tag{2}$$

Clearly the differential system (1) has the first integral

$$H_1 = H_1(x,y) = \frac{y^2}{2} + \sum_{\ell=0}^{m} \frac{b_\ell}{\ell+1} x^{\ell+1}.$$

Using the time-rescaling $ds = ydt$, the differential system (2) can be written as

$$x' = 1, \qquad y' = -f(x), \tag{3}$$

where the prime denotes derivative with respect to the new time s. Its first integral is

$$H_2 = H_2(x,y) = y + \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}.$$

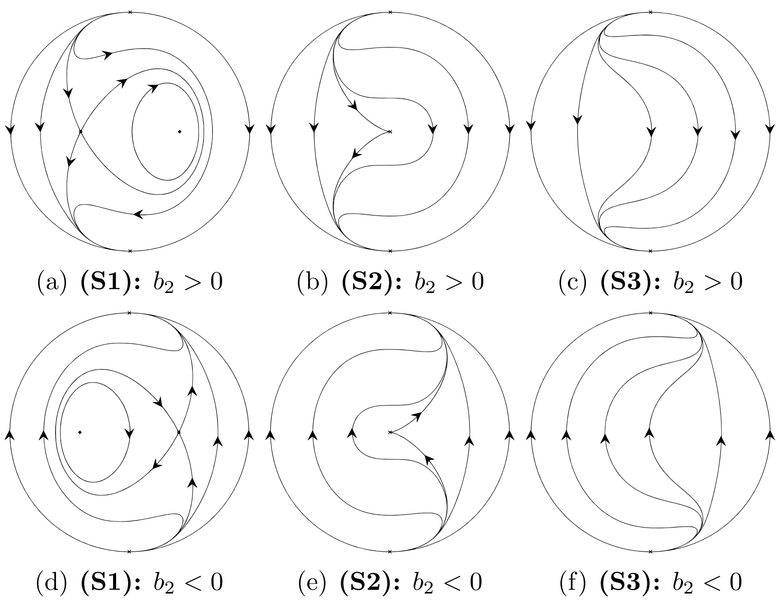
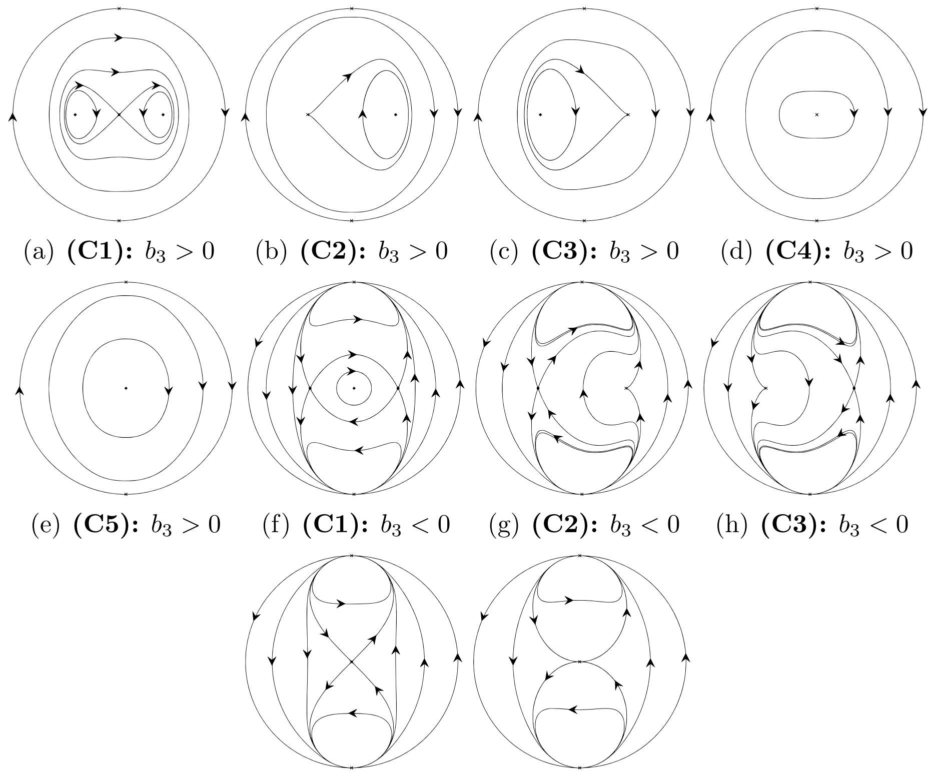


FIGURE 1. The phase portraits of the differential system (1) with $m = 2$ in the Poincaré disc in the three cases **(Sk)** for $k = 1, 2, 3$.

In the next two theorems we classify the phase portraits in the Poincaré disc of the differential system (1) when $m = 2$ and $m = 3$.

Theorem 1. *For $m = 2$ the phase portraits of the Liénard system* (1) *in the Poincaré disc are given in Figure 1.*

Theorem 2. *For $m = 3$ the phase portraits of the Liénard system* (1) *in the Poincaré disc are given in Figure 2.*



(a) **(C1):** $b_3 > 0$ (b) **(C2):** $b_3 > 0$ (c) **(C3):** $b_3 > 0$ (d) **(C4):** $b_3 > 0$

(e) **(C5):** $b_3 > 0$ (f) **(C1):** $b_3 < 0$ (g) **(C2):** $b_3 < 0$ (h) **(C3):** $b_3 < 0$

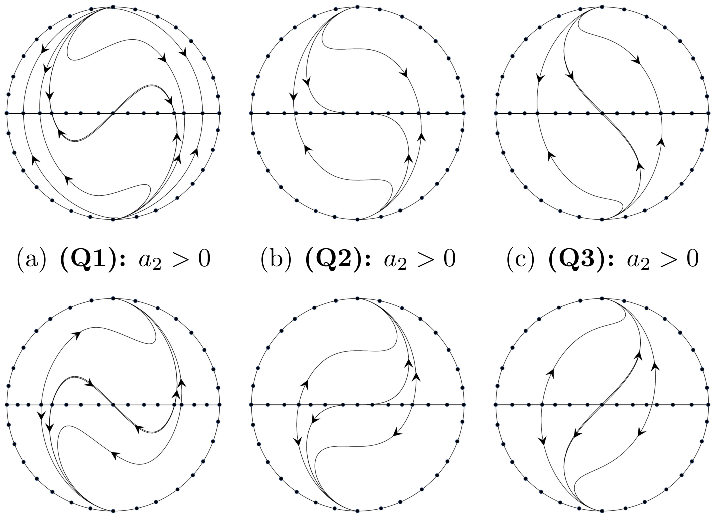
(i) **(C4):** $b_3 < 0$ (j) **(C5):** $b_3 < 0$

FIGURE 2. The phase portraits of the differential system (1) with $m = 3$ in the Poincaré disc in the five cases **(Ck)** for $k = 1, \dots, 5$.

In the two theorems we classify the phase portraits in the Poincaré disc of the differential system (2) with $n = 2$ and $n = 3$.

Theorem 3. *For $n = 2$ the phase portraits of the Liénard system* (2) *in the Poincaré disc are given in Figure 3.*

Theorem 4. *For $n = 3$ the phase portraits of the Liénard system* (2) *in the Poincaré disc are given in Figure 4.*



(a) **(Q1):** $a_2 > 0$ (b) **(Q2):** $a_2 > 0$ (c) **(Q3):** $a_2 > 0$

(d) **(Q1):** $a_2 < 0$ (e) **(Q2):** $a_2 < 0$ (f) **(Q3):** $a_2 < 0$

FIGURE 3. The phase portraits of the differential system (2) with $n = 2$ in the Poincaré disc in the three cases **(Qk)** for $k = 1, 2, 3$.

2. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. It is clear that system (1) has the finite equilibria $(x_0, 0)$ for each real root x_0 of the polynomial $g(x)$.

Here we use the notations of Chapter 5 of [1] for studying the Poincaré compactification of the polynomial differential systems in the local charts U_1 and U_2. First in the local chart U_1 the differential system (1) becomes

$$\dot{u} = -u^2 v^{m-1} - G_m(v), \qquad \dot{v} = -uv^m,$$

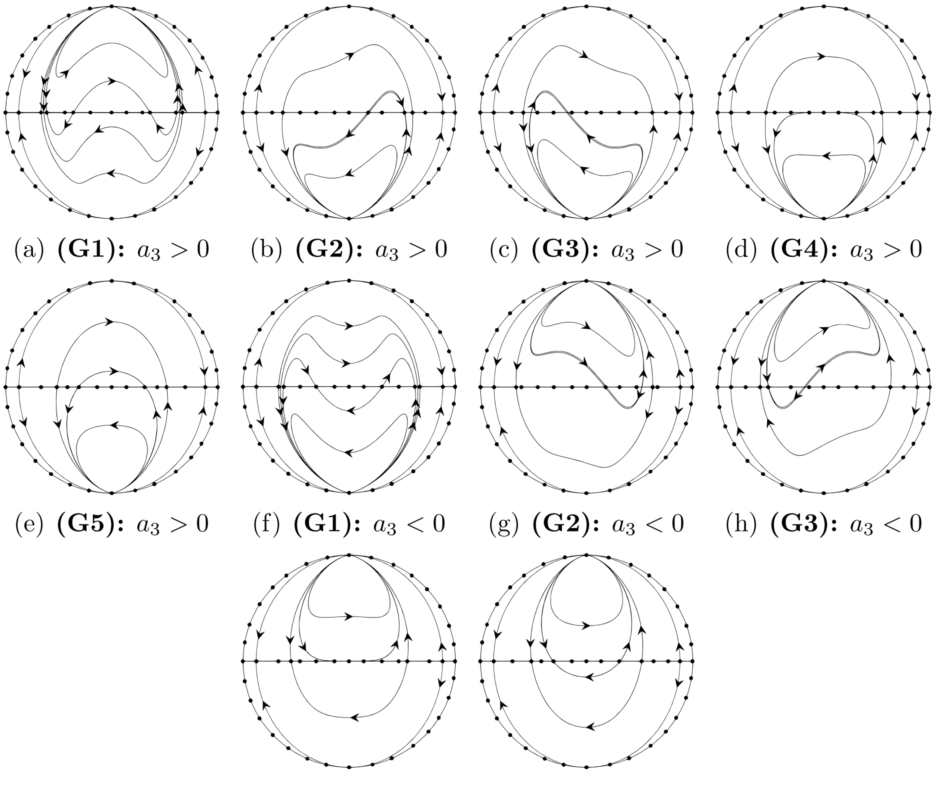
and in the local chart U_2 it is

$$\dot{u} = v^{m-1} + u\Phi_m(u, v), \qquad \dot{v} = v\Phi_m(u, v),$$

where

$$G_m(v) = \sum_{i=0}^{m} b_i v^{m-i}, \qquad \Phi_m(u, v) = \sum_{i=0}^{m} b_i u^i v^{m-i}.$$

Since $b_m \neq 0$ it follows that there are no infinite equilibrium points in the local chart U_1. Moreover the origin of the local chart U_2 is an equilibrium point. Note that $\dot{u} = b_m u^{m+1}$ and $\dot{v} = 0$ on the u-axis.



(a) **(G1):** $a_3 > 0$ (b) **(G2):** $a_3 > 0$ (c) **(G3):** $a_3 > 0$ (d) **(G4):** $a_3 > 0$

(e) **(G5):** $a_3 > 0$ (f) **(G1):** $a_3 < 0$ (g) **(G2):** $a_3 < 0$ (h) **(G3):** $a_3 < 0$

(i) **(G4):** $a_3 < 0$ (j) **(G5):** $a_3 < 0$

FIGURE 4. The phase portraits of the differential system (2) with $n = 3$ in the Poincaré disc in the five cases **(Gk)** for $k = 1, \cdots, 5$.

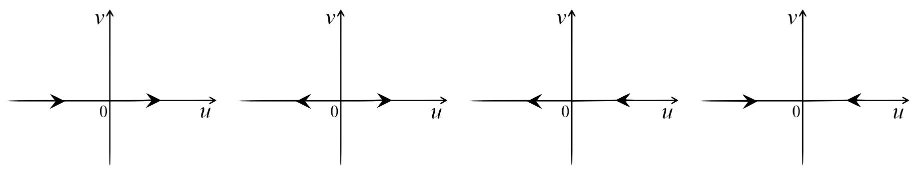
Then there are four possibilities:

$$(\mathbf{P1})\ b_m > 0,\ m = 2\ell + 1,$$
$$(\mathbf{P2})\ b_m > 0,\ m = 2\ell,$$
$$(\mathbf{P3})\ b_m < 0,\ m = 2\ell + 1,\ \text{and}$$
$$(\mathbf{P4})\ b_m < 0,\ m = 2\ell.$$

We get the orbits on the u-axis in the (u, v)-coordinates, as shown in Figure 5. We shall study the local phase portraits at the origin of the local chart U_2.

We first consider a general quadratic polynomial

$$g(x) = b_2 x^2 + b_1 x + b_0, \quad b_2 \neq 0.$$



(a) **(P1)** (b) **(P2)** (c) **(P3)** (d) **(P4)**

FIGURE 5. The orbits on the u-axis in the local chart U_1 for the differential system (1).

Then

$$H_1 = H_1(x, y) = \frac{y^2}{2} + \frac{b_2}{3}x^3 + \frac{b_1}{2}x^2 + b_0 x.$$

By the Poincaré transformation $x = u/v, y = 1/v$, we change the first integral to the following

$$\tilde{H}_1 = \tilde{H}_1(u, v) = \frac{3v + 2b_2u^3 + 3b_1u^2v + 6b_0uv^2}{6v^3}.$$

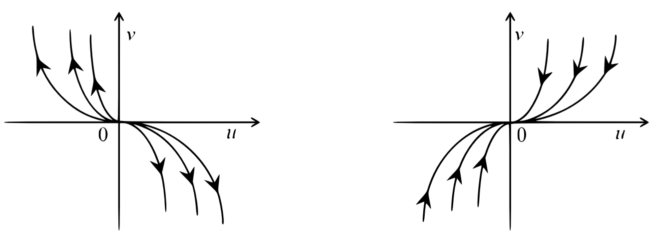
Let $\tilde{H}_1 = h$ varying $h \in \mathbb{R}$. Then

$$U(x, y) = 6v^3(\tilde{H}_1(u, v) - h) = 3v + 2b_2u^3 + 3b_1u^2v + 6b_0uv^2 - 6hv^3 = 0.$$

From $U(0, 0) = 0$ and $\partial U/\partial v|_{(0,0)} = 3$, by the Implicit Function Theorem [2, Theorem 1.3.1] we obtain the solution of $U(x, y) = 0$ as follows

$$\begin{aligned} v(u) = &-\frac{2}{3}b_2u^3 + \frac{2}{3}b_1b_2u^5 - \frac{2}{9}(4b_0b_2^2 + 3b_1^2b_2)u^7 + \frac{2}{27}(36b_0b_1b_2^2 + 9b_1^3b_2 \\ &- 8b_2^3h)u^9 + o(u^9). \end{aligned}$$

Thus we have the invariant curves $\tilde{H}_1(u, v) = h$ at the origin of the local chart U_2. Drawing the orbits living on these invariant curves for different values of h we obtain Figure 6.



(a) $b_2 > 0$ (b) $b_2 < 0$

FIGURE 6. The local phase portraits at the origin of the local chart U_2 for the differential system (1) when $m = 2$.

Next we shall study the local phase portraits at the finite equilibrium points when $g(x)$ is a polynomial of degree 2. Consequently it can be written as one of the following three forms

$$
\begin{aligned}
&(\mathbf{S1}) : g(x) = b_2(x - r_1)(x - r_2), \quad r_1 < r_2 \\
&(\mathbf{S2}) : g(x) = b_2(x - r_0)^2, \\
&(\mathbf{S3}) : g(x) = b_2(x^2 - 2ax + a^2 + b^2).
\end{aligned}
$$

For convenience we use the notations

$$
\phi(y) = \frac{y^2}{2}, \quad \psi(x) = b_0 x + \frac{b_1}{2} x^2 + \frac{b_2}{3} x^3.
$$

Assume $b_2 > 0$. Let r be a general real zero of g. Then there are three possibilities: the zero r is either an inflection point, or a minimum, or a maximum of the function $\psi(x)$. We shall study the local phase portraits at the equilibrium point $(r, 0)$ for the differential system (1).

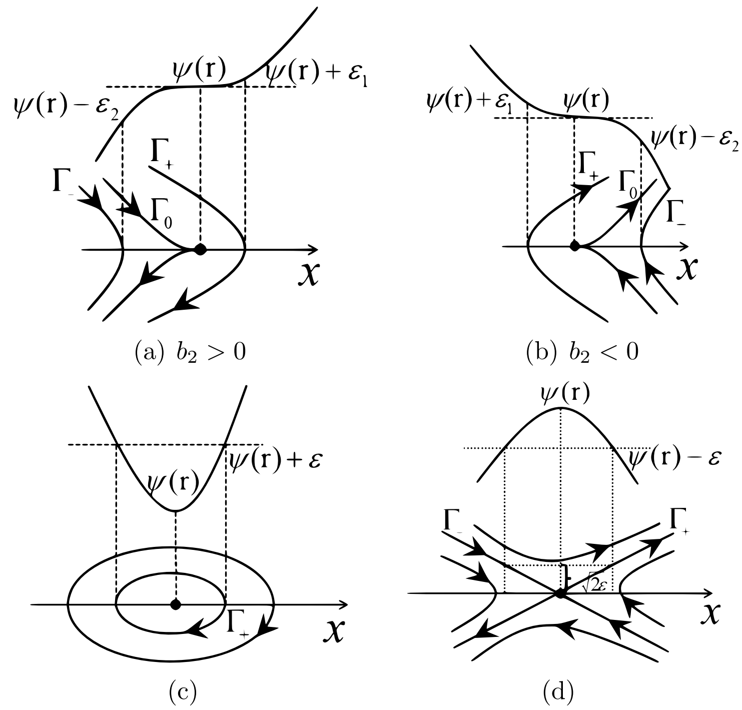


FIGURE 7. The local phase portraits at an inflection point in the subfigures (a) and (b), at a minimum in the subfigure (c), and at a maximum in the subfigure (d).

When the zero r is an inflection point of ψ, i.e. $\psi'(x)|_{x=r} = \psi''(x)|_{x=r} = 0$, we see that $b_1 = b_0 = 0$. Further we have that

$$H_1 = H_1(r, 0) = \psi(r) = \frac{b_2 r^3}{3}.$$

Consider the orbits contained in the curves $H_1 = b_2 r^3/3$. We only concentrate to the half plane $y > 0$ because of the symmetry. First we do the change of variables $x = u + r$ and $y = v$, and get

$$H_1(u, v) = \frac{b_2}{3} u^3 + \frac{v^2}{2} = 0. \tag{4}$$

Solving the equation when $v > 0$ we get

$$v = f(u) = \sqrt{\frac{2b_2}{3}}(-u)^{3/2}$$

as $u \leq 0$. Further we see

$$f'(u) = -\sqrt{\frac{-3b_2 u}{2}}.$$

It follows that $f'(u) = 0$ at $u = 0$ and $f'(u) < 0$ as $u < 0$. Thus we see that the orbit Γ_0 living on the curve $y = \sqrt{2b_2/3}(r - x)^{3/2}$ goes to the equilibrium point $(r, 0)$ tangent to the x-axis. For $\varepsilon_1, \varepsilon_2 > 0$, we consider the orbits living on the curves $H_1 = b_2 r_0^3/3 + \varepsilon_1$ and $H_1 = b_2 r_0^3/3 - \varepsilon_2$. Solving the former equation we consider the point $(x, 0)$ on the x-axis and get $x = x_+ > r$. For the latter equation we similarly get $x = x_- < r$. Starting the points $(x_\pm, 0)$ we respectively get the orbits Γ_+ and Γ_- because $\dot{x} = y > 0$ and $\dot{y} = -\psi'(x) < 0$. By the symmetry we obtain the local phase portrait of Figure 7(a).

When $b_2 < 0$ from the equation (4) we get

$$v = f(u) = \sqrt{\frac{-2b_2}{3}} u^{3/2}$$

as $u \geq 0$. Further we see

$$f'(u) = \sqrt{\frac{-3b_2 u}{2}}.$$

It follows that $f'(u) = 0$ at $u = 0$ and $f'(u) > 0$ as $u > 0$. As it is discussed in the case $b_2 > 0$ we obtain the local phase portrait Figure 7(b) because $\dot{x} = y > 0$ and $\dot{y} = -\psi'(x) > 0$.

When the zero r is a minimal point of ψ we consider the invariant curves $H_1 = \psi(r) + \varepsilon$ for $\varepsilon > 0$. Solving the solutions on the x-axis we get two zeros $x_\pm$ satisfying that $x_- < r < x_+$. Since $\dot{x} = y > 0$ and $\dot{y} = -\psi'(x) > 0$ in the interval (x_-, r) but $\dot{y} < 0$ in the interval

(r, x_+) we get the periodic orbit Γ_+ starting $(x_+, 0)$ or $(x_-, 0)$ by the symmetry. Thus we obtain a continuum of periodic orbits near the equilibrium point $(r, 0)$, see Figure 7(c).

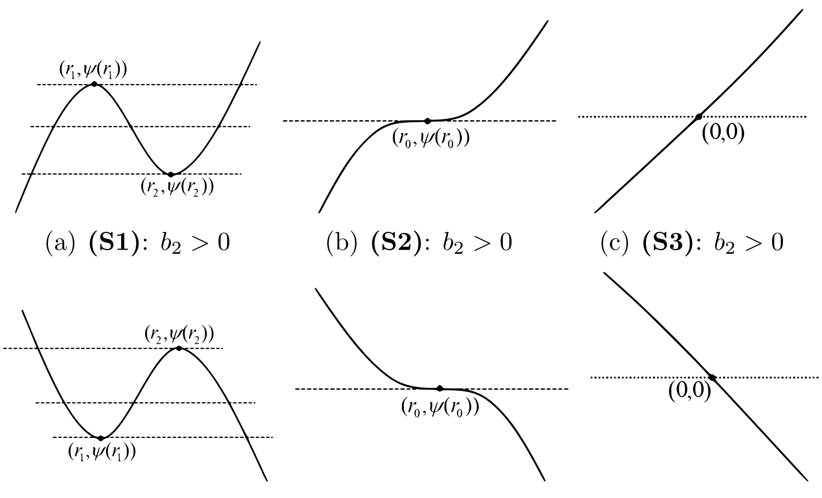
When the zero r is a maximal point of ψ we consider the orbits living on the curve $H_1(x, y) = \psi(r)$. First we solve the equations $\phi(y) = \varepsilon$ and $\psi(x) = \psi(r) - \varepsilon$ for $\varepsilon > 0$, and get $y = \sqrt{2\varepsilon}$ and $x = x_\pm$ satisfying that $x_- < r < x_+$ in the half plane $y \geq 0$. On the other hand we do the change of variables $x = u + r$ and $y = v$, and get

$$\frac{b_1}{2}u^2 + b_2 r u^2 + \frac{b_2}{3}u^3 + \frac{v^2}{2} = 0.$$

In the neighbourhood of the point $(r, 0)$ we have

$$v = \pm\sqrt{-b_1 - 2b_2 r}\,u + o(|u|),$$

since $\psi''(r) = b_1 + 2b_2 r < 0$. Thus we obtain the orbits Γ_- (respectively Γ_+) starting at $(x_-, \sqrt{2\varepsilon})$ (respectively $(x_+, \sqrt{2\varepsilon})$) tangent to the line $y = -\sqrt{-b_1 - 2b_2 r}(x - r)$ (respectively $y = \sqrt{-b_1 - 2b_2 r}(x - r)$), because $\dot{x} = y > 0$ and $\dot{y} = -\psi'(x) < 0$ as $x < r$, and $\dot{x} = y > 0$ and $\dot{y} = -\psi'(x) > 0$ as $x > r$. We obtain Figure 7(d) near the equilibrium point $(r, 0)$ by the symmetry.



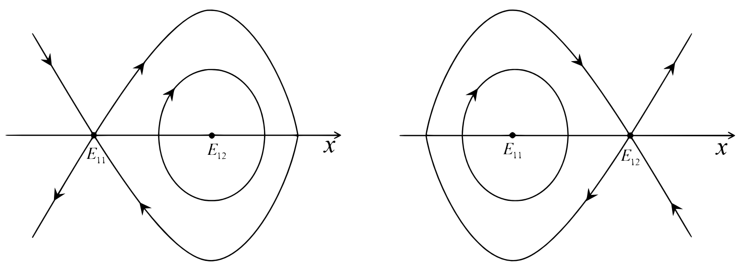
(a) **(S1)**: $b_2 > 0$ (b) **(S2)**: $b_2 > 0$ (c) **(S3)**: $b_2 > 0$

(d) **(S1)**: $b_2 < 0$ (e) **(S2)**: $b_2 < 0$ (f) **(S3)**: $b_2 < 0$

FIGURE 8. The graph of ψ in **(Sk)** for $k = 1, 2, 3$.

Assume **(S1)** with $b_2 > 0$, then

$$\psi(x) = \frac{b_2}{3}x^3 - \frac{b_2(r_1 + r_2)}{2}x^2 + b_2 r_1 r_2 x.$$



(a) $b_2 > 0$ (b) $b_2 < 0$

FIGURE 9. The local phase portraits at the equilibrium points E_{11} and E_{12} for the differential system (1) with $g(x) = b_2(x - r_1)(x - r_2)$.

Clearly the differential system (1) has exactly two equilibrium points $E_{11}(r_1, 0)$ and $E_{12}(r_2, 0)$. Moreover, ψ reaches the maximal value at $x = r_1$ and reaches the minimal value at $x = r_2$, see Figure 8(a). Thus by subfigures (c) and (d) in Figure 7 we get the local phase portrait of E_{11} and E_{12}, as shown in Figure 9(a). Assume $b_2 < 0$. Then ψ reaches the minimal value at $x = r_1$ and reaches the maximal value at $x = r_2$, see Figure 8(d). It follows that the local phase portrait of E_{11} and E_{12} is shown in Figure 9(b). Combining the orbits at infinity in Figure 5(b)(d) and Figure 6 we obtain the phase portraits of the differential system (1) with a general quadratic polynomial $g(x)$ in **(S1)**, see Figure 1(a)(d).

Assume **(S2)** then

$$\psi(x) = \frac{b_2 x(x^2 - 3r_0 x + 3r_0^2)}{3}.$$

The differential system (1) has exactly one equilibrium point $E_{21}(r_0, 0)$. Because r_0 is the inflection point of ψ displayed in Figure 8(b)(e) we see the local phase portraits near E_{21} as subfigures (a) and (b) of Figure 7. Combining the orbits at infinity we obtain the phase portraits of the differential system (1) with a general quadratic polynomial $g(x)$ in **(S2)**, see Figure 1(b)(e).

Assume **(S3)** with $b_2 > 0$. Clearly there is no equilibrium points for the differential system (1). On the other hand we see that $\dot{x} = y > 0$ and $\dot{y} = -\psi'(x) < 0$. By the orbits at infinity we obtain the phase portrait of the differential system (1) with a general quadratic polynomial $g(x)$ in **(S3)**, see Figure 1(c). Similarly when $b_2 < 0$ we obtain the phase portrait of the differential system (1) with a general quadratic polynomial $g(x)$ in **(S3)**, see Figure 1(f). This completes the proof of the theorem. □

Proof of Theorem 2. From the subfigures (a) and (c) in Figure 5 we get the orbits on the u-axis in the (u, v)-coordinates. On the other hand the origin of the local chart U_2 is an equilibrium point. So we shall study the local phase portraits at the origin of the local chart U_2.

Consider a generic cubic polynomial

$$g(x) = b_3x^3 + b_2x^2 + b_1x + b_0, \quad b_3 \neq 0.$$

Then the differential system (1) has a first integral $H_1 = H_1(x, y) = \phi(y) + \psi(y) = h$, where

$$\phi(y) = \frac{y^2}{2}, \qquad \psi(x) = \frac{b_3x^4}{4} + \frac{b_2x^3}{3} + \frac{b_1x^2}{2} + b_0x.$$

By the Poincaré transformation $x = u/v$ and $y = 1/v$ it becomes

$$\hat{H}_1 = \hat{H}_1(u, v) = \frac{12b_0uv^3 + 6b_1u^2v^2 + 4b_2u^3v + 3b_3u^4 + 6v^2}{12v^4} = h.$$

Varying $h \in \mathbb{R}$ we have the orbits living on the curves $\hat{H}_1 = h$ near the origin of the local chart U_2. Let

$$\begin{aligned} V(u, v) = 12v^4(\hat{H}_1 - h) =& 12b_0uv^3 + 6b_1u^2v^2 + 4b_2u^3v + 3b_3u^4 + 6v^2 \\ &- 12hv^4 = 0. \end{aligned}$$

By the Newton-Puiseux algorithm [5, 8] we solve $V(u, v) = 0$ as follows

$$\begin{aligned} v(u) =& \pm \frac{\sqrt{-b_3}u^2}{\sqrt{2}} - \frac{b_2}{3}u^3 \pm \frac{(9b_1b_3 + 2b_2^2)}{18\sqrt{-2b_3}}u^4 + \frac{3b_0b_3 + 2b_1b_2}{6}u^5 \\ &\pm \frac{(4b_2^4 + 324b_3^3h - 648b_0b_2b_3^2 - 243b_1^2b_3^2 - 108b_1b_2^2b_3)}{648\sqrt{2}(-b_3)^{3/2}}u^6 + o(u^6). \end{aligned}$$

Because $\dot{x} = y$ and $\dot{y} = -\psi'(x)$ we obtain the directions of those orbits, see Figure 10. When $b_3 > 0$ the local phase portrait at the origin of U_2 is formed by two hyperbolic sectors separated by the line of the infinity.

Next we shall study the local phase portraits at the finite equilibrium points when $g(x)$ is a polynomial of degree 3. First it can be written

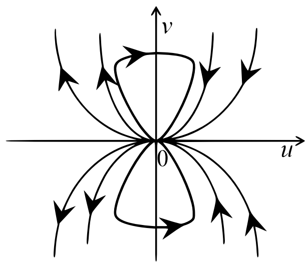


FIGURE 10. The local phase portrait of the origin of the local chart U_2 for the differential system (1) when $b_3 < 0$.

as one of the following five polynomials

$$
\begin{aligned}
&(\mathbf{C1}) : g(x) = b_3(x - r_1)(x - r_2)(x - r_3), \quad r_1 < r_2 < r_3\\
&(\mathbf{C2}) : g(x) = b_3(x - r_1)^2(x - r_2), \quad r_1 < r_2\\
&(\mathbf{C3}) : g(x) = b_3(x - r_1)(x - r_2)^2, \quad r_1 < r_2\\
&(\mathbf{C4}) : g(x) = b_3(x - r_1)^3, \text{ and}\\
&(\mathbf{C5}) : g(x) = b_3(x - r_1)(x^2 - 2\alpha x + \alpha^2 + \beta^2).
\end{aligned}
$$

Correspondingly we have the first integrals $H_1(x, y) = H_1(x, -y)$. Then the phase portraits of the differential system (1) are symmetric with respect to the x-axis.

Assume **(C1)** then

$$\psi(x) = \frac{b_3x(3x^3 - 4(r_1 + r_2 + r_3)x^2 + 6(r_1r_2 + r_1r_3 + r_2r_3)x - 12r_1r_2r_3)}{12}.$$

Clearly the differential system (1) has exactly three equilibrium points $E_{11}(r_1, 0)$, $E_{21}(r_2, 0)$ and $E_{31}(r_3, 0)$. When $b_3 > 0$ note that ψ reaches a minimal value at r_1 or r_3, and reaches a maximal value at r_2, see Figure 11(a). As it is surveyed for subfigures (c) and (d) in Figure 7, we obtain the local phase portrait at $E_{11}(r_1, 0)$, $E_{21}(r_2, 0)$ and $E_{31}(r_3, 0)$, see Figure 12(a). Considering any $h > \psi(r_2)$ we have two zeros on the x-axis and get a periodic orbit. When $b_3 < 0$ we have that ψ reaches a maximal value at r_1 or r_3, and reaches a minimal value at r_2, see Figure 11(f). We similarly obtain the local phase portrait at E_{11}, E_{12} and E_{13}, see Figure 12(b).

Assume **(C2)** then there are two equilibrium points $E_{21}(r_1, 0)$ and $E_{22}(r_2, 0)$. On the other hand

$$\psi(x) = \frac{b_3x(3x^3 - 4(2r_1 + r_2)x^2 + 6r_1(r_1 + 2r_2)x - 12r_1^2r_2)}{12}.$$

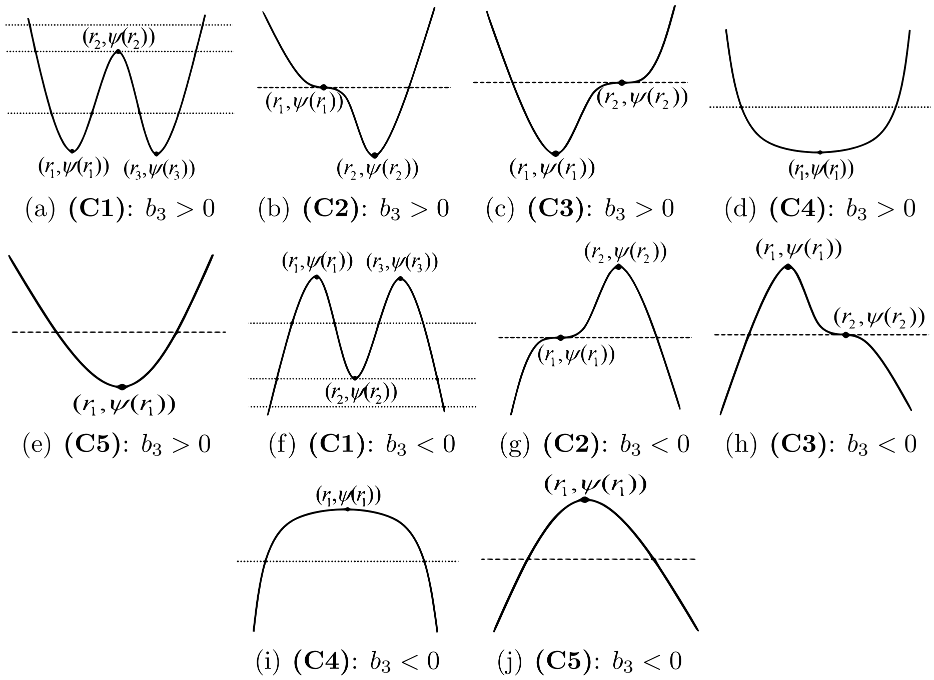


FIGURE 11. The graph of ψ in **(Ck)** for $k = 1, \cdots, 5$.

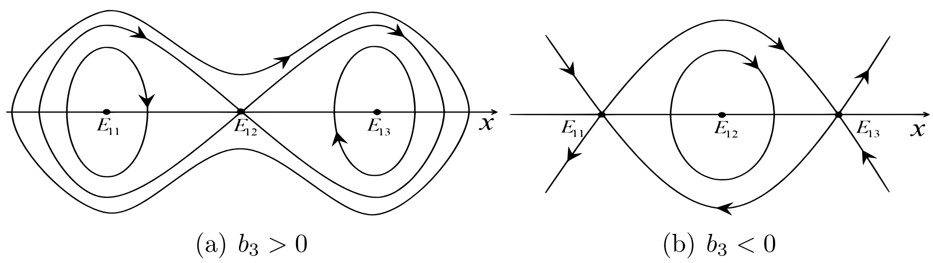
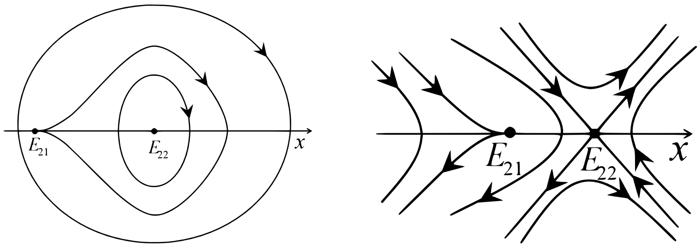


FIGURE 12. The local phase portraits at the equilibrium points $E_{1i}, i = 1, 2, 3$, for the differential system (1) in **(C1)**.

and its graph is shown in subfigures (b) and (g) of Figure 11. As it is done for subfigures (a), (b), (c) and (d) of Figure 7, we obtain the local phase portraits at E_{21} and E_{22}, see Figure 13.

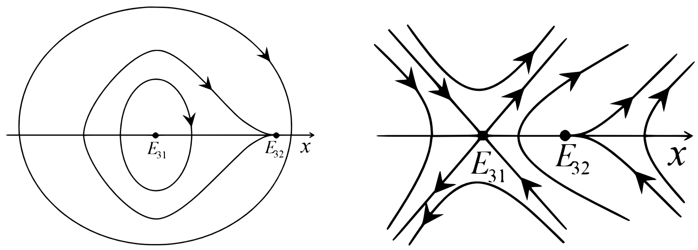
Assume **(C3)** then

$$\psi(x) = \frac{b_3 x(3x^3 - 4(r_1 + 2r_2)x^2 + 6r_2(2r_1 + r_2)x - 12r_1 r_2^2)}{12},$$



(a) $b_3 > 0$ (b) $b_3 < 0$

FIGURE 13. The local phase portraits at the equilibrium points $E_{1i}, i = 1, 2$, for the differential system (1) in **(C2)**.



(a) $b_3 > 0$ (b) $b_3 < 0$

FIGURE 14. The local phase portraits at the equilibrium points $E_{1i}, i = 1, 2, 3$ for the differential system (1) in **(C3)**.

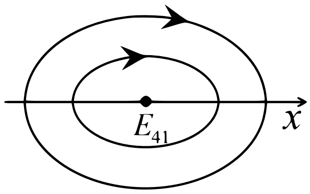
and its graph is shown in subfigures (c) and (h) of Figure 11. Similar to case **(C2)** we obtain the local phase portraits near E_{21} and E_{22}, see Figure 14.

Assume **(C4)** then

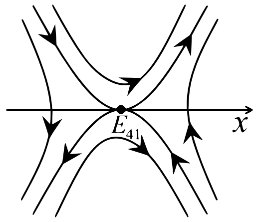
$$\psi(x) = \frac{y^2}{2} + \frac{b_3(r_1 - x)^4}{4}.$$

The differential system (1) has exactly one equilibrium point $E_{41}(r_1, 0)$. Note that ψ reaches a minimal value at $x = r_1$, which is displayed in the subfigure (d) of Figure 11. Then there is a continuum of periodic orbits near E_{41} when $b_3 > 0$, see Figure 15(a). When $b_3 < 0$ because ψ reaches a maximal value at $x = r_1$ shown in Figure 11(i) we see the trend of orbits near E_{41}. On the other hand we have $H_1 = \phi(y) + \psi(x) = 0$ by taking $x = r_1$ and $y = 0$. Thus solving the equation $H_1 = 0$ we have

$$y = f(x) = \sqrt{\frac{-b3}{2}}(r_1 - x)^2.$$



(a) $b_3 > 0$



(b) $b_3 < 0$

FIGURE 15. The local phase portraits at the equilibrium point E_{41} for the differential system (1) in **(C4)**.

which is a parabolic curve at E_{41}. Then we obtain the local phase portrait at E_{41}, see Figure 15(b).

Assume **(C5)** then

$$\psi(x) = \frac{b_3 x(3x^3 - 4(2a + r_1)x^2 + 6(a^2 + b^2 + 2ar_1)x - 12(a^2 + b^2)r_1)}{12}$$

whose graph is shown in subfigure (e) of Figure 11 when $b_3 > 0$ and in subfigure (j) of Figure 11 when $b_3 < 0$. The differential system (1) has exactly one equilibrium point $E_{51}(r_1, 0)$. Similar to the subfigure (c) of Figure 7 we get a continuum of periodic orbits near E_{51} when $b_3 > 0$. When $b_3 < 0$ we obtain the local phase portrait near E_{51} as done in Figure 7(d).

In summary when $m = 3$ we obtain the phase portraits of the Liénard system (1) in the Poincaré disc by combining the orbits at infinity in Figure 10, see Figure 2 □

Proof of Theorem 3. The differential system (2) has the straight line $y = 0$ filled up with finite equilibrium points. As it was mentioned in (3), we see that system (2) is orbitally equivalent to system (3). We shall study the orbits at infinity for the differential system (2) using the Poincaré compactification.

First in the local chart U_1 the differential system (2) becomes

$$\dot{u} = -u^2 v^n - uF_n(v), \qquad \dot{v} = -uv^{n+1},$$

and in the local chart U_2 it becomes

$$\dot{u} = v^n + u\Psi_n(u, v), \qquad \dot{v} = v\Psi_n(u, v),$$

where

$$F_n(v)=\sum_{i=0}^{n} a_i v^{n-i}, \qquad \Psi_n(u,v)=\sum_{i=0}^{n} a_i u^i v^{n-i}.$$

It follows that the u-axis are filled up with equilibrium points in the local chart U_1 and the origin of the local chart U_2 is an equilibrium point.

On the other hand for the differential system (2) with a general quadratic polynomial $f(x)=a_2x^2+a_1x+a_0, a_2\neq 0$, we obtain a first integral

$$H_2=H_2(x,y)=y+\frac{a_2}{3}x^3+\frac{a_1}{2}x^2+a_0x.$$

By the Poincaré transformation $x=1/v, y=u/v$ the first integral becomes

$$\hat{H}_2=\hat{H}_2(u,v)=\frac{6a_0v^2+3a_1v+2a_2+6uv^2}{6v^3},$$

and by the Poincaré transformation $x=u/v, y=1/v$ it becomes

$$\tilde{H}_2=\tilde{H}_2(u,v)=\frac{6v^2+2a_2u^3+3a_1u^2v+6a_0uv^2}{6v^3}.$$

Considering $\hat{H}_2=h$, where $h\in\mathbb{R}$, we see that

$$W(u,v)=6v^3(\hat{H}_2-h)=6a_0v^2+3a_1v+2a_2-6hv^3+6uv^2=0.$$

Since $a_2\neq 0$ we have $W(u,0)=a_2\neq 0$, which follows that at each one of these equilibrium points on the u-axis there are two hyperbolic sectors separated by the u-axis.

Let $\tilde{H}_2=h$ varying $h\in\mathbb{R}$. Then

$$U_1(x,y)=6v^3(\tilde{H}_2(u,v)-h)=6v^2+2a_2u^3+3a_1u^2v+6a_0uv^2-6hv^3=0.$$

By the Newton-Puiseux algorithm [5, 8] we solve two explicit solutions of $U_1(x,y)=0$ as follows

$$\begin{aligned}v(u)=&-\frac{\sqrt{-a_2}}{\sqrt{3}}u^{3/2}-\frac{a_1}{4}u^2+\Big(\frac{a_0\sqrt{-a_2}}{2\sqrt{3}}+\frac{\sqrt{3}a_1^2\sqrt{-a_2}}{32a_2}\Big)u^{5/2}\\&+\Big(\frac{a_0a_1}{4}-\frac{a_2h}{6}\Big)u^3+o(u^3) \qquad \text{for } x>0,\\ v(u)=&\frac{\sqrt{a_2}}{\sqrt{3}}(-u)^{3/2}-\frac{a_1}{4}u^2-\Big(\frac{a_0\sqrt{a_2}}{2\sqrt{3}}+\frac{\sqrt{3}a_1^2\sqrt{a_2}}{32a_2}\Big)(-u)^{5/2}\\&+\Big(\frac{a_0a_1}{4}-\frac{a_2h}{6}\Big)u^3+o(u^3) \qquad \text{for } x<0.\end{aligned}$$

Thus we obtain the invariant curves $\tilde{H}_2(u,v) = h$ at the origin of the local chart U_2. Drawing these invariant curves for different values of h we obtain the local phase portraits at the origin of the local chart U_2 as Figure 6.

In what follows we discuss the orbits living on the invariant curves $H_2 = h$ for the differential system (3). Similarly we consider a general quadratic polynomial in the three cases

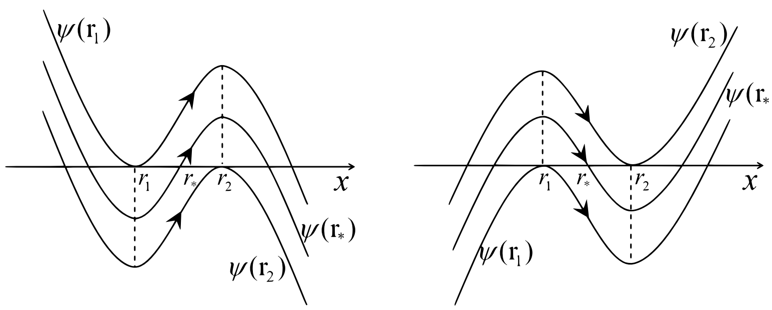
$$\begin{aligned}
&(\mathbf{Q1}) : f(x) = a_2(x - r_1)(x - r_2), \quad r_1 < r_2 \\
&(\mathbf{Q2}) : f(x) = a_2(x - r_0)^2, \\
&(\mathbf{Q3}) : f(x) = a_2(x^2 - 2ax + a^2 + b^2).
\end{aligned}$$

In the case that $a_2 > 2$ in **(Q1)** we see $H_2 = H_2(x,y) = y + \psi_1(x)$, where

$$\psi_1(x) = \frac{a_2}{6} x \left(2x^2 - 3(r_1 + r_2)x + 6r_1r_2\right).$$

Then we have the graph of ψ_1, which is displayed in Figure 8(a). On the other hand we obtain the invariant curves $y = -\psi_1(x) + h$. Note that $x' = 1$ and $y' = -\psi_1'(x) = -a_2(x - r_1)(x - r_2)$. Choosing $h = \psi_1(r_1), \psi_1(r_*)$ and $\psi_1(r_2)$, where $r_1 < r_* < r_2$, we get three invariant curves, which also are three orbits, see Figure 16. Bringing back to the differential system (2) we obtain the phase portrait, see subfigure (a) in Figure 3.

In a similar way we have subfigure (d) of Figure 3 for the case that $a_2 < 0$ in **(Q1)** and obtain the phase portraits for the differential system (2) in **(Q2)** and **(Q3)**, see the remain subfigures (b), (c), (e) and (f) in Figure 3. □



(a) $a_2 > 0$ (b) $a_2 < 0$

FIGURE 16. The local phase portraits for the differential system (3) in **(Q1)**.

Proof of Theorem 4. We have known that all points on the u-axis are equilibrium point in the local chart U_1 and the origin of the local chart U_2 is also an equilibrium point. Now we study the local phase portraits at each these equilibrium points.

Consider a generic cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0, a_3 \neq 0$. Then the differential system (3) has a first integral $H_2 = H_2(x, y) = y + \psi_1(y) = h$, where

$$\psi_1(x) = \frac{a_3x^4}{4} + \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x.$$

By the Poincaré transformation $x = 1/v$ and $y = u/v$, the first integral becomes

$$\hat{H}_2^* = \hat{H}_2^*(u, v) = \frac{12a_0v^3 + 6a_1v^2 + 4a_2v + 3a_3 + 12uv^3}{12v^4},$$

and by the Poincaré transformation $x = u/v$ and $y = 1/v$ it becomes

$$\tilde{H}_2^* = \tilde{H}_2^*(u, v) = \frac{12a_0uv^3 + 6a_1u^2v^2 + 4a_2u^3v + 3a_3u^4 + 12v^3}{12v^4}.$$

As it is done in the case that f is a quadratic polynomial, we have that there are two hyperbolic sectors at each one of these equilibrium point on the u-axis because $a_3 \neq 0$.

In order to clarify how the orbits connect to the origin of the local chart U_2, we vary $h \in \mathbb{R}$ to get the orbits living on the curves $\tilde{H}_2^* = h$. Let

$$\begin{aligned} V_1(u, v) &= 12v^4(\tilde{H}_2^* - h) \\ &= 12a_0uv^3 + 6a_1u^2v^2 + 4a_2u^3v + 3a_3u^4 + 12v^3 - 12hv^4 = 0. \end{aligned}$$

From the equation we obtain a explicit solution in a form of rational series

$$\begin{aligned} v(u) = &- \frac{a_3^{1/3}}{\sqrt[3]{4}}u^{4/3} + \frac{\sqrt[3]{4}a_2}{9a_3^{1/3}}u^{5/3} - \frac{a_1}{6}u^2 + \frac{729a_0a_3^2 + 162a_1a_2a_3 - 16a_2^3}{2187\sqrt[3]{4}a_3^{5/3}}u^{7/3} \\ &- \frac{5832a_0a_2a_3^2 + 2187a_1^2a_3^2 - 648a_1a_2^2a_3 + 64a_2^4 - 6561a_3^3h}{39366\sqrt[3]{2}a_3^{7/3}}u^{8/3} \\ &+ o(u^{8/3}). \end{aligned}$$

Since $\dot{x} = y$ and $\dot{y} = -\psi_1'(x)$ we obtain the directions of those orbits, see Figure 10.

For the local phase portraits at those equilibrium point we can easily obtain the phase portraits in the following five cases

$$(\mathbf{G1}): f(x) = a_3(x-r_1)(x-r_2)(x-r_3), \quad r_1 < r_2 < r_3$$
$$(\mathbf{G2}): f(x) = a_3(x-r_1)^2(x-r_2), \quad r_1 < r_2$$
$$(\mathbf{G3}): f(x) = a_3(x-r_1)(x-r_2)^2, \quad r_1 < r_2$$
$$(\mathbf{G4}): f(x) = a_3(x-r_1)^3,$$
$$(\mathbf{G5}): f(x) = a_3(x-r_1)(x^2-2\alpha x+\alpha^2+\beta^2).$$

The invariant curves $y = -\psi_1(x) + h$ are the orbits of the differential system (3) as displayed in Figure 11. Going back to the differential system (2), we obtain the phase portraits of Figure 4. □

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