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ON THE INTEGRABILITY OF A FOUR-PROTOTYPE RÖSLER SYSTEM[1]

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Abstract. We consider a four-prototype Rossler system introduced by Otto Rösler among others as prototypes of the simplest autonomous differential equations (in the sense of minimal dimension, minimal number of parameters, minimal number of non-linear terms) having chaotic behavior. We contribute towards the understanding of its chaotic behavior by studying its integrability from different points of view. We show that it is neither Darboux integrable, nor C^1-integrable.

1. Introduction and statement of the main results

The question whether a differential system admits a first integral (see section 2 for its precise definition) is of fundamental nature, first because the first integrals provide conservation laws for the system that enables to lower its dimension (once restricted to a precise value of it), and second because knowing a sufficient number of them allows to determine its orbits (solving it), at least in an explicit functional form because to compute the intersection of the different constant hypersurfaces defined by the first integrals is in general difficult to do. The distinction between integrable and nonintegrable systems has the qualitative implication of regular motion versus chaotic motion and is an intrinsic property of the system, is not a matter of whether a system can be integrated or not. However, the study of the existence or non-existence of first integrals is in general a difficult problem.

In this paper we deal with the four-prototype Rossler system given by the following non-linear differential system

$$(1) \qquad \dot{x} = -y - z, \quad \dot{y} = x, \quad \dot{z} = \alpha y(1-y) - \beta z,$$

where $\alpha, \beta \in \mathbb{R}$. This system was proposed and studied by Otto Rössler among others as prototypes of the simplest autonomous differential equations having chaotic behavior. More precisely, in the Rössler prototypes the chaos is minimal mainly due to three reasons: their nonlinearity terms are minimal because they consist in a unique

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quadratic term. In Rössler systems a chaotic attractor is generated with only one quadratic monomial, in contrast with the Lorenz system in which it is generated by two quadratic monomials. As in Lorenz system the phase-space of the Rössler systems has the minimal dimension three. For these three reasons the Rössler prototypes are paradigmatic problems to start with if one wants to understand the born/die of chaos in dynamical systems. Therefore, these models have been intensively studied by several authors (in the moment that this paper has been written there were more than 170 articles in MathSciNet related with these Rössler's prototypes) mainly with numerical studies on the presence of attractors, but not much studies are done regarding analytical proofs for the presence/absence of chaos. This is the main motivation for this paper and we focus on the study of their regular behaviour by studying their integrability.

More precisely, there are chaotic systems as for instance the Lorenz system, that for some values of the parameters exhibit chaos and for other values of the parameters have first integrals (see [8]), or it is completely integrable (see [5]). The objective of this paper is to see if the Rössler system here studied also present or not a similar behaviour as the Lorenz system which can exhibit chaos and integrability for different values of the parameters. The result will be that this Rössler system is more chaotic than the Lorenz system in the sense that it is neiher Darboux integrable, nor C^1 integrable for any values of the parameters, except for the trivial value $\alpha = 0$, but for this value the Rössler system becomes a linear differential system.

The differential system (1) was introduced in [15] (see also [16]) where it is proved that this system is chaotic with the presence of an strange attractor when the parameters take the values close to $\alpha = \beta = 1/2$. Moreover in [4, 13] the authors proved the existence of periodic orbits and study their stability and instability. Similar results concerning the existence/non-existence of first integrals for other Rössler systems are [6, 9]. More precisely, in these two last papers the Rössler system studied was $\dot{x} = -y-z$, $\dot{y} = x+ay$ and $\dot{z} = b-cz+xz$, in [6] the authors studied the formal and the analytic integrability of that system, while in [9] the authors studied its Darboux integrability. The definitions and basic results on the Darboux theory of integrability are the same in the paper [11] than in the present paper, but the proofs for studying the existence or non-existence of Darboux first integrals are completely different in both papers.

Note that when $\alpha = 0$ system (1) is linear and has the independent first integrals

$$H_1 = z, \quad H_2 = x^2 + y^2 + 2yz$$

when $\beta = 0$, and the independent first integrals

$$\begin{aligned} H_1 &= \left(x - \frac{\beta z}{1+\beta^2}\right)\cos\left(\frac{\log z}{\beta}\right) - \left(y + \frac{z}{1+\beta^2}\right)\sin\left(\frac{\log z}{\beta}\right), \\ H_2 &= \left(y + \frac{z}{1+\beta^2}\right)\cos\left(\frac{\log z}{\beta}\right) + \left(x - \frac{\beta z}{1+\beta^2}\right)\sin\left(\frac{\log z}{\beta}\right), \end{aligned}$$

when $\beta \neq 0$.

Since in both cases H_1 and H_2 are functionally independent system (1) when $\alpha = 0$ is completely integrable (see section 2 for the precise definitions). More precisely, with a (complex) linear combination of H_1 and H_2 we can build two independent Darbouxian first integrals. So, from now on we will restrict our study when $\alpha \neq 0$. Since in this case, as pointed out before it has been showed in [16] that the differential system (1) defines a dynamical continuous system with presence of chaos, the existence or not of first integrals as well as invariant algebraic surfaces is of great importance to contribute towards its non-chaotic behavior. The main theorem of this paper is the following one.

See the definitions of invariant algebraic surface, exponential factors and Darboux first integrals in section 2.

Theorem 1. *The following statements hold for system* (1) *with* $\alpha \neq 0$*:*

(i) *It has no invariant algebraic surfaces.*
(ii) *The unique exponential factors are of the form* $F = e^{\alpha_0+\alpha_1 x+\alpha_2 y}$ *with* $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$.
(iii) *It has no Darboux first integrals.*

Note that (see section 2) the non-existence of Darboux first integrals prevent the existence of either polynomial first integrals, or rational first integrals.

So the rest of the paper is as follows: we have section 2 where the notions needed for proving Theorem 1 are provided. In section 3 we give the proof of Theorem 1. Finally in section 4 we provide some comments about the C^1 first integrability of system (1).

2. Preliminary results

Let $U \subset \mathbb{R}^3$ be an open subset. We say that the C^1 non-constant function $H\colon U \to \mathbb{R}$ is a C^1*-first integral* of system (1) if $H(x(t), y(t), z(t))$ is constant for all values of t for which the solution $(x(t), y(t), z(t))$ is defined on U. When H is a polynomial we say that H is a *polynomial first integral*, when it is a rational function we say that it is a *rational first integral*, and when it is a Darboux function (see below) we say that it is a *Darboux first integral*. We will say that system (1) is *completely integrable* if it admits two functionally independent first integrals. Since each level surface of a first integral is invariant under the flow induced by the system, it is clear that if this system is completely integrable then the intersection of its two first integrals determines an invariant curve, which is formed by the orbits of the system.

Let $h\colon \mathbb{C}^3 \to \mathbb{C}$ be a nonconstant polynomial. We say that $h(x, y, z) = 0$ is *an invariant algebraic surface* of system (1) if it satisfies

$$-(y+z)\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} + (\alpha y(1-y) - \beta z)\frac{\partial h}{\partial z} = Kh \tag{2}$$

for some polynomial $K \in \mathbb{C}[x, y, z]$ called the *cofactor* of the invariant algebraic surface $h = 0$. Note that K has degree at most one and that an invariant algebraic surface with zero cofactor determines a polynomial first integral. For more information on the invariant algebraic surfaces see [2, 11, 12].

On the other hand we say that a nonconstant function $F = e^{g/h}$ where $g, h \in \mathbb{C}[x, y, z]$ are coprime polynomials it is an *exponential factor* of system (1) if it satisfies

$$-(y+z)\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y} + (\alpha y(1-y) - \beta z)\frac{\partial F}{\partial z} = LF$$

for some polynomial $L \in \mathbb{C}[x, y, z]$ of the degree at most one, called the *cofactor* of the exponential factor F. For a geometric and algebraic meaning of the exponential factor we refer the reader to [1]. It is well-known that $h = 0$ (whenever it is non-constant) is an invariant algebraic surface of the system (for a proof see for instance [1]). In the case in which h is constant then e^g can be an exponential factor which comes from the multiplicity of the infinity. For more information on exponential factors see again [2, 10, 11, 12].

A first integral of the form

$$G = f_1^{\lambda_1} \cdots f_k^{\lambda_k} [\exp[g_1/h_1]]^{\mu_1} \cdots [\exp[g_s/h_s]]^{\mu_s}$$

where $f_j = 0$ are invariant algebraic surfaces, $\exp[g_j/h_j]$ are exponential factors, and the constants $\lambda_1, \cdots, \lambda_k, \mu_1, \cdots \mu_s \in \mathbb{C}$ is called a *Darboux first integral.* For more information on Darboux first integrals see again [2].

3. Proof of Theorem 1

3.1. **Proof of statement** (i)**.** We change the variable z by the new variable u taking $u = z - \beta x$. Then the differential system (1) becomes

$$\dot{x} = -\beta x - y - u, \qquad \dot{y} = x, \qquad \dot{u} = (\alpha + \beta)y - \alpha y^2. \tag{3}$$

Since the differential system (3) is of degree two if it has an invariant surface $F = F(x, y, u) = 0$ it must satisfy

$$-(\beta x + y + u)\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y} + ((\alpha+\beta)y - \alpha y^2)\frac{\partial F}{\partial u} = (k_0 + k_1 x + k_2 y + k_3 u)F \tag{4}$$

Assume that the polynomial F has degree n. We denote by F_k the homogeneous part of the polynomial F of degree k. From (4) we obtain that F_n must satisfy

$$-\alpha y^2 \frac{\partial F_n}{\partial u} = (k_1 x + k_2 y + k_3 u)F_n.$$

Solving this partial differential equation we have that

$$F_n = f(x, y) e^{\frac{-(k_1 x + 2k_2 y + k_3 u)u}{2\alpha y^2}}.$$

Since F_n must be a homogeneous polynomial of degree n it follows that $k_1 = k_2 = k_3 = 0$. Then

$$F_n = \sum_{j=0}^{n} a_j x^{n-j} y^j.$$

From (4) the homogeneous terms of degree n satisfy

$$\sum_{j=0}^{n}(n-j)a_j x^{n-j-1}y^j + x\sum_{j=0}^{n} j a_j x^{n-j}y^{j-1} + k_0\sum_{j=0}^{n} a_j x^{n-j}y^j.$$

Solving this partial differential equations we get that

$$F_{n-1} = -\frac{u^2}{2\alpha}\sum_{j=0}^{n}(n-j)a_j x^{n-j-1}y^{j-2} + O(u),$$

where $O(u)$ is a polynomial in the variable u of degree at most one. Since F_{n-1} must be a homogeneous polynomial of degree $n-1$ we have that $a_0 = a_1 = 0$.

Now we shall prove by induction over k that assuming $a_0 = a_1 = \ldots = a_{2k-1} = 0$ and

$$F_{n-k} = \frac{(-1)^k u^{2k}}{\alpha^k \prod_{\ell=1}^{k} 2\ell}\sum_{j=2k}^{n} a_j x^{n-j-k}y^{j-2k}\prod_{\ell=0}^{k-1}(n-j-\ell) + O(u^{2k-1}),$$

then $a_{2k} = a_{2k+1} = 0$ and

$$F_{n-k-1} = \frac{(-1)^{k+1} u^{2k+2}}{\alpha^{k+1} \prod_{\ell=1}^{k+1} 2\ell}\sum_{j=2k+2}^{n} a_j x^{n-j-k-1}y^{j-2k-2}\prod_{\ell=0}^{k}(n-j-\ell) + O(u^{2k+1}).$$

Now from (4) the homogeneous terms of degree $n-k$ satisfy

$$-\alpha y^2\frac{\partial F_{n-k-1}}{\partial u} = (\beta x + y + u)\frac{\partial F_{n-k}}{\partial x} + x\frac{\partial F_{n-k}}{\partial y} + (\alpha+\beta)y\frac{\partial F_{n-k}}{\partial u} + k_0 F_{n-k}. \tag{5}$$

By induction assumption we note that

$$(\beta x + y)\frac{\partial F_{n-k}}{\partial x} + x\frac{\partial F_{n-k}}{\partial y} + (\alpha+\beta)y\frac{\partial F_{n-k}}{\partial u} + k_0 F_{n-k} = O(u^{2k}),$$

and that

$$u\frac{\partial F_{n-k}}{\partial x} = \frac{(-1)^k u^{2k+1}}{\alpha^k \prod_{\ell=1}^{k} 2\ell}\sum_{j=2k}^{n}(n-j-k)a_j x^{n-j-k-1}y^{j-2k}\prod_{\ell=0}^{k-1}(n-j-\ell) + O(u^{2k}).$$

From (5) we must solve the differeential equation

$$-\alpha y^2\frac{\partial F_{n-k-1}}{\partial u} = \frac{(-1)^k u^{2k+1}}{\alpha^k \prod_{\ell=1}^{k} 2\ell}\sum_{j=2k}^{n} a_j x^{n-j-k-1}y^{j-2k}\prod_{\ell=0}^{k}(n-j-\ell) + O(u^{2k}).$$

Its solutions is solution is

$$F_{n-k-1} = \frac{(-1)^{k+1} u^{2k+2}}{\alpha^{k+1} \prod_{\ell=1}^{k+1} 2\ell}\sum_{j=2k}^{n} a_j x^{n-j-k-1}y^{j-2k-2}\prod_{\ell=0}^{k}(n-j-\ell) + O(u^{2k+1}).$$

Since F_{n-k-1} must be a homogeneous polynomial of degree $n-k-1$, it follows that $a_{2k} = a_{2k+1} = 0$. So the induction process is proved. Hence we obtain that $a_0 = a_1 = \ldots = a_n = 0$, consequently $F_n = 0$, a contradiction with the fact that there is an invariant surface $F = 0$ of degree n.

3.2. **Proof of statement** (ii)**.** In view of statement (i) of Theorem 1, any exponential factor of system (1) is of the form e^g with $g \in \mathbb{C}[x, y, z]$. If g is constant then statement (ii) holds. Assume that g is not constant. Clearly $F = e^{\alpha_0+\alpha_1 x+\alpha_2 y}$ with $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ is an exponential factor with cofactor $-\alpha_1(y + z) + \alpha_2 x$. Taking into account that computing the cofactor of an exponential factor is a logarithmic derivation and so an additive morphism, and that $e^{\alpha_0+\alpha_1 x+\alpha_2 y}$ is an exponential factor, any other exponential factor $F_1 = e^g$ with g a polynomial in $\mathbb{C}[x, y, z]$ different from F can be taken with cofactor of the form $K = \beta_0 + \beta_1 z$ with $\beta_0, \beta_1 \in \mathbb{C}$ and so it satisfies

$$-(y+z)\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} + (\alpha y(1-y) - \beta z)\frac{\partial g}{\partial z} = \beta_0 + \beta_1 z. \tag{6}$$

We will see that there are no exponential factors of such form F_1.

We consider two different cases.

Case 1*:* $\beta \neq -\alpha$. In this case system (1) has two finite singular points which are

$$p_1 = (0,0,0) \quad \text{and} \quad p_2 = \Big(0, \frac{\alpha+\beta}{\alpha}, -\frac{\alpha+\beta}{\alpha}\Big).$$

Evaluating (6) on p_1 we get that $\beta_0 = 0$ and evaluating (6) on p_2 we get that $\beta_1 = 0$. Therefore it follows from (6) that g is either a constant or a polynomial first integral. Since none of the two are possible, we reach to a contradiction.

Case 2*:* $\beta = -\alpha$. In this case the unique finite singular point of system (1) is the origin. Evaluating (6) on $x = y = z = 0$ we get that $\beta_0 = 0$. On the other hand, evaluating it on $x = 0, y = -z$ we obtain

$$-\alpha z^2 \frac{\partial g}{\partial z}\Big|_{x=0,y=-z} = \beta_1 z,$$

which yields that $\beta_1 = 0$. Therefore it follows from (6) that g is either a constant or a polynomial first integral. Since none of the two are possible, we reach to a contradiction, and the proof of statement (ii) of Theorem 1 is complete.

3.3. **Proof of statement** (iii)**.** Using the definition of Darboux first integral provided in section 2 it follows from statements (i) and (ii) that the unique Darboux first integrals can be of the form

$$G = e^{\alpha_0+\alpha_1 x+\alpha_2 y} \quad \text{with cofactor} \quad -\alpha_1(y+z) + \alpha_2 x.$$

Since such a cofactor must be zero we get that $\alpha_2 = \alpha_1 = 0$, but then G is a constant which is not possible. In short there are no Darboux first integrals and the proof of the theorem is completed.

4. Comments about the C^1-integrability of system (1)

In this section we study the non-existence of the global C^1 first integrals using the fact that the birth of convenient isolated periodic orbits can be regarded as an obstacle to C^1-integrability, see below for details.

The main result concerning the C^1-integrability of system (1).

Theorem 2. *The following hold for the differential system* (1).

(a) *Assume* $\alpha(\alpha + 2\beta) > 0$. *Consider the new parameters* (a, b) *defined as follows* $\alpha = \varepsilon a$ *and* $\beta = -\varepsilon(b + a/(1 + \varepsilon^2 b^2))$. *Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there exists a periodic solution* γ_ε *of system* (1) *tending to* $\{z = -1/2\} \cap \{x^2 + (y + z)^2 = -(a + 2b)/(2a)\}$ *as* $\varepsilon \to 0$ *which is asymptotically stable when* $a + b < 0$ *and unstable when* $a + b > 0$. *Moreover, system* (1) *has no* C^1 *first integrals* $H(x, y, z)$ *defined in a neighborhood of the periodic orbit* γ_ε *satisfying that* $\nabla H(x, y, z)$ *and* $(-y - z, x, \alpha y(1 - y) - \beta z)$ *are linearly independent on the points of* γ_ε.

(b) *Assume* $\alpha > 2\beta^3$. *Consider the new parameters* (a, ε) *defined as follows* $\alpha = \varepsilon(2a - 1 + a^2\varepsilon^2)$ *and* $\beta = \varepsilon(1 - a)$. *Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and* $a > 1/2$ *there exists a periodic solution* γ_ε *of* (1) *tending to* $\{z = -1/2\} \cap \{x^2 + (y + z)^2 = 1/(2(2a - 1))\}$ *as* $\varepsilon \to 0$ *which is asymptotically stable when* $1/2 < a < 1$ *and unstable when* $a > 1$. *Moreover, system* (1) *has no* C^1 *first integrals* $H(x, y, z)$ *defined in a neighborhood of the periodic orbit* γ_ε *satisfying that* $\nabla H(x, y, z)$ *and* $(-y - z, x, \alpha y(1 - y) - \beta z)$ *are linearly independent on the points of* γ_ε.

The part of Theorem 2 regarding the existence and stability of periodic orbits is taken verbatim from the two main results in [4] and the conclusion of the theorem follows then from the well-known result (which goes back to Poincaré, see [14] and whose proof can be found in [7]) which states that if system (1) has a periodic orbit γ having only one multiplier equal to one, then it has no C^1 first integrals $H(X)$ defined in a neighborhood of γ such that $\nabla H(X)$ and $f(X)$ are linearly independent on the points $X \in \gamma$.

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Declaration section

Not applicable.

Conflict of interest

Not applicable.

Data availability

Not applicable.

References

[1] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. Math. **229** (2007), 63–117.

[2] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, UniversiText, Springer-Verlag, New York, 2006.

[3] S. D. Furta, *On non–integrability of general systems of differential equations*, Z. angew Math. Phys. **47** (1996), 112–131.

[4] I. Garcia, J. Llibre and S. Maza, *Periodic orbits and their stability in the Rössler prototype*-4 *system*, Physics Letters A, **376** (2012), 2234–2237.

[5] J. Llibre and C. Valls, *Formal and analytic integrability of the Lorenz system*, J. Phys. A: Math. Gen. **38** (2005) 2681–2686.

[6] J. Llibre and C. Valls, *Formal and analytic integrability of the Rössler system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **17** (2007), 3289–3293.

[7] J. Llibre and C. Valls, *On the* C^1 *non-integrability of differential systems via periodic orbits*, European J. Appl. Math. **22** (2011), 381–391.

[8] J. Llibre and X. Zhang, Invariant algebraic surfaces of the Lorenz systems, J. Mathematical Physics **43** (2002), 1622–1645.

[9] J. Llibre and X. Zhang, *Darboux integrability for the Rössler system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **12** (2002), 421–428.

[10] J. Llibre and X. Zhang, *Darboux theory of integrability for polynomial vector fields in* $\mathbb{R}^n$ *taking into account the multiplicity at infinity*, Bull. Sci. math. **133** (2009), 765–778.

[11] J. Llibre and X. Zhang, *Darboux theory of integrability in* $\mathbb{C}^n$ *taking into account the multiplicity*, J. Differential Equations **246** (2009), 541–551.

[12] J. Llibre and X. Zhang, *Rational first integrals in the Darboux theory of integrability in* $\mathbb{C}^n$, Bull. Sci. math. **134** (2010), 189–195.

[13] J. Mendoza, *Ciclos l'imite en un sistema* 4 *prototipo Rössler*, Scientia **23** vol. XXIII (2021), 193–198.

[14] H. Poincaré, *Sur l'intégration des équations differentielles du premier order et du premier degrée I e II*, Rend. Circ. Mat. Palermo **5** (1891), 161–191; **11** (1897), 193–239.

[15] O.E. Rössler, *Continuous chaos–four prototype equations*, Ann. New York Acad. Sci. **316** (1979), 376.

[16] J. C. Sprott, *Elegant Chaos. Algebraically Simple Chaotic Flows*, World Scientific, 2010.