



Dynamics of the Painlevé-Ince Equation

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Abstract. The Painlevé-Ince differential equation $y'' + 3yy' + y^3 = 0$ has been studied from many points of view. Here we complete its study providing its phase portrait in the Poincaré disc.

Mathematics Subject Classification. 34C05.

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1. Introduction and Statement of the Main Results

The Painlevé-Ince differential equation

$$\frac{d^2 y}{dt^2} + 3y \frac{dy}{dt} + y^3 = 0, \quad (1)$$

has been studied for several authors due to its interesting properties:

- (i) It has eight Lie point symmetries with the Lie Algebra $sl(2, R)$ consequently through a point transformation it is linearisable, see [12].
- (ii) It has a Riccati hierarchy based on the Riccati differential equation with the operator $\frac{d}{dy} + y$, see [5].
- (iii) It satisfies the Painlevé property, see [11].
- (iv) Its left Painlevé series together with its well known right Painlevé series have been studied in [6, 7, 9].
- (v) Its mixed Painlevé series together with their geometric interpretations were studied in [2, 8].

Extensions of the Painlevé-Ince differential Eq. (1) can be found in [10].

It is easy to check that the general solution of the Painlevé-Ince differential Eq. (1) is

$$y(t) = \frac{2a(b+t)}{3+ab^2+2abt+at^2},$$

where a and b are arbitrary constants. From this general solution we can determine the constants a and b for each particular solution $y(t)$ such that $y(t_0) = y_0$ and $y'(t_0) = y'_0$. But to know these explicit solutions, it is not easy to determine the qualitative properties of these solutions, i.e. where they are born, where they die tells, if they define homoclinic orbits, or heteroclinic orbits, ...

In order to describe the dynamics of the Painlevé-Ince differential Eq. (1) we write this second order differential equation as the system of first order

$$\begin{aligned} \frac{dx}{dt} &= x' = -3yx - y^3, \\ \frac{dy}{dt} &= y' = x. \end{aligned} \quad (2)$$

This differential system has the first integral

$$H(x, y) = \frac{(x + y^2)^2}{2x + y^2}, \quad (3)$$

as it is easy to verify. But again it is not trivial to describe the dynamics of the orbits of the Painlevé-Ince differential Eq. (1) using this first integral.

Here we one to complete the studies on the Painlevé-Ince differential equation describing the dynamics of its differential system (2) in the Poincaré disc. The Poincaré disc is the closed unit disc centered at the origin of coordinates of \mathbb{R}^2 , where its interior is identified with the whole plane \mathbb{R}^2 and its boundary (the circle \mathbb{S}^1) is identified with the infinity of \mathbb{R}^2 . Note that in the plane \mathbb{R}^2 we can go to infinity in as many as directions as points has the circle \mathbb{S}^1 . For a detailed introduction of the Poincaré disc see subject. 2.2.

Our main result is the following one.

Theorem 1. *The phase portrait of the Painlevé-Ince differential system (2) in the Poincaré disc is shown in Fig. 1.*

Theorem 1 is proved in the next section. From Fig. 1 we can see that the Painlevé-Ince differential system (2) has exactly five different kind of orbits:

- (i) A unique equilibrium point $(0, 0)$.
- (ii) A continuum of homoclinic orbits starting and ending at the equilibrium point $(0, 0)$.
- (iii) A continuum of heteroclinic orbits starting at the equilibrium point $(0, 0)$ and ending at infinity at the equilibrium point localized at the origin of the local chart V_1 (see the proof of Theorem 1 for more details).

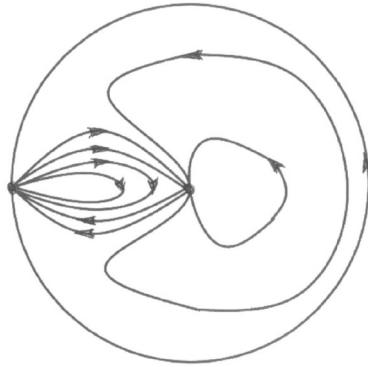


FIGURE 1. The phase portrait of the differential system (2) in the Poincaré disc

- (iv) A continuum of heteroclinic orbits starting at infinity at the equilibrium point localized at the origin of the local chart V_1 and ending at the equilibrium point $(0, 0)$.
- (v) A continuum of homoclinic orbits starting and ending at the equilibrium point localized at the origin of the local chart V_1 .

2. Proof of Theorem 1

2.1. Finite Equilibrium Points

Clearly that the differential system (2) has a unique finite equilibrium, the origin of coordinates. Since the linear part of this differential system at the origin has the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the origin is a nilpotent equilibrium point. By applying Theorem 3.5 of [4] its local phase portrait is formed by an elliptic and a hyperbolic sector, separated by two parabolic sectors. Using the first integral (3) it follows easily that the boundary of the elliptic sector is the parabola $x + y^2 = 0$, and that the boundary of the hyperbolic sector is the parabola $2x + y^2 = 0$. Between these two parabolas there are the two parabolic sectors. For a picture of the local phase portrait at this finite equilibrium point see the neighborhood of the origin in the Poincaré disc of Fig. 1.

2.2. Infinite Equilibrium Points

In this subsection we shall use the Poincaré compactification, done by Poincaré in [13, 14]. This compactification allows to study the dynamics near the infinity of a polynomial differential system in the plane \mathbb{R}^2 . See Chapter 5 of [4] or

the added Data File for additional information on this compactification and for the expressions in the local charts that we shall use in what follows for studying the phase portrait of system (2) the Poincaré disc.

We remark that for studying the infinite equilibrium points we only need to study the infinite equilibrium points of the local chart U_1 , and to verify if the origin of U_2 is an infinite equilibrium point.

Now we shall use the local charts U_1 and U_2 for studying the infinite singular points of the polynomial differential system (2) extended to the Poincaré disc. Thus the polynomial differential system (2) in the local chart U_1 is

$$\begin{aligned} u' &= v^2 + 3u^2v + u^4, \\ v' &= uv(3v + u^2). \end{aligned} \quad (4)$$

The unique infinite equilibrium of system (4), i.e. the unique equilibrium point (u, v) with $v = 0$, is the origin $(0, 0)$. Since the linear part of system (4) at the origin is the matrix identically zero, in order to study its local phase portrait we need to do changes of variables called blow ups, see [1] for more details on these changes of variables.

We do the blow up $(u, v) = (u_1, u_1v_1)$. Then in the new variables (u_1, v_1) the differential system (4) becomes

$$\begin{aligned} u_1' &= u_1^2(u_1^2 + 3u_1v_1 + v_1^2), \\ v_1' &= -u_1v_1^3. \end{aligned} \quad (5)$$

Doing the rescaling of the independent variable $t \rightarrow \tau$ through the change $d\tau = u_1 dt$ system (5) writes

$$\begin{aligned} \frac{du_1}{d\tau} &= u_1(u_1^2 + 3u_1v_1 + v_1^2), \\ \frac{dv_1}{d\tau} &= -v_1^3. \end{aligned} \quad (6)$$

The differential system (6) is a polynomial homogeneous differential system of degree 3, the phase portraits of such systems have been studied in [3]. Thus system (6) has the following four invariant straight lines through the origin of coordinates $u_1v_1(u_1 + v_1)(u_1 + 2v_1) = 0$, studying the motion on these straight lines we obtain that the local phase portrait at the origin of the differential system (6) is given in Fig. 2a.

Going back to the differential system (5) we get that the local phase portrait at the origin of the differential system (5) is given in Fig. 2b. Finally going back to the differential system (4) we get that the local phase portrait at the origin of the differential system (4) is given in Fig. 2c.

Now we study if the origin of the local chart U_2 is an equilibrium point. Then the differential system (2) in the local chart U_2 writes

$$\begin{aligned} u' &= -1 - 3uv - u^2v^2, \\ v' &= -uv^3. \end{aligned} \quad (7)$$

Hence the origin of the local chart U_2 is not an infinite equilibrium points.

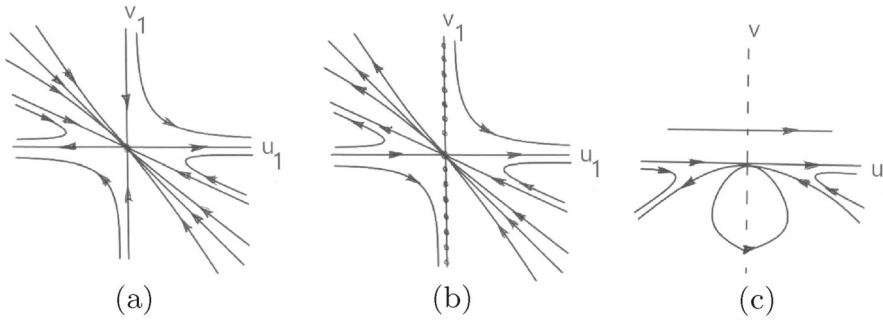


FIGURE 2. The local phase portraits of the blow up of the origin of the local chart U_1

In summary the unique infinite equilibrium points are the origins of the local charts U_1 and U_2 . In the Poincaré disc at the origin of U_1 we see the half phase portrait in $v \geq 0$ of Fig. 2c, and at the origin of V_1 the half phase portrait in $v \leq 0$ of Fig. 2c. See Fig. 1. This completes the phase portrait in the Poincaré disc of the differential system (2). Therefore Theorem 1 is proved.

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Data Availability I have added an attachment to the manuscript where I summarize the standard equations of the Poincaré compactification that we use in our proof, and that in general are well known.

Declarations

Conflicts of interest The authors have no relevant financial or non-financial interests to disclose.

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