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The solution of the Poincaré problem on the rational first integral for the Liénard polynomial differential equations

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ABSTRACT

In this work we classify the polynomial Liénard differential equations $\ddot{x}+f(x)\dot{x}+x=0$, having a rational first integral. Such classification was asked by Poincaré in 1891 for any general polynomial differential systems in the plane $\mathbb{R}^2$. As far as we know it is the first time that the complete classification is given for a relevant class of polynomial differential equations of arbitrary degree.

1. Introduction and statement of the main results

We consider a *polynomial differential system* that we can write as

$$\frac{dx}{dt}=\dot{x}=P(x,y), \qquad \frac{dy}{dt}=\dot{y}=Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y, and t is the independent variable. The degree of the polynomial differential system (1) is the maximum degree of the polynomials P and Q. The polynomial differential system (1) has associated the *polynomial vector field* $\mathcal{X}=P(x,y)\partial/\partial x+Q(x,y)\partial/\partial y$.

Let U be an open subset of $\mathbb{R}^2$. A *first integral* is defined as a C^1 non–locally constant function $H: U\to\mathbb{R}$ such that it is constant on the solutions $(x(t),y(t))$ of the polynomial differential system (1) contained in U, that is, satisfies $\mathcal{X}H=P(x,y)\partial H/\partial x+Q(x,y)\partial H/\partial y\equiv 0$ in U. We say that H is a *rational first integral* when the function H is rational.

Let $F(x,y)$ be a real polynomial in the variables x and y. The algebraic curve $F(x,y)=0$ is an *invariant algebraic curve* of a polynomial differential system (1) if for some polynomial $K=K(x,y)$ the equation

$$\mathcal{X}F=P\frac{\partial F}{\partial x}+Q\frac{\partial F}{\partial y}=KF, \tag{2}$$

is satisfied. The curve $F=0$ is formed by trajectories of the vector field $\mathcal{X}$ because on the points of the algebraic curve $F=0$ the gradient $(\partial F/\partial x,\partial F/\partial y)$ of the curve $F(x,y)=0$ is orthogonal to the vector field $\mathcal{X}=(P,Q)$. Consequently on the points of $F=0$ the vector field $\mathcal{X}$ is tangent to the curve $F=0$. Therefore the algebraic curve $F=0$ is invariant under the flow defined by $\mathcal{X}$, i.e. $F=0$ is formed by orbits of the differential system, see for instance [1].

One of the oldest problems in qualitative theory of differential systems on the plane is provide necessary and sufficient conditions in order that a polynomial differential system has a rational first integral. This problem goes back to the beginning of the qualitative theory and was stated by Poincaré in 1891, see [2]. In fact is a problem of global nature because the rational first integral is defined in all the plane except on the curves where the denominator of the rational first integral vanishes. This question involves the whole classes of polynomial differential systems and for this reason is being so hard.

One of the most studied polynomial differential equations are the *Liénard differential equations* given by

$$\ddot{x}+f(x)\dot{x}+x=0, \tag{3}$$

where $f(x)$ is a polynomial. This differential equation was considered by Liénard [3] during the development of radio and vacuum tube technology. Later on this equation and its generalizations was intensely studied as can be used to model oscillating circuits, see for instance [4] and the references therein.

The second order differential equations (3) can be write as the planar differential systems

$$\dot{x}=y, \qquad \dot{y}=-f(x)y-x, \tag{4}$$

where $f(x)$ is a polynomial in x of degree $m\geq 0$, and consequently the degree of the polynomial differential system (4) is $m+1$.

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The objective of this work is to classify the polynomial Liénard differential systems (4) having a rational first integral.

Our main results are the following two theorems.

Theorem 1. *The linear Liénard differential systems* $\dot{x} = y$, $\dot{y} = -ay - x$ *has a rational first integral if and only if* $a = \pm 2k/\sqrt{k^2-1}$ *where* $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

Theorem 2. *The polynomial Liénard differential systems (4) of degree* > 1 *has no invariant algebraic curves. Therefore they do not have rational first integrals.*

We prove Theorems 1 and 2 in the next section.

We note that Theorems 1 and 2 characterize all the Liénard differential equations (3) which have rational first integrals. As far as we know it is the first time that all rational first integrals of a relevant class of polynomial differential equations of arbitrary degree has been classified. We remark that Theorems 1 and 2 solved the problem stated by Poincaré about the characterization of the rational first integrals for the class of polynomial Liénard differential equations.

In fact, in Theorem 2 of the paper [5] it was stated that the polynomial Liénard differential systems (4) of degree > 1 have no rational first integrals, but the proof provided there was not correct because it uses a result of Hayashi [6] which is incorrect as it was proved in [7,8]. More precisely, the result stated by Hayashi in [6] is:

Theorem 3. *The generalized Liénard polynomial differential system*

$$\dot{x} = y, \qquad \dot{y} = -f_m(x)y - g_n(x), \tag{5}$$

where $m = \deg f_m$, $n = \deg g_n$, $f_m \not\equiv 0$ *and* $\deg f + 1 \geq n$ *has an invariant algebraic curve if and only if there is an invariant curve* $y - P(x) = 0$ *satisfying* $g_n(x) = -(f_m(x) + P'(x))P(x)$, *where* $P(x)$ *or* $P(x) + F(x)$ *is a polynomial of degree at most one, such that* $F(x) = \int_0^x f(s)ds$.

In fact the correct statement of Theorem 3 is the following theorem proved in [8].

Theorem 4. *The generalized Liénard polynomial differential system (5) with* $f_m \not\equiv 0$ *and* $m + 1 \geq n$ *has the invariant algebraic curve* $y - P(x) = 0$ *if* $g_n(x) = -(f_m(x) + P'(x))P(x)$, *being* $P(x)$ *or* $P(x) + F(x)$ *a polynomial of degree at most one, where* $F(x) = \displaystyle\int_0^x f(s)ds$.

In summary, the claim that Theorem 3 characterize all the invariant algebraic curves of the generalized Liénard polynomial differential system is false.

On the other hand Theorem 2 applied to the van der Pol differential equation, $\dot{x} = y$, $\dot{y} = -\mu(x^2-1)y - x$, shows that the limit cycle of this equation is not algebraic. This result already was proved by Odani in [9] studying the generalized Liénard system $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$ using complete different arguments. In fact Odani proved the following result.

Theorem 5. *If the generalized polynomial system* $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$ *satisfies* $\deg f \geq \deg g$ *and* $fg(f/g)' \neq 0$, *then it has no invariant algebraic curves.*

In fact the result obtained here is an extension of Theorem 5 for all the classical polynomial Liénard differential equations, i.e. when $g(x) = x$ and without the extra condition $fg(f/g)' \neq 0$ of Theorem 5.

We note that Theorem 2 cannot be extended to any generalized Liénard system $\dot{x} = y$, $\dot{y} = -g(x) - f(x)y$ with f and g polynomials as the following system shows. The differential system $\dot{x} = y$, $\dot{y} = -(4x+x^3)/4 - 3xy/2$ has the rational first integral $H = (4+x^2+2y)^2/(4x^2+x^4+4x^2y+4y^2)$.

We remark that the limit cycles of the polynomial Liénard differential equations (3) have been intensively studied by several authors, see for instance [10–14]. However many open questions about these limit cycles remain still open.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. The linear Liénard differential system of degree 1 is system (4) with $f(x) = a$, that is,

$$\dot{x} = y, \qquad \dot{y} = -ay - x. \tag{6}$$

It is easy to verify that system (6) has the first integral

$$H = \left(axy + x^2 + y^2\right)\left(\frac{\left(a^2-2-\sqrt{a^2-4}a\right)x^2 + 2\left(a-\sqrt{a^2-4}\right)xy + 2y^2}{axy+x^2+y^2}\right)^{-\frac{a}{\sqrt{a^2-4}}}.$$

Note that H is a rational first integral if and only if $a = \pm 2k/\sqrt{k^2-1}$ with $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. □

Proof of Theorem 2. We assume that $F(x, y) = 0$ is an invariant algebraic curve of degree n of the polynomial differential system (4) of degree $m + 1$. We expand $F(x, y)$ in its homogeneous parts as $F(x, y) = \sum_{j=0}^n F_j(x, y)$, where F_j are homogeneous polynomials of degree j in (x, y). Of course $F_n(x, y) \neq 0$.

We also write the polynomial $f(x)$ of degree m as $f(x) = \sum_{j=0}^m a_i x^i$, with $a_m \neq 0$. Moreover it is easy to see from (2) that the cofactor K of the invariant algebraic curve $F(x, y) = 0$ is a polynomial in x of degree at most the degree of the polynomial $f(x)$. So we write $K(x) = \sum_{j=0}^m k_i x^i$.

Substituting $F(x, y)$, $f(x)$ and $K(x)$ in Eq. (2) we obtain that the highest homogeneous part of that equation has degree $m + n$ and is

$$x^m\left(a_m y\frac{\partial F_n}{\partial y} + k_m F_n\right) = 0 \tag{7}$$

Solving this differential equation we get $F_n(x, y) = y^{-\frac{k_m}{a_m}} C_n(x)$, where $C_n(x)$ is an arbitrary function in the variable x. Since $F_n(x, y)$ must be a homogeneous polynomial of degree n we have $k_m = -a_m k$ with $k \in \{0, 1, \ldots, n\}$ and $F_n(x, y) = c_n x^{n-k} y^k$ for some constant $c_n \neq 0$ that we can take equals 1 because this does not change the invariant algebraic curve $F(x, y) = 0$.

Now the homogeneous part of Eq. (2) of degree $m + n - 1$ is

$$yx^m\frac{\partial F_{n-1}}{\partial y} - kx^m F_{n-1}(x, y) + \frac{1}{a_m}(a_{m-1}k + k_{m-1})y^k x^{m+n-k-1} = 0. \tag{8}$$

Solving this differential equation we obtain

$$F_{n-1}(x, y) = y^k C_{n-1}(x) - \frac{1}{a_m}(a_{m-1}k + k_{m-1})y^k x^{n-k-1}\log y,$$

where $C_{n-1}(x)$ is an arbitrary function of x. Since $F_{n-1}(x, y)$ must be a homogeneous polynomial of degree $n-1$ we must have $k_{m-1} = -a_{m-1}k$ and $F_{n-1}(x, y) = c_1 x^k y^{n-k-1}$ where c_1 is a constant.

Case 1: $m \geq n \geq 1$. We shall prove by induction the following expressions for the invariant algebraic curve $F(x, y) = 0$ and the cofactor $K(x)$

$$F(x, y) = y^k(x^{n-k} + c_1 x^{n-k-1} + \cdots + c_{n-k-1}x) + F_0,$$
$$K(x) = -k(a_m x^m + a_{m-1}x^{m-1} \cdots + a_{m-n+k}x^{m-n+k}),$$

where F_0 is a constant.

We have seen that the induction assumptions hold for the homogeneous parts of Eq. (2) of degrees $m+n$ and $m+n-1$, now we assume that they hold for the homogeneous parts of Eq. (2) of degree $m+n-(\ell-1)$ with $\ell = \{2, \ldots, n-k+1\}$ and we shall prove them for the homogeneous part of Eq. (2) of degree $m + n - \ell$.

Substituting

$$F(x, y) = y^k\left(x^{n-k} + c_1 x^{n-k-1} + \cdots + c_{m-(\ell-1)}x^{n-k-m+\ell-1}\right),$$
$$K(x) = -k(a_m x^m + a_{m-1}x^{m-1} \cdots + a_{\ell-1}x^{\ell-1}),$$

in (2) we get that its highest homogeneous part has degree $m + n - \ell$ and is

$$yx^m\frac{\partial F_{n-\ell}}{\partial y} - kx^m F_{n-\ell} + \frac{1}{a_m}(a_{m-\ell}k + k_{m-\ell})y^k x^{m+n-k-\ell} = 0.$$

The solution of this differential equation is

$$F_{n-\ell}(x,y) = y^k C_{n-\ell}(x) - \frac{1}{a_m}(a_{m-\ell}k + k_{m-\ell})y^k x^{n-k-\ell}\log y,$$

where $C_{n-\ell}(x)$ is an arbitrary function of x. Since $F_{n-\ell}(x,y)$ must be a homogeneous polynomial of degree $n-\ell$ we must have $k_{m-\ell} = -a_{m-\ell}k$ and $F_{n-\ell}(x,y) = c_\ell\, y^k x^{n-k-\ell}$. Hence the induction is proved until

$$F(x,y) = y^k(x^{n-k} + c_1 x^{n-k-1} + \cdots + c_{n-k-1}x),$$
$$K(x) = -k(a_m x^m + a_{m-1}x^{m-1}\cdots + a_{m-n+k+1}x^{m-n+k+1}),$$

Substituting these last expressions for $F(x,y)$ and $K(x)$ in Eq. (2) its highest homogeneous part has degree m and is

$$a_m k\, x^m F_0 - (a_{m-n+k}k - k_{m-n+k})y^k x^{m-k} = 0.$$

Therefore

$$F_0 = \frac{(a_{m-n+k}k - k_{m-n+k})y^k x^{-k}}{a_m k},$$

but as F_0 must be a constant we must to impose $k_{m-n+k} = -a_{m-n+k}k$ and we get $F_0 = 0$.

This implies that $F(x,y) = y^k(x^{n-k} + c_1x^{n-k-1} + \cdots c_{n-k-1}x)$ with $0 \le k \le n$. If $k \ne 0$ then $y = 0$ is an invariant straight line, but this is not possible for the Liénard system (3) because $\dot{y}|_{y=0} \ne 0$. If $k = 0$ then the invariant solution is function of x, i.e., $F(x,y) = F(x)$ and Eq. (2) becomes $F'(x)y = 0$ which implies that $F(x)$ is constant, again a contradiction with the hypothesis that $F(x,y) = F(x) = 0$ be an invariant algebraic curve. In summary, in case 1 the polynomial Liénard differential systems (4) have no invariant algebraic curves.

Case 2: $m < n$. This case can be solved in a similar way to case 1. We divide it into two subcases.

Subcase 2.1: $n-k \le m$. Then the induction process is exactly the same than in case 1, and consequently the polynomial Liénard differential systems (4) have no invariant algebraic curves.

Subcase 2.2: $m < n-k$. In this case we also will see that there are no invariant algebraic curves. The highest homogeneous part of Eq. (2) of degree $n+m$ is given in (7) which implies $F_n(x,y) = x^{n-k}y^k$ with $k < n-m$ where we have to impose $k_m = -a_m k$. The next highest homogeneous part of Eq. (2) given in (8) determines $F_{n-1}(x,y)$ which takes the value $F_{n-1}(x) = c_1 x^{n-k-1}y^k$ imposing $k_{m-1} = -a_{m-1}k$. The next highest homogeneous parts of Eq. (2) give the same results than in Case 1, i.e. $F_{n-i} = c_i x^{n-k-i}y^k$ for $i = 1,\dots,m-1$. The next homogeneous part of Eq. (2) is of degree m and is the differential equation

$$a_m k x^2 F_{n-m} - a_m x^2 y \frac{\partial F_{n-m}}{\partial y} - k\, x^{1-k+n}y^{k-1}$$
$$- (a_0 k + k_0)x^{n-k}y^k + (n-k)x^{n-k-1}y^{k+1} = 0,$$

whose solution is

$$F_{n-m}(x,y) = \frac{1}{a_m}\left(x^{n-m-k-1}y^k\left(\frac{kx^2}{y} - ky + ny\right) - (a_0k + k_0)x\log y\right) + c_m x^{n-k-m}y^k.$$

Therefore we have take $k_0 = -a_0 k$ and consequently the cofactor $K(x)$ becomes $K(x) = -kf(x)$. Now we consider the term of F_{n-m} given by

$$\mathcal{T} = \frac{1}{a_m}(n-k)x^{n-m-k-1}y^{k+1}.$$

This term substituted into Eq. (2), that is, $\mathcal{X}\mathcal{T} = (\partial\mathcal{T}/\partial x)y + (\partial\mathcal{T}/\partial y)(-x - f(x)y)$ which gives always the isolate term

$$-\frac{1}{a_m}(n-k)\,x^{n-m-k}y^k.$$

a term of degree $n-m$. Therefore affect the homogeneous part of degree $n-m$ of Eq. (2) which is

$$-\frac{1}{a_m}(n-k)(m+1+k-n)x^{n-m-2-k}y^{k+2} - \frac{1}{a_m}(k-1)k\,x^{n-m+2-k}y^{k-2}$$
$$-\frac{1}{a_m^3}A(a_{m-i},c_i,k,n)\,x^{n-m-1-k}y^{k+1} - \frac{1}{a_m^3}B(a_{m-i},c_i,k,n)\,x^{n-m+1-k}y^{k+1}$$
$$-\frac{1}{a_m}(n-k)x^{n-m-k}y^k + a_m k x^m F_{n-2m} - a_m x^m y\frac{\partial F_{n-2m}}{\partial y} = 0,$$

where A and B are functions of the parameters k, n, a_{m-i} and c_i for $i = 1,\dots,m$. The resolution of this equation determines F_{n-2m} which contains the logarithmic term

$$-\frac{1}{a_m^2}(n-k)x^{n-2\,m-k}y^k\log y,$$

which implies $n-k = 0$ which is a contradiction because the unique possibility is $k = n$ and then $F_n = y^n$, $F_{n-i} = 0$ for $i = 1,\dots,m-1$ and $F_{n-m} = ny^{n-1}/(a_m x^{m-1})$ which is not a homogeneous polynomial except if $m = 1$. But for $m = 1$ $F_{n-m-1} = (ny^{n-2}(nx - x - 2a_{m-1}y))/(2a_m^2 x)$ which is not a polynomial. □

CRediT authorship contribution statement

Jaume Giné: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Jaume Llibre:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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