This is a preprint of: "Limit cycles of continuous piecewise smooth differential systems", Joyce Casimiro, Jaume Llibre, *Results Math.*, vol. 78(173), 14 pp, 2023.
DOI: [10.1007/s00025-023-01948-w]

LIMIT CYCLES OF CONTINUOUS PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS

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ABSTRACT. From the beginning of this century many articles have been published on the continuous and discontinuous piecewise differential systems specially in the plane. The big interest on these piecewise differential systems mainly comes from their increasing number of applications for modelling many natural phenomena. As it is usual in the planar differential systems one of the main difficulties for understanding their dynamics consists in controlling their limit cycles. The major part of papers studying continuous piecewise differential systems has an straight line as the line of separation between the differential systems forming the continuous piecewise differential systems.

In this work we consider continuous piecewise differential systems separated by a circle and formed by one linear differential center and one quadratic differential center. We study the maximum number of limit cycles that such kind of continuos piecewise differential system can exhibit.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Around 1920's started the interest for studying the piecewise differential systems mainly in the works of Andronov, Vitt and Khaikin, see the book [17]. Nowadays this interest is increasing due to the fact that the piecewise differential systems model many processes appearing in mechanics, electronics, economy, etc., see for more details the books of Simpson [18], di Bernardo et al. [2] and, the survey of Makarenkov and Lamb [16], and the hundreds of references which appear in the references of these last citations.

The easiest continuous piecewise differential systems are the ones having only two pieces separated by a straight line in the plane $\mathbb{R}^2$ and formed by two linear differential systems. Lum and Chua in 1990 conjectured in [14, 15] that such piecewise differential systems have at most one limit cycle. We recall that a limit cycle is an isolated periodic orbit in the set of all periodic orbits of a differential system. The previous conjecture was proved in 1990 by Freire et al. [5]. Later on a distinct and shorter proof was given in 2013 by Llibre, Ordóñez and E. Ponce [11], and more recently in 2021 a new proof has been given by Carmona, Fernández-Sánchez and Novaes [3].

In the paper [10] the authors studied the discontinuous piecewise differential systems separated by a circle and formed by two linear differential systems, and proved that those systems can have at most 3 limit cycles, and that there are systems of this type having 3 limit cycles. But the same kind of piecewise differential systems being continuous on the circle has no limit cycles.

In this work before studying the limit cycles of the discontinuous piecewise differential systems separated by a circle and formed by one linear differential system and a quadratic differential system,

2010 *Mathematics Subject Classification.* 34A36, 34C07, 37G15.
Key words and phrases. Continuous piecewise differential systems, linear center, quadratic center, limit cycle.

we shall study the limit cycles of the easier continuous piecewise differential systems separated by a circle and formed by one linear differential system and a quadratic differential system.

In [9] it was proved, (see Theorem 1.1) that a piecewise differential system separated by a parabola, and formed by a linear differential center and a quadratic differential center, has at most one limit cycle, and that there exist such kind of piecewise differential systems with one limit cycle.

In this paper we study the continuous piecewise differential systems separated by the circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ and given by

$$Z = \begin{cases} Z_1(x,y), & \text{if } x^2+y^2 \le 1, \\ Z_2(x,y), & \text{if } x^2+y^2 \ge 1, \end{cases} \tag{1}$$

where Z_1, Z_2 are centers, one is linear and the other is quadratic. To ensure that we have a linear and a quadratic center we shall use the following to propositions.

Proposition 1 ([13, Lemma 1]). *A linear differential system having a center can be written as*

$$\dot{x} = -Bx - \frac{4B^2+\omega^2}{4A} y + D \quad \dot{y} = Ax + By + C, \tag{2}$$

with $A > 0$ and $\omega > 0$.

Proposition 2 is taken from book [4], but can also be found in the original works of Kapteyn and Bautin [1, 6, 7].

Proposition 2 ([4, Theorem 8.15]). *A quadratic differential system that has a center at the origin can be written in the form*

$$\dot{x} = -y - bx^2 - cxy - dy^2 \quad \dot{y} = x + ax^2 + exy - ay^2. \tag{3}$$

This system has a center at the origin if and only if at least one of the four following conditions hold

(i) $e - 2b = c + 2a = 0$,
(ii) $b + d = 0$,
(iii) $c + 2a = e + 3b + 5d = a^2 + bd + 2d^2 = 0$,
(iv) $c = a = 0$.

Our main result is the following theorem.

Theorem 3. *Consider the differential system* (1) *formed by a linear differential center* (2) *and a quadratic differential center* (3) *after the change of variables $x = kX + \alpha \quad y = MY + \beta$, with $k, M \neq 0$. Then the following statements hold.*

(a) There are no continuous piecewise differential systems (1) *with quadratic differential center of type* (i), (ii), (iii), *with $d \neq 0$. When $d = 0$ system* (iii) *becomes system* (iv).
(b) The continuous piecewise differential system (1) *with a quadratic differential center of type* (iv) *has at most three limit cycles.*

Theorem 3 is proved in Section 3.

The reason of consider in the statement of Theorem 3 quadratic differential centers after the mentioned change of variables, is that we increase the classes of quadratic differential centers described in Proposition 2 in four additional parameters. Unfortunately at this moment we cannot increases that class of quadratic differential centers doing a general affine transformation which will increase in six the number of parameters, but the computations necessary for studying this class cannot be done for now.

In summary, for the class of continuous piecewise differentials systems here studied we provide the upper bound of three for their maximum number of limit cycles. So we have solved the extension of the 16th Hilbert problem to this class of piecewise differential systems. For the moment is unknown if this upper bound is reached. We only know examples of these piecewise differential systems with one limit cycle.

In order to simplify the notation in what follows instead of linear differential center and quadratic differential center we shall write linear center and quadratic center, respectively.

2. Preliminaries

Let $I \subset \mathbb{R}$ be an open interval and let $f_0, f_1, ..., f_n : I \to \mathbb{R}$. We say that $f_0, f_1, ..., f_n$ are linearly independent functions if and only if when

$$\sum_{i=0}^{n} \lambda_i f_i(x) = 0, \ \forall\, x \in I \implies \lambda_0 = ... = \lambda_n. \tag{4}$$

The following result which can be found in [12], will be used in the proof of statement (b) of Theorem 3.

Proposition 4. *Let $f_0, f_1, ..., f_n : I \to \mathbb{R}$ analytic functions. If $f_0, f_1, ..., f_n$ are linearly independent then there exists $s_1, ..., s_n \in I$ and $\lambda_0, ..., \lambda_n \in \mathbb{R}$ such that for every $j \in \{1, ..., n\}$ we have $\sum_{i=0}^{n} \lambda_i f_i(s_j) = 0$.*

Let I be an open interval and $f_0, ..., f_n$ functions defined on I. We say that $(f_0, ..., f_n)$ forms an Extended Chebyshev system (ET-system) on I, if and only if, any non-trivial linear combination of these functions has at most n zeros counting their multiplicities and this number is reached. The functions $(f_0, ..., f_n)$ are an Extended Complete Chebyshev system (ECT-system) on I if and only if for any $j \in \{0, 1, ..., n\}$, $(f_0, ..., f_j)$ form an ET-system.

The next result can be found in [8].

Proposition 5. *Let $f_0, ..., f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then $(f_0, ..., f_n)$ is an ECT-system on I if and only if for each $j \in \{0, 1, ..., n\}$ and all $y \in I$ the Wronskian*

$$W(f_0, ..., f_j)(y) = \begin{bmatrix} f_0(y) & f_1(y) & \cdots & f_j(y) \\ f_0'(y) & f_1'(y) & \cdots & f_j'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(j)}(y) & f_1^{(j)}(y) & \cdots & f_j^{(j)}(y) \end{bmatrix} \tag{5}$$

is different from zero.

3. Proof of Theorem 3

Proof of statement (a) of Theorem 3. Under the assumptions of statement (a) of Theorem 3 there are no continuous piecewise differential systems (1) such that one of the differential systems (3) satisfying (i), (ii) and (iii) with $d \neq 0$ have a quadratic center. We will prove statement (a) for the quadratic centers (i),(ii) and (iii) for $d \neq 0$, separately.

Indeed, in case (i) we have that $e - 2b = c + 2a = 0$, then system (3) after the change of variables $x = kX + \alpha$, $y = MY + \beta$, can be write as

(6)
$$\begin{aligned} \dot{x} &= -\frac{(\beta + My)(-2a(\alpha + kx) + d(\beta + My) + 1) + b(\alpha + kx)^2}{k}, \\ \dot{y} &= x\left(\frac{k(2a\alpha + 2b\beta + 1)}{M} + 2bky\right) + \frac{a\alpha^2 - a\beta^2 + \alpha + 2\alpha b\beta}{M} + 2y(\alpha b - a\beta) + \frac{ak^2x^2}{M} - aMy^2. \end{aligned}$$

In order that this piecewise differential system to be continuous on the circle $x^2 + y^2 = 1$, the differential systems (2) and (6) must coincide on this circle, this implies that

$$a = 0,\ A = \frac{k}{M},\ b = 0,\ B = 0,\ C = \frac{\alpha}{M},\ d = 0,\ D = \frac{-\beta}{k},\ \omega = 2.$$

Under these conditions system (6) becomes

$$\dot{x} = -\frac{\beta}{k} - \frac{My}{k}, \quad \dot{y} = \frac{kx}{M} + \frac{\alpha}{M}. \tag{7}$$

Since (7) is not a quadratic system we do not have a continuous piecewise differential system with a quadratic center of type (i).

In case (ii) we have $b = -d$, then system (3) after the change of variables $x = kX + \alpha$, $y = MY + \beta$, writes

(8)
$$\begin{aligned} \dot{x} &= -\frac{b(\alpha + kx)^2 + (\beta + My)(c(\alpha + kx) + \beta d + dMy + 1)}{k}, \\ \dot{y} &= \frac{kx(2a\alpha + e(\beta + My) + 1) - a(-\alpha + \beta + My)(\alpha + \beta + My)}{M} \\ &\quad + \frac{ak^2x^2 + \alpha + \alpha e(\beta + My)}{M}. \end{aligned}$$

In order that this piecewise differential system to be continuous on the circle $x^2 + y^2 = 1$, the differential systems (2) and (8) must coincide on this circle this, implies that

$$a = 0,\ A = \frac{k}{M},\ b = 0,\ B = 0,\ c = 0,\ C = \frac{\alpha}{M},\ D = 0,\ e = 0,\ \omega = \frac{2\sqrt{k}}{\sqrt{M}}.$$

Under these conditions the systems (6) and (8) are the same and are given in (7). We do not have a continuous piecewise differential system with a quadratic center of type (ii).

In **case** (iii) we have that $c+2a = e+3b+5d = a^2+bd+2d^2 = 0$, so system (3) after applying the change of variables $x = kX+\alpha$, $y = MY+\beta$, is

$$
\begin{aligned}
\dot{x} &= \frac{-d\left(\beta+d\left(-2\alpha^2+\beta^2-2k^2x^2-4\alpha kx+M^2y^2+2\beta My\right)+My\right)}{dk} \\
&\quad +\frac{2ad(\alpha+kx)(\beta+My)+a^2(\alpha+kx)^2}{dk}, \\
\dot{y} &= \frac{d\left(ak^2x^2+k(2a\alpha x+x)-a(-\alpha+\beta+My)(\alpha+\beta+My)+\alpha\right)}{dM} \\
&\quad +\frac{3a^2(\alpha+kx)(\beta+My)+d^2(\alpha+kx)(\beta+My)}{dM}.
\end{aligned} \tag{9}
$$

However in order that this system be continuous on the circle $x^2+y^2 = 1$, the solutions obtained, that are not complex, are such that either $d = 0$ or $k = M = 0$. As this contradicts the hypotheses, we have that there is no continuos piecewise differential system in this case. □

Proof of statement (b) of Theorem 3. Now we will work with the quadratic center of type (iv), that is, $c = a = 0$. Under this condition system (3) is

$$
\dot{x} = -dy^2 - bx^2 - y, \quad \dot{y} = exy + x. \tag{10}
$$

Doing a rescaling of the time we can, without loss of generality, work with two subcases $b = 0$ and $b = 1$. In the first case, that is, $b = 0$, there is no continuous piecewise differential system on the circle $x^2+y^2 = 1$. Indeed, with the change of variables $x = kX+\alpha$, $y = MY+\beta$, we can writte system (10) as

$$
\begin{aligned}
\dot{x} &= -\frac{\beta(\beta d+1)}{k} - \frac{dM^2y^2}{k} - \frac{My(2\beta d+1)}{k}, \\
\dot{y} &= x\left(\frac{k(\beta e+1)}{M}+eky\right) + \frac{\alpha(\beta e+1)}{M} + \alpha ey.
\end{aligned} \tag{11}
$$

In order that this piecewise differential system to be continuous on the circle $x^2+y^2 = 1$, the differential system (11) and (2) must coincide on the circle, this implies that

$$
A = \frac{k}{M},\ B = 0,\ C = \frac{\alpha}{M},\ d = 0,\ D = \frac{-\beta}{k},\ e = 0,\ \omega = 2. \tag{12}
$$

But this solution is such that system (14) becomes non quadratic, because rewriting (11) under these conditions, we get the system

$$
\dot{x} = -\frac{\beta}{k} - \frac{My}{k}, \quad \dot{y} = \frac{kx}{M} + \frac{\alpha}{M}. \tag{13}
$$

It remains to study the case $b = 1$. Then with the change of variables $x = kX + \alpha$, $y = MY + \beta$ and $b = 1$, we can write system (10) as

$$\begin{aligned} \dot{x} &= -\frac{\alpha^2 + \beta + \beta^2 d}{k} - \frac{dM^2y^2}{k} - \frac{My(2\beta d + 1)}{k} - kx^2 - 2\alpha x, \\ \dot{y} &= x\left(\frac{k(\beta e + 1)}{M} + eky\right) + \frac{\alpha(\beta e + 1)}{M} + \alpha e y. \end{aligned} \tag{14}$$

In order that this piecewise differential system to be continuous on the circle $x^2 + y^2 = 1$, the differential system (14) and (2) must coincide on this circle. This implies that

$$A = \frac{k}{M},\ B = C = 0,\ d = \frac{k^2}{M^2},\ D = \frac{-\beta^2k^2 - k^2M^2 - \beta M^2}{kM^2},\ e = \alpha = 0,\ \omega = \frac{2\sqrt{2\beta k^2 + M^2}}{M}. \tag{15}$$

Under these conditions system (2) becomes

$$\begin{aligned} \dot{x} &= \frac{-\beta^2k^2 - k^2M^2 - \beta M^2}{kM^2} - \frac{y\left(2\beta k^2 + M^2\right)}{kM}, \\ \dot{y} &= \frac{kx}{M}. \end{aligned} \tag{16}$$

with the first integral

$$H_1(x,y) = 4k^2x^2M + 4y^2M\left(2\beta k^2 + M^2\right) - 8y\left(-\beta^2k^2 - k^2M^2 - \beta M^2\right), \tag{17}$$

and system (14) becomes

$$\begin{aligned} \dot{x} &= -\frac{k\left(\beta^2 + M^2\left(x^2 + y^2\right) + 2\beta My\right)}{M^2} - \frac{\beta + My}{k}, \\ \dot{y} &= \frac{kx}{M}. \end{aligned} \tag{18}$$

System (18) has the first integral $H_2(x,y)$ equal to

$$\frac{\exp(2My)\left(2\beta^2k^2 - 2\beta k^2 + 4\beta k^2My - 2k^2My + k^2 + 2M^3y + M^2(2k^2x^2 + 2k^2y^2 + 2\beta - 1)\right)}{2k^2M^2}. \tag{19}$$

Assume that the continuous piecewise differential system has a limit cycle which intersects the circle $x^2 + y^2 = 1$ in the two points (x_1, y_1) and (x_2, y_2). To determine how many limit cycles can exist for this continuous piecewise differential system formed by systems (16) and (18) we will analyse how many solutions the system below can have

$$\begin{aligned} e_1(x,y) &:= H_1(x_1,y_1) - H_1(x_2,y_2) = 0, \\ e_2(x,y) &:= H_2(x_1,y_1) - H_2(x_2,y_2) = 0. \end{aligned} \tag{20}$$

Consider the change of variables given by

$$x_i = \sin t_i \quad y_i = \cos t_i. \tag{21}$$

with the change variable (21), system (20) becomes

(22)
$$
\begin{aligned}
e_1(t_1,t_2) &= \frac{4(\sin t_1-\sin t_2)\left(2\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)+M\left((2\beta-1)k^2+M^2\right)(\sin t_1+\sin t_2)\right)}{M^3},\\
e_2(t_1,t_2) &= \frac{\left(k^2\left(2(\beta-1)\beta+2M^2+1\right)+(2\beta-1)M^2\right)\left(\exp(2M\sin t_1)-\exp(2M\sin t_2)\right)}{2k^2M^2}\\
&\quad+\frac{2M\left((2\beta-1)k^2+M^2\right)\left(\sin t_1\exp(2M\sin t_1)-\sin t_2\exp(2M\sin t_2)\right)}{2k^2M^2}.
\end{aligned}
$$

Notice that $e_1=0$ when $\sin t_1=\sin t_2$, or

$$
2\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)+M\left((2\beta-1)k^2+M^2\right)(\sin t_1+\sin t_2)=0 \tag{23}
$$

If $\sin t_1 = \sin t_2$ then e_2 is identically zero. Therefore, if there are periodic orbits, we will have a continuum of periodic orbits for system (22), that is, no limit cycles.

Now we consider the second case, when (23) holds. We obtain two solutions for the variable t_1 given by

(24)
$$
\begin{aligned}
t_1^1 &= \pi-\sin^{-1}\left(\frac{-2\beta^2k^2-2k^2M^2-2\beta k^2M\sin t_2+k^2M\sin t_2+M^3(-\sin t_2)-2\beta M^2}{M\left(2\beta k^2-k^2+M^2\right)}\right),\\
t_1^2 &= \sin^{-1}\left(\frac{-2\beta^2k^2-2k^2M^2-2\beta k^2M\sin t_2+k^2M\sin t_2+M^3(-\sin t_2)-2\beta M^2}{M\left(2\beta k^2-k^2+M^2\right)}\right).
\end{aligned}
$$

Substituting t_1^1 in e_2, we obtain the equation

$$
e_2(t_1^1,t_2)=\sum_{i=0}^{3}a_if_i(t_2)=0, \tag{25}
$$

where the coefficients are given by

(26)
$$
\begin{aligned}
a_0 &= \frac{2M\left((2\beta-1)k^2+M^2\right)\left(-2\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)\right)}{(2\beta-1)k^2M+M^3},\\
a_1 &= \frac{2M\left((2\beta-1)k^2+M^2\right)\left(-M\left((2\beta-1)k^2+M^2\right)\right)}{(2\beta-1)k^2M+M^3},\\
a_2 &= k^2\left(2(\beta-1)\beta+2M^2+1\right)+(2\beta-1)M^2,\\
a_3 &= 2M\left((2\beta-1)k^2+M^2\right).
\end{aligned}
$$

and the functions are

$$
\begin{aligned}
f_0(t_2) &= \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right),\\
f_1(t_2) &= \sin t_2 \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right),\\
f_2(t_2) &= \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right)-\exp(2M\sin t_2),\\
f_3(t_2) &= \sin t_2\left(-\exp(2M\sin t_2)\right).
\end{aligned} \tag{27}
$$

We compute the following Wronskians

$$W(f_0)(t_2) = \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right), \tag{28}$$

$$W(f_0,f_1)(t_2) = \cos t_2 \exp\left(-\frac{8\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-4M\sin t_2\right), \tag{29}$$

$$W(f_0,f_1,f_2)(t_2) = -16M^2\cos^3 t_2 \exp\left(-\frac{8\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right), \tag{30}$$

$$W(f_0,f_1,f_2,f_3)(t_2) = 256M^4\cos^6 t_2 \exp\left(-\frac{8\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}\right). \tag{31}$$

In the interval $(-\pi/2, \pi/2)$ the Wronskians (28), (29), (30) and (31) are not zero. Then (f_0, f_1, f_2, f_3) is an Extended Chebyshev system that have at most three solutions. This means that we can have at most three limit cycles.

Since the rank of the 4×3 matrix

$$\begin{bmatrix} \frac{\partial a_0}{\partial k} & \frac{\partial a_0}{\partial M} & \frac{\partial a_0}{\partial \beta} \\ \frac{\partial a_1}{\partial k} & \frac{\partial a_1}{\partial M} & \frac{\partial a_1}{\partial \beta} \\ \frac{\partial a_2}{\partial k} & \frac{\partial a_2}{\partial M} & \frac{\partial a_2}{\partial \beta} \\ \frac{\partial a_3}{\partial k} & \frac{\partial a_3}{\partial M} & \frac{\partial a_3}{\partial \beta} \end{bmatrix} \tag{32}$$

cannot be four, the four coefficients are not independent, and consequently we cannot guarantee that the system has three solutions, we only can say that it has at most three solutions. □

4. EXAMPLES

We will present two examples, both with only one limit cycle. The first one is composed by a quadratic center inside the circle $x^2 + y^2 = 1$ and a linear center outside the circle. In the second example, we obtain the limit cycle regardless of which system is defined inside (or outside) the circle $x^2 + y^2 = 1$.

As we saw in the proof of Theorem 3, the existence of limit cycles is possible for quadratic centers of type (iv). So in the examples we will be under the condition $c = a = 0$.

Example 1. Consider a continuous piecewise differential system separated by the unit circle centered at the origin of coordinates

$$Z = \begin{cases} Z_1(x,y) = \left(-\dfrac{2x^2}{3} - \dfrac{2y^2}{3} - \dfrac{77y}{60} - \dfrac{61}{150}, \dfrac{4x}{3}\right), \text{ if } x^2+y^2 \le 1, \\ \\ Z_2(x,y) = \left(-\dfrac{77y}{60} - \dfrac{161}{150}, \dfrac{4x}{3}\right), \text{ if } x^2+y^2 \ge 1, \end{cases} \tag{33}$$

note that $Z_1(x,y)$ is the quadratic center (18), and $Z_2(x,y)$ is the linear center (16), both with $\beta = 1/5$, $k = 2/3$, and $M = 1/2$. The first integrals of Z_1 and Z_2 are respectively

$$\begin{aligned} H_1(x,y) &= \frac{64x^2}{9} + \frac{308y^2}{45} + \frac{2576y}{225}, \\ H_2(x,y) &= \frac{9}{2}\exp(y)\left(\frac{2x^2}{9} + \frac{2y^2}{9} - \frac{y}{60} + \frac{137}{900}\right), \end{aligned} \tag{34}$$

This continuous piecewise differential system (33) has exactly one limit cycle, because the unique real solution (x_1, y_1, x_2, y_2) of the system

$$\begin{aligned} &H_1(x_1,y_1) - H_1(x_2,y_2) = 0 \\ &H_2(x_1,y_1) - H_1(x_2,y_2) = 0, \\ &x_1^2 + y_1^2 = 1, \\ &x_2^2 + y_2^2 = 1, \end{aligned} \tag{35}$$

is

$$(x_1, y_1, x_2, y_2) = (0.886447.., 0.462831.., -0.886447.., 0.462831..). \tag{36}$$

See this limit cycle in Figure 1.

Example 2. Let the linear center

$$\dot{x} = -\frac{49y}{30} - \frac{97}{150}, \qquad \dot{y} = \frac{2x}{3}. \tag{37}$$

with the first integral

$$H_1(x,y) = \frac{16x^2}{9} + \frac{196y^2}{45} + \frac{776y}{225}. \tag{38}$$

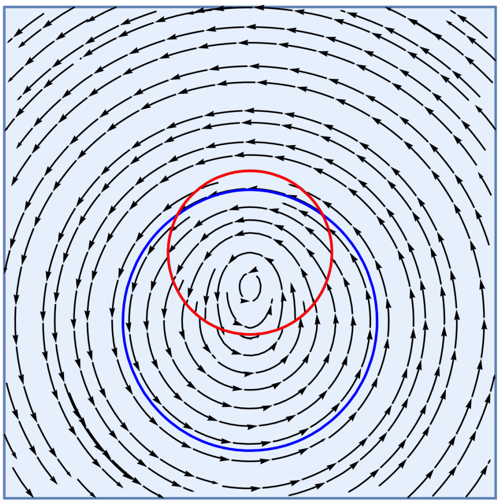

FIGURE 1. System (33) with its limit cycle, which looks as the big circle in the figure, passing through the points (x_1, y_1) and (x_2, y_2) given by the solution (36). The small circle of the figure is the circle $x^2 + y^2 = 1$.

And the quadratic center

$$\dot{x} = -\frac{x^2}{3} - \frac{y^2}{3} - \frac{49y}{30} - \frac{47}{150}, \qquad \dot{y} = \frac{2x}{3}. \tag{39}$$

with the first integral

$$H_2(x,y) = \frac{81}{8}\exp(y)\left(\frac{8x^2}{81} + \frac{8y^2}{81} + \frac{116y}{405} - \frac{392}{2025}\right). \tag{40}$$

This continuous piecewise differential system (33) has exactly one limit cycle, because the unique real solution (x_1, y_1, x_2, y_2) of the system

$$\begin{aligned} &H_1(x_1,y_1) - H_1(x_2,y_2) = 0\\ &H_2(x_1,y_1) - H_1(x_2,y_2) = 0,\\ &x_1^2 + y_1^2 = 1,\\ &x_2^2 + y_2^2 = 1, \end{aligned} \tag{41}$$

is

$$(x_1, y_1, x_2, y_2) = (1, 0, -1, 0). \tag{42}$$

Note that we have two possible configurations, one for the linear center inside the circle $x^2 + y^2 = 1$ and the quadratic center outside this circle, and vice versa. In both cases we have one unique limit cycle passing through the solution (42). See this two possibilities in Figure 2 and Figure 3.

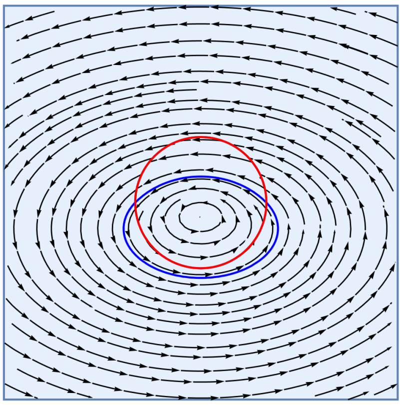

FIGURE 2. The limit cycle of the continuous piecewise differential system formed by the linear center (37) outside the circle $x^2 + y^2 = 1$, and the quadratic center (39) inside the circle.

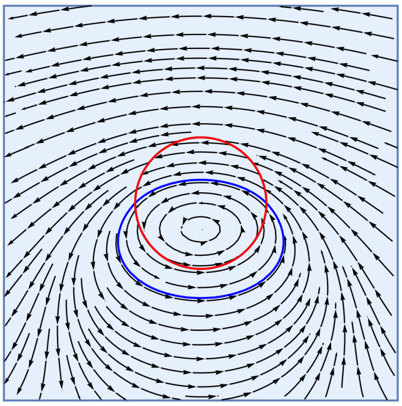

FIGURE 3. The limit cycle of the continuous piecewise differential system formed by the linear center (37) inside the circle $x^2 + y^2 = 1$, and the quadratic center (39) outside the circle.

ACKNOWLEDGMENTS

The first author is partially supported by the grant #2022/01375-7 of the São Paulo Research Foundation (FAPESP). The second author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

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