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LIMIT CYCLES IN PIECEWISE POLYNOMIAL HAMILTONIAN SYSTEMS ALLOWING NONLINEAR SWITCHING BOUNDARIES

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Abstract. This paper aims to study the limit cycles of planar piecewise polynomial Hamiltonian systems of degree n with the switching boundary $y = x^m$, where m and n are positive integers. We answer the extension of the 16th Hilbert problem for such systems, providing an upper bound for the maximum number of limit cycles in function of m and n. We also are devoted to giving a lower bound via perturbing piecewise linear Hamiltonian systems having at the origin a global center. For this a complete classification on the center conditions is established. In pursuit of a better lower bound, we require that this global center is nonlinear induced by the piecewise linearity, instead of the normal linear differential center considered in most of the existing articles. This renders that the traveling time of the unperturbed periodic orbits in each smooth zone is not calculable explicitly. Thus it is difficult to use the known Melnikov functions and averaged functions for studying the limit cycles. To overcome this difficulty, we develop an arbitrary order Melnikov-like function, which does not depend on the travelling time, for general d-dimensional piecewise smooth integrable systems allowing nonlinear switching boundaries. Finally, employing the new Melnikov-like function to the considered perturbation problem, we obtain a lower bound and an upper bound for the maximum number of limit cycles bifurcating from the unperturbed periodic orbits up to any order.

1. Introduction

Numerous phenomena in the fields of mechanics [37, 39], neurosciences [42], electronics [6, 12] and epidemiology [40] show that non-smoothness or even discontinuity is one of the main theme of the real world. This inspires many researchers to study the dynamics of piecewise smooth differential systems, which consist of distinct smooth ordinary differential equations defined in different domains separated by some smooth manifolds, called *switching boundaries* [6]. For these systems, once they are discontinuous, the definition of solutions and basic theories have been established by A.F. Filippov in [16]. Because of switching boundaries, piecewise smooth differential systems possess a wealth of novel dynamics that do not exist in smooth differential systems, such as sliding motion [16], discontinuity-induced bifurcation [6], and an abrupt transition from strongly stable periodic motion to full scale chaotic motion under perturbations [6, 7].

Another interesting dynamical phenomenon is that piecewise smooth Hamiltonian systems can have limit cycles, more precisely, *crossing limit cycles* [24], as revealed in [26, 45] and others. This means that it is reasonable to extend the famous 16th Hilbert problem to piecewise polynomial Hamiltonian systems (PWPHSs), i.e. ask for the maximum number of

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crossing limit cycles that PWPHSs of a given degree may have. This extended problem heavily depends on the shape of switching boundaries, which have a significant impact on the number of crossing limit cycles as shown in [10, 27, 46] and others. When the switching boundary is a straight line, [45] gives lower bounds and upper bounds for the maximum number of the small amplitude crossing limit cycles appearing in the Hopf bifurcations of PWPHSs, and [26] obtains a sharp upper bound for the maximum number of the large amplitude crossing limit cycles bifurcating from the periodic orbits of the harmonic oscillator under the first order piecewise polynomial Hamiltonian perturbations. When the switching boundary is no longer a straight line, except for piecewise linear Hamiltonian systems, see [3, 4, 13] for conic or irreducible curves as switching boundaries, it seems that there exist no results for general degree. Generally speaking, nonlinear switching boundaries lead to more complex dynamics and greatly increase the difficulty. At present, as far as we know, the extended 16th Hilbert problem for PWPHSs is far from being solved, no matter what the shape of the switching boundary is.

In this paper we represent a step forward in solving the extended 16th Hilbert problem for PWPHSs. In particular, we consider the PWPHSs

$$(\dot{x},\dot{y}) = \begin{cases} \left(-\partial_y H^+(x,y), \partial_x H^+(x,y)\right), & y > x^m, \\ \left(-\partial_y H^-(x,y), \partial_x H^-(x,y)\right), & y < x^m, \end{cases} \tag{1}$$

where $H^+(x,y)$ and $H^-(x,y)$ are real polynomials of degree $n+1$, $m, n \in \mathbb{N}^+$, and $\mathbb{N}^+$ denotes the set of positive integers. We will prove the following theorem.

Theorem 1. *Let $\mathcal{H}(m,n)$ be the maximum number of crossing limit cycles that the PWPHS (1) may have for given $m \in \mathbb{N}^+$ and $n \in \mathbb{N}^+$. Then*

$$\mathcal{H}(m,n) \leq \sum_{p=1}^{\left[\frac{mn+m}{2}\right]} \left[\frac{(mn+m-1)^{2p}}{2p}\right],$$

where $[\,\cdot\,]$ denotes the integer part function. In particular, $\mathcal{H}(1,2) \leq 1$.

On the other hand, we are also interested in providing a lower bound of $\mathcal{H}(m,n)$. To this end, we study the bifurcation of periodic orbits in the perturbation problem

$$(\dot{x},\dot{y}) = \begin{cases} \left(-\partial_y H_0^+(x,y) - \sum_{i=1}^{\infty} \partial_y H_i^+(x,y)\varepsilon^i, \partial_x H_0^+(x,y) + \sum_{i=1}^{\infty} \partial_x H_i^+(x,y)\varepsilon^i\right), & y > x^m, \\ \left(-\partial_y H_0^-(x,y) - \sum_{i=1}^{\infty} \partial_y H_i^-(x,y)\varepsilon^i, \partial_x H_0^-(x,y) + \sum_{i=1}^{\infty} \partial_x H_i^-(x,y)\varepsilon^i\right), & y < x^m, \end{cases} \tag{2}$$

where

$$H_0^\pm(x,y) = a_1^\pm x + a_2^\pm y + a_3^\pm x^2 + a_4^\pm xy + a_5^\pm y^2,$$

$H_i^\pm(x,y)$ for $i = 1, 2, \cdots$ are real polynomials of degree $n+1$, and we assume that the origin is a global center of the unperturbed system, i.e. $\varepsilon = 0$. Here and in the sequel the shorthand notation $\pm$ is used to represent both the $+$ and $-$ functions and equations. Besides in this paper we only consider the non-degenerate case of the unperturbed system in the following sense that the Jacobian matrix of each subsystem has no zero eigenvalues. The next theorem gives a complete classification on the center conditions.

Theorem 2. *For the non-degenerate system* (2) *with* $\varepsilon = 0$ *the origin is a global center if and only if the parameters* a_j^+ *and* a_j^-, $j = 1,2,3,4,5$, *satisfy I for* $m = 1$, *one of II–IX for* $m = 2$, *X for* $m = 2k+1, k \in \mathbb{N}^+$, *and one of XI and XII for* $m = 2k+2, k \in \mathbb{N}^+$, *where*

- I: $\Delta^+ < 0, \Delta^- < 0, a_1^+ + a_2^+ = a_1^- + a_2^- = 0, a_1^+(a_3^+ + a_4^+ + a_5^+) \le 0 \le a_1^-(a_3^- + a_4^- + a_5^-), (a_3^+ + a_4^+ + a_5^+)(a_3^- + a_4^- + a_5^-) > 0$;
- II: $\Delta^+ < 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ a_3^+ \ge 0 \ge a_2^- a_3^-, (a_2^+ + a_3^+)(a_2^- + a_3^-) > 0, a_5^+(a_2^+ + a_3^+) > 0, a_5^-(a_2^- + a_3^-) > 0, a_4^+ = a_4^- = 0$;
- III: $\Delta^+ < 0, \Delta^- < 0, a_1^+ = a_1^- = 0, (a_2^+ a_4^+)^2 - 2a_2^+ a_3^+ \Delta^+ \ge 0 \ge (a_2^- a_4^-)^2 - 2a_2^- a_3^- \Delta^-, (a_2^+ + a_3^+)(a_2^- + a_3^-) > 0, 9(a_4^+)^2 - 32a_5^+(a_2^+ + a_3^+) < 0, a_4^+ a_4^- \ne 0, a_4^+(a_2^- + a_3^-) = a_4^-(a_2^+ + a_3^+), a_5^+(a_2^- + a_3^-) = a_5^-(a_2^+ + a_3^+)$;
- IV: $\Delta^+ > 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ a_3^+ < 0, a_2^- a_3^- \le 0, (a_2^+ + a_3^+)(a_2^- + a_3^-) > 0, a_5^+(a_2^+ + a_3^+) > 0, a_5^-(a_2^- + a_3^-) > 0, a_4^+ = a_4^- = 0$;
- V: $\Delta^+ > 0, \Delta^- < 0, a_1^+ = a_1^- = 0, (a_2^+ a_4^+)^2 - 2a_2^+ a_3^+ \Delta^+ > 0 \ge (a_2^- a_4^-)^2 - 2a_2^- a_3^- \Delta^-, (a_2^+ + a_3^+)(a_2^- + a_3^-) > 0, 9(a_4^+)^2 - 32a_5^+(a_2^+ + a_3^+) < 0, a_4^+ a_4^- \ne 0, a_4^+(a_2^- + a_3^-) = a_4^-(a_2^+ + a_3^+), a_5^+(a_2^- + a_3^-) = a_5^-(a_2^+ + a_3^+)$;
- VI: $\Delta^+ > 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ + a_3^+ = 0, a_2^+ a_3^+ < 0, a_2^- a_3^- \le 0, a_5^+(a_2^- + a_3^-) > 0, a_5^-(a_2^- + a_3^-) > 0, a_4^+ = a_4^- = 0$;
- VII: $\Delta^+ < 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^- + a_3^- = 0, a_2^+ a_3^+ \ge 0 \ge a_2^- a_3^-, a_5^-(a_2^+ + a_3^+) > 0, a_5^+(a_2^+ + a_3^+) > 0, a_4^+ = a_4^- = 0$;
- VIII: $\Delta^+ > 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^- + a_3^- = 0, a_2^+ a_3^+ < 0, a_2^- a_3^- \le 0, a_5^-(a_2^+ + a_3^+) > 0, a_5^+(a_2^+ + a_3^+) > 0, a_4^+ = a_4^- = 0$;
- IX: $\Delta^+ > 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ + a_3^+ = 0, a_2^- + a_3^- = 0, a_2^+ a_3^+ < 0, a_2^- a_3^- \le 0, a_5^+ a_5^- > 0, a_4^+ = a_4^- = 0$;
- X: $\Delta^+ < 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ = a_2^- = 0$, $a_3^+ a_3^- > 0$, $(a_3^+, a_4^+, a_5^+) \in \Xi_{2k+1}^+$, $(a_3^-, a_4^-, a_5^-) \in \Xi_{2k+1}^-$;
- XI: $\Delta^+ < 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ a_3^+ \ge 0 \ge a_2^- a_3^-, a_3^+ a_3^- > 0, (a_2^-, a_3^-, a_5^-) \in \Xi_{2k+2}^-, a_4^+ = a_4^- = 0$;
- XII: $\Delta^+ < 0, \Delta^- < 0, a_1^+ = a_1^- = 0, a_2^+ = a_2^- = 0, a_3^+ a_3^- > 0, (2k+3)^2(a_4^+)^2 - 32(k+1)a_3^+ a_5^+ < 0, a_4^+ a_4^- \ne 0, a_4^+ a_3^- = a_4^- a_3^+, a_5^+ a_3^- = a_5^- a_3^+$;

where $\Delta^\pm = (a_4^\pm)^2 - 4a_3^\pm a_5^\pm$,

$$\begin{aligned}\Xi_{2k+1}^\pm =& \{(a_3^\pm, a_4^\pm, a_5^\pm) \in \mathbb{R}^3 : (2k+2)^2(a_4^\pm)^2 - 16(2k+1)a_3^\pm a_5^\pm < 0\} \\ & \cup \{(a_3^\pm, a_4^\pm, a_5^\pm) \in \mathbb{R}^3 : (2k+2)^2(a_4^\pm)^2 - 16(2k+1)a_3^\pm a_5^\pm \ge 0, a_4^\pm a_5^\pm > 0\},\end{aligned}$$

and

$$\Xi^-_{2k+2} = \{(a_2^-, a_3^-, a_5^-) \in \mathbb{R}^3 : a_2^- = 0\} \cup \Bigg\{ (a_2^-, a_3^-, a_5^-) \in \mathbb{R}^3 :$$
$$a_2^- \left(a_3^- + (k+1)a_2^- \left(\frac{-k a_2^-}{(4k+2)a_5^-} \right)^{\frac{2k}{2k+2}} + (2k+2)a_5^- \left(\frac{-k a_2^-}{(4k+2)a_5^-} \right)^{\frac{4k+2}{2k+2}} \right) < 0 \Bigg\}.$$

In addition, for $m \neq 2k$, $k \in \mathbb{N}^+$, the global center is Σ-equidistant, i.e. the distances from the two intersections of each crossing periodic orbit and the switching boundary to the global center are equal, while for $m = 2$ (resp. $m = 2k+2$, $k \in \mathbb{N}^+$), it is Σ-equidistant if and only if one of II, IV and VI–IX (resp. XI) holds.

A majority of articles such as [1,2,8,9,18,28,36] on the bifurcation of periodic orbits prefer to consider that the unperturbed periodic orbits are formed by the normal linear center $\dot{x} = -y, \dot{y} = x$. However, some prior knowledge tells that more crossing limit cycles may bifurcate from irregular periodic orbits. Therefore, in order to pursue a better lower bound of $\mathcal{H}(m,n)$, we allow the unperturbed periodic orbits to be formed by a global piecewise linear center at the origin. The cost of the extension from the normal linear center to piecewise linear centers is that we have to face some new difficulties. One of them is that the travelling time of the unperturbed periodic orbits in each smooth zone is not calculable, particularly for nonlinear switching boundaries. After carefully checking almost all of the existing Melnikov functions and averaged functions, e.g. [11, 14, 15, 19, 25, 28, 29, 41, 44] and [1, 2, 11, 14, 15, 19, 22, 25, 28–32, 34, 41, 43, 44], we notice that they, particularly the higher order ones, heavily depend on the travelling time in each smooth zone, and thus they are only well-suited for the case where the explicit expression of the travelling time can be computed. This is why the investigation for the perturbations of the linear center $\dot{x} = -y, \dot{y} = x$ is very popular. Therefore, the existing Melnikov functions and averaged functions are not applicable for our purpose. To overcome this difficulty, in Section 2 we develop an arbitrary order Melnikov-like function for general d-dimensional piecewise smooth integrable systems allowing nonlinear switching boundaries. The new function avoids computing the travelling time. Eventually employing it to the perturbation problem (2) we obtain the following theorem.

Theorem 3. *The maximum number of crossing limit cycles of system* (2) *bifurcating from the periodic orbits of any Σ-equidistant global center obtained in Theorem 2 is at most $k(mn + 7m - 4) - 4m + 3$ up to order k, taking into account the multiplicities. Moreover, at least $\zeta(1,n), \zeta(2,n)+1$ and $\zeta(m,n)+2$ crossing limit cycles can bifurcate for $m = 1$, $m = 2$ and $m \geq 3$ respectively, where*

$$\zeta(m,n) = \begin{cases} \left[\frac{n+2}{2}\right]\left[\frac{n+4}{2}\right] - 1, & m > n+1 \text{ and odd } m, \\ \left[\frac{n+2}{2}\right]\left[\frac{n+3}{2}\right] - 1, & m > n+1 \text{ and even } m, \\ \left[\frac{n+2}{2}\right]\left[\frac{n+4}{2}\right] - \left[\frac{n-m+3}{2}\right]\left[\frac{n-m+3}{2}\right] - 1, & m \leq n+1 \text{ and odd } m, \\ \left[\frac{n+2}{2}\right]\left[\frac{n+3}{2}\right] - \left[\frac{n-m+2}{2}\right]\left[\frac{n-m+3}{2}\right] - 1, & m \leq n+1 \text{ and even } m. \end{cases}$$

Actually the second part of Theorem 3 provides a lower bound of $\mathcal{H}(m,n)$. Besides, since a planar PWPHS with a linear switching boundary always can be transformed into the form of system (1) with $m = 1$, preserving the degree of the system, it directly follows from Theorem 1 and Theorem 3 that

Corollary 4. *The maximum number of crossing limit cycles that piecewise linear* (*resp. quadratic*) *Hamiltonian systems with a linear switching boundary may have is* 0 (*resp. 1*).

It is worth mentioning that the nonexistence of crossing limit cycles in piecewise linear Hamiltonian systems with a linear switching boundary also can be obtained from [35, Theorem 1, Theorem 4] with the help of Poincaré map and the theory of Chebyshev systems.

The paper is organized as follows. In order to prove the main theorems, we develop two preliminary results in Section 2. One of them aims at establishing an arbitrary order Melnikov-like function for general piecewise smooth integrable systems allowing nonlinear switching boundaries. The other gives a formula for computing the higher order derivatives of implicit functions. Using the two results we prove Theorems 1, 2 and 3 in Sections 3, 4 and 5 respectively. We summarize the contributions of this article and introduce some directions for future research in Section 6.

2. Preliminaries

The goals of this section are twofold, including establishing an arbitrary order Melnikov-like function for general piecewise smooth integrable systems that allow nonlinear switching boundaries in Section 2.1, and deriving a formula for computing the higher order derivatives of implicit functions in Section 2.2.

We start by reviewing some basics notions on piecewise smooth differential systems. Consider

$$\dot{\mathbf{z}} = \begin{cases} \mathbf{F}^+(\mathbf{z}, \varepsilon), & \mathbf{z} \in \Sigma^+, \\ \mathbf{F}^-(\mathbf{z}, \varepsilon), & \mathbf{z} \in \Sigma^-, \end{cases} \tag{3}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_d)^\top \in \mathbb{R}^d$, $d \geq 2$, $\varepsilon \in \mathbb{R}$ is a parameter, $\mathbf{F}^+ : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $\mathbf{F}^- : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ are C^∞ with respect to the variables and the parameter, Σ^+ and Σ^- are characterized as

$$\Sigma^+ = \{\mathbf{z} \in \mathbb{R}^d : \phi(\mathbf{z}) > 0\}, \qquad \Sigma^- = \{\mathbf{z} \in \mathbb{R}^d : \phi(\mathbf{z}) < 0\}$$

with the C^∞ function $\phi(\mathbf{z}) : \mathbb{R}^d \to \mathbb{R}$ having the zero as a regular value, i.e. the gradient $\partial_{\mathbf{z}}\phi(\mathbf{z})$ does not vanish for all $\mathbf{z}$ satisfying $\phi(\mathbf{z}) = 0$. In this setting the switching boundary

$$\Sigma = \{\mathbf{z} \in \mathbb{R}^d : \phi(\mathbf{z}) = 0\}$$

is a C^∞ manifold. If $\mathbf{F}^+(\mathbf{z}, \varepsilon) = \mathbf{F}^-(\mathbf{z}, \varepsilon)$ for all $\mathbf{z} \in \Sigma$, we usually regard (3) as a continuous system by defining the vector field on Σ as $\mathbf{F}^+(\mathbf{z}, \varepsilon)$. Otherwise we adopt the *Filippov convention* to define its solutions, see [16, 24] for details.

In addition to the standard periodic orbits totally located in Σ^+ or Σ^-, system (3) possesses two new types of periodic orbits, one of which is called *crossing periodic orbit*, a periodic orbit that intersects Σ only at the *crossing set*

$$\Sigma^c = \{\mathbf{z} \in \Sigma : \langle \mathbf{F}^+(\mathbf{z}, \varepsilon), \partial_{\mathbf{z}}\phi(\mathbf{z})\rangle \langle \mathbf{F}^-(\mathbf{z}, \varepsilon), \partial_{\mathbf{z}}\phi(\mathbf{z})\rangle > 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. The other is called *sliding periodic orbit*, a periodic orbit that has a segment on the *sliding set*

$$\Sigma^s = \{\mathbf{z} \in \Sigma : \langle \mathbf{F}^+(\mathbf{z}, \varepsilon), \partial_{\mathbf{z}}\phi(\mathbf{z})\rangle \langle \mathbf{F}^-(\mathbf{z}, \varepsilon), \partial_{\mathbf{z}}\phi(\mathbf{z})\rangle \leq 0\}.$$

According to [20], the points satisfying $\langle \mathbf{F}^+(\mathbf{z},\varepsilon), \partial_{\mathbf{z}}\phi(\mathbf{z})\rangle = 0$ (resp. $\langle \mathbf{F}^-(\mathbf{z},\varepsilon), \partial_{\mathbf{z}}\phi(\mathbf{z})\rangle = 0$) are called *tangency points* of subsystem $(3)_+$ (resp. $(3)_-$).

Furthermore, a crossing (resp. sliding) periodic orbit is said to be a *crossing limit cycle* (resp. *sliding limit cycle*) if it is isolated in the set of all crossing (resp. sliding) periodic orbits of the system. In this paper we are only interested in the crossing case.

The following notation will be used repeatedly. For a sufficiently smooth function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$, we follow [33] to define a symmetric L-multilinear mapping $\partial^L_{\mathbf{x}}\mathbf{g}(\mathbf{x})$, which acts on a 'product' of L n-dimensional vectors, as

$$\partial^L_{\mathbf{x}}\mathbf{g}(\mathbf{x}) \bigodot_{j=1}^{L} \mathbf{u}_j = \sum_{i_1,\cdots,i_L=1}^{n} \frac{\partial^L \mathbf{g}(\mathbf{x})}{\partial x_{i_1}\cdots \partial x_{i_L}} u_{1i_1}\cdots u_{Li_L},$$

where L is a positive integer, $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$ and $\mathbf{u}_j = (u_{j1}, u_{j2}, \cdots, u_{jn})^\top \in \mathbb{R}^n$. Moreover, denote $\mathbf{u}^b = \bigodot_{j=1}^{b} \mathbf{u} \in \mathbb{R}^{nb}$ for a positive integer b and an n-dimensional vector $\mathbf{u}$.

The Faá di Bruno's formula [23,33] will be used to compute the higher order derivatives of the composite functions involved in this paper. For the convenience of the readers, we state it here.

Theorem 5. *Let $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{f}(y) : \mathbb{R} \to \mathbb{R}^n$ be two functions with a sufficient number of derivatives. Then*

$$\frac{d^l \mathbf{g}(\mathbf{f}(y))}{dy^l} = \sum_{S_l} \frac{l!}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \partial^L_{\mathbf{x}}\mathbf{g}(\mathbf{f}(y)) \bigodot_{i=1}^{l} \left(\frac{d^i\mathbf{f}(y)}{dy^i}\right)^{b_i},$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

2.1. **Melnikov analysis for piecewise smooth integrable systems.** Assume that the piecewise smooth differential system (3) satisfies the following.

(H1) The function $\phi(\mathbf{z})$ can be written in the form

$$\phi(\mathbf{z}) = z_q - \psi(z_1, \cdots, z_{q-1}, z_{q+1}, \cdots, z_d).$$

(H2) For $\varepsilon \in \mathbb{R}$, subsystem $(3)_+$ (resp. $(3)_-$) has $d-1$ C^∞ first integrals $I_i^+(\mathbf{z},\varepsilon)$ (resp. $I_i^-(\mathbf{z},\varepsilon)$), $i = 1, 2, \cdots, d-1$, such that for each $\mathbf{z} \in \Sigma^+$ (resp. Σ^-), the associated $d-1$ gradients

$$\partial_{\mathbf{z}} I_1^+(\mathbf{z},\varepsilon), \cdots, \partial_{\mathbf{z}} I_{d-1}^+(\mathbf{z},\varepsilon) \quad (\text{resp. } \partial_{\mathbf{z}} I_1^-(\mathbf{z},\varepsilon), \cdots, \partial_{\mathbf{z}} I_{d-1}^-(\mathbf{z},\varepsilon))$$

are linearly independent.

(H3) For $\varepsilon = 0$, there is an open set $U \subset \mathbb{R}^{d-1}$ such that system (3) has a family of unimodal crossing periodic orbits $\mathcal{A} = \{\Gamma_{\mathbf{h}} : \mathbf{h} \in U\}$ parameterized by $\mathbf{h}$, i.e. each crossing periodic orbit exactly has two different intersections with Σ, denoted by $\mathbf{z}_0(\mathbf{h})$ and $\mathbf{z}(\mathbf{h})$. Moreover, both $\mathbf{z}_0(\mathbf{h})$ and $\mathbf{z}(\mathbf{h})$ are one-to-one functions.

(H4) The d gradients $\partial_{\mathbf{z}}\phi(\mathbf{z}), \partial_{\mathbf{z}} I_1^+(\mathbf{z},0), \cdots, \partial_{\mathbf{z}} I_{d-1}^+(\mathbf{z},0)$ (resp. $\partial_{\mathbf{z}}\phi(\mathbf{z}), \partial_{\mathbf{z}} I_1^-(\mathbf{z},0), \cdots, \partial_{\mathbf{z}} I_{d-1}^-(\mathbf{z},0)$) are linearly independent at $\mathbf{z} = \mathbf{z}(\mathbf{h})$.

For any given $\mathbf{h}_* \in U$, we next establish a criterion for determining the persistence of the crossing periodic orbit $\Gamma_{\mathbf{h}_*}$ as ε varies in a sufficiently small neighborhood of 0.

By (H2)–(H3) and the C^∞ dependency of solutions of an ordinary differential system on initial values and parameters, there exists a neighborhood $V \subset U$ of $\mathbf{h}_*$ such that for all $\mathbf{h} \in V$ and $|\varepsilon|$ sufficiently small, the orbit of subsystem $(3)_+$ (resp. $(3)_-$) with the initial value $\mathbf{z}_0(\mathbf{h}) \in \Sigma$ evolves in Σ^+ (resp. Σ^-) until it reaches Σ again at a point $\mathbf{z}^+(\mathbf{h},\varepsilon)$ (resp. $\mathbf{z}^-(\mathbf{h},\varepsilon)$) after a positive or negative travelling time.

Lemma 6. *The functions $\mathbf{z}^\pm(\mathbf{h},\varepsilon)$, defined for $\mathbf{h} \in V$ and $|\varepsilon|$ sufficiently small, are C^∞ with respect to ε, and we can write them as*

$$\mathbf{z}^\pm(\mathbf{h},\varepsilon) = \mathbf{z}(\mathbf{h}) + \sum_{k=1}^{\infty} \frac{\mathbf{z}_k^\pm(\mathbf{h})}{k!}\varepsilon^k, \tag{4}$$

where $\mathbf{z}_k^\pm(\mathbf{h}) : V \to \mathbb{R}^d$ for $k \geq 1$ are recurrently defined by

$$\begin{aligned} \mathbf{z}_1^\pm(\mathbf{h}) &= \left(\partial_{\mathbf{z}}\mathbf{I}_0^\pm(\mathbf{z}(\mathbf{h}))\right)^{-1}\left(\mathbf{I}_1^\pm(\mathbf{z}_0(\mathbf{h})) - \mathbf{I}_1^\pm(\mathbf{z}(\mathbf{h}))\right), \\ \mathbf{z}_k^\pm(\mathbf{h}) &= \left(\partial_{\mathbf{z}}\mathbf{I}_0^\pm(\mathbf{z}(\mathbf{h}))\right)^{-1}\left(\mathbf{I}_k^\pm(\mathbf{z}_0(\mathbf{h})) - \mathbf{I}_k^\pm(\mathbf{z}(\mathbf{h})) - \mathbf{J}_k^\pm(\mathbf{h})\right), \end{aligned} \tag{5}$$

$\mathbf{I}_k^\pm(\mathbf{z})$ are the k-th $(k \geq 0)$ order partial derivatives of

$$\mathbf{I}^\pm(\mathbf{z},\varepsilon) = \left(\phi(\mathbf{z}), I_1^\pm(\mathbf{z},\varepsilon), \cdots, I_{d-1}^\pm(\mathbf{z},\varepsilon)\right)^\top$$

with respect to ε evaluated at $\varepsilon = 0$, and

$$\begin{aligned} \mathbf{J}_k^\pm(\mathbf{h}) =& k!\sum_{l=1}^{k-1}\frac{1}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_{\mathbf{z}}^L\mathbf{I}_{k-l}^\pm(\mathbf{z}(\mathbf{h}))\bigodot_{i=1}^{l}\mathbf{z}_i^\pm(\mathbf{h})^{b_i} \\ &+ k!\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\partial_{\mathbf{z}}^L\mathbf{I}_0^\pm(\mathbf{z}(\mathbf{h}))\bigodot_{i=1}^{k}\mathbf{z}_i^\pm(\mathbf{h})^{b_i}, \end{aligned}$$

S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1+2b_2+\cdots+lb_l = l$, $\sigma = \{(0,0,\cdots,0,1)\}$ and $L = b_1 + b_2 + \cdots + b_l$.

Proof. The C^∞ smoothness of $\mathbf{z}^\pm(\mathbf{h},\varepsilon)$ with respect to ε directly follows from the C^∞ dependency of solutions on parameters, because each subsystem of (3) is assumed to be C^∞ with respect to ε.

In order to obtain (4), it is enough to prove that

$$\begin{aligned} \partial_\varepsilon\mathbf{z}^\pm(\mathbf{h},\varepsilon)\big|_{\varepsilon=0} &= \mathbf{z}_1^\pm(\mathbf{h}), \\ \partial_\varepsilon^k\mathbf{z}^\pm(\mathbf{h},\varepsilon)\big|_{\varepsilon=0} &= \mathbf{z}_k^\pm(\mathbf{h}), \qquad k \geq 2, \end{aligned} \tag{6}$$

due to $\mathbf{z}^\pm(\mathbf{h},0) = \mathbf{z}(\mathbf{h})$.

Let

$$\widetilde{\mathbf{I}}^\pm(\mathbf{z},\varepsilon) = \left(0, I_1^\pm(\mathbf{z},\varepsilon), \cdots, I_{d-1}^\pm(\mathbf{z},\varepsilon)\right)^\top$$

and $\widetilde{\mathbf{I}}_k^\pm(\mathbf{z})$ be the k-th $(k \geq 0)$ order partial derivative with respect to ε evaluated at $\varepsilon = 0$. To alleviate the notation, we will drop the superscript $\pm$ in the rest. Considering the functions $\widetilde{\mathbf{I}}(\mathbf{z}_0(\mathbf{h}),\varepsilon)$ and $\mathbf{I}(\mathbf{z}(\mathbf{h},\varepsilon),\varepsilon)$, for $k \geq 1$ we get

$$\partial_\varepsilon^k\widetilde{\mathbf{I}}(\mathbf{z}_0(\mathbf{h}),\varepsilon)\Big|_{\varepsilon=0} = \widetilde{\mathbf{I}}_k(\mathbf{z}_0(\mathbf{h})) \tag{7}$$

and

$$\begin{aligned}
&\partial_\varepsilon^k \mathbf{I}(\mathbf{z}(\mathbf{h},\varepsilon),\varepsilon)\big|_{\varepsilon=0}\\
&=\partial_\varepsilon^k\left(\sum_{i=0}^{\infty}\frac{1}{i!}\mathbf{I}_i(\mathbf{z}(\mathbf{h},\varepsilon))\varepsilon^i\right)\Bigg|_{\varepsilon=0}\\
&=\sum_{i=0}^{\infty}\frac{1}{i!}\left(\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\partial_\varepsilon^l\mathbf{I}_i(\mathbf{z}(\mathbf{h},\varepsilon))\partial_\varepsilon^{k-l}\varepsilon^i\right)\Bigg|_{\varepsilon=0}\\
&=\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\,\partial_\varepsilon^l\mathbf{I}_{k-l}(\mathbf{z}(\mathbf{h},\varepsilon))\big|_{\varepsilon=0}\\
&=\mathbf{I}_k(\mathbf{z}(\mathbf{h}))+\sum_{l=1}^{k}\frac{k!}{l!(k-l)!}\,\partial_\varepsilon^l\mathbf{I}_{k-l}(\mathbf{z}(\mathbf{h},\varepsilon))\big|_{\varepsilon=0}\\
&=\mathbf{I}_k(\mathbf{z}(\mathbf{h}))+\sum_{l=1}^{k}\frac{k!}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_{\mathbf{z}}^L\mathbf{I}_{k-l}(\mathbf{z}(\mathbf{h}))\bigodot_{i=1}^{l}\left(\partial_\varepsilon^i\mathbf{z}(\mathbf{h},\varepsilon)\big|_{\varepsilon=0}\right)^{b_i}.
\end{aligned}\tag{8}$$

We used Theorem 5 in the last equality of (8).

Since $\mathbf{z}(\mathbf{h},\varepsilon)\in\Sigma$ and $\mathbf{z}_0(\mathbf{h})\in\Sigma$ are connected by the same orbit,

$$\widetilde{\mathbf{I}}(\mathbf{z}_0(\mathbf{h}),\varepsilon)=\mathbf{I}(\mathbf{z}(\mathbf{h},\varepsilon),\varepsilon),$$

so that

$$\partial_\varepsilon^k\widetilde{\mathbf{I}}(\mathbf{z}_0(\mathbf{h}),\varepsilon)\Big|_{\varepsilon=0}=\partial_\varepsilon^k\mathbf{I}(\mathbf{z}(\mathbf{h},\varepsilon),\varepsilon)\big|_{\varepsilon=0},\qquad k\geq 1.$$

This together with (7) and (8) conclude that

$$\begin{aligned}
\widetilde{\mathbf{I}}_1(\mathbf{z}_0(\mathbf{h}))=&\mathbf{I}_1(\mathbf{z}(\mathbf{h}))+\partial_{\mathbf{z}}\mathbf{I}_0(\mathbf{z}(\mathbf{h}))\ \partial_\varepsilon\mathbf{z}(\mathbf{h},\varepsilon)\big|_{\varepsilon=0},\\
\widetilde{\mathbf{I}}_k(\mathbf{z}_0(\mathbf{h}))=&\mathbf{I}_k(\mathbf{z}(\mathbf{h}))+\sum_{l=1}^{k}\frac{k!}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_{\mathbf{z}}^L\mathbf{I}_{k-l}(\mathbf{z}(\mathbf{h}))\bigodot_{i=1}^{l}\left(\partial_\varepsilon^i\mathbf{z}(\mathbf{h},\varepsilon)\big|_{\varepsilon=0}\right)^{b_i}\\
=&\mathbf{I}_k(\mathbf{z}(\mathbf{h}))+\sum_{l=1}^{k-1}\frac{k!}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_{\mathbf{z}}^L\mathbf{I}_{k-l}(\mathbf{z}(\mathbf{h}))\bigodot_{i=1}^{l}\left(\partial_\varepsilon^i\mathbf{z}(\mathbf{h},\varepsilon)\big|_{\varepsilon=0}\right)^{b_i}\\
&+k!\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\partial_{\mathbf{z}}^L\mathbf{I}_0(\mathbf{z}(\mathbf{h}))\bigodot_{i=1}^{k}\left(\partial_\varepsilon^i\mathbf{z}(\mathbf{h},\varepsilon)\big|_{\varepsilon=0}\right)^{b_i}\\
&+\partial_{\mathbf{z}}\mathbf{I}_0(\mathbf{z}(\mathbf{h}))\ \partial_\varepsilon^k\mathbf{z}(\mathbf{h},\varepsilon)\big|_{\varepsilon=0},\qquad k\geq 2.
\end{aligned}$$

It follows from (H4) that $\partial_{\mathbf{z}}\mathbf{I}_0(\mathbf{z}(\mathbf{h}))$ is invertible. Solving the above equations and using $\widetilde{\mathbf{I}}_k(\mathbf{z}_0(\mathbf{h}))=\mathbf{I}_k(\mathbf{z}_0(\mathbf{h}))$ for $k\geq 1$, we finally get (6). This completes the proof of Lemma 6. □

Define a k-th order Melnikov-like function $\mathcal{M}_k(\mathbf{h}):V\to\mathbb{R}^{d-1}$ by

$$\mathcal{M}_k(\mathbf{h})=\pi\mathbf{z}_k^+(\mathbf{h})-\pi\mathbf{z}_k^-(\mathbf{h}),\qquad k\geq 1,\tag{9}$$

where $\pi:\mathbb{R}^d\to\mathbb{R}^{d-1}$ is the projection onto the coordinates $(z_1,\cdots,z_{q-1},z_{q+1},\cdots,z_d)$. The following theorem states that the first non-vanishing Melnikov-like function can be applied to control the persistence of the unperturbed crossing periodic orbit $\Gamma_{\mathbf{h}_*}$.

Theorem 7. *Let the piecewise smooth differential system* (3) *satisfy* (H1)–(H4)*. Then the k-th order Melnikov-like function $\mathcal{M}_k(\mathbf{h})$ defined in* (9) *is C^∞ in V. In addition, assuming that k_0 is the first positive integer such that $\mathcal{M}_{k_0}(\mathbf{h}) \not\equiv \mathbf{0}$ in V, we have the following statements.*

(i) *If $\mathbf{h}_* \in V$ is not a zero of $\mathcal{M}_{k_0}(\mathbf{h})$, then there exist no crossing periodic orbits near $\Gamma_{\mathbf{h}_*}$ for $|\varepsilon| > 0$ sufficiently small.*
(ii) *If $\mathbf{h}_* \in V$ is a simple zero of $\mathcal{M}_{k_0}(\mathbf{h})$, i.e. $\mathcal{M}_{k_0}(\mathbf{h}_*) = \mathbf{0}$ and the Jacobian matrix of $\mathcal{M}_{k_0}(\mathbf{h})$ at $\mathbf{h}_*$ has no zero eigenvalues, then there exists a unique crossing periodic orbit near $\Gamma_{\mathbf{h}_*}$ for $|\varepsilon| > 0$ sufficiently small.*

Proof. For $|\varepsilon| > 0$ sufficiently small, system (3) has a crossing periodic orbit near $\Gamma_{\mathbf{h}_*}$ if and only if the function $\mathbf{z}^+(\mathbf{h}, \varepsilon) - \mathbf{z}^-(\mathbf{h}, \varepsilon)$ has a zero in $\mathbf{h}$ near $\mathbf{h}_*$. With the help of (H1), this is equivalent to that

$$\mathcal{D}(\mathbf{h}, \varepsilon) = \pi\mathbf{z}^+(\mathbf{h}, \varepsilon) - \pi\mathbf{z}^-(\mathbf{h}, \varepsilon)$$

has a zero in $\mathbf{h}$ near $\mathbf{h}_*$ for $|\varepsilon| > 0$ sufficiently small.

By (4) and (9) we obtain

$$\begin{aligned}\mathcal{D}(\mathbf{h}, \varepsilon) &= \pi \sum_{k=1}^{\infty} \frac{\mathbf{z}_k^+(\mathbf{h})}{k!}\varepsilon^k - \pi \sum_{k=1}^{\infty} \frac{\mathbf{z}_k^-(\mathbf{h})}{k!}\varepsilon^k \\ &= \sum_{k=1}^{\infty} \frac{\pi\mathbf{z}_k^+(\mathbf{h}) - \pi\mathbf{z}_k^-(\mathbf{h})}{k!}\varepsilon^k \\ &= \sum_{k=1}^{\infty} \frac{\mathcal{M}_k(\mathbf{h})}{k!}\varepsilon^k.\end{aligned}$$

Under the assumption that $\mathcal{M}_k(\mathbf{h}) \equiv \mathbf{0}$ for $k = 1, 2, \cdots, k_0 - 1$, $\mathcal{D}(\mathbf{h}, \varepsilon)$ has the same zeros with

$$\widetilde{\mathcal{D}}(\mathbf{h}, \varepsilon) = \frac{\mathcal{M}_{k_0}(\mathbf{h})}{k_0!} + \sum_{k=k_0+1}^{\infty} \frac{\mathcal{M}_k(\mathbf{h})}{k!}\varepsilon^{k-k_0}.$$

If $\mathbf{h}_*$ is not a zero of $\mathcal{M}_{k_0}(\mathbf{h})$, then $\widetilde{\mathcal{D}}(\mathbf{h}, \varepsilon) \neq \mathbf{0}$ for $\|\mathbf{h} - \mathbf{h}_*\|$ and $|\varepsilon| > 0$ sufficiently small. This means that there exist no crossing periodic orbits near $\Gamma_{\mathbf{h}_*}$ for $|\varepsilon| > 0$ sufficiently small, i.e. statement (i) holds.

If $\mathbf{h}_*$ is a simple zero of $\mathcal{M}_{k_0}(\mathbf{h})$, then $\widetilde{\mathcal{D}}(\mathbf{h}_*, 0) = \mathbf{0}$ and the Jacobian matrix of $\widetilde{\mathcal{D}}(\mathbf{h}, \varepsilon)$ with respect to $\mathbf{h}$ evaluated at $(\mathbf{h}, \varepsilon) = (\mathbf{h}_*, 0)$ has no zero eigenvalues. Employing the Implicit Function Theorem, we conclude that there exists a unique function $\mathbf{h} = \mathbf{h}(\varepsilon)$, defined in a small neighborhood of $0 \in \mathbb{R}$, such that $\mathbf{h}(0) = \mathbf{h}_*$ and $\widetilde{\mathcal{D}}(\mathbf{h}(\varepsilon), \varepsilon) \equiv \mathbf{0}$. Consequently, we obtain a unique crossing periodic orbit near $\Gamma_{\mathbf{h}_*}$ for $|\varepsilon| > 0$ sufficiently small, i.e. statement (ii) holds. □

2.2. **Higher order derivatives of implicit functions.** As we all know, the Implicit Function Theorem is a powerful tool for mathematical analysis. But it seems that there is no a formula for computing the higher order derivatives of implicit functions. We fill this gap in the following theorem.

Theorem 8. *For a C^r $(r \geq 1)$ function $F(x, y) : \mathbb{R}^2 \to \mathbb{R}$ we assume that $F(0, 0) = 0$ and $\partial_y F(0, 0) \neq 0$. Then there exits a unique function $f(x)$, defined in a small neighborhood I of*

$0 \in \mathbb{R}$, such that $f(0) = 0$ and $F(x, f(x)) = 0$ for all $x \in I$. In addition, $f(x)$ is C^r and the l-th order derivative $f^{(l)}(0)$ at $x = 0$ is recurrently given by

$$f^{(1)}(0) = -\,\partial_x F(0,0)/\partial_y F(0,0),$$

$$f^{(l)}(0) = -\frac{1}{\partial_y F(0,0)} \sum_{S_l \setminus \sigma} \frac{l!(f^{(2)}(0))^{b_2}(f^{(3)}(0))^{b_3}\cdots(f^{(l)}(0))^{b_l}}{b_1! b_2! 2!^{b_2}\cdots b_l! l!^{b_l}} \sum_{i=0}^{b_1}\left(\frac{\partial^L F(0,0)}{\partial x^{b_1-i}\partial y^{L-b_1+i}} \frac{b_1!}{i!(b_1-i)!}(f^{(1)}(0))^i\right), \qquad 2 \le l \le r, \tag{10}$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, $\sigma = \{(0, 0, \cdots, 0, 1)\}$ and $L = b_1 + b_2 + \cdots + b_l$.

Proof. The existence, uniqueness and smoothness of $f(x)$ is the conclusion of the classical Implicit Function Theorem. The contribution of this theorem is to provide a formula for computing the higher order derivatives of $f(x)$ at $x = 0$.

Regarding $F(x, f(x))$ as the composition of F and (id, f) and taking the l-th order derivative at $x = 0$, by Theorem 5 and the meaning of the notation $\bigodot$ we get

$$\begin{aligned}
\left.\frac{dF(x,f(x))}{dx}\right|_{x=0} &= \partial_x F(0,0) + \partial_y F(0,0) f^{(1)}(0),\\
\left.\frac{d^l F(x,f(x))}{dx^l}\right|_{x=0} &= \sum_{S_l} \frac{l!}{b_1! b_2! 2!^{b_2}\cdots b_l! l!^{b_l}} \frac{\partial^L F(0,0)}{\partial(x,y)^L} \bigodot_{i=1}^{l} \left(\left.\frac{d^i x}{dx^i}\right|_{x=0}, f^{(i)}(0)\right)^{b_i}\\
&= \sum_{S_l} \frac{l!}{b_1! b_2! 2!^{b_2}\cdots b_l! l!^{b_l}} \sum_{i=0}^{b_1}\left(\frac{\partial^L F(0,0)}{\partial x^{b_1-i}\partial y^{L-b_1+i}} \frac{b_1!}{i!(b_1-i)!}(f^{(1)}(0))^i \right.\\
&\qquad \left. (f^{(2)}(0))^{b_2}(f^{(3)}(0))^{b_3}\cdots(f^{(l)}(0))^{b_l}\right), \qquad 2 \le l \le r.
\end{aligned} \tag{11}$$

Besides, from $F(x, f(x)) = 0$ for all $x \in I$, it follows that

$$\left.\frac{d^l F(x,f(x))}{dx^l}\right|_{x=0} = 0, \qquad 1 \le l \le r.$$

This together with (11) yield the formula (10). □

Applying Theorem 8 to the real polynomial

$$G(x,y) = x + y + \alpha \sum_{i=0}^{m-1} x^{m-1-i} y^i + \beta \sum_{i=0}^{m} x^{m-i} y^i + \gamma \sum_{i=0}^{2m-1} x^{2m-1-i} y^i,$$

where $m \ge 2$, we obtain

Lemma 9. *There exists a unique function $g(x)$ defined in a small neighborhood I of $0 \in \mathbb{R}$ such that $g(0) = 0$ and $G(x, g(x)) = 0$ for all $x \in I$. In addition, the following statements hold.*

(i) *For $m = 2k$ we can write $g(x)$ near $x = 0$ in the form*

$$g(x) = -x - \beta \sum_{l=2}^{\infty} c_l x^l \tag{12}$$

with

$$c_2 = \frac{1}{1+\alpha}, \qquad c_3 = \frac{\beta}{(1+\alpha)^2}, \qquad c_4 = \frac{2\beta^2}{(1+\alpha)^3} - \frac{2\gamma}{(1+\alpha)^2}$$

for $m = 2$, *while for* $m \geq 4$, $c_2 = c_3 = \cdots = c_{m-1} = 0$,

$$c_m = 1, \quad c_{2m-2} = -\frac{m}{2}\alpha, \quad c_{3m-2} = \begin{cases} -8\alpha^3 + 8\beta^2 - 4\gamma & for\ m = 4, \\ \dfrac{m^2}{2}\beta^2 - m\gamma & for\ m > 4. \end{cases}$$

(ii) *For* $m = 2k+1$ *we can write* $g(x)$ *near* $x = 0$ *in the form*

$$g(x) = -x - \alpha \sum_{l=2}^{\infty} d_l x^l \tag{13}$$

with

$$d_2 = 1, \qquad d_3 = \alpha, \qquad d_4 = 2\alpha^2 - 2\beta,$$
$$d_5 = 4\alpha^3 - 6\alpha\beta, \qquad d_6 = 9\alpha^4 - 19\alpha^2\beta + 4\beta^2 - 3\gamma$$

for $m = 3$, *while for* $m \geq 5$,

$$d_{m-1} = 1, \qquad d_{2m-2} = -\frac{(m+1)}{2}\beta, \qquad d_{3m-3} = \frac{(m+1)^2}{4}\beta^2 - m\gamma.$$

Proof. The existence, uniqueness and smoothness of $g(x)$ is a direct conclusion of the Implicit Function Theorem. Next we discuss the Taylor expansion of $g(x)$ near $x = 0$.

For $m = 2k$ (resp. $2k+1$), if $\beta = 0$ (resp. $\alpha = 0$), then $y = -x$ solves $G(x, y) = 0$, which implies $g(x) = -x$ because $G(x, y) = 0$ determines a unique function passing through $(0, 0)$. This together with the first derivative $g^{(1)}(0) = -1$ mean that we can write $g(x)$ near $x = 0$ in the form of (12) (resp. (13)) for $m = 2k$ (resp. $2k+1$).

Taking $F = G$ and $f = g$, we use formula (10) to compute the desired coefficients in (12) and (13). The computations of $c_2, c_3, c_4, d_2, d_3, d_4, d_5, d_6$ for $m = 2$ and $m = 3$ are direct and we neglect them. The key to obtaining the remaining coefficients is to determine which l-tuples $(b_1, b_2, \cdots, b_l)$ in $S_l \setminus \sigma$ contribute to the computation of $g^{(l)}(0)$. Note that

$$\frac{\partial^L G(0,0)}{\partial(x,y)^L} = 0 \qquad \text{for} \quad L \notin \{m-1, m, 2m-1\},$$

and $g^{(l)}(0)$ depends on $g^{(i)}(0)$ with $i \leq l-1$. Then, according to formula (10), for given $l \geq 2$ only the l-tuples $(b_1, b_2, \cdots, b_{l-1}, 0)$ satisfying $L = b_1 + b_2 + \cdots + b_{l-1} \in \{m-1, m, 2m-1\}$ and $b_i = 0$ for $i \leq l-1$ that makes $g^{(i)}(0) = 0$ contribute to the computation of $g^{(l)}(0)$.

Collecting these l-tuples into a set $\widetilde{S}_l$, we obtain

$$\begin{aligned}
\widetilde{S}_2 = \cdots = \widetilde{S}_{m-2} &= \emptyset,\\
\widetilde{S}_{m-1} =&\{(m-1,0,\cdots,0)\},\\
\widetilde{S}_{m} =&\{(m,0,\cdots,0),(m-2,1,0,\cdots,0)\},\\
\widetilde{S}_{m+1} =&\{(m-1,1,0,\cdots,0),(m-2,0,1,0,\cdots,0)\},\\
\widetilde{S}_{m+2} =&\{(2,0,\cdots,0,1_m,0,0)\},\\
\widetilde{S}_{2m-2} =&\{(m-2,0,\cdots,0,1_m,0,\cdots,0)\},\\
\widetilde{S}_{2m-1} =&\{(m-1,0,\cdots,0,1_m,0,\cdots,0)\},\\
\widetilde{S}_{2m} =&\{(m-2,0,\cdots,0,1_{m+2},0,\cdots,0)\},\\
\widetilde{S}_{3m-2} =&\{(m-2,0,\cdots,0,1_{2m},0,\cdots,0),(m-2,0,\cdots,0,2_m,0,\cdots,0),\\
&(m-1,0,\cdots,0,1_{2m-1},0,\cdots,0),(2m-2,0,\cdots,0,1_m,0,\cdots,0)\}
\end{aligned}$$

for $m = 2k \geq 4$, and

$$\begin{aligned}
\widetilde{S}_2 = \cdots = \widetilde{S}_{m-2} = \widetilde{S}_{m+1} &= \emptyset,\\
\widetilde{S}_{m-1} =&\{(m-1,0,\cdots,0)\},\\
\widetilde{S}_{m} =&\{(m,0,\cdots,0)\},\\
\widetilde{S}_{2m-2} =&\{(m-1,0,\cdots,0,1_{m-1},0,\cdots,0)\},\\
\widetilde{S}_{2m-1} =&\{(2m-1,0,\cdots,0)\},\\
\widetilde{S}_{3m-3} =&\{(m-1,0,\cdots,0,1_{2m-2},0,\cdots,0),(2m-2,0,\cdots,0,1_{m-1},0,\cdots,0)\}
\end{aligned}$$

for $m = 2k+1 \geq 5$. Here the subscript j of 1_j and 2_j denotes that 1 is the j-th component of the corresponding l-tuple. Finally, using the formula (10) and the facts that $c_l = -g^{(l)}(0)/(l!\beta)$ and $d_l = -g^{(l)}(0)/(l!\alpha)$, we get the coefficients $c_2,\cdots,c_m,c_{2m-2},c_{3m-2}$ in (12) and $d_{m-1},d_{2m-2},d_{3m-3}$ in (13). $\square$

3. Proof of Theorem 1

We start the proof of Theorem 1 by discussing the maximum number of isolated tangency points that each subsystem of (1) may have. By the definition a tangency point (x,x^m) of subsystem $(1)_+$ satisfies

$$\partial_x H^+(x,x^m) + mx^{m-1}\partial_y H^+(x,x^m) = 0. \tag{14}$$

The left-hand side is a polynomial of degree $mn+m-1$ because $H^+(x,y)$ is a polynomial of degree $n+1$. Thus equation (14) has at most $mn+m-1$ isolated solutions, which implies that subsystem $(1)_+$ has at most $mn+m-1$ isolated tangency points. Similarly, subsystem $(1)_-$ also has at most $mn+m-1$ isolated tangency points.

On the other hand, let Γ be a crossing limit cycle of system (1) intersecting $y = x^m$ with $2p$ different points. Note that the number of intersections of a crossing periodic orbit and $y = x^m$ must be even, because a crossing limit cycle is defined as a periodic orbit that intersects with the switching boundary only at crossing sets. Then the maximum number of intersections of

Γ and $y = x^m$ does not exceed the maximum number of isolated tangency points that each subsystem may have plus one, i.e.

$$1 \le p \le \left[\frac{mn+m}{2}\right]. \tag{15}$$

Denote the $2p$ intersections by $(u_1, u_1^m), (u_2, u_2^m), \cdots, (u_{2p}, u_{2p}^m)$ in the order of intersection and, without loss of generality, assume that (u_1, u_1^m) and (u_2, u_2^m) are connected by the orbit of subsystem $(1)_+$. Then they satisfy the system that is composed of the following $2p$ equations

$$H^+(u_{2k-1}, u_{2k-1}^m) = H^+(u_{2k}, u_{2k}^m), \qquad H^-(u_{2k}, u_{2k}^m) = H^-(u_{2k+1}, u_{2k+1}^m), \tag{16}$$

$k = 1, 2, \cdots, p$, where we are assuming that $u_{2p+1} = u_1$ for convenience.

Let $H^+(x, y)$ and $H^-(x, y)$ be characterized by

$$H^+(x,y) = \sum_{i+j=1}^{n+1} a_{i,j}^+ x^i y^j, \qquad H^-(x,y) = \sum_{i+j=1}^{n+1} a_{i,j}^- x^i y^j,$$

respectively. We can write (16) as

$$\sum_{i+j=1}^{n+1} a_{i,j}^+ \left(u_{2k-1}^{i+mj} - u_{2k}^{i+mj}\right) = 0, \qquad \sum_{i+j=1}^{n+1} a_{i,j}^- \left(u_{2k}^{i+mj} - u_{2k+1}^{i+mj}\right) = 0,$$

$k = 1, 2, \cdots, p$. Due to $u_{2k-1} \ne u_{2k}$ and $u_{2k} \ne u_{2k+1}$ for all $k = 1, 2, \cdots, p$, the system can be reduced to

$$\sum_{i+j=1}^{n+1} a_{i,j}^+ \sum_{l=0}^{i+mj-1} u_{2k-1}^l u_{2k}^{i+mj-1-l} = 0, \qquad \sum_{i+j=1}^{n+1} a_{i,j}^- \sum_{l=0}^{i+mj-1} u_{2k}^l u_{2k+1}^{i+mj-1-l} = 0, \tag{17}$$

$k = 1, 2, \cdots, p$, after cancelling factors $u_{2k-1} - u_{2k}$ and $u_{2k} - u_{2k+1}$ respectively. Clearly, the left-hand side of each equation of (17) is a polynomial of degree $mn + m - 1$. Therefore, by Bézout Theorem [38] (17) has at most $(mn + m - 1)^{2p}$ isolated real solutions. Using the symmetry of solutions of (16), we get that system (1) has at most

$$\left[\frac{(mn+m-1)^{2p}}{2p}\right]$$

crossing limit cycles that intersect with $y = x^m$ at $2p$ different points.

Finally, combined with (15), we conclude

$$\mathcal{H}(m,n) \le \sum_{p=1}^{\left[\frac{mn+m}{2}\right]} \left[\frac{(mn+m-1)^{2p}}{2p}\right].$$

In particular, if $m = 1$ and $n = 2$, we have $p = 1$ from (15), and that system (17) reads

$$\begin{aligned} &\sum_{i+j=1}^{2} a_{i,j}^+ \sum_{l=0}^{i+j-1} u_1^l u_2^{i+j-1-l} + \sum_{i+j=3} a_{i,j}^+ \sum_{l=0}^{2} u_1^l u_2^{2-l} = 0, \\ &\sum_{i+j=1}^{2} a_{i,j}^- \sum_{l=0}^{i+j-1} u_2^l u_1^{i+j-1-l} + \sum_{i+j=3} a_{i,j}^- \sum_{l=0}^{2} u_2^l u_1^{2-l} = 0, \end{aligned} \tag{18}$$

where we used $u_{2p+1}=u_1$. When $\sum_{i+j=3}a^+_{i,j}\sum_{i+j=3}a^-_{i,j}=0$, Bézout Theorem says that (18) has at most 2 isolated real solutions. When $\sum_{i+j=3}a^+_{i,j}\sum_{i+j=3}a^-_{i,j}\neq 0$, then (18) has the same solutions with

$$
\begin{aligned}
&\sum_{i+j=1}^{2}a^+_{i,j}\sum_{l=0}^{i+j-1}u_1^l u_2^{i+j-1-l}+\sum_{i+j=3}a^+_{i,j}\sum_{l=0}^{2}u_1^l u_2^{2-l}=0,\\
&\sum_{i+j=3}a^+_{i,j}\sum_{i+j=1}^{2}a^-_{i,j}\sum_{l=0}^{i+j-1}u_2^l u_1^{i+j-1-l}-\sum_{i+j=3}a^-_{i,j}\sum_{i+j=1}^{2}a^+_{i,j}\sum_{l=0}^{i+j-1}u_1^l u_2^{i+j-1-l}=0.
\end{aligned}\tag{19}
$$

By Bézout Theorem again (18) or equivalently (19) has at most 2 isolated real solutions. Using the symmetry of solutions of (18), we get that system (1) with $m=1,n=2$ has at most one crossing limit cycles, i.e. $\mathcal{H}(1,2)\leq 1$. The proof of Theorem 1 is completed.

4. Proof of Theorem 2

Consider the non-degenerate piecewise linear Hamiltonian system

$$
(\dot{x},\dot{y})=\begin{cases}\left(-\partial_y H_0^+(x,y),\partial_x H_0^+(x,y)\right), & y>x^m,\\ \left(-\partial_y H_0^-(x,y),\partial_x H_0^-(x,y)\right), & y<x^m,\end{cases}\tag{20}
$$

i.e. system (2) with $\varepsilon=0$. Let

$$
E^\pm=\left(\frac{2a_1^\pm a_5^\pm-a_2^\pm a_4^\pm}{\Delta^\pm},\frac{2a_2^\pm a_3^\pm-a_1^\pm a_4^\pm}{\Delta^\pm}\right)
$$

be the unique equilibria of subsystem $(20)_\pm$, where $\Delta^\pm$ are defined in Theorem 2. Define

$$
\Theta_m^\pm=\left(\frac{2a_1^\pm a_5^\pm-a_2^\pm a_4^\pm}{\Delta^\pm}\right)^m-\frac{2a_2^\pm a_3^\pm-a_1^\pm a_4^\pm}{\Delta^\pm}
$$

for determining the positional relationship of $E^\pm$ and the switching boundary $y=x^m$. Clearly, $\Theta_m^\pm>0$ (resp. $=0$, <0) imply that $E^\pm$ are located below (resp. in, above) $y=x^m$. We set

$$
\mathcal{C}_m^+=\{(x,y)\in\mathbb{R}^2: y=x^m, x>0\},\qquad \mathcal{C}_m^-=\{(x,y)\in\mathbb{R}^2: y=x^m, x<0\}.
$$

The proof of Theorem 2 is composed of the proofs of Propositions 10, 11, 12 and 13.

Proposition 10. *Theorem 2 holds for $m=1$.*

Proof. Suppose that the origin is a global center of the non-degenerate system (20) with $m=1$. Then E^+ and E^- must be centers, i.e.

$$
\Delta^+<0,\qquad \Delta^-<0.\tag{21}
$$

Otherwise there is at least one invariant straight line that transversally intersects the switching boundary $y=x$, which yields a contradiction. From the uniqueness of singularities of system (20) it follows that E^+ lies in $y\leq x$ and E^- lies in $y\geq x$, i.e.

$$
\Theta_1^+\geq 0\geq\Theta_1^-.
$$

The globality of the center implies that $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$ are crossing sets with opposite directions. By the definition of crossing sets this implies

$$
a_1^++a_2^+=0,\qquad a_1^-+a_2^-=0,\qquad (a_3^++a_4^++a_5^+)(a_3^-+a_4^-+a_5^-)>0.\tag{22}
$$

Under (21) and (22) we can simplify $\Theta_1^+ \geq 0 \geq \Theta_1^-$ as

$$a_1^+(a_3^+ + a_4^+ + a_5^+) \leq 0 \leq a_1^-(a_3^- + a_4^- + a_5^-). \tag{23}$$

Consequently, collecting (21), (22) and (23) together, we obtain that the condition I is necessary in order that the origin is a global center of the non-degenerate system (20) with $m = 1$.

On the other hand, the condition I ensures that all orbits of system (20) with $m = 1$ starting from $\mathcal{C}_1^+$ rotate around the origin. Thus, using the forward or backward orbit of subsystem $(20)_+$ (resp. $(20)_-$) with the initial value $(x, x) \in \mathcal{C}_1^+$, we can define a transition mapping $x_1^+(x)$ (resp. $x_1^-(x)$) from $x > 0$ to $x < 0$ such that $(x_1^+(x), x_1^+(x)) \in \mathcal{C}_1^-$ (resp. $(x_1^-(x), x_1^-(x)) \in \mathcal{C}_1^-$) and the zeros of $x_1^+(x) - x_1^-(x)$ in $x > 0$ are in one-to-one correspondence with the crossing periodic orbits of system (20) with $m = 1$. By the Hamiltonian property, $x_1^+(x)$ and $x_1^-(x)$ satisfy

$$H_0^+(x, x) = H_0^+\left(x_1^+(x), x_1^+(x)\right) \qquad \text{and} \qquad H_0^-(x, x) = H_0^-\left(x_1^-(x), x_1^-(x)\right),$$

respectively. Solving the two equations, we get that $x_1^+(x) = x_1^-(x) = -x$ for all $x > 0$ when the condition I holds. As a result, the origin is a Σ-equidistant global center, and then the sufficiency is obtained. □

Proposition 11. *Theorem 2 holds for $m = 2$.*

Proof. We first prove the necessity. In fact, to make the origin become a global center of the non-degenerate system (20) with $m = 2$, E^+ must lie in $y \leq x^2$ and E^- must lie in $y \geq x^2$, i.e.

$$\Theta_2^+ \geq 0 \geq \Theta_2^-, \tag{24}$$

from the uniqueness of singularities of system (20). Besides there must be no invariant straight lines with an infinite length segment that separates $y > x^2$ or $y < x^2$ into two unbounded zones. Thus E^- is a center, and E^+ is either a center or a saddle in $y < x^2$ with the stable manifold and unstable manifold that are not parallel to the y-axis. This yields either

$$\Delta^- < 0, \qquad \Delta^+ < 0, \tag{25}$$

or

$$\Delta^- < 0, \qquad \Delta^+ > 0, \qquad \Theta_2^+ \neq 0, \qquad a_5^+ \neq 0. \tag{26}$$

Since $\mathcal{C}_2^+$ and $\mathcal{C}_2^-$ must be crossing sets with opposite directions, it follows from the definition of crossing sets that one of the following four conditions hold

$$\begin{aligned} &a_1^+ = a_1^- = 0, && (a_2^+ + a_3^+)(a_2^- + a_3^-) > 0, \\ &9(a_4^+)^2 - 32a_5^+(a_2^+ + a_3^+) < 0, && 9(a_4^-)^2 - 32a_5^-(a_2^- + a_3^-) < 0; \end{aligned} \tag{27}$$

$$\begin{aligned} &a_1^+ = a_1^- = 0, \qquad && a_2^+ + a_3^+ = 0, \qquad && a_4^+ = 0, \\ &a_5^+(a_2^- + a_3^-) > 0, && 9(a_4^-)^2 - 32a_5^-(a_2^- + a_3^-) < 0; \end{aligned} \tag{28}$$

$$\begin{aligned} &a_1^+ = a_1^- = 0, \qquad && a_2^- + a_3^- = 0, \qquad && a_4^- = 0, \\ &a_5^-(a_2^+ + a_3^+) > 0, && 9(a_4^+)^2 - 32a_5^+(a_2^+ + a_3^+) < 0; \end{aligned} \tag{29}$$

$$a_1^+ = a_1^- = 0, \quad a_2^+ + a_3^+ = 0, \quad a_2^- + a_3^- = 0, \quad a_4^+ = 0, \quad a_4^- = 0, \quad a_5^+ a_5^- > 0. \tag{30}$$

Using the forward or backward orbit of subsystem $(20)_+$ (resp. $(20)_-$) with the initial value $(x, x^2) \in \mathcal{C}_2^+$, we can define a transition mapping $x_2^+(x)$ (resp. $x_2^-(x)$) from $x > 0$ to $x < 0$ such that

$$\left(x_2^+(x), x_2^+(x)^2\right) \in \mathcal{C}_2^- \qquad \left(\text{resp. } \left(x_2^-(x), x_2^-(x)^2\right) \in \mathcal{C}_2^-\right),$$

and the zeros of $x_2^+(x) - x_2^-(x)$ in $x > 0$ are in one-to-one correspondence with the crossing periodic orbits of system (20) with $m = 2$. The Hamiltonian property says that $x_2^+(x)$ and $x_2^-(x)$ obey

$$H_0^+\left(x, x^2\right) = H_0^+\left(x_2^+(x), x_2^+(x)^2\right), \qquad H_0^-\left(x, x^2\right) = H_0^-\left(x_2^-(x), x_2^-(x)^2\right). \tag{31}$$

If (27) holds, then (31) can be reduced to

$$x + x_2^+(x) + \frac{a_4^+}{a_2^+ + a_3^+}\sum_{i=0}^{2} x^i x_2^+(x)^{2-i} + \frac{a_5^+}{a_2^+ + a_3^+}\sum_{i=0}^{3} x^i x_2^+(x)^{3-i} = 0,$$

$$x + x_2^-(x) + \frac{a_4^-}{a_2^- + a_3^-}\sum_{i=0}^{2} x^i x_2^-(x)^{2-i} + \frac{a_5^-}{a_2^- + a_3^-}\sum_{i=0}^{3} x^i x_2^-(x)^{3-i} = 0.$$

Applying Lemma 9(ii) to solve $x_2^+(x)$ and $x_2^-(x)$ around $x = 0$ we get that $x_2^+(x) - x_2^-(x) = 0$ for all $x > 0$ if and only if either

$$a_4^+ = a_4^- = 0 \tag{32}$$

or

$$a_4^+ a_4^- \neq 0, \qquad \frac{a_4^+}{a_2^+ + a_3^+} = \frac{a_4^-}{a_2^- + a_3^-}, \qquad \frac{a_5^+}{a_2^+ + a_3^+} = \frac{a_5^-}{a_2^- + a_3^-}. \tag{33}$$

After simplification, we obtain the condition II from (24), (25), (27) and (32), the condition III from (24), (25), (27) and (33), the condition IV from (24), (26), (27) and (32), the condition V from (24), (26), (27) and (33).

If (28) holds, then (31) can be reduced to

$$\sum_{i=0}^{3} x^i x_2^+(x)^{3-i} = 0,$$

$$x + x_2^-(x) + \frac{a_4^-}{a_2^- + a_3^-}\sum_{i=0}^{2} x^i x_2^-(x)^{2-i} + \frac{a_5^-}{a_2^- + a_3^-}\sum_{i=0}^{3} x^i x_2^-(x)^{3-i} = 0.$$

Clearly, $x_2^+(x) = -x$ is the solution of the first equation. Applying Lemma 9(ii) to solve $x_2^-(x)$ around $x = 0$ we get that $x_2^+(x) - x_2^-(x) = 0$ for all $x > 0$ if and only if

$$a_4^- = 0. \tag{34}$$

Since it follows from $\Theta_2^+ \geq 0, a_1^+ = 0, a_2^+ + a_3^+ = 0, a_4^+ = 0$ that $\Delta^+ > 0$, condition VI is obtained from (24), (26), (28) and (34).

If (29) holds, then (31) can be reduced to

$$x + x_2^+(x) + \frac{a_4^+}{a_2^+ + a_3^+}\sum_{i=0}^{2} x^i x_2^+(x)^{2-i} + \frac{a_5^+}{a_2^+ + a_3^+}\sum_{i=0}^{3} x^i x_2^+(x)^{3-i} = 0,$$

$$\sum_{i=0}^{3} x^i x_2^-(x)^{3-i} = 0.$$

Similarly, $x_2^-(x) = -x$ is the solution of the second equation, and applying Lemma 9(ii) to solve $x_2^+(x)$ around $x = 0$ we get that $x_2^+(x) - x_2^-(x) = 0$ for all $x > 0$ if and only if

$$a_4^+ = 0. \tag{35}$$

Thus we get the condition VII from (24), (25), (29) and (35), and the condition VIII from (24), (26), (29) and (35).

If (30) holds, then (31) can be reduced to

$$\sum_{i=0}^{3} x^i x_2^+(x)^{3-i} = 0, \qquad \sum_{i=0}^{3} x^i x_2^-(x)^{3-i} = 0.$$

So $x_2^+(x) = x_2^-(x) = -x$. Moreover, it follows from $\Theta_2^+ \geq 0, a_1^+ = 0, a_2^+ + a_3^+ = 0, a_4^+ = 0$ that $\Delta^+ > 0$. Thus condition IX is obtained from (24), (26) and (30).

In summary, the above analysis yields the necessity for the origin being a global center of the non-degenerate system (20) with $m = 2$.

If one of the conditions II–IX holds, tracing back to the proof process of the necessity, it is not difficult to verify that (24), (25) or (26), and one of (27)-(30) ensure that all orbits of system (20) starting from $\mathcal{C}_2^+$ rotate around the origin. Thus the displacement mapping $x_2^+(x) - x_2^-(x)$ for $x > 0$ as shown below (30) is still valid to capture the crossing periodic orbits of system (20), where $x_2^+(x)$ and $x_2^-(x)$ obey (31). Then we can solve that $x_2^+(x) = x_2^-(x)$ for all $x > 0$ under given conditions. As a result, the sufficiency also holds. In particular, we find that $x_2^+(x) = x_2^-(x) = -x$ if one of the conditions II, IV, VI–IX holds, while $x_2^+(x) = x_2^-(x) \neq -x$ if one of the conditions III and V holds, which conclude that the global center is Σ-equidistant if and only if one of the conditions II, IV, VI–IX holds. ☐

Proposition 12. *Theorem 2 holds for $m = 2k + 1$, $k \in \mathbb{N}^+$.*

Proof. To make the origin be a global center of the non-degenerate system (20) with $m = 2k + 1$, $k \in \mathbb{N}^+$, we similarly obtain that E^+ is a center in $y \leq x^{2k+1}$ and E^- is a center in $y \geq x^{2k+1}$ as in the proof of the previous propositions. Hence

$$\Delta^+ < 0, \qquad \Delta^- < 0, \qquad \Theta_{2k+1}^+ \geq 0 \geq \Theta_{2k+1}^-. \tag{36}$$

Again, $\mathcal{C}_{2k+1}^+$ and $\mathcal{C}_{2k+1}^-$ are crossing sets with opposite directions. This is equivalent to

$$a_1^+ = a_1^- = 0, \quad a_3^+ a_3^- > 0, \quad (a_2^+, a_3^+, a_4^+, a_5^+) \in \widetilde{\Xi}_{2k+1}^+, \quad (a_2^-, a_3^-, a_4^-, a_5^-) \in \widetilde{\Xi}_{2k+1}^-, \tag{37}$$

where $\widetilde{\Xi}_{2k+1}^\pm$ are the parameter sets such that the equations

$$2a_3^\pm + (2k+1)a_2^\pm x^{2k-1} + (2k+2)a_4^\pm x^{2k} + (4k+2)a_5^\pm x^{4k} = 0 \tag{38}$$

have no solutions in $\mathbb{R} \setminus \{0\}$.

Using the forward or backward orbit of subsystem $(20)_+$ (resp. $(20)_-$) with the initial value $(x, x^{2k+1}) \in \mathcal{C}_{2k+1}^+$, we can define a transition mapping $x_{2k+1}^+(x)$ (resp. $x_{2k+1}^-(x)$) from $x > 0$ to $x < 0$ such that

$$\left(x_{2k+1}^+(x), x_{2k+1}^+(x)^{2k+1}\right) \in \mathcal{C}_{2k+1}^- \qquad \left(\text{resp. } \left(x_{2k+1}^-(x), x_{2k+1}^-(x)^{2k+1}\right) \in \mathcal{C}_{2k+1}^-\right),$$

and the zeros of $x_{2k+1}^+(x) - x_{2k+1}^-(x)$ in $x > 0$ are in one-to-one correspondence with the crossing periodic orbits of system (20) with $m = 2k + 1$. By the Hamiltonian property,

$x^+_{2k+1}(x)$ and $x^-_{2k+1}(x)$ satisfy

$$H_0^+\left(x, x^{2k+1}\right) = H_0^+\left(x^+_{2k+1}(x), x^+_{2k+1}(x)^{2k+1}\right),$$
$$H_0^-\left(x, x^{2k+1}\right) = H_0^-\left(x^-_{2k+1}(x), x^-_{2k+1}(x)^{2k+1}\right),$$

respectively. Due to $a_3^+ a_3^- > 0$ in (37), the two equations can be reduced to

$$\begin{aligned} x + x^+_{2k+1}(x) + \frac{a_2^+}{a_3^+}\sum_{i=0}^{2k} x^i x^+_{2k+1}(x)^{2k-i} \\ + \frac{a_4^+}{a_3^+}\sum_{i=0}^{2k+1} x^i x^+_{2k+1}(x)^{2k+1-i} + \frac{a_5^+}{a_3^+}\sum_{i=0}^{4k+1} x^i x^+_{2k+1}(x)^{4k+1-i} = 0, \\ x + x^-_{2k+1}(x) + \frac{a_2^-}{a_3^-}\sum_{i=0}^{2k} x^i x^-_{2k+1}(x)^{2k-i} \\ + \frac{a_4^-}{a_3^-}\sum_{i=0}^{2k+1} x^i x^-_{2k+1}(x)^{2k+1-i} + \frac{a_5^-}{a_3^-}\sum_{i=0}^{4k+1} x^i x^-_{2k+1}(x)^{4k+1-i} = 0. \end{aligned}$$

Using Lemma 9(ii) to solve $x^+_{2k+1}(x)$ and $x^-_{2k+1}(x)$ around $x = 0$ we obtain that $x^+_{2k+1}(x) - x^-_{2k+1}(x) = 0$ for all $x > 0$ if and only if either

$$a_2^+ = a_2^- = 0, \tag{39}$$

or

$$a_2^+ a_2^- \neq 0, \qquad \frac{a_2^+}{a_3^+} = \frac{a_2^-}{a_3^-}, \qquad \frac{a_4^+}{a_3^+} = \frac{a_4^-}{a_3^-}, \qquad \frac{a_5^+}{a_3^+} = \frac{a_5^-}{a_3^-}.$$

However, we find that the latter contradicts (36) and (37).

Therefore, collecting (36), (37) and (39) together and solving the two equations in (38), we get that the condition X is necessary for the origin being a global center of the non-degenerate system (20) with $m = 2k + 1, k \in \mathbb{N}^+$.

On the contrary, if the condition X holds, by tracing the above process (36) and (37) ensure that all orbits of system (20) starting from $\mathcal{C}^+_{2k+1}$ rotate around the origin, and (39) ensures that the displacement mapping $x^+_{2k+1}(x) - x^-_{2k+1}(x)$ defined below (38) vanishes in $x > 0$. Indeed, $x^+_{2k+1}(x) = x^-_{2k+1}(x) = -x$ for all $x > 0$. Consequently, the origin is a Σ-equidistant global center, provided the condition X. □

Proposition 13. *Theorem 2 holds for $m = 2k + 2$, $k \in \mathbb{N}^+$.*

Proof. Suppose the origin is a global center of the non-degenerate system (20) with $m = 2k+2$, $k \in \mathbb{N}^+$. We similarly obtain

$$\Delta^- < 0, \qquad \Theta^-_{2k+2} \leq 0, \tag{40}$$

as in the proof of Proposition 11. Moreover, using the forward or backward orbit of subsystem $(20)_+$ (resp. $(20)_-$) with the initial value $(x, x^{2k+2}) \in \mathcal{C}^+_{2k+2}$, we can define a transition mapping $x^+_{2k+2}(x)$ (resp. $x^-_{2k+2}(x)$) from $x > 0$ to $x < 0$ such that

$$\left(x^+_{2k+2}(x), x^+_{2k+2}(x)^{2k+2}\right) \in \mathcal{C}^-_{2k+2} \qquad \left(\text{resp. } \left(x^-_{2k+2}(x), x^-_{2k+2}(x)^{2k+2}\right) \in \mathcal{C}^-_{2k+2}\right),$$

and the zeros of $x^+_{2k+2}(x) - x^-_{2k+2}(x)$ in $x > 0$ are in one-to-one correspondence with the crossing periodic orbits of system (20) with $m = 2k + 2$. By the Hamiltonian property, $x^+_{2k+2}(x)$ and $x^+_{2k+2}(x)$ obey

$$\begin{aligned} H_0^+\left(x, x^{2k+2}\right) &= H_0^+\left(x^+_{2k+2}(x), x^+_{2k+2}(x)^{2k+2}\right), \\ H_0^-\left(x, x^{2k+2}\right) &= H_0^-\left(x^-_{2k+2}(x), x^-_{2k+2}(x)^{2k+2}\right). \end{aligned} \tag{41}$$

We claim that $a_3^+ \neq 0$. In fact, if $a_3^+ = 0$, then E^+ is a saddle by the non-degeneracy, and it cannot lie in the switching boundary $y = x^{2k+2}$ in order to the origin be a center. Moreover,

$$a_1^+ = a_1^- = 0, \tag{42}$$

because both vector fields are not transversal to $y = x^{2k+2}$ at the origin. So $a_2^+ \neq 0$, and then the first equation of (41) can be simplified to

$$\sum_{i=0}^{2k+1} x^i x^+_{2k+2}(x)^{2k+1-i} + \frac{a_4^+}{a_2^+}\sum_{i=0}^{2k+2} x^i x^+_{2k+2}(x)^{2k+2-i} + \frac{a_5^+}{a_2^+}\sum_{i=0}^{4k+3} x^i x^+_{2k+2}(x)^{4k+3-i} = 0. \tag{43}$$

Due to $\Delta^- < 0$, the second equation of (41) can be simplified to

$$\begin{aligned} x + x^-_{2k+2}(x) &+ \frac{a_2^-}{a_3^-}\sum_{i=0}^{2k+1} x^i x^-_{2k+2}(x)^{2k+1-i} \\ &+ \frac{a_4^-}{a_3^-}\sum_{i=0}^{2k+2} x^i x^-_{2k+2}(x)^{2k+2-i} + \frac{a_5^-}{a_3^-}\sum_{i=0}^{4k+3} x^i x^-_{2k+2}(x)^{4k+3-i} = 0. \end{aligned} \tag{44}$$

By Lemma 9(i) we know that $x^-(x)$ around $x = 0$ can be written in the form

$$x^-_{2k+2}(x) = -x - \frac{a_4^-}{a_3^-}x^{2k+2} + \mathcal{O}(x^{2k+3}).$$

Replacing $x^+_{2k+2}(x)$ in the equation (43) with $x^-_{2k+2}(x)$, we find that $x^-_{2k+2}(x)$ is not a solution of (43) because it follows from the non-degeneracy of system that $a_4^+ \neq 0$ if $a_3^+ = 0$. This contradicts the fact that $x^+(x) - x^-(x) = 0$ for all $x > 0$ under the assumption that the origin is a global center. The proof of the claim is completed.

Therefore, under the conditions $a_3^+ \neq 0$, (40) and (42), the fact that $\mathcal{C}^+_{2k+2}$ and $\mathcal{C}^-_{2k+2}$ are crossing sets with opposite directions is equivalent to

$$a_3^+ a_3^- > 0, \quad (a_2^+, a_3^+, a_4^+, a_5^+) \in \widetilde{\Xi}^+_{2k+2}, \quad (a_2^-, a_3^-, a_4^-, a_5^-) \in \widetilde{\Xi}^-_{2k+2}, \tag{45}$$

where $\widetilde{\Xi}^\pm_{2k+2}$ are the parameter sets such that the equations

$$2a_3^\pm + (2k+2)a_2^\pm x^{2k} + (2k+3)a_4^\pm x^{2k+1} + (4k+4)a_5^\pm x^{4k+2} = 0 \tag{46}$$

have no solutions in $\mathbb{R} \setminus \{0\}$.

Besides, due to $a_3^+ \neq 0$, the simplified equation (43) of $(41)_+$ can be replaced by

$$\begin{aligned} x + x^+_{2k+2}(x) &+ \frac{a_2^+}{a_3^+}\sum_{i=0}^{2k+1} x^i x^+_{2k+2}(x)^{2k+1-i} \\ &+ \frac{a_4^+}{a_3^+}\sum_{i=0}^{2k+2} x^i x^+_{2k+2}(x)^{2k+2-i} + \frac{a_5^+}{a_3^+}\sum_{i=0}^{4k+3} x^i x^+_{2k+2}(x)^{4k+3-i} = 0. \end{aligned} \tag{47}$$

Applying Lemma 9(i) to solve $x^+(x)$ and $x^-(x)$ around $x = 0$ from (47) and (44) respectively, we get that $x^+(x) - x^-(x) = 0$ for all $x > 0$ if and only if either

$$a_4^+ = a_4^- = 0 \tag{48}$$

or

$$a_4^+ a_4^- \neq 0, \qquad \frac{a_2^+}{a_3^+} = \frac{a_2^-}{a_3^-}, \qquad \frac{a_4^+}{a_3^+} = \frac{a_4^-}{a_3^-}, \qquad \frac{a_5^+}{a_3^+} = \frac{a_5^-}{a_3^-}. \tag{49}$$

Note that $\Delta^+ = -4a_3^+ a_5^+$ (resp. $\Delta^+ = (a_3^+/a_3^-)^2 \Delta^-$) when (48) (resp. (49)) holds, and $a_3^+ a_5^+ > 0$ from the nonexistence of solutions of (46) in $\mathbb{R} \setminus \{0\}$ and $\Delta^- < 0$ from (40). Thus

$$\Delta^+ < 0, \tag{50}$$

i.e. E^+ is a center. According to the uniqueness of singularities of system (20), E^+ must lie in $y \leq x^{2k+2}$, so that

$$\Theta_{2k+2}^+ \geq 0. \tag{51}$$

By (42) we can reduce $\Theta_{2k+2}^+ \geq 0 \geq \Theta_{2k+2}^-$ obtained in (40) and (51) to

$$a_2^+ a_3^+ \geq 0 \geq a_2^- a_3^-, \tag{52}$$

when (48) holds, and

$$a_2^+ = a_2^- = 0, \tag{53}$$

when (49) holds.

Finally, solving the two equations in (46) by standard analysis, we obtain the condition XI from (40), (42), (45), (48), (50) and (52), the condition XII from (40), (42), (45), (49), (50) and (53). This gives the necessity in order that the origin is a global center of the non-degenerate system (20) with $m = 2k + 2, k \in \mathbb{N}^+$.

On the contrary, if one of the conditions XI and XII holds, tracing back to the proof process of the necessity, we see that (40), (42), (45), (50) and (52) or (53) ensure that all orbits of system (20) starting from $\mathcal{C}_{2k+2}^+$ rotate around the origin, and (48) or (49) ensure that the displacement mapping $x_{2k+2}^+(x) - x_{2k+2}^-(x)$ defined below (40) vanishes in $x > 0$, where $x_{2k+2}^+(x)$ and $x_{2k+2}^-(x)$ obey (47) and (44) respectively. This implies the sufficiency. In particular, from (47) and (44) it follows that $x_{2k+2}^+(x) = x_{2k+2}^-(x) = -x$ if (48) holds, while if (49) holds, $x_{2k+2}^+(x) = x_{2k+2}^-(x) \neq -x$. Thus the global center is Σ-equidistant if and only if the condition XI holds. □

5. Proof of Theorem 3

In this section we apply the new arbitrary order Melnikov-like method developed in Section 2.1 to prove Theorem 3. To this end, we first derive the expression of the k-th Melnikov-like function associated to the perturbed piecewise polynomial Hamiltonian system (2).

Lemma 14. *Assume that the origin is a Σ-equidistant global center obtained in Theorem 2 for the non-degenerate system (2) with $\varepsilon = 0$. Then restricting the k-th order Melnikov-like function defined in (9) to system (2) we get*

$$\mathcal{M}_k(h) = M_k^+(h) - M_k^-(h), \qquad h \in \mathbb{R}^+, \tag{54}$$

where $M_k^{\pm}(h)$ for $k \geq 1$ are recurrently defined by

$$M_1^{\pm}(h) = \frac{H_1^{\pm}(-h,\psi(-h)) - H_1^{\pm}(h,\psi(h))}{\varphi^{\pm}(h)},$$

$$\begin{aligned}
M_k^{\pm}(h) =& \frac{k!}{\varphi^{\pm}(h)}\Bigg(H_k^{\pm}(-h,\psi(-h)) - H_k^{\pm}(h,\psi(h)) \\
&+ \sum_{l=1}^{k-1}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L_{(x,y)}H^{\pm}_{k-l}(-h,\psi(-h))\bigodot_{i=1}^{l} \\
&\left(M_i^{\pm}(h), \sum_{S_i}\frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}}\frac{d^L\psi(-h)}{dx^L}\bigodot_{j=1}^{i}M_j^{\pm}(h)^{b_j}\right)^{b_i} \\
&+ \sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\partial^L_{(x,y)}H^{\pm}_{0}(-h,\psi(-h))\bigodot_{i=1}^{k} \\
&\left(M_i^{\pm}(h), \sum_{S_i}\frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}}\frac{d^L\psi(-h)}{dx^L}\bigodot_{j=1}^{i}M_j^{\pm}(h)^{b_j}\right)^{b_i} \\
&+ \partial_y H_0^{\pm}(-h,\psi(-h))\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\frac{d^L\psi(-h)}{dx^L}\bigodot_{j=1}^{k}M_j^{\pm}(h)^{b_j}\Bigg), \quad k \geq 2,
\end{aligned}$$

and $\psi(x) = x^m$,

$$\varphi^{\pm}(h) = -\frac{d\psi(-h)}{dx}\partial_y H_0^{\pm}(-h,\psi(-h)) - \partial_x H_0^{\pm}(-h,\psi(-h)) \neq 0, \qquad \forall h \in \mathbb{R}^+.$$

Proof. Now we have $d = 2$, $\mathbf{z} = (x,y)^\top$, $q = 2$, $\phi(\mathbf{z}) = y - \psi(x)$, $I_1^{\pm}(\mathbf{z},\varepsilon) = H_0^{\pm}(x,y) + \sum_{i=1}^{\infty} H_i^{\pm}(x,y)\varepsilon^i$ to get hypotheses (H1) and (H2). Note that each orbit except for the origin is a crossing periodic orbit and $\varphi^{\pm}(h) \neq 0$ for all $h \in \mathbb{R}^+$ from the assumption of lemma. Thus hypotheses (H3) and (H4) also hold by taking $U = \mathbb{R}^+$, $\mathbf{z}_0(h) = (h,\psi(h))^\top$ and $\mathbf{z}(h) = (-h,\psi(-h))^\top$. Furthermore, identifying the notations in Lemma 6 with $\mathbf{I}_0^{\pm}(\mathbf{z}) = \left(y - \psi(x), H_0^{\pm}(x,y)\right)^\top$, $\mathbf{I}_k^{\pm}(\mathbf{z}) = (0, k!H_k^{\pm}(x,y))^\top$ for $k \geq 1$, we get

$$\left(\partial_{\mathbf{z}}\mathbf{I}_0^{\pm}(\mathbf{z}(h))\right)^{-1} = \frac{1}{\varphi^{\pm}(h)}\begin{pmatrix} \partial_y H_0^{\pm}(-h,\psi(-h)) & -1 \\ -\partial_x H_0^{\pm}(-h,\psi(-h)) & -\frac{d\psi(-h)}{dx}\end{pmatrix},$$

and

$$\mathbf{I}_k^{\pm}(\mathbf{z}_0(h)) = \left(0, k!H_k^{\pm}(h,\psi(h))\right)^\top, \qquad \mathbf{I}_k^{\pm}(\mathbf{z}(h)) = \left(0, k!H_k^{\pm}(-h,\psi(-h))\right)^\top.$$

Hence, after standard calculations for $\mathbf{z}_k^{\pm}(h)$ in (5), we obtain this lemma from (9). □

Remark that $\varphi^+(h) \equiv \varphi^-(h)$ when the origin is a linear center of system (2) with $\varepsilon = 0$. In this case, we can obtain a particular k-th order Melnikov-like function that has less isolated zeros. To a certain extent, this provides evidence for the assertion that more crossing limit cycles may bifurcate from the periodic orbits of global piecewise linear centers in comparison with linear centers.

Next we study the number of isolated zeros of the k-th order Melnikov-like function $\mathcal{M}_k(h)$ in $\mathbb{R}^+$ for $k \geq 1$.

Lemma 15. *The k-th order Melnikov-like function $\mathcal{M}_k(h)$ for $k \geq 1$ given in* (54) *has at most $k(mn + 7m - 4) - 4m + 3$ isolated zeros in $\mathbb{R}^+$, taking into account their multiplicities.*

Proof. We first prove that $M_k^+(h)$ for $k \geq 1$ can be written in the form of a rational polynomial of degree

$$\deg M_k^+(h) = \Big(kmn + (3k-2)m - 2(k-1), (2k-1)(2m-1)\Big). \tag{55}$$

Here the first and second components are degrees of the numerator and denominator of a rational polynomial respectively. This representation of the degree of rational polynomials will be repeatedly used in the rest. In fact, it is not difficult to see that $M_k^+(h)$ is defined as the multiplication and addition of a series of rational polynomials, because $\psi(x)$, $H_i^+(x,y)$ and $H_i^-(x,y)$ for $i \geq 0$ are polynomials. Thus $M_k^+(h)$ always can be written in the form of a rational polynomial, and we only need to prove that its degree satisfies (55). Obviously, (55) holds for $k=1$, because $H_0^+(x,y), H_1^+(x,y)$ and $\psi(x)$ are polynomials of degrees $2, n+1$ and m respectively.

Now assuming that (55) holds for $k = 1, 2, \cdots, k_0-1$, $k_0 \geq 2$, it remains to prove that (55) also holds for $k = k_0$ by the mathematical induction. Recalling the meaningful of the notation $\bigodot$, we get

$$\deg \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} = (0,0)$$

if $L > m$, and

$$\begin{aligned}
&\deg \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \\
=&\left((m-L) + \sum_{j=1}^{i} b_j(jmn + (3j-2)m - 2(j-1)), \sum_{j=1}^{i} b_j(2j-1)(2m-1)\right) \\
=&\Big(m - L + imn + (3i-2L)m - 2(i-L), (2i-L)(2m-1)\Big)
\end{aligned}$$

if $L \leq m$. Thus

$$\begin{aligned}
&\deg \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \\
=&\Big(m - 1 + imn + (3i-2)m - 2(i-1), (2i-1)(2m-1)\Big)
\end{aligned} \tag{56}$$

for $i = 1, 2, \cdots, k_0 - 1$, and

$$\begin{aligned}
&\deg \partial_y H_0^+(-h, \psi(-h)) \sum_{S_{k_0}\setminus\sigma} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_{k_0}!k_0!^{b_{k_0}}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{k_0} M_j^+(h)^{b_j} \\
=&\Big(m + m - 2 + k_0mn + (3k_0-4)m - 2(k_0-2), (2k_0-2)(2m-1)\Big).
\end{aligned} \tag{57}$$

It follows from (56) that

$$\begin{aligned}
&\deg\partial_{(x,y)}^L H_{k_0-l}^+(-h, \psi(-h)) \\
&\bigodot_{i=1}^{l} \left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j}\right)^{b_i} = (0,0)
\end{aligned}$$

if $L > n+1$, and

$$\begin{aligned}&\deg \partial^L_{(x,y)} H^+_{k_0-l}(-h,\psi(-h)) \bigodot_{i=1}^{l} \left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \right)^{b_i} \\ =&\left(m(n+1-L) + \sum_{i=1}^{l} b_i(m-1+imn+(3i-2)m-2(i-1)), \sum_{i=1}^{l} b_i(2i-1)(2m-1) \right) \\ =&\Big(m(n+1) - L + lmn + (3l-2L)m - 2(l-L), (2l-L)(2m-1) \Big)\end{aligned}$$

if $L \leq n+1$. This concludes

$$\begin{aligned}&\deg \sum_{S_l} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \partial^L_{(x,y)} H^+_{k_0-l}(-h,\psi(-h)) \bigodot_{i=1}^{l} \\ &\left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \right)^{b_i} \\ =&\Big(m(n+1) - 1 + lmn + (3l-2)m - 2(l-1), (2l-1)(2m-1) \Big).\end{aligned}$$

Thus

$$\begin{aligned}&\deg \sum_{l=1}^{k_0-1} \sum_{S_l} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \partial^L_{(x,y)} H^+_{k_0-l}(-h,\psi(-h)) \bigodot_{i=1}^{l} \\ &\left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \right)^{b_i} \\ =&\Big(m(n+1) - 1 + (k_0-1)mn + (3k_0-5)m - 2(k_0-2), (2k_0-3)(2m-1) \Big).\end{aligned} \tag{58}$$

Again by (56) we have

$$\begin{aligned}&\deg \partial^L_{(x,y)} H^+_0(-h,\psi(-h)) \\ &\bigodot_{i=1}^{k_0} \left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \right)^{b_i} = (0,0)\end{aligned}$$

if $L > 2$, and

$$\begin{aligned}&\deg \partial^L_{(x,y)} H^+_0(-h,\psi(-h)) \bigodot_{i=1}^{k_0} \left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \right)^{b_i} \\ =&\Big(2m-2 + k_0mn + (3k_0-4)m - 2(k_0-2), (2k_0-2)(2m-1) \Big)\end{aligned}$$

if $L=2$. So

$$\deg \sum_{S_{k_0}\setminus\sigma} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_{k_0}!k_0!^{b_{k_0}}} \partial^L_{(x,y)} H_0^+(-h,\psi(-h)) \bigodot_{i=1}^{k_0} \left(M_i^+(h), \sum_{S_i} \frac{i!}{b_1!b_2!2!^{b_2}\cdots b_i!i!^{b_i}} \frac{d^L\psi(-h)}{dx^L} \bigodot_{j=1}^{i} M_j^+(h)^{b_j} \right)^{b_i} \tag{59}$$
$$=\Big(2m-2+k_0mn+(3k_0-4)m-2(k_0-2),(2k_0-2)(2m-1)\Big).$$

Note that $H^+_{k_0}(h,\psi(h))$, $H^+_{k_0}(-h,\psi(-h))$ and $\varphi^+(h)$ are polynomials of degrees $m(n+1)$, $m(n+1)$ and $2m-1$, respectively. Consequently, combining (57), (58), (59) and the definition of $M^+_{k_0}(h)$, we get (55) for $k=k_0$.

In a similar way doing almost the same analysis we can prove that $M_k^-(h)$ for $k\geq 1$ also can be written in the form of a rational polynomial of degree

$$\deg M_k^-(h)=\Big(kmn+(3k-2)m-2(k-1),(2k-1)(2m-1)\Big).$$

In summary, according to the definition (54), $\mathcal{M}_k(h)$ for $k\geq 1$ can be written in the form of a rational polynomial of degree

$$\deg \mathcal{M}_k(h)=\Big(kmn+(3k-2)m-2(k-1)+(2k-1)(2m-1),2(2k-1)(2m-1)\Big).$$

On the other hand, observe that the denominators of $M_k^+(h)$ and $M_k^-(h)$ in the form of rational polynomials are polynomials in $\varphi^+(h)$ and $\varphi^-(h)$, respectively. Thus the denominator of $\mathcal{M}_k(h)$ in the form of rational polynomials is a polynomial in $\varphi^+(h)$ and $\varphi^-(h)$. Since $\varphi^+(h)\neq 0$ and $\varphi^-(h)\neq 0$ for all $h\in\mathbb{R}^+$, the maximum number of isolated zeros of $\mathcal{M}_k(h)$ in $\mathbb{R}^+$ is determined by the numerator, and is no more than $kmn+(3k-2)m-2(k-1)+(2k-1)(2m-1)=k(mn+7m-4)-4m+3$, taking into account their multiplicities. This completes the proof. □

Lemma 16. *The first order Melnkov-like function $\mathcal{M}_1(h)$ given in* (54) *has at least $\zeta(1,n)$, $\zeta(2,n)+1$ and $\zeta(m,n)+2$ simple zeros in $\mathbb{R}^+$ for $m=1$, $m=2$ and $m\geq 3$, respectively, where $\zeta(m,n)$ is defined in Theorem 3.*

Proof. Take

$$H_0^+(x,y)=\begin{cases} x^2+y^2 & for\ m=1,\\ \frac{1}{4}x^2+y^2 & for\ m=2,\\ \frac{3}{8}x^2+\frac{1}{2}xy+\frac{3}{8}y^2 & for\ m\geq 3\ odd,\\ y+\frac{1}{4}x^2+\frac{1}{4}y^2 & for\ m\geq 4\ even,\end{cases}$$

and

$$H_0^-(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2,\qquad H_1^+(x,y)=H_1^-(x,y)=\sum_{i+j=1}^{n+1} b_{i,j}x^iy^j.$$

We easily verify that the conditions I, II, X and XI of Theorem 2 hold for $m=1$, $m=2$, $m\geq 3$ odd and $m\geq 4$ even, respectively. Thus the origin is a Σ-equidistant global center of the unperturbed system in the above settings. By Lemma 14 we compute the corresponding first order Melnikov-like function

$$\mathcal{M}_1(h) = \mathcal{M}_{11}(h)\mathcal{M}_{12}(h),$$

where

$$\begin{aligned}
\mathcal{M}_{11}(h) &= \frac{\varphi^-(h) - \varphi^+(h)}{\varphi^+(h)\varphi^-(h)} \\
&= \begin{cases}
-\dfrac{1}{4h} & \text{for } m = 1, \\
\dfrac{-2h^2 + \frac{1}{2}}{\left(4h^3 + \frac{h}{2}\right)(2h^2+1)} & \text{for } m = 2, \\
\dfrac{\frac{m}{4}h^{2m-2} - \frac{m+1}{2}h^{m-1} + \frac{1}{4}}{\left(\frac{3m}{4}h^{2m-1} + \frac{m+1}{2}h^m + \frac{3}{4}h\right)(mh^{2m-2}+1)} & \text{for } m \geq 3 \text{ odd}, \\
\dfrac{\frac{m}{2}h^{2m-2} - mh^{m-2} + \frac{1}{2}}{\left(\frac{m}{2}h^{2m-1} + mh^{m-1} + \frac{1}{2}h\right)(mh^{2m-2}+1)} & \text{for } m \geq 4 \text{ even},
\end{cases}
\end{aligned}$$

and

$$\mathcal{M}_{12}(h) = \sum_{i+j=1}^{n+1} ((-1)^{i+mj} - 1)b_{i,j}h^{i+mj}.$$

Thus the simple zeros of $\mathcal{M}_1(h)$ in $\mathbb{R}^+$ are the ones of $\mathcal{M}_{11}(h)$ and $\mathcal{M}_{12}(h)$ in $\mathbb{R}^+$.

Clearly $\mathcal{M}_{11}(h)$ exactly has zero and one simple zero in $\mathbb{R}^+$ for $m = 1$ and $m = 2$, respectively. For $m \geq 3$ odd or even, we let $\widetilde{\mathcal{M}}_{11}(h)$ be the numerator of $\mathcal{M}_{11}(h)$. Then the zeros of $\mathcal{M}_{11}(h)$ in $\mathbb{R}^+$ are totally determined by the zeros of $\widetilde{\mathcal{M}}_{11}(h)$ because the denominator is always positive in $\mathbb{R}^+$. Observe that $\widetilde{\mathcal{M}}_{11}(h) \to 1/4$ or $1/2$ as $h \to 0$, $\widetilde{\mathcal{M}}_{11}(1) < 0$ and $\widetilde{\mathcal{M}}_{11}(h) \to +\infty$ as $h \to +\infty$, which implies that $\widetilde{\mathcal{M}}_{11}(h)$ or equivalently $\mathcal{M}_{11}(h)$ has exactly two zeros in $\mathbb{R}^+$. Moreover, we compute that the zero of $d\widetilde{\mathcal{M}}_{11}(h)/dh$ in $\mathbb{R}^+$ is

$$h_0 = \begin{cases}
\left(\dfrac{m+1}{m}\right)^{\frac{1}{m-1}} & \text{for } m \geq 3 \text{ odd}, \\
\left(\dfrac{m-2}{m-1}\right)^{\frac{1}{m}} & \text{for } m \geq 4 \text{ even}.
\end{cases}$$

Substituting h_0 into $\widetilde{\mathcal{M}}_{11}(h)$ we get that

$$\widetilde{\mathcal{M}}_{11}(h_0) = \frac{m - (1+m)^2}{4m} < 0$$

for $m \geq 3$ odd, and

$$\begin{aligned}
\widetilde{\mathcal{M}}_{11}(h_0) &= \frac{m}{2}\left(\frac{m-2}{m-1}\right)^{\frac{2m-2}{m}} - m\left(\frac{m-2}{m-1}\right)^{\frac{m-2}{m}} + \frac{1}{2} \\
&< \frac{m}{2}\frac{m-2}{m-1} - m\frac{m-2}{m-1} + \frac{1}{2} \\
&= \frac{-m^3 + 3m - 1}{2(m-1)} \\
&< 0
\end{aligned}$$

for $m \geq 4$ even. Thus the two zeros of $\widetilde{\mathcal{M}}_{11}(h)$ or equivalently $\mathcal{M}_{11}(h)$ are simple.

Regarding $\mathcal{M}_{12}(h)$ all its monomials of even degree vanish, and there are up to $\zeta(m,n)+1$ distinct monomials of odd degree. This implies that $\mathcal{M}_{12}(h)$ has at most $\zeta(m,n)$ isolated zeros in $\mathbb{R}^+$ by Descartes Theorem [5, 17]. Since the coefficients of all monomials can be chosen arbitrarily, the maximum number is reachable. In particular, we can adjust the $\zeta(m,n)$ isolated zeros of $\mathcal{M}_{12}(h)$ such that they are simple and different from the zeros of $\mathcal{M}_{11}(h)$. In conclusion, in the above settings, $\mathcal{M}_1(h)$ has $\zeta(1,n)$, $\zeta(2,n)+1$ and $\zeta(m,n)+2$ simple zeros in $\mathbb{R}^+$ for $m=1$, $m=2$ and $m\geq 3$. This proves this lemma. □

Having these preliminary lemmas we now can prove Theorem 3.

Proof of Theorem 3. As we have seen in the proof of Theorem 7, the maximum number of crossing limit cycles of system (2) that bifurcate from the periodic orbits of any Σ-equidistant global center obtained in Theorem 2 is totally controlled by the corresponding displacement function $\mathcal{D}(h,\varepsilon)$, more precisely,

$$\widetilde{\mathcal{D}}(h,\varepsilon)=\frac{\mathcal{M}_k(h)}{k!}+\mathcal{O}(\varepsilon),$$

when the first $k-1$ Melnikov-like functions vanish. From Lemma 15 we know that $\mathcal{M}_k(h)$ has at most $k(mn+7m-4)-4m+3$ isolated zeros in $\mathbb{R}^+$, taking into account their multiplicities. By [21, Theorem 3.1] this concludes that $\widetilde{\mathcal{D}}(h,\varepsilon)$ or equivalently $\mathcal{D}(h,\varepsilon)$ has at most $k(mn+7m-4)-4m+3$ isolated zeros in $\mathbb{R}^+$, taking into account their multiplicities, i.e. the upper bound in Theorem 3 is obtained. By Theorem 7 (ii) the lower bound is a direct conclusion of Lemma 16, because of the finiteness of the simple zeros of $\mathcal{M}_1(h)$ in $\mathbb{R}^+$. □

6. Conclusions and future directions

In this paper we studied the crossing limit cycles of planar piecewise polynomial Hamiltonian systems. Unlike most articles that only consider the switching boundary to be a straight line, we allow nonlinear switching boundaries in the form $y=x^m$, $m\in\mathbb{N}^+$. Generally speaking, nonlinear switching boundaries lead to more complex dynamics, such as increasing the number of crossing limit cycles. Firstly, we provided an upper bound for the maximum number $\mathcal{H}(m,n)$ of crossing limit cycles that such piecewise polynomial Hamiltonian systems of a given degree n may have, see Theorem 1. This upper bound counts the maximum number of all possible multi-crossing limit cycles, not just two-crossing limit cycles. Thus we answered the extended 16th Hilbert problem for planar piecewise polynomial Hamiltonian systems with the switching boundary $y=x^m$. Secondly, applying the new Melnikov-like function developed in Theorem 7 to the piecewise polynomial Hamiltonian perturbations of the piecewise linear Hamiltonian systems with the origin as a global center, we gave a lower bound of $\mathcal{H}(m,n)$ in Theorem 3. It is worth mentioning that we did not restrict that the global center of the unperturbed system is linear as in the papers published until now. On the contrary, in order to pursue a better lower bound of $\mathcal{H}(m,n)$, we allow the global center to be piecewise linear. For this we established a complete classification on the center conditions in Theorem 2. Notice that there is a gap between the upper bound and the lower bound that we have obtained, except for a few small m and n as we saw in Corollary 4. Therefore, a future research topic is to narrow this gap until a sharp upper bound is provided.

In Theorem 3 we also presented an upper bound for the maximum number of crossing limit cycles which can be obtained with the Melnikov-like function of order k developed in this paper for the considered perturbation problem. This upper bound is a function of m,n

and k. Because we have proved in Theorem 1 that the maximum number of crossing limit cycles of the general planar piecewise polynomial Hamiltonian systems with the switching boundary $y = x^m$ is bounded, there must be a value k_* of k such that the number of crossing limit cycles of the perturbation problem bifurcating from the unperturbed periodic orbits no longer increases as $k > k_*$. This gives rise to another research topic, namely to determine how the order k depends on m and n.

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