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ABSTRACT

We consider a system of n coupled oscillators described by the Kuramoto model with the dynamics given by $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta})$. In this system, an equilibrium solution $\boldsymbol{\theta}^*$ is considered stable when $\boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}^*) = \mathbf{0}$, and the Jacobian matrix $D\mathbf{f}(\boldsymbol{\theta}^*)$ has a simple eigenvalue of zero, indicating the presence of a direction in which the oscillators can adjust their phases. Additionally, the remaining eigenvalues of $D\mathbf{f}(\boldsymbol{\theta}^*)$ are negative, indicating stability in orthogonal directions. A crucial constraint imposed on the equilibrium solution is that $|\Gamma(\boldsymbol{\theta}^*)| \leq \pi$, where $|\Gamma(\boldsymbol{\theta}^*)|$ represents the length of the shortest arc on the unit circle that contains the equilibrium solution $\boldsymbol{\theta}^*$. We provide a proof that there exists a unique solution satisfying the aforementioned stability criteria. This analysis enhances our understanding of the stability and uniqueness of these solutions, offering valuable insights into the dynamics of coupled oscillators in this system.

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Synchronization in ensembles of network-coupled heterogeneous oscillators is crucial in various natural and engineered phenomena, ranging from cell cycles to robust power systems. One of the most prominent and elegant models for studying synchronization is the Kuramoto model,[1] which provides a mathematically tractable description of this phenomenon. Kuramoto recognized the mean-field approach as the most suitable method for analytical treatment and introduced an all-to-all purely sinusoidal coupling scheme, deriving the governing equations for each oscillator in the system. The Kuramoto model is a mathematically tractable description of synchronization in network-coupled heterogeneous oscillators. We investigate the conditions for stable equilibrium solutions in this model. Our main finding is that a stable equilibrium solution $\boldsymbol{\theta}^*$ exists when $\boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}^*) = \mathbf{0}$, and the Jacobian matrix $D\mathbf{f}(\boldsymbol{\theta}^*)$ has zero as a simple eigenvalue and negative eigenvalues in orthogonal directions. We also establish the constraint $|\Gamma(\boldsymbol{\theta}^*)| \leq \pi$, indicating that the equilibrium lies within the shortest arc on the unit circle containing it. This analysis contributes to understanding the dynamics and stability of the Kuramoto model.

I. INTRODUCTION

The Kuramoto model is a widely studied mathematical model that describes the synchronization behavior in a system of n coupled oscillators. It was introduced by Kuramoto in his seminal works[1,2] and has since become a fundamental framework for understanding synchronization phenomena in various fields. The model assumes n connected oscillators, each characterized by a phase variable θ_i and a natural frequency ω_i. The dynamics of each i-oscillator, coupled with the rest, is described by the equation

$$\dot{\theta}_i = \omega_i + \frac{K}{n}\sum_{j=1}^{n}\sin(\theta_j - \theta_i), \tag{1}$$

where $\dot{\theta}_i$ denotes the time derivative of θ_i and K represents the coupling strength among the oscillators.

The Kuramoto model has been extensively studied due to its ability to capture and explain synchronization phenomena in a wide range of systems, including biological, physical, and social systems.[3–6] It has been applied to understand phenomena, such

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as neuronal synchronization,[7] power grid synchronization,[8] and opinion formation in social networks.[9]

In this paper, we focus on investigating the stability and properties of equilibrium solutions in the Kuramoto model for a finite system of n oscillators. We analyze the conditions under which stable synchronization emerges and explore the dynamics of the system as the coupling strength K varies. Our findings contribute to a deeper understanding of synchronization mechanisms in finite oscillator networks.

II. PRELIMINARIES AND MAIN RESULT

In this work, we focus on the finite Kuramoto model, which consists of n oscillators arranged on the unit circle. The phase space for this model is the n-torus, defined as

$$\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1. \tag{2}$$

We use the usual model in each circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, identifying two angles θ_1 and θ_2 in $\mathbb{R}$ if and only if $\theta_1 - \theta_2 = 2\pi k$ for some index $k \in \mathbb{Z}$. Thus, given any point $\boldsymbol{\theta} \in \mathbb{T}^n$, we can always express $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, where $\theta_i \in (-\pi, \pi]$. This phase space serves as the domain for a system of first-order ordinary differential equations, initially introduced by Kuramoto,[1]

$$\begin{cases} \dot{\theta}_1 = & \omega_1 + Kf_1(\theta_1, \ldots, \theta_n), \\ \dot{\theta}_2 = & \omega_2 + Kf_2(\theta_1, \ldots, \theta_n), \\ \quad \vdots & \quad \vdots \\ \dot{\theta}_n = & \omega_n + Kf_n(\theta_1, \ldots, \theta_n), \end{cases} \tag{3}$$

where

$$f_i(\theta_1, \ldots, \theta_n) = \sum_{j=1, j\neq i}^{n} a_{ij} \sin(\theta_j - \theta_i) \quad \text{for} \quad 1 \leq i \leq n. \tag{4}$$

The network's topology is represented by an adjacency matrix A of size $n \times n$. Specifically, $a_{ii} = 0$, and $a_{ij} = a_{ji} = 1$ if nodes i and j are connected, while $a_{ij} = a_{ji} = 0$ otherwise. Furthermore, we require the network to be connected. Systems (3) and (4) described above are known as the "Kuramoto model" of synchronization associated with the network defined by the adjacency matrix A. However, we refer to the specific case where $a_{ij} = 1$ for all $i \neq j$ and $a_{ii} = 0$ as the "classic Kuramoto model" since it corresponds to the original formulation.

Using the compact notation $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, and $\mathbf{f} = (f_1, \ldots, f_n)$. The Kuramoto model can be expressed as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}), \tag{5}$$

where $K \geq 0$ is a real parameter. For any fixed initial condition $\boldsymbol{\theta}_0$ on the torus, there exists a unique solution $\varphi : [0, \infty) \to \mathbb{T}^n$ of (5), defined for all $t \geq 0$, passing through $\boldsymbol{\theta}_0$ at $t = 0$. An $\boldsymbol{\theta}^*$ is referred to as an *equilibrium point* of (5) if $\boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}^*) = \mathbf{0}$. In this case, the solution $\varphi(t) \equiv \boldsymbol{\theta}^*$, defined for all $t \geq 0$, satisfies the ordinary differential equation (ODE), and the local behavior of this solution is determined by the spectrum of $D\mathbf{f}(\boldsymbol{\theta}^*)$, i.e., the eigenvalues of the differential matrix. We denote the differential matrix as $\mathcal{H} := D\mathbf{f}$. Thus,

$$\mathcal{H}(\boldsymbol{\theta}) = \begin{pmatrix} h_{1,1}(\boldsymbol{\theta}) & h_{1,2}(\boldsymbol{\theta}) & \ldots & h_{1,n}(\boldsymbol{\theta}) \\ h_{1,2}(\boldsymbol{\theta}) & h_{2,2}(\boldsymbol{\theta}) & \ldots & h_{2,n}(\boldsymbol{\theta}) \\ \vdots & \vdots & \ddots & \vdots \\ h_{1,n}(\boldsymbol{\theta}) & \ldots & h_{n-1,n}(\boldsymbol{\theta}) & h_{n,n}(\boldsymbol{\theta}) \end{pmatrix}, \tag{6}$$

where

$$\begin{aligned} h_{i,i}(\boldsymbol{\theta}) &= \frac{\partial f_i}{\partial \theta_i}(\boldsymbol{\theta}) = -\sum_{j=1, j\neq i}^{n} a_{ij} \cos(\theta_j - \theta_i), \\ h_{i,j}(\boldsymbol{\theta}) &= \frac{\partial f_i}{\partial \theta_j}(\boldsymbol{\theta}) = a_{ij} \cos(\theta_j - \theta_i) \quad i \neq j. \end{aligned} \tag{7}$$

We observe that the matrix $\mathcal{H}(\boldsymbol{\theta})$ is symmetric, which implies that its eigenvalues are real. This property allows us to compute the derivative of the eigenvalues of $\mathcal{H}(\boldsymbol{\theta})$ and control how they evolve (as demonstrated in Proposition III.3). Notably, the spectrum of $\mathcal{H}(\boldsymbol{\theta})$ remains unaffected by variations in $\boldsymbol{\omega}$ or K.

Before delving into the analysis of the spectrum of $\mathcal{H}(\boldsymbol{\theta})$, we present an alternative approach to this problem. The symmetry of the matrix $\mathcal{H}$ enables us to find a primitive of $\mathbf{f}$. More precisely, we can consider the scalar map

$$V : [-\pi, \pi]^n \subset \mathbb{R}^n \to \mathbb{R}$$

given by

$$V(\boldsymbol{\theta}) = C - \boldsymbol{\omega} \cdot \boldsymbol{\theta} - K \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_{ij} \cos(\theta_i - \theta_j), \tag{8}$$

where the expression $\mathbf{a} \cdot \mathbf{b}$ denotes the inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ and C is any constant value. It is important to note that the function V is defined within the hypercube $[-\pi, \pi]^n$ but not on the torus $\mathbb{T}^n$, except in the case when $\boldsymbol{\omega} = \mathbf{0}$. This is due to the fact that the inner product $\boldsymbol{\omega} \cdot \boldsymbol{\theta}$ is not a 2π-periodic map.

To illustrate this claim, let us consider the case where $\boldsymbol{\omega} \neq \mathbf{0}$, and without loss of generality, assume that $\omega_1 \neq 0$. We observe that the inner product $\boldsymbol{\omega} \cdot (\pi, 0, \ldots, 0) = \pi\omega_1$, whereas $\boldsymbol{\omega} \cdot (-\pi, 0, \ldots, 0) = -\pi\omega_1$. This demonstrates that the inner product does not exhibit 2π periodicity, leading to the conclusion that V is not well-defined on the torus $\mathbb{T}^n$ in the presence of non-zero $\boldsymbol{\omega}$.

Using the expression for V given above, it is evident that the dynamics of the Kuramoto model (5) can be represented as the potential flow generated by the scalar map V (8). This can be seen by observing that

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}) = -\nabla V(\boldsymbol{\theta}). \tag{9}$$

Therefore, locally, the flow of the Kuramoto model can be interpreted as the potential flow induced by the scalar map V.

The Hessian matrix of V, denoted as $\mathbf{HV}$, and the differential of $\mathbf{f}$, denoted as $D\mathbf{f}$, are related by

$$\mathbf{HV}(\boldsymbol{\theta}) = -KD\mathbf{f}(\boldsymbol{\theta}) = -K\mathcal{H}(\boldsymbol{\theta}). \tag{10}$$

It is worth noting that the symmetry of $\mathcal{H}(\boldsymbol{\theta})$ is not surprising, as it is a multiple of the Hessian matrix of V. This relationship establishes the connection between the symmetry of $\mathcal{H}(\boldsymbol{\theta})$ and the potential flow representation of the Kuramoto model.

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The Kuramoto model exhibits a finite number of equilibrium points. This can be demonstrated by rewriting the nonlinear system $\boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$ as a quadratic system and applying Bézout's theorem (for more details, refer to Ref. 10). In previous discussions, we have defined an equilibrium solution of the Kuramoto system (5). Now, we introduce the concept of a stable equilibrium solution, as defined in Ref. 11.

Definition II.1: *We say that* $\boldsymbol{\theta}^* = (\theta_1^*, \ldots, \theta_n^*)$ *is a stable solution of (5) if and only if*

(a) $\boldsymbol{\omega} + K\mathbf{f}(\boldsymbol{\theta}^*) = \mathbf{0}$,
(b) $\mathcal{H}(\boldsymbol{\theta}^*)$ *is negative semi-definite, and*
(c) $\dim(Ker(\mathcal{H}(\boldsymbol{\theta}^*))) = 1$.

One important remark about the above definition is that the requirement for negative definiteness of $\mathcal{H}(\boldsymbol{\theta}^*)$ is not feasible in the Kuramoto model. Due to the nature of the equation driving system (5), stable solutions are not isolated. Specifically, if $\boldsymbol{\theta}^*$ is an equilibrium point, then for any angle $\alpha \in \mathbb{S}^1$, $\boldsymbol{\theta}^* + \alpha = (\theta_1^* + \alpha, \ldots, \theta_n^* + \alpha)$ is also an equilibrium point. We denote all these solutions as $[\boldsymbol{\theta}^*] = \{\theta^* + \alpha, \alpha \in \mathbb{S}^1\}$. This fact is evident in the eigenvalues of $\mathcal{H}(\boldsymbol{\theta})$, as $\mathbf{1} = (1, \ldots, 1)$ is an eigenvector of $\mathcal{H}(\boldsymbol{\theta})$ with eigenvalue $\lambda = 0$ for all $\boldsymbol{\theta} \in \mathbb{T}^n$. Thus, $\lambda = 0$ always appears in the spectrum of $\mathcal{H}(\boldsymbol{\theta})$. The third condition requires that the remaining eigenvalues of $\mathcal{H}(\boldsymbol{\theta}^*)$ are strictly negative.

A second important remark concerns the stability of the equilibrium point $\boldsymbol{\theta}^*$. On one hand, the existence of a strictly positive eigenvalue of $D\mathbf{f}(\boldsymbol{\theta}^*)$ implies that the unstable manifold of θ^* has a dimension greater than or equal to one. On the other hand, the stability of $\boldsymbol{\theta}^*$ can also be linked to the local behavior of V near $\boldsymbol{\theta}^*$. Thus, we impose that V exhibits a local minimum at $\boldsymbol{\theta}^*$. Consequently, the Hessian map $\mathbf{H}V(\boldsymbol{\theta}^*)$ is positive semi-definite, and, therefore, $\mathcal{H}(\boldsymbol{\theta}^*)$ is negative semi-definite since $\mathbf{H}V(\boldsymbol{\theta}^*) = -K\mathcal{H}(\boldsymbol{\theta}^*)$ [see Eq. (10)].

The following definition will play a fundamental role in the classification of stable solutions of the Kuramoto model.

Definition II.2: *For any point* $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ *on* $\mathbb{T}^n$, *we denote* $\Gamma(\boldsymbol{\theta})$ *as a closed shortest arc on* $\mathbb{S}^1$ *containing all angles. The length of this arc is denoted by* $|\Gamma(\boldsymbol{\theta})|$ *(see Fig. 1).*

The literature on the Kuramoto model is extensive, and it has served as the foundation for the study of various synchronization phenomena. Providing a comprehensive list of all contributions would be impractical. However, we can outline some key findings in the finite Kuramoto model based on different scenarios involving the frequency vector $\boldsymbol{\omega}$.

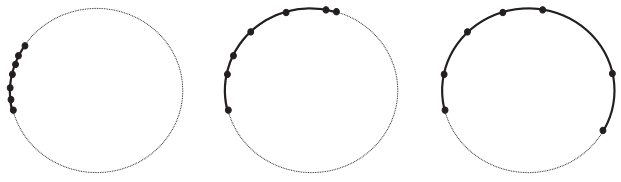

FIG. 1. Three different examples of $\theta = (\theta_1, \ldots, \theta_7) \in \mathbb{T}^7$. We show the closed shortest arc $\Gamma(\theta)$ containing all angles and their length $|\Gamma(\theta)|$.

The first case corresponds to $\boldsymbol{\omega} = \mathbf{0}$. Taylor[12] demonstrated that the origin $\boldsymbol{\theta}^* = (0, \ldots, 0)$ is the only stable solution in the classic Kuramoto model. Subsequently, several authors[12–14] showed that for sufficiently dense networks, the origin is the unique stable solution. Network density is measured using a parameter μ, which indicates that each oscillator has at least $\mu(n-1)$ connections with other oscillators. The classic Kuramoto model corresponds to $\mu = 1$. Taylor[12] proved that the origin is the unique stable solution for networks with density parameter $\mu \geq 0.9395$. Ling *et al.*[13] established the same result for $\mu \geq (3 - \sqrt{2})/2 \sim 0.7929$ and Kassabov *et al.*[14] for $\mu \geq 0.75$. Therefore, it is possible to have multiple stable solutions for small values of μ.[15] When $\boldsymbol{\omega} = \mathbf{0}$, the Kuramoto model can be analyzed using Morse theory developed by Milnor.[16] In this framework, the system of ODEs (5) represents the downhill flow of the map $V : \mathbb{T}^n \to \mathbb{R}$ (8). However, the global behavior becomes more complex, as Morse theory reveals the existence of multiple unstable solutions. The number of these unstable solutions is related to the Betti numbers of $\mathbb{T}^n$.[10] For instance, since $V : \mathbb{T}^n \to \mathbb{R}$ is a continuous map defined on a compact manifold, it must reach at least one local maximum. Consequently, there always exist initial conditions that do not converge to the stable solution $\boldsymbol{\theta}^* = \mathbf{0}$. Additionally, it should be noted that in the case of $\boldsymbol{\omega} = \mathbf{0}$, the Kuramoto model is independent of parameter K.

In the case where $\boldsymbol{\omega} \neq \mathbf{0}$, parameter K plays a crucial role in the dynamics of the Kuramoto model (5). Numerical simulations reveal the existence of a critical parameter $K_c > 0$, such that for $0 < K < K_c$, the Kuramoto model does not possess any stable solutions. This fact can be easily demonstrated. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \neq \mathbf{0}$, and assume without loss of generality that $\omega_1 \neq 0$. Then, there exists ϵ_0 such that for $0 < K < \epsilon_0$, we have $\omega_1 + Kf_1(\boldsymbol{\theta}) \neq 0$, since f_1 is a bounded map. Therefore, the Kuramoto model (5) does not exhibit any stable solutions for $0 < K < \epsilon_0$. Numerous results have been obtained regarding the estimation of the critical parameter K_c (see Refs. 3 and 17, and references therein).

On the other hand, when parameter K is sufficiently large, the Kuramoto model becomes similar to the case when $\boldsymbol{\omega} = \mathbf{0}$. Thus, for large enough K, the Kuramoto model possesses a stable equilibrium $\boldsymbol{\theta}_K$ that converges to $\mathbf{0}$ as K tends to infinity. However, the number of stable equilibrium points in the Kuramoto model is still unknown. The stability of solutions in the Kuramoto model has been extensively studied by various authors, including Refs. 11, 18, and 19.

The objective of this work is to investigate the number of stable solutions in the Kuramoto model. Our main result can be stated as follows:

Theorem A: *Let* $\boldsymbol{\theta}^*$ *and* $\boldsymbol{\eta}^*$ *be two stable solutions of the Kuramoto model (5) satisfying* $|\Gamma(\boldsymbol{\theta}^*)| \leq \pi$ *and* $|\Gamma(\boldsymbol{\eta}^*)| \leq \pi$. *Then,* $[\boldsymbol{\theta}^*] = [\boldsymbol{\eta}^*]$.

It is important to note that the above result does not impose any explicit assumptions on the system variables, such as the frequencies $\boldsymbol{\omega}$, the parameter K, or the adjacency matrix A.

The Kuramoto model is known to exhibit the possibility of multiple stable solutions (see Ref. 15, and references therein). In Sec. III, we provide a concrete example where the Kuramoto model demonstrates two stable solutions. Furthermore, we discuss how this example relates to the tools utilized in the proof of Theorem A.

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III. PROOF OF THEOREM A AND EXAMPLES

The following lemma is well known and can be proven using the Gershgorin circle theorem (see Chap. 6 of Ref. 20). We include it here for completeness.

Lemma III.1: *Let $M = (m_{i,j})$ be an $n \times n$ real, symmetric and diagonal dominant matrix, i.e., $|m_{i,i}| \geq \sum_{j=1, j\neq i}^{n} |m_{i,j}|$ for $1 \leq i \leq n$. The following conditions hold:*

- *Assume that all the entries in the diagonal verify $m_{i,i} \geq 0$. Then, all the eigenvalues of M are non-negative, and, therefore, M is positive semi-definite and $\mathbf{x}^T M \mathbf{x} \geq 0$ for all vectors $\mathbf{x} \in \mathbb{R}^n$.*
- *Assume that all the entries in the diagonal verify $m_{i,i} \leq 0$. Then, all eigenvalues of M are non-positive, and, therefore, M is negative semi-definite and $\mathbf{x}^T M \mathbf{x} \leq 0$ for all vectors $\mathbf{x} \in \mathbb{R}^n$.*

In our work, the derivative of an eigenvalue with respect to a real parameter will play a fundamental role. The following result can be used to compute the derivative of a simple or multiple eigenvalue with respect to a real parameter. It is a combination of two theorems: Theorem 5 of Ref. 21 for the derivative of a simple eigenvalue and Theorem 2.3 of Ref. 22 for the derivative of a multiple eigenvalue depending on a single real parameter. Additional references related to this result include Refs. 22 and 23.

Lemma III.2: *Let $A(t)$ be a $n \times n$ real and symmetric matrix and such that $t \mapsto A(t)$ is a real analytic function of all $t \in (a, b) \subset \mathbb{R}$. Suppose that $\lambda(t_0)$ is an eigenvalue of $A(t_0)$ with multiplicity $r \geq 1$, where $t_0 \in (a, b)$. Then, there exists $\epsilon > 0$, and real analytic functions $\lambda_1(t), \ldots, \lambda_r(t)$ and $\mathbf{v}_1(t), \ldots, \mathbf{v}_r(t)$, such that*

$$\begin{aligned} A(t)\mathbf{v}_s(t) &= \lambda_s(t)\mathbf{v}_s(t), \quad \forall t \in (t_0 - \epsilon, t_0 + \epsilon), \\ \lambda_s(t_0) &= \lambda(t_0), \quad s = 1, \ldots, r, \\ \mathbf{v}_s(t)^T \mathbf{v}_s(t) &= 1, \end{aligned} \tag{11}$$

and we have that

$$\lambda_s'(t_0) = \frac{d\lambda_s}{dt}(t_0) = \mathbf{v}_s(t_0)^T A'(t_0)\, \mathbf{v}_s(t_0), \quad s = 1, \ldots, r.$$

Using the above result, we can prove that, under certain conditions, the derivative of the eigenvalues of the matrix $\mathcal{H}(t\boldsymbol{\theta})$ is not negative. In the following lemma, we collect this result.

Proposition III.3: *Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ be any point in $\mathbb{T}^n$ (2). For any $t \geq 0$, we define the $n \times n$ matrix $A(t) = \mathcal{H}(t\boldsymbol{\theta})$. Let $\lambda(t_0)$ be an eigenvalue of $A(t_0)$. Then, $\lambda'(t_0) \geq 0$ for all $0 < t_0 \leq 1/2$. Moreover, assuming that $\theta_i \in [0, \pi]$ for all $1 \leq i \leq n$, then $\lambda'(t_0) \geq 0$ for all $0 < t_0 \leq 1$.*

Proof. We consider the $n \times n$ symmetric matrix given by $A(t) = \mathcal{H}(t\boldsymbol{\theta})$. From the definition of $\mathcal{H}(\boldsymbol{\theta})$ [see Eqs. (6) and (7)], we have that $A(t) = (h_{i,j}(t\boldsymbol{\theta}))$ for $1 \leq i, j \leq n$, where the functions $h_{i,j}(t\boldsymbol{\theta})$ are given by

$$\begin{aligned} h_{i,i}(t\boldsymbol{\theta}) &= -\sum_{j=1, j\neq i}^{n} a_{ij} \cos\left(t\left(\theta_j - \theta_i\right)\right), \\ h_{i,j}(t\boldsymbol{\theta}) &= a_{ij} \cos\left(t\left(\theta_j - \theta_i\right)\right). \end{aligned}$$

Computing the derivative with respect to the real parameter t, we obtain

$$\begin{aligned} \frac{dh_{i,i}(t\boldsymbol{\theta})}{dt} &= \sum_{j=1, j\neq i}^{n} a_{ij}(\theta_j - \theta_i) \sin\left(t(\theta_j - \theta_i)\right), \\ \frac{dh_{i,j}(t\boldsymbol{\theta})}{dt} &= -a_{ij}(\theta_j - \theta_i) \sin\left(t(\theta_j - \theta_i)\right). \end{aligned} \tag{12}$$

Thus, we have obtained an explicit expression of the derivative $A'(t)$ given by

$$A'(t) = \begin{pmatrix} \frac{dh_{1,1}(t\boldsymbol{\theta})}{dt} & \frac{dh_{1,2}(t\boldsymbol{\theta})}{dt} & \cdots & \frac{dh_{1,n}(t\boldsymbol{\theta})}{dt} \\ \frac{dh_{2,1}(t\boldsymbol{\theta})}{dt} & \frac{dh_{2,2}(t\boldsymbol{\theta})}{dt} & \cdots & \frac{dh_{2,n}(t\boldsymbol{\theta})}{dt} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{dh_{n,1}(t\boldsymbol{\theta})}{dt} & \frac{dh_{n,2}(t\boldsymbol{\theta})}{dt} & \cdots & \frac{dh_{n,n}(t\boldsymbol{\theta})}{dt} \end{pmatrix}. \tag{13}$$

We introduce the auxiliary function $g_t(x) = x \sin(tx)$ for $t > 0$. We left to the reader to check the following properties of the map g_t (see Fig. 2). For any value of $t > 0$, the function g_t is an even map, i.e., $g_t(x) = g_t(-x)$ for all $x \in \mathbb{R}$ and $g_t(x) \geq 0$ for all $x \in [-\pi/t, \pi/t]$. Moreover, if we pick any value of $t \in (0, 1/2]$, then $g_t(x) \geq 0$ for all $x \in [-2\pi, 2\pi]$.

For any $t > 0$, the matrix $A'(t)$ is given by

$$A'(t) = \begin{pmatrix} \sum_{j=1, j\neq 1}^{n} a_{1j} g_t(\theta_j - \theta_1) & -a_{12} g_t(\theta_2 - \theta_1) & \cdots & -a_{1n} g_t(\theta_n - \theta_1) \\ -a_{21} g_t(\theta_1 - \theta_2) & \sum_{j=1, j\neq 2}^{n} a_{2j} g_t(\theta_j - \theta_2) & \cdots & -a_{2n} g_t(\theta_n - \theta_2) \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} g_t(\theta_1 - \theta_n) & -a_{n2} g_t(\theta_2 - \theta_n) & \cdots & \sum_{j=1, j\neq n}^{n} a_{nj} g_t(\theta_j - \theta_n) \end{pmatrix}. \tag{14}$$

We claim that $A'(t)$ is a symmetric, diagonally dominant, and positive semi-definite matrix for all $t \in (0, 1/2]$. The symmetry of $A'(t)$ follows from the evenness of the function $g_t(x)$ and the symmetry of the adjacency matrix $A = (a_{ij})$.

By hypothesis, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ is a point in the n-torus $\mathbb{T}^n$, where $-\pi < \theta_i \leq \pi$ for all $1 \leq i \leq n$. Consequently, $-2\pi \leq \theta_i - \theta_j \leq 2\pi$ for all $1 \leq i, j \leq n$. Therefore, for any $t \in (0, 1/2]$, we have $g_t(\theta_i - \theta_j) \geq 0$ for all $1 \leq i, j \leq n$. It follows that for any

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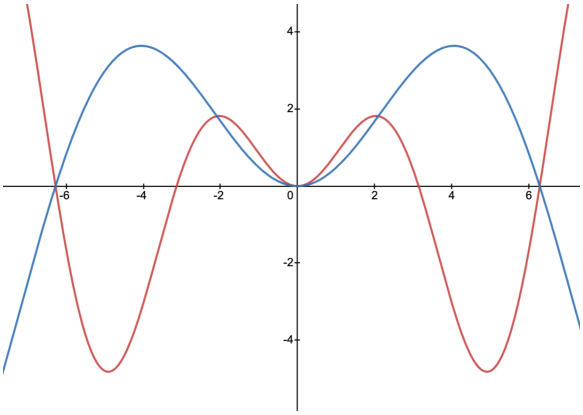

FIG. 2. Graph of $g_t(x) = x\sin(tx)$ for $t = 1/2$ (blue) and $t = 1$ (red).

$1 \le k \le n$,

$$\left|\sum_{j=1,j\neq k}^{n} a_{kj} g_t(\theta_j - \theta_k)\right| = \sum_{j=1,j\neq k}^{n} a_{kj} g_t(\theta_j - \theta_k) = \sum_{j=1,j\neq k}^{n} \left|-a_{kj} g_t(\theta_j - \theta_k)\right|.$$

This proves that matrix $A'(t)$ is diagonally dominant for any $t \in (0, 1/2]$. Finally, matrix $A'(t)$ is positive semi-definite for any $t \in (0, 1/2]$ since it is diagonally dominant and all entries in its diagonal are non-negative (Lemma III.1).

We select $0 < t_0 \le 1/2$ and consider $\lambda(t_0)$ as a real eigenvalue of matrix $A(t_0)$. Since $A(t_0)$ is a symmetric matrix, all its eigenvalues are real. Applying Lemma III.2, we can compute its derivative as

$$\lambda'(t_0) = \mathbf{v}(t_0)^T A'(t_0)\,\mathbf{v}(t_0),$$

where $\mathbf{v}(t_0)$ is a normalized eigenvector of $A(t_0)$ corresponding to the eigenvalue $\lambda(t_0)$. By our previous argument that $A'(t)$ is positive semi-definite for all $t \in (0, 1/2]$, we conclude that $z^t A'(t) z \ge 0$ for all $z \in \mathbb{R}^n$.

Now, let us consider the case where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ satisfies $0 \le \theta_i \le \pi$ for all $1 \le i \le n$. From this assumption, we can easily conclude that $-\pi \le \theta_i - \theta_j \le \pi$ for all $1 \le i, j \le n$. As a result, $g_t(\theta_i - \theta_j) \ge 0$ for all $t \in (0, 1]$ where $g_t(x) = x\sin(tx)$ is the auxiliary function defined earlier (see Fig. 2). Consequently, matrix $A'(t)$ is symmetric, diagonally dominant, and positive semi-definite for all $t \in (0, 1]$.

Finally, let $\lambda(t_0)$ be any eigenvalue of $A(t_0)$. By applying Lemma III.2, we have $\lambda'(t_0) = \mathbf{v}(t_0)^T, A'(t_0), \mathbf{v}(t_0) \ge 0$, since $A'(t_0)$ is a positive semi-definite matrix for any $t_0 \in (0, 1]$. □

The previous proposition can be interpreted geometrically in the following sense. At $t = 0$, and independently on $\boldsymbol{\theta}$, the $n \times n$ matrix $A(0) = \mathcal{H}(0)$ is given by

$$A(0) = \begin{pmatrix} -\sum_{j=1,j\neq 1}^{n} a_{1j} & a_{12} & \cdots & a_{1n} \\ a_{21} & -\sum_{j=1,j\neq 2}^{n} a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n2} & -\sum_{j=1,j\neq n}^{n} a_{nj} \end{pmatrix}. \tag{15}$$

It is easy to see that $A(0)$ is a symmetric, diagonally dominant, and semi-definite negative matrix. So, all its eigenvalues are nonpositive (see Lemma III.1). As we travel through the ray $t\boldsymbol{\theta}$ for $t \in (0, 1/2]$, the spectrum of $A(t\boldsymbol{\theta})$ moves to the right with respect to the spectrum of $A(0)$.

Remark 1: *The Laplacian matrix is a matrix representation of a network. In particular, the rank of the Laplacian matrix is related to the number of connected components of the network (see Chap. 13 of Ref. 24 for details). Let L be the Laplacian matrix associated to the network of oscillators. From the expression of A(0) (15), we just observe that $L = -A(0)$. Moreover, it is well known that $\lambda = 0$ is an eigenvalue of L whose multiplicity coincides with the number of connected components of the graph (Lemma 13.1.1 of Ref. 24). In our case, we have assumed that our network of oscillators form a connected graph. So, we conclude that $\lambda = 0$ is a simple eigenvalue of $A(0)$ and the rest of its eigenvalues are strictly negative real numbers.*

Proof of Theorem A

We define set $\mathcal{C}$, related with the set of points where function V is convex,

$$\mathcal{C} = \{\boldsymbol{\theta} \in [-\pi, \pi]^n \mid \mathcal{H}(\boldsymbol{\theta}) \text{ is semi-definite negative and } \dim(Ker(\mathcal{H}(\boldsymbol{\theta}))) = 1\}. \tag{16}$$

We start showing that $\mathcal{C}$ is an open and nonempty set. We first prove that $\mathbf{0} = (0, \ldots, 0)$ belongs to $\mathcal{C}$. Taking $t = 0$ the matrix $\mathcal{H}(\mathbf{0}) = A(0)$ (15) is a symmetric, diagonally dominant, and semi-definite negative matrix. Moreover, $\lambda = 0$ is a simple eigenvalue of $\mathcal{H}(\mathbf{0})$ since the network formed by all the oscillators is connected (see Remark 1). We claim that $\mathcal{S}$ is an open set. Let $\boldsymbol{\theta}_0$ be a point in $\mathcal{C}$. We denote by $p_{\boldsymbol{\theta}_0}(x)$ the characteristic polynomial of $\mathcal{H}(\boldsymbol{\theta}_0)$. By hypothesis $p_{\boldsymbol{\theta}_0}(x) = x \cdot q_{\boldsymbol{\theta}_0}(x)$ with $q_{\boldsymbol{\theta}_0}(0) \neq 0$. Moreover, all the roots of $q_{\boldsymbol{\theta}_0}(x)$ are real and strictly negative numbers. Hence, in a sufficiently small neighborhood of $\boldsymbol{\theta}_0$, the roots of $q_{\boldsymbol{\theta}}(x)$ are still strictly negative, showing that $\mathcal{C}$ is an open set.

We denote by $\partial\mathcal{C}$ and $\overline{\mathcal{C}}$ the boundary and the closure of $\mathcal{C}$, respectively. Set $\partial\mathcal{C}$ contains all the $\boldsymbol{\theta}$'s such that $\mathcal{H}(\boldsymbol{\theta})$ is negative semi-definite and $\lambda = 0$ is a multiple eigenvalue. Furthermore, the set $\overline{\mathcal{C}}$ coincides with the set of points where the map V is a convex map. Finally, the open set $[-\pi, \pi]^n \setminus \overline{\mathcal{C}}$ contains all the $\boldsymbol{\theta}$'s such that $\mathcal{H}(\boldsymbol{\theta})$ has at least one strictly positive eigenvalue.

As we mention before $\mathcal{C} \neq \emptyset$ is an open set. Hence, we can decompose $\mathcal{C}$ into its disjoint connected components, and there are at most a countably many connected components of $\mathcal{C}$. So,

$$\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i \quad \text{with} \quad \mathcal{C}_i \cap \mathcal{C}_j = \emptyset \text{ for } i \neq j.$$

Without loss of generality, we can assume that $\mathcal{C}_1$ is the connected component of $\mathcal{C}$ containing the origin $\mathbf{0}$. *A priori* we do not know how many of those $\mathcal{C}_i$ for $i > 1$ are different from the

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empty set. We assume that $\boldsymbol{\theta}^* = (\theta_1, \ldots, \theta_n)$ and $\boldsymbol{\eta}^* = (\eta_1, \ldots, \eta_n)$ are two stable solutions of the Kuramoto model (5) with $|\Gamma(\boldsymbol{\theta}^*)| \leq \pi$ and $|\Gamma(\boldsymbol{\eta}^*)| \leq \pi$. As we mention before $\boldsymbol{\theta}^* + (\alpha, \ldots, \alpha)$ and $\boldsymbol{\eta}^* + (\beta, \ldots, \beta)$ are also stable solutions for all $\alpha \in (-\pi, \pi]$ and $\beta \in (-\pi, \pi]$. We select α_0 and β_0 such that $0 \leq \theta_i + \alpha_0 \leq \pi$ and $0 \leq \eta_i + \beta_0 \leq \pi$ for all $1 \leq i \leq n$. So, we just choose a stable solution in the upper half part of the n-torus. Thus, renaming $\boldsymbol{\theta}^* = (\theta_1, \ldots, \theta_n)$ and $\boldsymbol{\eta}^* = (\eta_1, \ldots, \eta_n)$, if necessary, we can assume without loss of generality that $\theta_i, \eta_i \in [0, \pi]$ for all $1 \leq i \leq n$.

We first assume that $\boldsymbol{\theta}^*$ and $\boldsymbol{\eta}^*$ belong to two different connected components. Thus, one of them, for example, $\boldsymbol{\eta}^*$ belongs to $\mathcal{C}_{i^*}$ with $i^* > 1$. We consider the ray $t\boldsymbol{\eta}^*$ for $t \in [0, 1]$. This ray crosses (at least) two different connected components $\mathcal{C}_1$ and $\mathcal{C}_{i^*}$. The first one since the origin $\mathbf{0}$ is contained in $\mathcal{C}_1$ and the second one since $\boldsymbol{\eta}^*$ belongs to $\mathcal{C}_{i^*}$. Thus, we can assume that $t\eta^* \in \mathcal{C}_1$ for $t \in [0, t_0)$ and $t\eta^* \in \mathcal{C}_{i^*}$ for $t \in (t_1, 1]$. Moreover, $t_0\eta^* \in \partial\mathcal{C}_1$ and $t_1\eta^* \in \partial\mathcal{C}_{i^*}$. We have proved that the derivative of any eigenvalue $\lambda(t)$ of $\mathcal{H}(t\eta^*)$ verifies $\lambda'(t) \geq 0$ (see Proposition III.3).

Now, suppose that the ray $t\boldsymbol{\eta}^*$ exits the set $\mathcal{C}$ and enters $[-\pi, \pi]^n \setminus \overline{\mathcal{C}}$ for some $t \in (t_0, t_1)$. This is a contradiction with the fact that $\lambda'(t) \geq 0$ for $t \in (0, 1]$, since at least one eigenvalue needs to be non-negative in $\mathcal{C}_1$, then positive in the complement of $\overline{\mathcal{C}}$ and then again non-negative in $\mathcal{C}_{i^*}$. On the other hand, suppose that the ray $t\boldsymbol{\eta}^*$ does not exist $\overline{\mathcal{C}}$. This could be the case, for example, if $t_0 = t_1$. We observe that in this case when $t \to t_0^-$ all the eigenvalues of $\mathcal{H}(t\eta^*)$ increase and (at least) one of them collides to $\lambda = 0$ since for $t = t_0$ and for $t \to t_1^-$, this eigenvalue needs to come back, so the derivative at this eigenvalue needs to be strictly negative.

We further assume that θ^* and η^* belong to the same connected component. In this scenario, both minima correspond to the same point, as it is implausible to have two distinct local minima in a region where the map is convex, unless $[\theta^*] = [\eta^*]$.

We conclude this section with two particular examples involving $n = 5$ oscillators as related to Theorem A. The first example illustrates a Kuramoto model with a unique stable solution $\boldsymbol{\theta}_1$ such that $|\Gamma(\boldsymbol{\theta}_1)| < \pi$. The second example demonstrates a Kuramoto model with two stable solutions, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, satisfying $|\Gamma(\boldsymbol{\theta}_1)| < \pi$ and $|\Gamma(\boldsymbol{\theta}_2)| > \pi$, respectively.

We first consider a Kuramoto model with five oscillators, where only the central oscillator is connected to the rest of the elements in the network (see Fig. 3 on the left). The corresponding adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

For any $0 < \varepsilon \leq 1$, we consider the Kuramoto model with adjacent matrix A and the following choice of parameters:

$$\boldsymbol{\omega}_\varepsilon = \left(0, \frac{\sqrt{3}}{2}, \sin\left(\frac{\pi}{2}(1-\varepsilon)\right), -\frac{\sqrt{3}}{2}, -\sin\left(\frac{\pi}{2}(1-\varepsilon)\right)\right)$$

and $K = 1$.

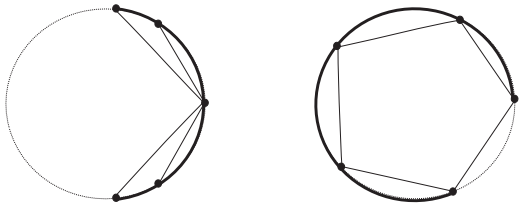

FIG. 3. Two stable configurations in the Kuramoto model. On the left hand side with adjacent matrix A (17) and on the right hand side with adjacent matrix A (18). We also show the length of the shortest arc containing the five oscillators.

We claim that the point

$$\boldsymbol{\theta}^*_\varepsilon = \left(0, \frac{\pi}{3}, \frac{\pi}{2}(1-\varepsilon), -\frac{\pi}{3}, -\frac{\pi}{2}(1-\varepsilon)\right)$$

is the unique stable equilibrium point of the Kuramoto model for all $0 < \varepsilon \leq 1$. To verify this claim, we must ensure that $\boldsymbol{\omega}_\varepsilon + \mathbf{f}(\boldsymbol{\theta}^*_\varepsilon) = \mathbf{0}$ and that the matrix $\mathcal{H}(\boldsymbol{\theta}^*_\varepsilon)$ has four negative eigenvalues for all $0 < \varepsilon \leq 1$. In this specific case, we can explicitly compute the five eigenvalues of $\mathcal{H}(\boldsymbol{\theta}^*_\varepsilon)$, denoted by $\lambda_i(\varepsilon)$ for $i = 1, \ldots, 5$. Simple computations reveal that

$$\lambda_1(\varepsilon) = 0, \quad \lambda_2(\varepsilon) = -\frac{1}{2}, \quad \lambda_3(\varepsilon) = -\tau(\varepsilon),$$
$$\lambda_4(\varepsilon) = \frac{1}{4}\left(-3 - 6\tau(\varepsilon) - \sqrt{9 - 4\tau(\varepsilon) + 36\tau(\varepsilon)^2}\right),$$
$$\lambda_5(\varepsilon) = \frac{1}{4}\left(-3 - 6\tau(\varepsilon) + \sqrt{9 - 4\tau(\varepsilon) + 36\tau(\varepsilon)^2}\right),$$

where $\tau(\epsilon) = \cos\left(\frac{\pi}{2}(1-\varepsilon)\right)$. Thus, there are always four negative eigenvalues for any $0 < \varepsilon \leq 1$. Moreover, $|\Gamma(\boldsymbol{\theta}^*_\varepsilon)| = \pi(1-\varepsilon) < \pi$, which tends to π as $\varepsilon \to 0$.

We next consider a specific example where the Kuramoto model exhibits more than one stable solution, as demonstrated in Ref. 15 and other references. We study a Kuramoto model with five oscillators in which each oscillator is connected to its two nearest neighbors (see Fig. 3 on the right). The corresponding adjacency matrix is presented as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{18}$$

In this example, we assume that the vector of frequencies $\boldsymbol{\omega}$ is zero and $K = 1$. The Kuramoto model has two equilibrium points located at $\mathbf{0} = (0, 0, 0, 0, 0)$ and $\boldsymbol{\theta}^* = \left(0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}\right)$. These equilibrium points are stable since the matrices $\mathcal{H}(\mathbf{0})$ and $\mathcal{H}(\boldsymbol{\theta}^*)$ have 4 strictly negative eigenvalues, as shown in Ref. 15.

Although this case is not covered by Theorem A since $|\Gamma(\boldsymbol{\theta}^*)| = \frac{8\pi}{5} > \pi$ (see Fig. 3 right), we can apply Proposition III.3 to this case since it is applicable in a general context. We consider the matrix $A(t) = \mathcal{H}(t\boldsymbol{\theta}^*)$ given by

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$$A(t) = \begin{pmatrix} -\cos\left(\frac{8\pi t}{5}\right) - \cos\left(\frac{2\pi t}{5}\right) & \cos\left(\frac{2\pi t}{5}\right) & 0 & 0 & \cos\left(\frac{8\pi t}{5}\right) \\ \cos\left(\frac{2\pi t}{5}\right) & -2\cos\left(\frac{2\pi t}{5}\right) & \cos\left(\frac{2\pi t}{5}\right) & 0 & 0 \\ 0 & \cos\left(\frac{2\pi t}{5}\right) & -2\cos\left(\frac{2\pi t}{5}\right) & \cos\left(\frac{2\pi t}{5}\right) & 0 \\ 0 & 0 & \cos\left(\frac{2\pi t}{5}\right) & -2\cos\left(\frac{2\pi t}{5}\right) & \cos\left(\frac{2\pi t}{5}\right) \\ \cos\left(\frac{8\pi t}{5}\right) & 0 & 0 & \cos\left(\frac{2\pi t}{5}\right) & -\cos\left(\frac{2\pi t}{5}\right) - \cos\left(\frac{8\pi t}{5}\right) \end{pmatrix}. \tag{19}$$

In this particular example, it is possible to obtain the explicit expression of the eigenvalues $\lambda_k(t)$ for $k = 1, \ldots, 5$ of $A(t)$. More precisely, we have that

$$\lambda_1(t) = 0,$$

$$\lambda_2(t) = -\frac{1}{2}\left(\sqrt{5}+5\right)\cos\left(\frac{2\pi t}{5}\right),$$

$$\lambda_3(t) = \frac{1}{2}\left(\sqrt{5}-5\right)\cos\left(\frac{2\pi t}{5}\right),$$

$$\lambda_4(t) = -\frac{3}{2}\cos\left(\frac{2\pi t}{5}\right) - \cos\left(\frac{8\pi t}{5}\right) + \frac{\sqrt{2}}{4}\sqrt{9 + 5\cos\left(\frac{4\pi t}{5}\right) - 4\cos\left(\frac{6\pi t}{5}\right) - 4\cos(2\pi t) + 4\cos\left(\frac{16\pi t}{5}\right)},$$

$$\lambda_5(t) = -\frac{3}{2}\cos\left(\frac{2\pi t}{5}\right) - \cos\left(\frac{8\pi t}{5}\right) - \frac{\sqrt{2}}{4}\sqrt{9 + 5\cos\left(\frac{4\pi t}{5}\right) - 4\cos\left(\frac{6\pi t}{5}\right) - 4\cos(2\pi t) + 4\cos\left(\frac{16\pi t}{5}\right)}.$$

In Fig. 4, we have to plot the graph of the five eigenvalues of $A(t) = \mathcal{H}(t\boldsymbol{\theta}^*)$ for $0 \le t \le 1$. Thus, for $t = 0$, the eigenvalues of $\mathcal{H}(\mathbf{0})$ are given by $\lambda_1(0) = 0$, $\lambda_2(0) = \lambda_5(0) = -\frac{1}{2}\left(\sqrt{5}+5\right) \sim -3.618\,034$, and $\lambda_3(0) = \lambda_4(0) = \frac{1}{2}\left(\sqrt{5}-5\right) \sim -1.381\,966$, proving, thus, that $\mathbf{0}$ is a stable equilibrium point. Similarly, for $t = 1$, the eigenvalues of $\mathcal{H}(\boldsymbol{\theta}^*)$ are given by $\lambda_1(1) = 0$, $\lambda_2(1) = \lambda_5(1) = -\frac{1}{2}\left(\sqrt{5}+5\right)\cos\left(\frac{2\pi}{5}\right) \sim -1.118\,034$ and $\lambda_3(1) = \lambda_4(1) = \frac{1}{2}\left(\sqrt{5}-5\right)\cos\left(\frac{2\pi}{5}\right) \sim -0.427\,051$ showing that $\boldsymbol{\theta}^*$ is also a stable equilibrium solution of the Kuramoto model.

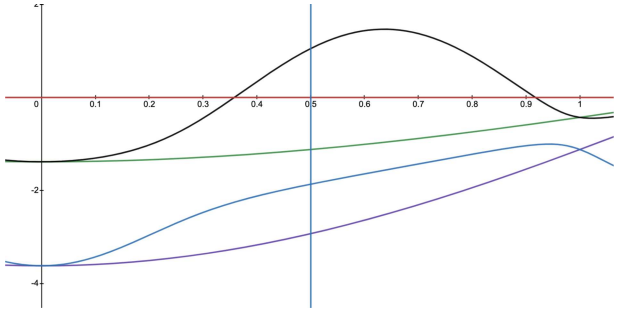

FIG. 4. Graph of the eigenvalues $\lambda_1(t)$ (red), $\lambda_2(t)$ (dark lilac), $\lambda_3(t)$ (green), $\lambda_4(t)$ (black), and $\lambda_5(t)$ (dark blue) of $A(t)$ (19) for $0 \le t \le 1$. It is also shown the vertical line $t = 1/2$.

In Proposition III.3, we have proved that all the eigenvalues verify $\lambda_k'(t) \ge 0$ for $0 < t \le 1/2$. In Fig. 4, we also have to plot the vertical line $t = 1/2$ and we can see that the functions $\lambda_k(t)$ are increasing for all $k = 1, \ldots, 5$ as it is proved in Proposition III.3. Moreover, it is possible to prove that $\lambda_4'(t) < 0$ and $\lambda_5'(t) < 0$ for some value of $1/2 < t \le 1$.

Although Theorem A does not apply to this particular example (since $|\Gamma(\boldsymbol{\theta}^*)| > \pi$), the ideas used in the proof of Theorem A can used to understand the existence of these two stable equilibrium solutions. Thus, these two stable equilibrium solutions $\mathbf{0}$ and $\boldsymbol{\theta}^*$ belong to two different connected components of set $\mathcal{C}$ [see (16)]. We recall that $\mathcal{C}$ is the set of points $\boldsymbol{\theta}$ where four of the eigenvalues of $\mathcal{H}(\boldsymbol{\theta})$ are strictly negative. We denote by $\mathcal{C}_1$ the connected component of $\mathcal{C}$ containing $\mathbf{0}$ and $\mathcal{C}_2$ the connected component of $\mathcal{C}$ containing $\boldsymbol{\theta}^*$. In Fig. 4, it is shown the evolution of all the eigenvalues from $t = 0$ to $t = 1$. We observe that λ_4 is the only eigenvalue that changes their sign. More precisely, $\lambda_4(t) \ge 0$ for $t \in [t_0, t_1]$ and $\lambda_4(t) \le 0$ for $t \in [0, t_0] \cup [t_1, 1]$ (see Fig. 4). Hence, at $t = t_0$, point $t_0\boldsymbol{\theta}^*$ belongs to the boundary of $\mathcal{C}_1$, and for $t \in (t_0, t_1)$, point $t\boldsymbol{\theta}^*$ is the complement of $\overline{\mathcal{C}}$ and at $t = t_1$, point $t_1\boldsymbol{\theta}^*$ belongs to the boundary of $\mathcal{C}_2$ and for $t_1 < t \le 1$, point $t\boldsymbol{\theta}^*$ belongs to $\mathcal{C}_2$. Thus, eigenvalue $\lambda_4(t)$ needs to increase to exit $\mathcal{C}_1$ and decrease to enter into $\mathcal{C}_2$ or in other words their derivative change their sign.

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IV. DISCUSSION

In this paper, we have demonstrated that the Kuramoto model yields a unique stable solution $[\theta^*]$ satisfying $|\Gamma(\theta^*)| \leq \pi$. This implies that half of the unit circle can be selected to include all the oscillators. Such solutions can be perceived as a type of "entrained" solution, characterizing what is commonly seen as a cluster of entrained oscillators around a mutual phase. This is distinct from other solution types, such as the splay state solution depicted in Fig. 3. The stability of any equilibrium solution of (5) is captured within the symmetric matrix $\mathcal{H}(\theta^*)$ as detailed in Eq. (6). Our proof of this principal finding hinges on controlling the derivative of the eigenvalues of the function $t \mapsto \lambda(t)$, where $\lambda(t)$ denotes an eigenvalue of the matrix $A(t) = \mathcal{H}(t\theta)$.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Alex Arenas: Conceptualization (lead); Formal analysis (equal); Supervision (lead); Writing – review & editing (equal). **Antonio Garijo:** Formal analysis (lead); Supervision (equal); Writing – original draft (lead). **Sergio Gómez:** Formal analysis (equal); Writing – review & editing (equal). **Jordi Villadelprat:** Formal analysis (equal); Supervision (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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