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GLOBAL ATTRACTOR IN THE POSITIVE QUADRANT OF THE LOTKA–VOLTERRA SYSTEMS IN $\mathbb{R}^2$

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ABSTRACT. We characterize when the unique equilibrium point of the positive quadrant of a 2–dimensional Lotka–Volterra system is a global attractor in that quadrant. Additionally we classify the phase portraits of these class of Lokta-Volterra system.

1. Introduction and statement of the main results

A Lotka–Volterra system is given by a quadratic vector field $X = (P_1(x_1, x_2), P_2(x_1, x_2))$ in $\mathbb{R}^2$ where x_i is a factor of the polynomial P_i for $i = 1, 2$. The Lotka–Volterra systems, were initially proposed in $\mathbb{R}^2$ by Alfred J. Lotka in 1925 [16] and by Vito Volterra in 1926 [22] independently, as a model for studying the interactions between species, or the interactions in a predator-prey model. Later on in 1936 Kolmogorov [14] extended these type of systems to the also called Kolmogorov systems where they are considered in arbitrary dimension and with arbitrary degree.

Many natural phenomena can be modeled by the Lotka–Volterra systems such as the time evolution of conflicting species in biology, see for example [11, 17, 18, 19], in ecology [2, 12], chemical reactions [8], hydrodynamics [3], economics [21], the evolution of electrons, ions and neutral species in plasma physics [15], etc.

We want to emphasize that the study of the existence of a global attractor in a Lotka–Volterra system is very important in the applications to biology and ecology. The existence of a global attractor $e_1 \in \mathbb{R}^2_+$ (or in $\mathbb{R}^n_+$ in general) guarantees the no extinction of the species for initial conditions in the interior of $\mathbb{R}^2_+$.

Hou [9] did important contributions to the study of global attractors in $\mathbb{R}^n_+$ for the Lotka-Volterra systems. He in [10] stablishes a criterium with conditions for the existence of a unique global attractor.

In this work we characterize the Lotka–Volterra systems in $\mathbb{R}^2$ with one equilibrium point in the interior of the positive quadrant which is a global attractor in this quadrant, i.e. a system of the form

$$\dot{X} = X(\alpha_0 + \alpha_1 X + \alpha_2 Y), \quad \dot{Y} = Y(\beta_0 + \beta_1 X + \beta_2 Y), \tag{1}$$

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with a finite equilibrium (X_0, Y_0) with $X_0 > 0$ and $Y_0 > 0$. Here the dot denotes derivative with respect to the time t.

From works of [6], [7] and [13] it follows that a condition in order that the Lotka–Volterra system (1) has a global attractor is that

$$\frac{\alpha_2}{\alpha_1} < 1 \text{ and } \frac{\beta_2}{\beta_1} < 1.$$

Note that without loss of generality if system (1) has an equilibrium point (X_0, Y_0) in the positive quadrant we can consider that $(X_0, X_0) = (1, 1)$. Indeed, considering the change of variables $(X, Y) \to (X/X_0, Y/Y_0)$ the Lotka–Volterra system (1) becomes another Lotka–Volterra system having the equilibrium $(1, 1)$.

In what follows we consider an arbitrary Lotka–Volterra system with the equilibrium $(1, 1)$, which can be written as

$$\begin{aligned} \dot{x} &= x(a_1(x-1) + a_2(y-1)), \\ \dot{y} &= y(b_1(x-1) + b_2(y-1)). \end{aligned} \tag{2}$$

This system depends on four parameters a_1, a_2, b_1 and b_2. We assume that system (2) does not have a common factor between $\dot{x}$ and $\dot{y}$. We note that changing the sign of the time if necessary we can assume $a_1 \leq 0$.

We recall that a *limit cycle* for a planar differential system is a periodic orbit which is isolated in the set of all periodic orbits of the system.

Bautin in [1] proved that any 2–dimensional Lotka–Volterra system cannot have limit cycles, see also Coppel [4].

The objective of this work is to classify the Lotka–Volterra systems (2) for which the equilibrium $(1, 1)$ is a global attractor in the positive quadrant.

Roughly speaking we say that the Poincaré disc $\mathbb{D}^2$ is the closed unit disc centered at the origin of $\mathbb{R}^2$, where its interior is identified with $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ is identified with the infinity $\mathbb{R}^2$, in the sense that we can go to or come from the infinity in the plane $\mathbb{R}^2$ in as many directions as points has the circle $\mathbb{S}^1$. A polynomial differential system in $\mathbb{R}^2$, i.e. in the interior of $\mathbb{D}^2$ can be extended to its boundary $\mathbb{S}^1$ in a unique analytic way, this extension was done by first time by Poincaré in [20], and now it is called the Poincaré compactification of a polynomial differential system. For more details on the Poincaré compactification see for instance Chapter 5 of [5].

Let $p(X)$ be the compactified vector field in the Poincaré disc $\mathbb{D}^2$ of a polynomial vector field X. A *separatrix* of $p(X)$ is an orbit which is either an equilibrium point, or a trajectory which lies in the boundary of a hyperbolic sector of a finite or an infinite equilibrium point, or a limit cycle, or any orbit contained in $\mathbb{S}^1$ (the boundary of $\mathbb{D}^2$, i.e. the infinity of the plane), for more details see [5].

We denote the open positive quadrant by $\mathring{Q} = \{(x, y) : x > 0, y > 0\} \cap \mathbb{D}^2$ of the Poincaré disc, and by $Q = \{(x, y) : x \geq 0, y \geq 0\} \cap \mathbb{D}^2$ the closed positive quadrant.

Here we shall use the notion of *quadrant-topologically equivalent.* This definition says that two Lotka–Volterra systems (2) are quadrant-topologically equivalent if there exists a homeomorphism $h : Q \mapsto Q$ sending orbits to orbits and the boundaries $\{x = 0\} \cap \mathbb{D}^2$, $\{y = 0\} \cap \mathbb{D}^2$ and $\mathbb{S}^1 \cap \mathbb{D}^2$ to $\{x = 0\} \cap \mathbb{D}^2$, $\{y = 0\} \cap \mathbb{D}^2$ and $\mathbb{S}^1 \cap \mathbb{D}^2$ respectively, and which preserves or reverses the orientation of all orbits. Our main result is the following.

Theorem 1. *The following statements hold*

(a) *The phase portrait in the quadrant Q of the Poincaré disc of a 2–dimensional Lotka–Volterra system* (2) *having a global attractor in the interior Q is quadrant-topologically equivalent to one of the quadrants of Figure 1.*
(b) *In Tables 1 and 2 we provide the values of the parameters of system* (2) *having a global attractor in the quadrant Q.*

This work is organized as follows. In section 2 we study the local dynamics of the finite and infinite equilibria in Q when system (2) has at $(1, 1)$ a local attractor. In section 3 we prove Theorem 1.

2. Phase Portraits of system (2)

First note that system (2) has always the invariant straight lines $x = 0$ and $y = 0$. This implies that the quadrant Q is invariant by the flow of system (2), because the infinity $\mathbb{S}^1 \cap Q$ is always invariant by the Poincaré compactification.

System (2) has at most four finite equilibria in the quadrant Q and at least two. The origin $e_1 = (0, 0)$ and $e_2 = (1, 1)$ are always finite equilibria on Q, and under some conditions on the parameters we can have the finite equilibria $e_3 = (0, (b_1 + b_2)/b_2)$ and $e_4 = ((a_1 + a_2)/a_1, 0)$.

The linear part of system (2) is

$$\left(\begin{array}{cc} a_1(2x-1) + a_2(y-1) & a_2 x \\ b_1 y & b_1(x-1) + b_2(2y-1) \end{array} \right). \tag{3}$$

Therefore the eigenvalues for each finite equilibria are

$$\begin{array}{lll} e_1 & \longrightarrow & \lambda_1^1 = -(a_1 + a_2), \quad \lambda_2^1 = -(b_1 + b_2), \\ e_2 & \longrightarrow & \lambda_{1,2}^2 = \left(a_1 + b_2 \pm \sqrt{a_1^2 + 4a_2 b_1 + b_2(b_2 - 2a_1)} \right)/2, \\ e_3 & \longrightarrow & \lambda_1^3 = -a_1 + a_2 b_1/b_2, \quad \lambda_2^3 = b_1 + b_2, \\ e_4 & \longrightarrow & \lambda_1^4 = a_1 + a_2, \quad \lambda_2^4 = a_2 b_1/a_1 - b_2. \end{array} \tag{4}$$

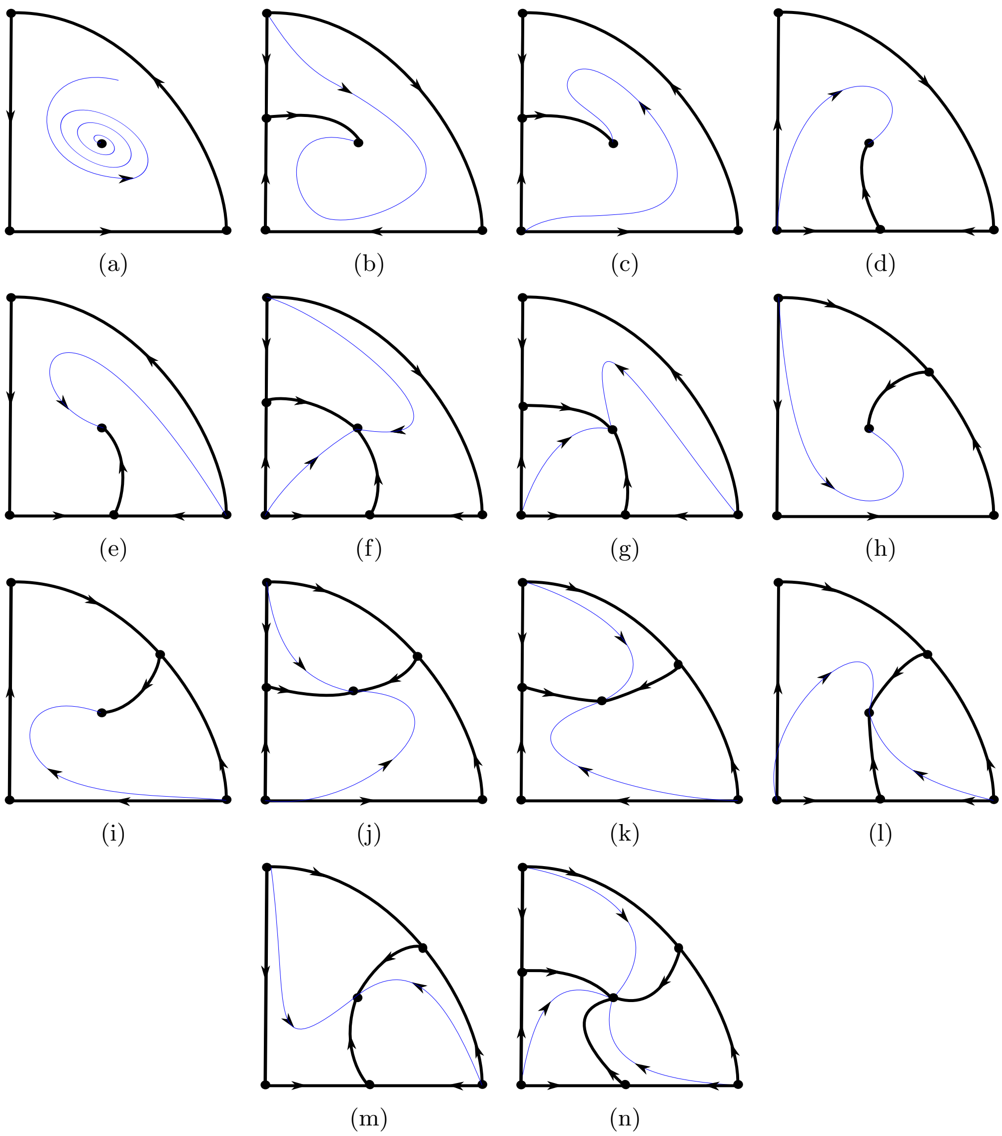


FIGURE 1. The distinct quadrant-topologically equivalent phase portraits of system (2) with a global attractor in the interior of the positive quadrant of the Poincaré disc. The thick lines are separatrices, and the thin lines are orbits, one orbit for each canonical region.

For the study of the infinite equilibria we use the Poincaré compactification (see details in Chapter 5 of [5]) and we get that the associated system in the local chart U_1 is

$$z_1' = -z_1(a_1 - b_1 + a_2 z_1 - b_2 z_1), \quad z_2' = z_2(-a_1 - a_2 z_1 + a_1 z_2 + a_2 z_2). \tag{5}$$

			$b_1 < -b_2$	$b_1 = -b_2$	$-b_2 < b_1 < 0$	$b_1 = 0$	$b_1 > 0$
$a_1 < 0$, $b_2 > 0$	$a_2 < -a_1$	$a_2 > b_2$, $a_1 < b_1$	1(l) N/F	x	x	x	L.A.
		$a_2 > b_2$, $a_1 \geq b_1$	1(d) N/F	x	x	x	L.A.
		$a_2 = b_2$, $a_1 \leq b_1$	x	x	x	x	L.A.
		$a_2 = b_2$, $a_1 > b_1$	1(d) N/F	x	x	x	L.A.
		$a_2 < b_2$, $a_1 \leq b_1$	x	x	x	x	L.A.
		$a_2 < b_2$, $a_1 > b_1$	L.A.	x	x	x	L.A.
	$a_2 = -a_1$	$a_1 \geq b_1$	1(a) F	x	x	x	x
		$a_1 < b_1$	1(i) N/F	x	x	x	x
	$a_2 > -a_1$	$a_1 \geq b_1$	1(a) F	1(i) N/F	L.A.	x	x
		$a_1 < b_1$	1(i) N/F	1(i) N/F	L.A.	x	x

TABLE 1. The classification of the Lotka–Volterra systems (2) having at $(1,1)$ a global attractor in the positive quadrant with $b_2 > 0$. As example $1(a)$ provides the values of the parameters of system (2) for which the phase portraits on Q is quadrant–topologically equivalent to Figure 1(a), N/F means that the attractor can be a node or a focus, N or F means that the attractor only can be a node or a focus, respectively; L.A. denotes that $(1,1)$ is just a local attractor but it is not a global attractor; x shows when for the relations of the parameters in the table the equilibrium $(1,1)$ is not an attractor.

Then at infinity, i.e. at $z_2 = 0$, we have the equilibria $(0,0)$ and $p_1 = ((b_1 - a_1)/(a_2 - b_2), 0)$. Furthermore the linear part at $z_2 = 0$ is

$$\begin{pmatrix} -a_1 + b_1 + 2(b_2 - a_2)z_1 & (a_1 + a_2 - b_1 - b_2)z_1 \\ 0 & -a_1 - a_2 z_1 \end{pmatrix}. \tag{6}$$

The associated system in the local chart U_2 is

$$z_1' = z_1(a_2 + (a_1 - b_1)(z_1 - z_2) + b_2(-1 + z_2) - a_2 z_2), \quad z_2' = z_2(-b_2 - b_1 z_1 + b_1 z_2 + b_2 z_2), \tag{7}$$

where the origin is an equilibrium and its eigenvalues are

$$a_2 - b_2 \text{ and } -b_2. \tag{8}$$

We are interested in the existence of a global attractor at $(1,1)$ in the positive quadrant, then we need that this equilibrium will be a local attractor, this will be satisfied, using the Routh-Hurwitz criterion, when the two coefficients of the characteristic polynomial at $(1,1)$ $p(\lambda) = \lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1)$ are

			$b_1 < 0$	$b_1 = 0$	$0 < b_1 < -b_2$	$b_1 = -b_2$	$b_1 > -b_2$
$a_1 < 0$, $b_2 = 0$, $a_2 < 0$			x	x	1(e)	1(e)	1(e)
$a_1 < 0$, $b_2 = 0$, $a_2 > 0$	$a_2 < -a_1$	$a_1 < b_1$	1(l)	x	x	x	x
		$a_1 > b_1$	1(d)	x	x	x	x
	$a_2 \geq -a_1$	$a_1 < b_1$	1(f)	x	x	x	x
		$a_1 > b_1$	1(n)	x	x	x	x
$a_1 < 0$, $b_2 < 0$, $a_2 \geq 0$	$a_2 \geq -a_1$	$a_1 \geq b_1$	1(b) N/F	1(k) N	1(k) N	x	x
		$a_1 < b_1$	1(k) N/F	1(k) N	1(k) N	x	x
	$a_2 = 0$	$a_1 \geq b_1$	1(f) N	1(n) N	1(n) N	1(m) N	1(m) N
		$a_1 < b_1$	1(n) N	1(n) N	1(n) N	1(m) N	1(m) N
	$0 \leq a_2 < -a_1$	$a_1 \geq b_1$	1(f) N/F	1(n) N/F	1(n) N/F	1(m) N /F	1(m) N /F
		$a_1 < b_1$	1(n) N/F	1(n) N/F	1(n) N/F	1(m) N /F	1(m) N /F
$a_1 < 0$, $b_2 < 0$, $a_2 < 0$	$a_1 < b_1$	$a_2 < b_2$	1(g) N	1(g) N	1(g) N/F	1(e)N/F	1(e) N/F
		$a_2 = b_2$	1(g) N	1(g) N	1(g) N/F	1(e) N/F	1(e) N/F
		$a_2 > b_2$	1(n) N	1(n) N	1(n) N/F	1(m) N/F	1(m) N/F
	$a_1 = b_1$	$a_2 < b_2$	x	x	x	x	x
		$a_2 = b_2$	x	x	x	x	x
		$a_2 > b_2$	1(n) N	x	x	x	x
	$a_1 > b_1$	$a_2 < b_2$	x	x	x	x	x
		$a_2 = b_2$	x	x	x	x	x
		$a_2 > b_2$	1(f) N	x	x	x	x
$a_1 = 0$, $b_2 < 0$	$a_2 < 0$	$a_2 \leq b_2$	x	x	1(c) N/F	1(a) N/F	1(a) N/F
		$a_2 > b_2$	x	x	1(j) N/F	1(h) N/F	1(h) N/F
	$a_2 > 0$		1(b) N/F	x	x	x	x

TABLE 2. The classification of the Lotka–Volterra systems (2) having at $(1, 1)$ a global attractor in the positive quadrant with $b_2 \leq 0$.

positive, i.e.

$$\tau = -a_1 - b_2 > 0 \text{ and } \delta = a_1 b_2 - a_2 b_1 > 0. \tag{9}$$

Hence in what follows we assume that the two inequalities of (9) hold. Note that the equilibrium $(1, 1)$ is a stable node if $\tau^2 - 4\delta \geq 0$, and a stable focus if $\tau^2 - 4\delta < 0$.

We separate the study of the local phase portraits at the finite and infinite equilibria according to the sign of a_1.

2.1. Case $\mathbf{a_1 = 0}$.

From (9) $b_2 < 0$ and $a_2 b_1 < 0$. Note that e_4 does not exist for $a_1 = 0$.

Subcase: $b_1 < 0$ and $a_2 > 0$. We study the local phase portraits of all equilibria in Q in order to stablish the conditions to have a global attractor at $(1, 1)$. In this sense we analize the finite equilibria. The origin $(0, 0)$ under these conditions is a hyperbolic saddle, this follows from (4) and Theorem 2.15 of [5], with local stable manifold on the x-axis (we denote as S_h this type of saddle where the stable manifold is the horizontal axis). The equilibrium e_3 belongs to Q and it is a hyperbolic saddle with their stable separatrices in the y-axis (we denote this type of saddle as S_v).

Now we are going to analize the local phase portrait at the infinite equilibria. In U_1 the origin becomes a semi-hyperbolic equilibrium with eigenvalues 0 and $b_1 < 0$. Applying Theorem 2.19 of [5] we obtain that it is a saddle-node with central manifold in the z_2-axis, and since $a_2 > 0$ the flow is stable in the z_1-axis, Figure 2(c) shows the local phase portrait at the origin of system (5). This implies than in the quadrant Q, the origin of U_1 consists in one hyperbolic sector. The infinite equilibria p_1 has the coordinate z_1 negative, so it is not in Q.

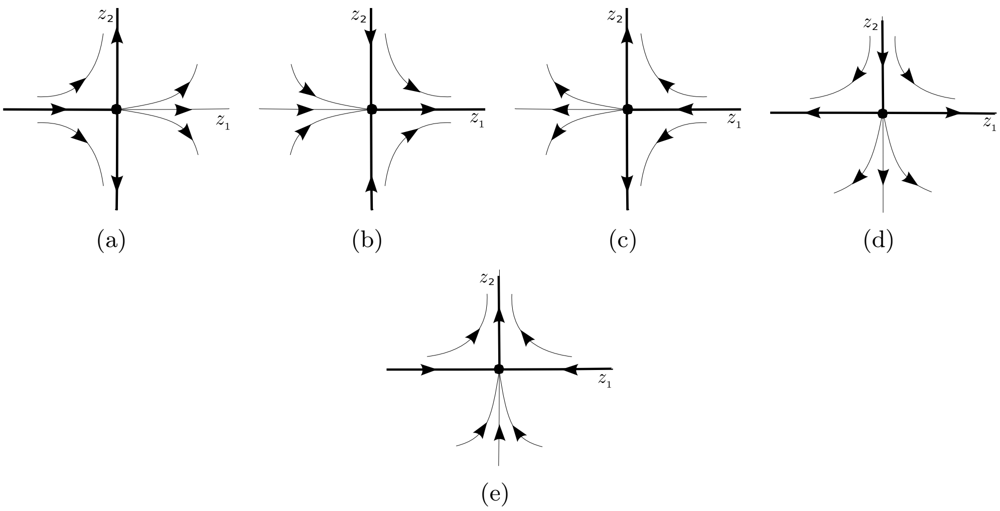


FIGURE 2. Some types of local phase portrait at $(0,0)$ in the plane (z_1, z_2) being a $(0,0)$ semi–hyperbolic saddle-node.

The origin of U_2 has the eigenvalues given in (8) both positives implying that it is a hyperbolic repeller (i.e. either an unstable node or an unstable focus). With these informations we obtain the local phase portrait of all the equilibria of system (2) for $a_1 = 0$, $a_2 > 0$ and $b_1 < 0$, see Figure 3.

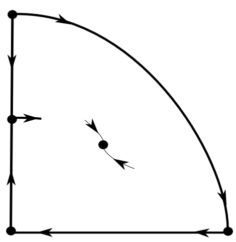


FIGURE 3. Local phase portrait at the equilibria in Q of system (2) for $a_1 = 0$, $a_2 > 0$, $b_1 < 0$ and $b_2 < 0$

Subcase: $b_1 > 0$ and $a_2 < 0$. In the finite region we can see from (4) that the origin has eigenvalues $\lambda_1^1 = -a_2 > 0$ and $\lambda_2^1 = -b_1 - b_2$, then we have three possibilities: if $b_1 < -b_2$ the origin will be a hyperbolic repeller, if $b_1 > -b_2$ the origin will be a hyperbolic saddle S_v with the local stable manifold on the

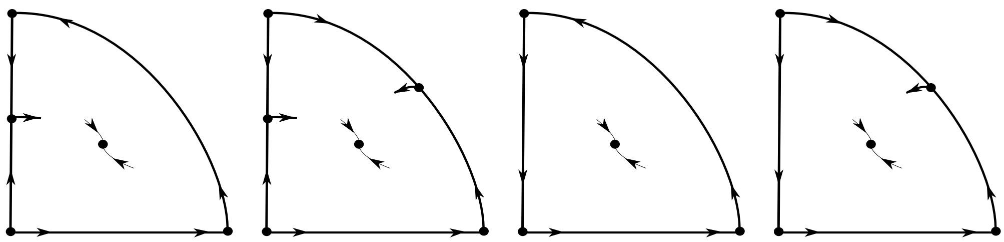
y-axis, and if $b_1 = -b_2$ the origin is a semi-hyperbolic equilibrium, its analysis using Theorem 2.19 of [5] gives that its local phase portrait is a saddle-node as the one shown in Figure 2(c). Note that in the quadrant Q we have the same dynamics for $b_1 > -b_2$ and $b_1 = -b_2$.

The equilibrium e_3 is in Q if and only if $b_1 < -b_2$, furthermore from (4) its eigenvalues $\lambda_1^3 = a_2b_1/b_2 > 0$ and $\lambda_2^3 = b_1 + b_2 < 0$ imply that it is a hyperbolic saddle of type S_v.

On the infinite region we have that the origin of U_1 (as for the previous subcase $b_1 < 0$ and $a_2 > 0$) is a semi-hyperbolic equilibrium and by Theorem 2.19 in [5] the origin of U_1 is a saddle-node, but since in this case $b_1 > 0$ it is as the one of Figure 2(d). The infinite equilibria p_1 is in Q only if $a_2 > b_2$, and from (6) we conclude that it is a hyperbolic saddle with its local stable separatrices in the z_1-axis (its eigenvalues are $-b_1 < 0$ and $a_2b_1/(b_2 - a_2) > 0$).

To complete the local analysis we have from (8) that the origin of U_2 has one eigenvalue positive, and the other eigenvalue depends on the sign of $a_2 - b_2$, then it is a hyperbolic unstable node if $a_2 > b_2$, a hyperbolic saddle with its stable separatrices in the z_1-axis if $a_2 < b_2$, and if $a_2 = b_2$ it is a semi-hyperbolic infinite equilibrium, according to Theorem 2.19 its local phase portrait is a saddle-node as the one of Figure 2(c). On the quadrant Q the origin of U_2 has the same dynamics for $a_2 < b_2$ and $a_2 = b_2$.

From the previous analysis we have that the local phase portrait of the equilibria in Q is as one of Figure 4.



(a) (b) (c) (d)

FIGURE 4. Local phase portraits at the equilibria in Q of system (2) for $a_1 = 0$, $b_2 < 0$, $a_2 < 0$ and $b_1 > 0$. (a) $0 < b_1 < -b_2$ and $a_2 \le b_2$, (b) $0 < b_1 < -b_2$ and $a_2 > b_2$, (c) $b_1 \ge -b_2$ and $a_2 \le b_2$, (d) $b_1 \ge -b_2$ and $a_2 > b_2$.

2.2. **Case $\mathbf{a_1 < 0}$.** Then from (9) it holds that $a_1 < -b_2$ and $b_2 < a_2b_1/a_1$.

In order to do a clear analysis we separate the study of this case in three sub-cases according the sign of b_2.

Subcase $b_2 > 0$. Then the origin of system (2) is a hyperbolic attractor if $-b_2 < b_1 < 0$, so we cannot have a global attractor at $(1, 1)$. A similar situation happens for the finite equilibria e_3 for $-b_2 < b_1$, because then system (2) has a hyperbolic saddle on Q and one of its a stable separatrices is in the region $x > 0$, $y > 0$. So we assume $b_1 \leq -b_2$.

First we study the finite equilibria. From (3) the origin is a hyperbolic repeller for $b_1 < -b_2$ and $a_2 < -a_1$; it is a hyperbolic saddle of type S_h for $b_1 < -b_2$ and $a_2 > -a_1$; it is a hyperbolic saddle of type S_v for $b_1 > 0$. For the remaining cases $a_1 = -a_2$ or $b_1 = -b_2$ the origin is a semi-hyperbolic equilibrium, applying Theorem 2.19 of [5] it is a saddle-node. More precisely it is as the one of Figure 2(c) if $a_2 = -a_1$ and as the one of Figure 2(e) if $b_2 = -b_1$, i.e. it has one hyperbolic sector on the region Q. Note that $(a_2 + a_2)^2 + (b_1 + b_2)^2 \neq 0$ from condition (9). The finite equilibria e_3 only is in Q if $-b_2 < b_1$, but we already stablish that under this condition e_3 is a hyperbolic saddle that has a stable separatrix in the region $x > 0$, $y > 0$, then $(1, 1)$ cannot be a global attractor. The other finite equilibrium e_4 is in Q if $a_2 < 0$, or $0 < a_2 < -a_1$ (note that a_2 cannot be zero due to the second condition of (9)). When e_4 is in Q it is always a hyperbolic saddle S_h.

Now the origin of U_1 has one eigenvalue $-a_1$ positive and the other, from (6) is $-a_1 + b_1$, then it is a hyperbolic unstable node if $a_1 < b_1$, a saddle S_h if $a_1 > b_1$, and for $a_1 = b_1$ the origin becomes semi-hyperbolic. Then we study its local phase portraits through Theorem 2.19 of [5], and it is a saddle-node, more precisely, (9) implies that $a_2 > b_2$ when $a_1 < 0$ and $b_2 > 0$, and consequently it is as the one of Figure 2(c). Thus, if $a_1 = b_1$ the origin of U_1 in Q consist of one hyperbolic sector. The other infinite equilibria located in U_1 is p_1, with eigenvalues $a_1 - b_1$ and $(a_1b_2 - a_2b_1)/(a_2 - b_2)$, it is in Q only for two combination of parameters, first for $a_1 < b_1$ and $a_2 > b_2$ being a hyperbolic saddle S_h, and second for $a_1 > b_1$ and $a_2 < b_2$ being a hyperbolic saddle of type S_v. This implies that a stable separatrix becomes from the finite region, then we cannot have a global attractor at $(1, 1)$ when $a_1 > b_1$ and $a_2 < b_2$.

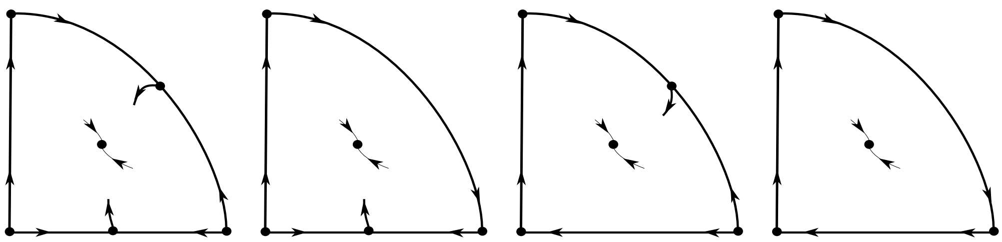
The origin of U_2 has one negative eigenvalue $-b_2$ and the other, from (8), is $a_2 - b_2$, then we have three possible local phase portraits for the origin of U_2. It is a hyperbolic unstable node if $a_2 < b_2$, it is a hyperbolic saddle S_v if $a_2 > b_2$, and for $a_2 = b_2$ it becomes in a semi-hyperbolic equilibrium, by Theorem 2.19 of [5], it is a saddle-node, more precisely, from (9) it holds that for $a_1 < 0$, $b_2 > 0$ and $a_2 = b_2$ we have $a_1 - b_1 > 0$, then its local phase portrait is as the one of Figure 2(b). Note that in the region Q of the Poincaré disc the origin of U_2 consists in one hyperbolic sector.

Note that if $a_1 < 0$ and $b_2 > 0$ we have that the finite equilibria $(1, 1)$ is a local hyperbolic attractor if $a_1 < -b_2$ and $b_2 < a_2b_1/a_1$ (from (9)), but we can have another attracting sector or a stable separatrix that prevents that $(1, 1)$ be a global attractor, this happens for $b_1 > 0$, for $-b_2 < b_1 < 0$, and for $a_2 < b_2$ and $a_1 > b_1$. Table 3 summarizes the cases when $(1, 1)$ is or not a local attractor, and shows the cases when it is just a local attractor but not a global attractor.

We summarize in Figure 5 the local phase portraits of the equilibria of system (2) in the quadrant Q for $a_1 < 0$ and $b_2 > 0$ when $e_2 = (1, 1)$ is a local attractor.

		$b_1 < -b_2$			$b_1 = -b_2$
		$a_1 < b_1$	$a_1 = b_1$	$a_1 > b_1$	
$a_2 < -a_1$	$a_2 > b_2$	Fig. 5(a)	Fig. 5(b)	Fig. 5(b)	x
	$a_2 = b_2$	x		Fig. 5(b)	
	$a_2 < b_2$			L.A.	
$a_2 = -a_1$		Fig. 5(c)	Fig. 5(d)	Fig. 5(d)	x
$a_2 > -a_1$		Fig. 5(c)	Fig. 5(d)	Fig. 5(d)	Fig. 5(c)

TABLE 3. About the finite equilibria $(1,1)$ for $a_1 < 0$ and $b_2 > 0$. x: $(1,1)$ is not an attractor or the *relations are not well defined.* L.A: $(1,1)$ is a local attractor but is not a global attractor. Fig. 5: $(1,1)$ is a local attractor as shows Figure 5 and can be a global attractor.



(a) (b) (c) (d)

FIGURE 5. Local phase portraits at the equilibria of system (2) for $a_1 < 0$ and $b_2 > 0$ with $b_2 < a_2b_1/a_1$. (a) if $b_1 < -b_2$, $a_2 > b_2$ and $a_1 < b_1$, (b) if $b_1 < -b_2$, $a_1 \geq b_1$ and $a_2 > b_2$; or if $a_1 > b_1$ and $a_2 = b_2$, (c) $b_1 < -b_2$, $a_1 < b_1$ and $a_2 \geq -a_1$; or if $b_1 = -b_2$ and $a_2 > -a_1$. (d) $b_1 < -b_2$, $a_1 \geq b_1$ and $a_2 \geq -a_1$.

Subcase $b_2 < 0$. In this case the condition on the parameters (9) is reduced to $b_2 < a_2b_1/a_1$. We start the study of the local phase portraits of the finite equilibria. Studying the eigenvalues of (4) we have that the origin is a hyperbolic repeller if $b_1 \leq b_2$ and $a_2 < -a_1$; it is a hyperbolic saddle of type S_h if $a_2 > 0$ and $-a_2 < a_1$; and it is a hyperbolic saddle of type S_v if $b_1 > -b_2$ and $-a_2 > a_1$. When one eigenvalue is zero (both eigenvalues cannot be zero from condition (9)) we have a semi-hyperbolic saddle-node, its local phase portrait is obtained applying Theorem 2.19 of [5] and it is as the one of Figure 2(c) if $a_1 = -a_2$, or as the one of Figure 2(b) if $b_1 = -b_2$. Note that the saddle-node in both cases has one hyperbolic sector on Q. The finite equilibria e_3 belongs to Q if $b_1 < -b_2$, and it is a hyperbolic saddle of type S_h (this not contradicts that e_2 can be a global attractor in $\mathring{Q}$). Finally the equilibrium e_4 belongs to Q if $a_2 < -a_1$, then e_4 is a hyperbolic saddle of type S_h, i.e its stable separatrices are locally on the x-axis.

	$b_1 < 0$	$0 \leq b_1 < -b_2$	$b_1 \geq -b_2$
$a_2 \geq -a_1$	Fig. 6(a) if $a_1 \geq b_1$ Fig. 6(b) if $a_1 < b_1$	Fig. 6(b)	x
$0 \leq a_2 < -a_1$	Fig. 6(c) if $a_1 \geq b_1$ Fig. 6(d) if $a_1 < b_1$	Fig. 6(d)	Fig. 6(e)

TABLE 4. About the finite equilibria $(1, 1)$ for $a_1 < 0$, $b_2 < 0$ and $a_2 \geq 0$ with $b_2 < a_2 b_1/a_1$.

		$b_1 < 0$	$0 \leq b_1 < -b_2$	$b_1 \geq -b_2$
$a_1 < b_1$	$a_2 \leq b_2$	Fig. 6(g)	Fig. 6(g)	Fig. 6(f)
	$a_2 > b_2$	Fig. 6(d)	Fig. 6(d)	Fig. 6(e)
$a_1 = b_1$	$a_2 \leq b_2$	x	x	x
	$a_2 > b_2$	Fig. 6(c)	x	x
$a_1 > b_1$	$a_2 \leq b_2$	x	x	x
	$a_2 > b_2$	Fig. 6(c)	x	x

TABLE 5. About the finite equilibria $(1, 1)$ for $a_1 < 0$, $b_2 < 0$ and $a_2 < 0$ with $b_2 < a_2 b_1/a_1$.

At infinity we study the origin of both local charts and p_1 when it is in Q. Since we are considering $b_2 < 0$ it follows from (8) that the origin of U_2 has one positive eigenvalue, then it is a hyperbolic unstable node if $a_2 - b_2$ is positive; it is a hyperbolic saddle of type S_h if $a_2 < b_2$; and for $a_2 = b_2$ it becomes a semi-hyperbolic equilibrium, doing the analysis through the theorem previously mentioned for semi-hyperbolic equilibria it is a saddle-node as the one of Figure 2(a) if $a_1 - b_1 > 0$, or it is a saddle-node as the one of Figure 2(c) if $a_1 - b_1 < 0$. Note that in Q there is a hyperbolic sector for $a_1 - b_1 > 0$, and a repelling parabolic sector for $a_1 - b_1 < 0$.

The origin of the local chart U_1 has the same local phase portrait that for the case $a_1 < 0$ and $b_2 > 0$. This means that it is a hyperbolic unstable node if $a_1 < b_1$, it is a hyperbolic saddle S_h if $a_1 > b_1$, and for $a_1 = b_1$ it is a saddle-node as the one of Figure 2(c) with one hyperbolic sector in Q. The remaining infinite equilibria p_1 is in Q for $a_2 \leq 0$, $b_1 > a_1$ and $b_2 < a_2$, or for $a_2 > 0$ and $b_1 > a_1$, in both cases it is a hyperbolic saddle S_h.

In Tables 4 and 5 are summarized the conditions for the existence of the local phase portraits in the quadrant Q shown in Figure 6 for system (2) with $a_1 < 0$ and $b_2 < 0$ being $(1, 1)$ a local attractor. Note that Figure 6(a) coincides exactly with Figure 3.

Subcase $b_2 = 0$. Then system (2) takes the form

$$\dot{x} = x(a_1(x - 1) + a_2(y - 1)), \quad \dot{y} = b_1\, y(x - 1), \tag{10}$$

and it has the finite equilibria e_1, e_2 and e_4. If $b_2 = 0$ and $a_1 < 0$, then e_2 is a local hyperbolic attractor if $a_2 b_1 < 0$ according to (9).

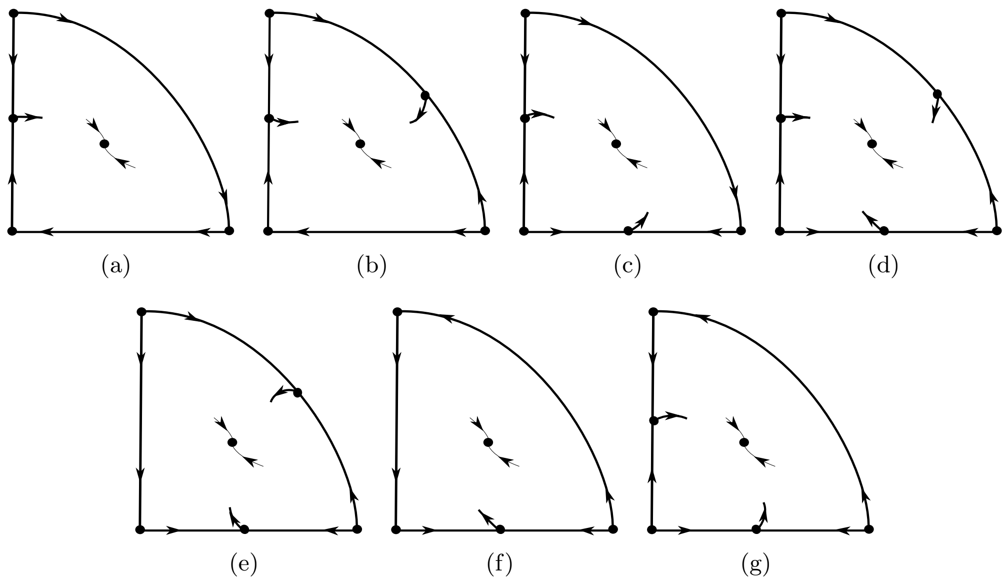


FIGURE 6. Local phase portraits of the equilibria in Q of system (2) for $a_1 < 0$ and $b_2 < 0$ being $(1, 1)$ a local hyperbolic attractor.

If $a_2 < 0$ and $b_1 > 0$ the origin is a hyperbolic saddle S_v, and e_4 is in Q and it is a hyperbolic saddle S_h. If $a_2 > 0$ and $b_1 < 0$ we need to consider two subcases: For $a_1 < -a_2$ the origin is a hyperbolic repeller and e_4 is a hyperbolic saddle S_h. If $a_1 \geq -a_2$ the local phase portrait at the origin is a hyperbolic saddle S_h if $a_1 > -a_2$, or is a saddle-node as the one of Figure 2(c) if $a_1 = -a_2$. Note that in both cases the origin has one hyperbolic sector in Q, with its local stable separatrices on the x-axis. When $a_1 \geq -a_2$ the finite equilibria e_4 does not belong to Q.

At infinity using the Poincaré compactification we have that the origin of U_1 for $b_1 > 0$ is a hyperbolic repeller, and for $b_1 < 0$ its local phase portrait changes with the sign of the eigenvalue $-a_1 + b_1$, if $a_1 < b_1$ it is a hyperbolic repeller, for $a_1 > b_1$ it is a hyperbolic saddle S_h, and for $a_1 = b_1$ it is a semi-hyperbolic equilibrium, since $a_2 > 0$ it is a saddle-node as the one of Figure 2(c). Then the local phase portrait at the origin of U_1 in Q when $b_1 < 0$ coincides with the phase portrait when $a_1 > b_1$ and $a_1 = b_1$.

The infinite equilibrium p_1 of U_1 is in Q only if $b_1 < 0, a_2 > 0$ and $a_1 < b_1$, then it is hyperbolic saddle S_h.

Finally the origin of U_2 has the eigenvalues (see (8)) 0 and a_2, then it is semi-hyperbolic. By Theorem 2.19 of [5] it is a saddle-node, even more, for $a_2 < 0$ and $b_1 > 0$ is as the one of Figure 2(e), and for $a_2 > 0$ and $b_1 < 0$ it is as the one of Figure 2(d).

These previous analysis allow to give all the local phase portraits of the equilibria in Q. The local phase portraits at the equilibria of system (10) in Q are as the ones of Figure (5)(a) if $a_2 > 0, b_1 < 0$, $a_1 < -a_2$ and $b_1 > a_1$; are as the ones of Figure (5)(b) if $a_2 > 0, b_1 < 0$, $a_1 < -a_2$ and $b_1 \leq a_1$; are as the ones of Figure (5)(c) if $a_2 > 0, b_1 < 0$, $a_1 \geq -a_2$ and $b_1 > a_1$; are as the ones of Figure (5)(d) if $a_2 > 0, b_1 < 0$, $a_1 \geq -a_2$ and $b_1 \leq a_1$; and are as the ones of Figure (6)(f) if $a_2 < 0, b_1 > 0$.

3. Global Phase Portraits

Now we shall provide the global phase portraits which are not quadrant–topologically equivalent of system (2) having at $(1,1)$ a global attractor in the open quadrant $\mathring{Q}$. It is important to remember that a Lotka-Volterra system (2) cannot have a limit cycle as it was establish in section 1.

We have fifteen local phase portraits in Figures 4, 5 and 6 which are different between them (we recall that Figure 3 coincides with Figure 6(a)). However Figure 4(c) and Figure 5(d) have the same equilibria on each boundary and they have the same local phase portraits but with reverse orientation, then according with our definition of quadrant-topologically equivalent they are equivalent. Therefore we analyse the global phase portrait of Figure 4(c) and of the remaining thirteen figures.

We started with Figure 4(c) its boundary contains only three equilibria, each one generating one hyperbolic sector in Q. So the boundary is a graphic formed by these three equilibria, the x-axis, the y-axis and $\mathbb{S}^1 \cap Q$. When the local phase portrait of system (2) is quadrant–topologically equivalent to 4(c) the parameters satisfy $\tau^2 - 4\delta < 0$, i.e. the finite equilibrium $(1,1)$ is always a stable focus. Then, by the Poincaré-Bendixon Theorem (see Corollary 1.30 of [5]) the α-limit of all the orbits in $\mathring{Q}$ different from the equilibrium $(1,1)$ is the graphic of the boundary of Q, and their ω-limit is the equilibrium $(1,1)$. We conclude that the global phase portrait associated to Figure 4(c) (and 5(d)) is quadrant-topologically equivalent to Figure 1(a).

We continue with the analysis when we have three finite equilibria and two infinite equilibria in Q. The finite equilibria are as usual the origin and $(1,1)$, and the third one can be either on $x = 0$, or on $y = 0$, and it is a saddle with its stable separatrices on one of the axis. If the third equilibrium is on $x = 0$ we are in the cases when the parameters are such that the local phase portraits are as the one of Figure 3 (or equivalent Figure 6(a)), or as the one of Figure 4(a). In both cases, and again by the Poincaré-Bendixon Theorem, the unstable separatrix of the third equilibrium located on $x = 0$ must go to $(1,1)$. Since does not exist more separatrices we can complete the global phase portrait considering the continuity of the solutions. The global phase portrait of system (2) with parameters having the local phase portraits showed in Figure 3 and Figure 6(a) is quadrant–topologically equivalent to Figure 1(b); and the global phase portrait of system (2) with parameters having the local phase portrait showed in Figure 4(a) is quadrant-topologically equivalent to Figure 1(c). For Figures 1(b) and 1(c) and the following ones the attractor $(1,1)$ can be a stable node when $\tau^2 - 4\delta \geq 0$, or a stable focus for $\tau^2 - 4\delta < 0$ (both behaviours can occur).

In a similar way if the third finite equilibria is located on $y = 0$ (local phase portrait as Figure 5(b) and Figure 6(f)), then its unstable separatrix in $\mathring{Q}$ must go to the attractor $(1, 1)$. From the Poincaré–Bendixson Theorem we complete the global phase portrait. Then the global phase portrait associated to parameters that generate the local phase portraits at the equilibria showed in Figures 5(b) and Figure 6(f) are quadrant-topologically equivalent to Figure 1(d) and Figure 1(e) respectively.

In the next we do the analysis considering the cases when we still have only two infinite equilibria, but now we have four finite equilibria as occurs when the parameters $a_1 > 0$, $b_2 < 0$ and a_2, b_1 satisfied the conditions for Figures 6(c) and 6(g). The differences between these two figures are that the local phase portraits of the origin of U_1 and U_2 are interchanged, in both cases we have one repelling parabolic sector at the origin of U_1 or at the origin of U_2 and the other consists in one hyperbolic sector, then the orbits coming from the infinity in this repelling sector must go to the attractor $(1, 1)$. On the other hand, the unstable separatrix of e_3 and of e_4 located in Q must go to the attractor $(1, 1)$. By the Poincaré–Bendixson Theorem the global phase portraits for system (2) when its parameters are satisfying Figures 6(c) and 6(g) are quadrant-topologically equivalent to Figures 1(f) and 1(g) respectively. This concludes the study of the global phase portraits having two infinite equilibria, the origin of U_1 and the origin of U_2.

We focus now in the study of the global phase portraits for the cases when there exist three infinite equilibria, and we separate the study in three cases according to the number of finite equilibria. First, we analyse when system (2) only has the attractor $(1, 1)$ and the origin as finite equilibria. This happens when the parameters satisfy the conditions for local phase portraits showed in Figures (4)(d) and Figure 5(c). In each case the local phase portrait of the infinite equilibrium p_1 consists in two hyperbolic sectors in Q with two stable separatrices at infinity and the unstable separatrix go to the finite region and must go to the attractor $(1, 1)$. By the Poincaré–Bendixson Theorem we can conclude that all orbits being born at the repeller located at the infinity must go to the attractor $(1, 1)$. The previous analysis implies that the global phase portraits of system (2) when the parameters satisfy the conditions of Figures (4)(d) and 5(c) are quadrant-topologically equivalent to Figure 1(h) and 1(i) respectively.

Second, for the cases with three finite equilibria in Q, i.e. when besides the origin and the attractor $(1, 1)$ also exist a third equilibrium which can be located on $x = 0$ or $y = 0$. As before, the infinite equilibrium p_1 is a saddle with its unstable separatrix going to $(1, 1)$. The finite equilibrium e_3 or e_4 consists in two hyperbolic sectors in Q, with stable separatrices on some axis. Then the unstable separatrix located in $\mathring{Q}$ must go to the attractor $(1, 1)$. Since we already have determined the α– and ω- limits of all the separatrices we obtain that the global phase portrait when the parameters satisfy Figure 4(b) is quadrant-topologically equivalent to Figure 1(j). If the parameters satisfy Figure 6(b) the global phase portrait is quadrant–topologically equivalent to

Figure 1(k). If they satisfy Figure 5(a) the global phase portrait is quadrant–topologically equivalent to Figure 1(l). If they satisfy Figure 6(e) then the global phase portrait is quadrant–topologically equivalent to Figure 1(m).

Finally we consider the case when the parameters are such that system (2) have three infinite equilibria and four finite equilibria in Q (see Figure 6(d)). In this case the finite equilibria e_3 and e_4 have one unstable separatrix in $\mathring{Q}$, and due to the fact that the only attractor is the finite equilibrium $(1,1)$, each one of these unstable separatrices must go to this attractor. In the same way the unstable separatrix of the infinite equilibria in $U_1 \cap V_1$ must go to $(1,1)$ and we established all the connections of the separatrices in Q, then we can complete the global phase portrait of system (2) when there exist the major quantity of possible equilibria in Q and it is quadrant-topologically equivalent to Figure 1(n).

In Tables 1 and 2 are summarized the relations of the parameters when $(1,1)$ is a global attractor of system (2) in the $\mathring{Q}$ and shows the quadrant-topologically equivalent global phase portraits.

This completes the proof of Theorem 1.

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