This is a preprint of: "Crossing limit cycles for discontinuous piecewise linear differential centers separated by three parallel straight lines", Maria Elisa Anacleto, Jaume Llibre, Clàudia Valls, Claudio Vidal, *Rend. Circ. Mat. Palermo (2)*, vol. 72(3), 1739–1750, 2023.
DOI: [10.1007/s12215-022-00766-3]

CROSSING LIMIT CYCLES FOR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL CENTERS SEPARATED BY THREE PARALLEL STRAIGHT LINES

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Abstract. In this paper we study the continuous and discontinuous planar piecewise differential systems formed by four linear centers separated by three parallel straight lines denoted by $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = -b, x = 0, x = a,\ a, b > 0\}$. We prove that when these piecewise differential systems are continuous they have no limit cycles. While for the discontinuous case we show that they can have at most four limit cycles and we also provide examples of such systems with zero, one, and two limit cycles. In particular we have solved the extension of the 16th Hilbert problem to this class of piecewise differential systems.

1. Introduction and statement of the main result

A *limit cycle* is a periodic orbit of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system. The study of the limit cycles goes back essentially to Poincaré [25] at the end of the nineteenth century. The existence of limit cycles became important in the applications to the real world, because many phenomena are related with their existence, see for instance the Van der Pol oscillator [28,29] or the Belousov-Zhabotinskii reaction which is a classical reaction of non-equilibrium thermodynamics appearing in a non-linear chemical oscillator, see [3,30].

The study of the continuous piecewise linear differential systems separated by one or two parallel straight lines appears in a natural way in the control theory, see for instance the books [2,10,13,14,19,24].

The easiest continuous piecewise linear differential systems are formed by two linear differential systems separated by one straight line. It is known that such systems have at most one limit cycle, see [8,16,21,22].

The study of the discontinuous piecewise linear differential systems separated by straight lines goes back to Andronov et al. [1] and until nowadays they had special attention from the mathematicians, mainly because these systems appear in mechanics, electrical circuits, economy, etc, see for instance the books [6,26] and the surveys [23,27].

2010 *Mathematics Subject Classification.* 34C05, 34C07.
Key words and phrases. Limit cycles, linear centers, continuous piecewise differential systems, discontinuous piecewise differential systems, first integrals.

Of course the discontinuous differential systems in principle are not well defined on the line of discontinuity, on this line the differential system is defined following the rules of Filippov [7].

In the planar discontinuous piecewise linear differential systems, the limit cycles can be of two kinds: sliding limit cycles or crossing limit cycles; the first ones contain some segment of the lines of discontinuity, and the second ones only contain isolated points of the lines of discontinuity. In Theorem 2 are studied the limit cycles which leaves only in three regions separated by two parallel lines. So in this paper we only study the crossing limit cycles or simply limit cycles which intersect the four component of $\mathbb{R}^2 \setminus \Sigma$. These components are denoted by

$$R_1 = \{(x, y) \in \mathbb{R}^2 : x < -b\},\ R_2 = \{(x, y) \in \mathbb{R}^2 : -b < x < 0\},$$
$$R_3 = \{(x, y) \in \mathbb{R}^2 : 0 < x < a\},\ R_4 = \{(x, y) \in \mathbb{R}^2 : x > a\}.$$

Again the easiest discontinuous piecewise linear differential systems are formed by two linear differential systems separated by one straight line. It is known that such systems can have three limit cycles, see [4,5,9,12,15,17,20]. It remains open to know if three is the maximum number of limit cycles that such systems can exhibit. In [18] it was proved:

Theorem 1. *A continuous piecewise differential system separated by one or two parallel straight lines and formed by linear differential centers has no limit cycles.*

Theorem 2. *A discontinuous piecewise differential system separated by*

(a) *one straight line and formed by two linear differential centers has no limit cycles;*
(b) *two parallel straight lines and formed by three linear differential centers can have at most one limit cycle, and there are examples of these piecewise differential systems having one limit cycle.*

In this work we extend the study of existence of crossing limit cycles to continuous and discontinuous piecewise linear differential systems separated by three arbitrary parallel straight lines Σ, and are formed by four linear differential centers. Our main results are the following.

Theorem 3. *A continuous piecewise differential system separated by three parallel straight lines, and formed by four linear differential centers has no limit cycles.*

Theorem 4. *A discontinuous piecewise differential system separated by three parallel straight lines, and formed by four linear differential centers can have at most four limit cycles intersecting the four pieces of these piecewise differential systems. Moreover there are systems in this class having zero, one, or two limit cycles.*

Theorems 3 and 4 are proved in section 3.

Note that providing the upper bound of four limit cycles in Theorem 4 we have solved the extension of the 16th Hilbert problem to the class of discontinuous piecewise differential systems studied in that theorem. See [11] for the most famous problem related with limit cycles, i.e. the 16th Hilbert problem .

We remark that in general it is very difficult to provide an explicit upper bound for the maximum number of limit cycles for a given class of piecewise differential systems and even harder to provide an upper bound that it is reached for that class. This is mainly due to the fact that our class of piecewise differential systems depends on 12 parameters, and that the solutions of the so-called *closing equations* (see below) do not need to correspond necessarily to periodic solutions of the discontinuous piecewise differential systems.

In view of Theorem 4 and from all studied discontinuous piecewise differential systems separated by three parallel straight lines, and formed by four linear differential centers, we state the following open question.

Open Problem. *It is an open problem to know if for a discontinuous piecewise differential system separated by three parallel straight lines, and formed by four linear differential centers the exact upper bound on the number of limit cycles is two, three or four.*

2. Preliminaries results

The following lemma, proved in [18], provides a normal form for an arbitrary linear differential system having a center.

Lemma 1. *A linear differential system having a center can be written as*

$$\dot{x} = -\beta x - \frac{4\beta^2 + \omega^2}{4\alpha} y + \delta, \quad \dot{y} = \alpha x + \beta y + \gamma, \tag{1}$$

with $\alpha > 0$ *and* $\omega \neq 0$.

Note that the linear center provided in (1) has the first integral

$$\begin{aligned} H &= 4(\alpha x + \beta y)^2 + 8\alpha(\gamma x - \delta y) + \omega^2 y^2 \\ &= 4\alpha^2 \left(\left(x + \frac{\beta}{\alpha} y \right)^2 + 2\Big(\frac{\gamma}{\alpha} x - \frac{\delta}{\alpha} y \Big) + \frac{\omega^2}{4\alpha^2} y^2 \right). \end{aligned}$$

Taking the notation $b = \beta/\alpha$, $c = \gamma/\alpha$, $d = \delta/\alpha$ and $w = \omega/(2\alpha)$ it is not restrictive to consider the first integral

$$H = (x + by)^2 + 2(cx - dy) + w^2 y^2.$$

Moreover since w is the velocity in which the orbits are travelled it does not interfere in the existence of periodic orbits and so it is not restrictive to take $w = 1$. In short, any arbitrary linear differential system having a center can be written as

$$\dot{x} = 2d - 2bx - 2(b^2 + 1)y, \quad \dot{y} = 2c + 2x + 2by,$$

with first integral

$$H = (x + by)^2 + 2(cx - dy) + y^2 = 2cx - 2dy + x^2 + 2byx + (b^2 + 1)y^2.$$

It will be convenient in this paper in order to simplify the computations to take the new variable $\tilde{b} = b^2 + 1$ and with this new variable we have that any arbitrary linear differential system having a center can be written as

$$\dot{x} = 2d - 2bx - 2\tilde{b}y, \quad \dot{y} = 2c + 2x + 2by$$

with first integral

$$H = 2cx - 2dy + x^2 + 2byx + \tilde{b}y^2.$$

3. Proof of Theorems 3 and 4

Suppose that we have a piecewise linear differential system separated by three parallel straight lines. Without loss of generality we can assume that these straight lines are $x = -b$, $x = 0$ and $x = a$ with $a > 0$ and $b > 0$.

In view of section 2 we can write the four linear centers of such a discontinuous piecewise linear differential system as

$$\dot{x} = 2d_i - 2b_i x - 2\tilde{b}_i y, \quad \dot{y} = 2c_i + 2x + 2b_i y \tag{2}$$

with first integrals

$$H = 2c_i x - 2d_i y + x^2 + 2b_i yx + \tilde{b}_i y^2,$$

for $i = 1, 2, 3, 4$ with $\tilde{b}_i = b_i^2 + 1$.

In view of Theorems 1 and 2 we want to study the limit cycles of these piecewise differential systems which intersect the four open regions $x < -b$, $-b < x < 0$, $0 < x < a$ and $x > a$. So a possible limit cycle must intersect each straight line $x = -b$, $x = 0$ and $x = a$ in exactly two points, namely $(-b, y_1)$, $(-b, y_2)$, $(0, y_3)$, (a, y_4), (a, y_5) and $(0, y_6)$ with $y_1 > y_2$, $y_3 < y_6$ and $y_4 < y_5$. Hence the y_i for $i = 1, \ldots, 6$ must satisfy the following system of six equations

$$\begin{aligned}
e_1 &= H_1(-b, y_1) - H_1(-b, y_2) = 0,\\
e_2 &= H_2(-b, y_2) - H_2(0, y_3) = 0,\\
e_3 &= H_3(0, y_3) - H_3(a, y_4) = 0,\\
e_4 &= H_4(a, y_4) - H_4(a, y_5) = 0,\\
e_5 &= H_3(a, y_5) - H_3(0, y_6) = 0,\\
e_6 &= H_2(0, y_6) - H_2(-b, y_1) = 0.
\end{aligned} \tag{3}$$

Since $y_1 > y_2$, $y_6 > y_3$ and $y_5 > y_4$ we can write

$$y_1 = y_2 + A, \quad y_6 = y_3 + B, \quad y_5 = y_4 + C,$$

with A, B and C positive.

Proof of Theorem 3. Since in Theorem 3 this piecewise differential system must be continuous, systems (2) with $i = 1$ and $i = 2$ must coincide on $x = -b$, systems (2) with $i = 2$ and $i = 3$ must coincide on $x = 0$ and systems (2) with $i = 3$ and $i = 4$ must coincide on $x = a$. Therefore,

$$b_2 = b_3 = b_4,\ \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = \tilde{b}_4,\ c_2 = c_3 = c_4,\ d_2 = d_3 = d_4,\ d_1 = -bb_1 + b_3 + bd_3.$$

Consequently system (3) reduces to

$$\begin{aligned}
E_1 &= -2(bb_4 + d_4) + \tilde{b}_4 A + 2\tilde{b}_4 y_2,\\
E_2 &= b^2 - 2bc_4 + 2d_4 B - 2(bb_4 + d_4)y_2 + 2d_4 y_6 - \tilde{b}_4 B^2 - 2\tilde{b}_4 B y_6 + \tilde{b}_4 y_2^2 - \tilde{b}_4 y_6^2,\\
E_3 &= -a^2 - 2ac_4 - 2d_4 B + 2(d_3 - ab_3)y_4 - 2d_3 y_6 + \tilde{b}_4 B^2 + 2\tilde{b}_4 B y_6 - \tilde{b}_4 y_4^2 + \tilde{b}_4 y_6^2,\\
E_4 &= 2(ab_4 - d_4) + 2\tilde{b}_4 C + 2\tilde{b}_4 y_4,\\
E_5 &= a^2 + 2ac_4 + 2(ab_4 - d_4)C + 2(ab_4 - d_4)y_4 + 2d_3 y_6 + \tilde{b}_4 C^2 + 2\tilde{b}_4 C y_4 + \tilde{b}_4 y_4^2 - \tilde{b}_4 y_6^2,\\
E_6 &= -b^2 + 2bc_4 + 2(b_4 + d_4)A + 2(bb_4 + d_4)y_2 - 2d_4 y_6 - \tilde{b}_4 A^2 - 2\tilde{b}_4 A y_2 - \tilde{b}_4 y_2^2 + \tilde{b}_4 y_6^2.
\end{aligned}$$

The solutions of $E_i = 0$, for $i = 1, \ldots, 6$ with A, B, C, y_2, y_4, y_6 being A, B, C positive are of the form

$$y_2 = \frac{d_3 - bb_3}{\tilde{b}_3} - \frac{A}{2}, \quad y_4 = \frac{d_3 - ab_3}{\tilde{b}_3} - \frac{C}{2}, \quad B = \frac{2d_3}{\tilde{b}_3} - 2y_6,$$

with

$$A = C = \frac{2}{|\tilde{b}_3|}\sqrt{(d_3 - \tilde{b}_3 y_6)^2 + bb_3(bb_3 + 2d_3) + b\tilde{b}_3(2c_3 - b)}$$

and $y_6 < d_3/\tilde{b}_3$. Hence there is a continuum of periodic orbits for the continuous piecewise linear differential system, and consequently this differential system has no limit cycles. □

Proof of Theorem 4. System (3) becomes

$$\begin{aligned}
e_1 =& -2(bb_1 + d_1) + \tilde{b}_1(y_1 + y_2) = 0,\\
e_2 =& b^2 - 2bc_2 - 2(bb_2 + d_2)y_2 + 2d_2y_3 + \tilde{b}_2y_2^2 - \tilde{b}_2y_3^2 = 0,\\
e_3 =& -a^2 - 2ac_3 - 2d_3y_3 + 2(d_3 - ab_3)y_4 + \tilde{b}_3y_3^2 - \tilde{b}_3y_4^2 = 0,\\
e_4 =& 2(ab_4 - d_4) + \tilde{b}_4(y_4 + y_5) = 0,\\
e_5 =& a^2 + 2ac_3 + 2(ab_3 - d_3)y_5 + 2d_3y_6 + \tilde{b}_3y_5^2 + -\tilde{b}_3y_6^2 = 0,\\
e_6 =& -b^2 + 2bc_2 + 2(bb_2 + d_2)y_1 - 2d_2y_6 - \tilde{b}_2y_1^2 + \tilde{b}_2y_6^2 = 0.
\end{aligned} \tag{4}$$

In order to lighten the computations we introduce a new notation. Let
$r_1 = b^2-2bc_2$, $r_2 = a^2+2ac_3$, $r_3 = bb_2+d_2$, $r_4 = d_3-ab_3$, $r_5 = d_1+bb_1$, $r_6 = ab_4-d_4$.
Note that to recover the original variables we just need to take $c_2 = (r_1 - b^2)/2b$, $d_2 = r_3 - bb_2$, $d_3 = r_4 + ab_3$ and $\tilde{b}_i = b_i^2 + 1$ for $i = 1, \ldots, 4$.

With this new notations system (4) becomes

$$\begin{aligned}
e_1 =& -2r_5 + \tilde{b}_1(y_1 + y_2) = 0,\\
e_2 =& r_1 - 2r_3y_2 + 2d_2y_3 + \tilde{b}_2(y_2^2 - y_3^2) = 0,\\
e_3 =& -r_2 - 2d_3y_3 + 2r_4y_4 + \tilde{b}_3(y_3^2 - y_4^2) = 0,\\
e_4 =& 2r_6 + \tilde{b}_4(y_4 + y_5) = 0,\\
e_5 =& r_2 - 2r_4y_5 + 2d_3y_6 + \tilde{b}_3(y_5^2 - y_6^2) = 0,\\
e_6 =& -r_1 + 2r_3y_1 - 2d_2y_6 - \tilde{b}_2(y_1^2 - y_6^2) = 0.
\end{aligned} \tag{5}$$

From the equations e_1 and e_4 we obtain

$$y_2 = \frac{r_5}{\tilde{b}_1} - \frac{A}{2} \quad \text{and} \quad y_4 = -\frac{r_6}{\tilde{b}_4} - \frac{C}{2},$$

respectively. Substituting these expressions in equations e_2, e_3, e_5, e_6 in (5), and denoting them as E_2, E_3, E_5, E_6 we get that

$$E_2 - E_6 = -4\tilde{b}_1\big(2A(\tilde{b}_1r_3 - \tilde{b}_2r_5) + B\tilde{b}_1(B\tilde{b}_2 - 2d_2 + 2\tilde{b}_2y_3)\big).$$

Solving $E_2 - E_6 = 0$ in the variable y_3 and since $B > 0$ we get

$$y_3 = \frac{d_2}{\tilde{b}_2} + \frac{\tilde{b}_2r_5 - \tilde{b}_1r_3}{B\tilde{b}_1\tilde{b}_2}A - \frac{B}{2}.$$

Now introducing y_3 in E_2, E_3, E_5 and denoting them as F_2, F_3, F_5 we get that

$$F_3 - F_5 = -8B^2\tilde{b}_1\tilde{b}_2\tilde{b}_4\big(\tilde{b}_3\tilde{b}_4(\tilde{b}_1 r_3 - \tilde{b}_2 r_5)A + \tilde{b}_1\tilde{b}_4(\tilde{b}_2 d_3 - \tilde{b}_3 d_2)B - \tilde{b}_1\tilde{b}_2(\tilde{b}_4 r_4 + \tilde{b}_3 r_6)C\big).$$

We consider two different cases.

Case 1*:* $\tilde{b}_4 r_4 + \tilde{b}_3 r_6 = 0$.

In this case we get that

$$F_3 - F_5 = -8B^2\tilde{b}_1\tilde{b}_2\tilde{b}_4\big(\tilde{b}_3\tilde{b}_4(\tilde{b}_1 r_3 - \tilde{b}_2 r_5)A + \tilde{b}_1\tilde{b}_4(\tilde{b}_2 d_3 - \tilde{b}_3 d_2)B\big).$$

If $\tilde{b}_2 d_3 - \tilde{b}_3 d_2 = 0$ then the solution would be either $A = 0$ which is not possible, or $A \neq 0$ and $\tilde{b}_1 r_3 - \tilde{b}_2 r_5 = 0$, but in this last case the solution of $F_2 = F_3 = 0$ leads to a continuum of solutions, because we have two equations $F_2 = F_3 = 0$ and three unknowns A, B, C.

In summary we can assume that $\tilde{b}_2 d_3 - \tilde{b}_3 d_2 \neq 0$ and solving $F_3 - F_5 = 0$ in the variable B we get

$$B = \frac{\tilde{b}_3(\tilde{b}_1 r_3 - \tilde{b}_2 r_5)}{\tilde{b}_1(\tilde{b}_3 d_2 - \tilde{b}_2 d_3)} A.$$

In this case introducing B into F_2 and F_3 and denoting them by G_2, G_3 we get that

$$\begin{aligned} G_2 = {} & (\tilde{b}_1 r_3 - \tilde{b}_2 r_5)^2 \Big(A^2\tilde{b}_2\tilde{b}_3^2 \Big(\tilde{b}_1^2 \Big(\tilde{b}_2^2 d_3^2 - 2\tilde{b}_2\tilde{b}_3 d_2 d_3 + \tilde{b}_3^2\left(d_2^2 - r_3^2\right)\Big) + 2\tilde{b}_1\tilde{b}_2\tilde{b}_3^2 r_3 r_5 \\ & -\tilde{b}_2^2\tilde{b}_3^2 r_5^2\Big) - 4(\tilde{b}_3 d_2 - \tilde{b}_2 d_3)^2 \Big(\tilde{b}_1^2\Big(\tilde{b}_2 d_3^2 + \tilde{b}_3^2 r_1 - 2\tilde{b}_3 d_2 d_3\Big) + 2\tilde{b}_1\tilde{b}_3^2 r_3 r_5 - \tilde{b}_2\tilde{b}_3^2 r_5^2\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} G_3 = {} & (\tilde{b}_1 r_3 - \tilde{b}_2 r_5)^2 \Big(-\tilde{b}_1^2 \Big(\tilde{b}_3^4 \Big(\tilde{b}_4^2\left(C^2 d_2^2 - A^2 r_3^2\right) - 4d_2^2 r_6^2\Big) + \tilde{b}_3^2 d_3 \Big(\tilde{b}_2^2 d_3 \Big(\tilde{b}_4^2 C^2 - 4r_6^2\Big) \\ & -8\tilde{b}_2\tilde{b}_4^2 d_2 r_2 + 4\tilde{b}_4^2 d_2^2 d_3\Big) + 4\tilde{b}_2^2\tilde{b}_4^2 d_3^4 + 2\tilde{b}_3^3 d_2\Big(-\tilde{b}_2\tilde{b}_4^2 C^2 d_3 + 4\tilde{b}_2 d_3 r_6^2 + 2\tilde{b}_4^2 d_2 r_2\Big) \\ & +4\tilde{b}_2\tilde{b}_3\tilde{b}_4^2 d_3^2(\tilde{b}_2 r_2 - 2d_2 d_3)\Big) - 2A^2\tilde{b}_1\tilde{b}_2\tilde{b}_3^4\tilde{b}_4^2 r_3 r_5 + A^2\tilde{b}_2^2\tilde{b}_3^4\tilde{b}_4^2 r_5^2\Big) \end{aligned}$$

Note that if $\tilde{b}_1 r_3 - \tilde{b}_2 r_5 = 0$ we have a continuum of solutions. On the other hand, G_2 is a polynomial that only depends on the variable A and is of degree two in this variable. Moreover, G_3 is a polynomial in the variables A, C and is of degree two in these variables. Since A and C must be positive, solving $G_2 = 0$ we get a unique positive value of A and introducing it in $G_3 = 0$ and solving in the variable C we get a unique positive value of C. In short, there is at most a unique solution of $e_i = 0$ for $i = 1, \ldots, 6$ and so at most a unique limit cycle in this case.

Case 2*:* $\tilde{b}_4 r_4 + \tilde{b}_3 r_6 \neq 0$. Solving in the variable C we get

$$C = \frac{\tilde{b}_4(A\tilde{b}_3(\tilde{b}_1 r_3 - \tilde{b}_2 r_5) + B\tilde{b}_1(\tilde{b}_2 d_3 - \tilde{b}_3 d_2))}{\tilde{b}_1\tilde{b}_2(\tilde{b}_3 r_6 + \tilde{b}_4 r_4)}.$$

Introducing C into F_2 and F_3 and denoting them by G_2 and G_3 respectively, we get

$$\begin{aligned} G_2 &= \beta_0 B^4 + \beta_1 B^2 + \beta_2 A^2 B^2 + \beta_3 A^2, \\ G_3 &= \alpha_0 B^4 + \alpha_1 A B^3 + \alpha_2 B^2 + \alpha_3 A^2 B^2 + \alpha_4 A B + \alpha_5 A^2, \end{aligned}$$

where

$$
\begin{aligned}
\beta_0 &= \tilde{b}_1^2\tilde{b}_2^2,\\
\beta_1 &= -4\tilde{b}_1^2\left(d_2^2-\tilde{b}_2r_1\right)+8\tilde{b}_1\tilde{b}_2r_3r_5-4\tilde{b}_2^2r_5^2,\\
\beta_2 &= -\tilde{b}_1^2\tilde{b}_2^2,\\
\beta_3 &= 4(\tilde{b}_1r_3-\tilde{b}_2r_5)^2,\\
\alpha_0 &= \tilde{b}_1^2\tilde{b}_3\tilde{b}_4^2\left(\tilde{b}_3^2\left(\tilde{b}_4^2d_2^2-\tilde{b}_2^2r_6^2\right)+\tilde{b}_2^2\tilde{b}_4^2\left(d_3^2-r_4^2\right)-2\tilde{b}_2\tilde{b}_3\tilde{b}_4(\tilde{b}_2r_4r_6+\tilde{b}_4d_2d_3)\right),\\
\alpha_1 &= -2\tilde{b}_1\tilde{b}_3^2\tilde{b}_4^4(\tilde{b}_1r_3-\tilde{b}_2r_5)(\tilde{b}_3d_2-\tilde{b}_2d_3),\\
\alpha_2 &= -4\tilde{b}_1^2(\tilde{b}_3r_6+\tilde{b}_4r_4)^2\left(\tilde{b}_3\left(\tilde{b}_4^2d_2^2-\tilde{b}_2^2r_6^2\right)-\tilde{b}_2\tilde{b}_4(\tilde{b}_2\tilde{b}_4r_2+2\tilde{b}_2r_4r_6+2\tilde{b}_4d_2d_3)\right),\\
\alpha_3 &= \tilde{b}_3^3\tilde{b}_4^4(\tilde{b}_1r_3-\tilde{b}_2r_5)^2,\\
\alpha_4 &= 8\tilde{b}_1\tilde{b}_4^2(\tilde{b}_1r_3-\tilde{b}_2r_5)(\tilde{b}_3d_2-\tilde{b}_2d_3)(\tilde{b}_3r_6+\tilde{b}_4r_4)^2,\\
\alpha_5 &= -4\tilde{b}_3\tilde{b}_4^2(\tilde{b}_1r_3-\tilde{b}_2r_5)^2(\tilde{b}_3r_6+\tilde{b}_4r_4)^2.
\end{aligned}
$$

We need to compute the common zeroes of $G_2 = 0$ and $G_3 = 0$. For doing so we will compute the resultant of G_2 and G_3 with respect to the variable A, that is, $\mathcal{R} = \text{Res}\,(G_2, G_3, A)$. This resultant is

$$
\mathcal{R} = \tilde{b}_2^2B^4(C_0 + C_1B^2 + C_2B^4 + C_3B^6 + C_4B^8), \tag{6}
$$

for some constants C_i for $i = 0, \ldots, 4$. Since the variable B must be positive, we get that there are at most four positive solutions of $\mathcal{R} = 0$. Note that $G_2 = G_2(A^2, B^2)$ being a polynomial of degree two in the variable A and of degree two in the variable B. Again since the variable A must be positive, for each B being a solution of $\mathcal{R} = 0$ there is a unique A solving $G_2 = 0$. In short, there is at most four positive solutions of $G_2 = G_3 = 0$, and so there are at most four crossing limit cycles in this case.

Remark 5. *We point out that if all the linear systems are homogeneous then there at most two limit cycles. In fact, that the polynomial $C_0 + C_1B^2 + C_2B^4 + C_3B^6 + C_4B^8$ in the resultant* (6) *assumes the form $C_0 + C_1B^2 + C_2B^4$. The same is true when three linear systems are homogeneous. But in the case of two homogeneous linear systems there are at most four limit cycles.*

Below are examples with zero, one and two limit cycles that will complete the proof of the theorem.

First we are going to exhibit an example with zero limit cycle and we consider $a = b = 1$. In this case we consider the systems

$$
\begin{aligned}
&\dot{x} = -y,\ \dot{y} = x - 1 \quad \text{in } x < -1,\\
&\dot{x} = -y,\ \dot{y} = 4x - 2 \quad \text{in } -1 < x < 0,\\
&\dot{x} = -y,\ \dot{y} = 9x - 2 \quad \text{in } 0 < x < 1,\\
&\dot{x} = -y,\ \dot{y} = x + 5 \quad \text{in } x > 1,
\end{aligned}
$$

with first integrals

$$\begin{aligned}H_1(x,y) =& x^2-2x+y^2,\\ H_2(x,y) =& 4x^2-4x+y^2,\\ H_3(x,y) =& 9x^2-4x+y^2,\\ H_4(x,y) =& x^2+10x+y^2.\end{aligned}$$

Then system (3) is equivalent to system

$$\begin{aligned}e_1 &= (y_2-y_1)(y_1+y_2)=0,\\ e_2 &= y_1^2-y_4^2+8=0,\\ e_3 &= y_4^2-y_6^2-5=0,\\ e_4 &= (y_6-y_5)(y_6+y_5)=0,\\ e_5 &= -y_3^2+y_5^2+5=0,\\ e_6 &= -y_2^2+y_3^2-8=0.\end{aligned} \tag{7}$$

Taking account that the solutions $(y_1,y_2,y_3,y_4,y_5,y_6)$ of system (7) must satisfies $y_1>y_2$, $y_3<y_4$ and $y_5<y_6$ we obtain the solution

$$(y_1,y_2,y_3,y_4,y_5,y_6)=\left(\sqrt{y_3^2-8},\ -\sqrt{y_3^2-8},\ y_3,\ -y_3,\ -\sqrt{y_3^2-5},\ \sqrt{y_3^2-5}\right),$$

with $y_3<0$, $y_3^2-8>0$ and $y_3^2-5>0$. Thus, we have a continuum of periodic orbits and, therefore system (7) has not limit cycle.

We consider the following planar discontinuous piecewise differential system with four zones separated by the three straight parallel lines $x=-1$, $x=0$ and $x=1$ and formed by the four linear differential centers

$$\begin{aligned}&\dot x=-y,\ \dot y=16x-2 \quad \text{in } x<-1,\\ &\dot x=-y-1,\ \dot y=\frac{1849}{100}x-3 \quad \text{in } -1<x<0,\\ &\dot x=-y-\frac{5}{2},\ \dot y=9x-2 \quad \text{in } 0<x<1,\\ &\dot x=-y-\frac{3}{20}, \dot y=\frac{729}{100}y+5 \quad \text{in } x>1,\end{aligned} \tag{8}$$

with first integrals

$$\begin{aligned}H_1(x,y) =& 16x^2-4x+y^2,\\ H_2(x,y) =& \frac{1849}{100}x^2-6x+y^2+2y,\\ H_3(x,y) =& 9x^2-4x+y^2+5y,\\ H_4(x,y) =& \frac{729}{100}x^2+10x+y^2+\frac{3}{10}y,\end{aligned}$$

respectively. This discontinuous piecewise differential system has a unique crossing limit cycle (see Figure 1), because the unique real solution $(y_1,y_2,y_3,y_4,y_5,y_6)$ of system (3) with $y_1>y_2,\ y_3<y_4,\ y_5<y_6$ is

$$(5.1732..,-5.1732..,-7.47345..,6.91191..,-6.94243..,6.64243..).$$

This limit crossing limit cycle is hyperbolic and stable because the derivative of the Poincaré map defined in a neighborhood of the point $(1, 6.64243..)$ on the line $x = 1$ is approximately $0.8985...$

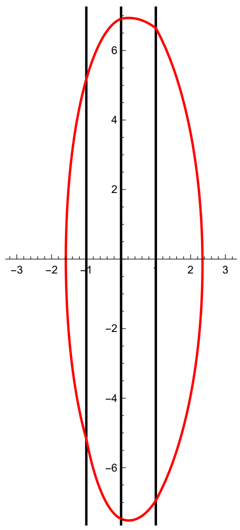

Figure 1: The unique crossing limit cycle of the discontinuous piecewise differential system formed by the centers (8). This limit cycle is traveled in counterclockwise.

Now we construct a piecewise differential system having exactly two crossing limit cycles. We consider $a = b = 1$ and the piecewise differential system

$$
\begin{aligned}
&\dot{x} = -y,\ \dot{y} = x + 1 \quad \text{in } x < -1,\\
&\dot{x} = -y - \frac{19}{5},\ \dot{y} = \frac{1369}{25}x + \frac{207}{10} \quad \text{in } -1 < x < 0,\\
&\dot{x} = -y - 1,\ \dot{y} = \frac{1681}{25}x - \frac{21}{2} \quad \text{in } 0 < x < 1,\\
&\dot{x} = -y - \frac{1}{8}, \dot{y} = 81x - 2 \quad \text{in } x > 1,
\end{aligned} \tag{9}
$$

with first integrals

$$
\begin{aligned}
H_1(x, y) =& x^2 + 2x + y^2,\\
H_2(x, y) =& \frac{1369}{25}x^2 + \frac{207}{5}x + y^2 + \frac{38}{5}y,\\
H_3(x, y) =& \frac{1681}{25}x^2 - 21x + y^2 + 2y,\\
H_4(x, y) =& 81x^2 - 4x + y^2 + \frac{1}{4}y,
\end{aligned}
$$

respectively. This discontinuous piecewise differential system formed by the previous four linear differential centers has two crossing limit cycles (see Figure 2), coming from the two real solutions $(y_1, y_2, y_3, y_4, y_5, y_6)$ of system (3) with $y_1 > y_2$, $y_3 < y_4$, $y_5 < y_6$ which are

$$(7.03696.., -7.03696.., -8.68241.., 7.63677.., -4.57483.., 4.32483..),$$

$$(5.47132.., -5.47132.., -7.81912.., 6.16581.., -1.51028.., 1.26028..).$$

These two crossing limit cycles are hyperbolic, being the biggest limit cycle stable because the derivative of the Poincaré map defined in a neighborhood of the point $(1, 4.32483..)$ on the line $x = 1$ is approximately 0.927034.., and the smallest limit cycle is unstable because the derivative of the Poincaré map defined in a neighborhood of the point $(1, 1.26028..)$ on the line $x = 1$ is approximately 1.88... □

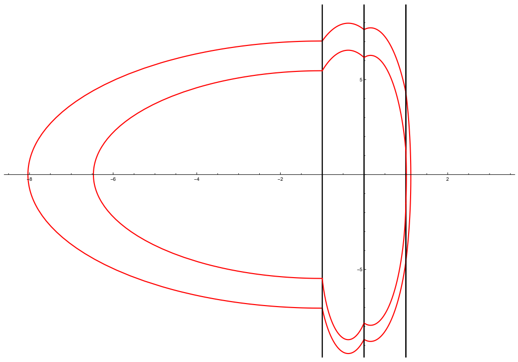

Figure 2: The two crossing limit cycles of the discontinuous piecewise differential system formed by the centers (9). These limit cycles are traveled in counterclockwise.

Acknowledgements

The second author is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The third author is partially supported by FCT/Portugal through UID/MAT/04459/2019. The present paper is part of the thesis of the first author.

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