# Mathematical objects that configure the partial area meanings mobilized in task-solving

The area concept may appear in a wide range of problem situations. The representations used (geometrical, numerical and algebraic) in such situations entail particular practices that condition the partial area meanings. Thus, this study aims to characterize the partial area meanings mobilized in the resolution of area tasks. Each of the area partial meanings is characterized by a configuration of mathematical objects, which compromised the resolution process through semiotic functions. A top-down / bottom-up approach is implemented to identify these mathematical objects and the relationship between them. First, an analysis of the specialized literature on learning the area concept is carried out in order to make an initial characterization of the partial area meanings. Subsequently, an analysis of resolutions of students aged 13-14 years old of a set of area tasks allows the first partial area meanings to be refined. The results showed three partial area meanings, which emerge from a detailed analysis of the mathematical practices involved in the students' resolutions. The relationships between the partial area meanings allow a better understanding of the global area meaning in a task-solving context.

Keywords: partial area meanings, mathematical objects, global area meaning.

Subject classification codes: include these here if the journal requires them



## 1. Introduction

Several types of research analyze the necessary aspects for the comprehension of the area and the comprehensive use of formulas understood as a strategical use based on the properties of the object to be measured. Some studies highlight cut, move and paste procedures, as well as certain other procedures based on Euclidean geometry to understand the different properties involved in area measurement processes (Douady & Perrin-Glorian, 1989; Freudenthal, 1983; Kordaki & Potari 1998; Kospentaris, Spyrou & Lappas, 2011; Mamona-Downs & Papadopoulos, 2006; Zacharos, 2006). Other research studies focus on the importance of units of measurement, the comprehension of the area as an attribute, and its geometric nature, and on the use of formulas in a comprehensive manner (Huang & Witz, 2013; Kamii & Kysh, 2006). Sarama and Clements (2009) emphasize the different concepts, properties, and procedures that must be coordinated to overcome the difficulties related to spatial structuring and the composition and iteration of units, all of which must be worked on continuously over time to develop a robust understanding in students (Barrett, Clements & Sarama, 2017). Similarly, Outhred and Mitchelmore (2000) emphasize the relationship that exists between the measuring unit and the surface that is going to be measured, because, although a surface quantity remains constant, the numerical magnitude varies according to the size of the measuring unit. If this relationship is worked out through partitioning and covering procedures, students are more likely to develop a better understanding of the area (Huang, 2017).

Previous studies allow us to infer that, although the difficulties of understanding the concept of the area have been widely reported, the factors responsible for the problem remain diffused, making it difficult to know with certainty where to focus efforts to promote an adequate understanding of both the concept of area and the measurement processes (Smith, Males & Gonulates, 2016). From our stance, these studies provide complementary and relevant information on the main aspects in comprehending the area and its measurement. However, none of them offers an integrated picture of the different mathematical objects involved in area measurement processes and how they configure partial area meanings. Since different objects are involved in area measurement processes, area task-solving presents particular challenges, which the authors noted themselves.

In Author 1, Author 2, and Author 3 (2020), we used four partial meanings of the area to analyze the solving profiles of 13-14 year old students when solving area tasks. The results showed that students who had an upper-level performance by using isometries to compare areas accomplish a wider variety of solving strategies and provide more elaborated justifications in solving different area tasks. These results suggest that the ability to relate different partial area meanings is crucial in solving area tasks, and allows students to justify what is done and why. These results also revealed the need to make a more detailed characterization of the partial area meanings, and the mathematical objects that students mobilize while solving area tasks. This is because the students' resolutions showed the emergence of some mathematical objects that had not been previously contemplated. Since we have not found any works detailing these objects, we pose the following research questions: (1) *What are the mathematical objects that are mobilized in the resolution of area tasks, and how are they related to configure the partial area meanings?* (2) *How are these partial meanings articulated to shape the global area meaning?* To answer these questions, we considered a first characterization of the partial area meanings, which emerge from the analysis of the specialized literature. This first characterization is used to analyze student resolutions and the mathematical practices which derive from them to resolve a set of area tasks. The analysis of the mathematical practices, and the mathematical objects involved in the resolutions, led to a new characterization (refinement) of the partial meanings. The different problem situations, registers of semiotic representation, properties, principles, concepts, procedures, and arguments, are considered to be analytical categories that allow us, as researchers, to explicitly identify the partial meanings that are mobilized and the relationships between them, accounting for the complexity that underlies the concept of area.

## 2. Theoretical framework

### *2.1 Elements involved in area measurement processes*

Sarama and Clements (2009) emphasize three elements that are necessary to execute precise area measurement processes. First, the concepts, which correspond to the area as an attribute related to assigning a number to a quantity of surface; and the measurement units, which must be easily reproducible and divisible, without leaving gaps when covering the surface. Second, the properties correspond to an accumulation and additivity, which involves composing and recomposing figures on surfaces with equivalent areas; conservation, which involves cutting a surface and reorganizing its parts, noting the immutability of the area; and transitivity, which involves comparing the area of two surfaces, turning the area into a third surface as a reference. Finally, the procedures correspond to the equitable partition, which requires cutting the two-dimensional space, physically or mentally, into parts of equal area and usually congruent; the iteration of units, which requires covering certain surfaces with a two-dimensional unit without gaps or overlaps; and the spatial structuring, which requires covering a surface with squares aligned in rows and columns. All the concepts, properties, and procedures described by the authors provide a basis for identifying and characterizing the elements involved in the complexity of the area. However, it is necessary to make other elements associated with such complexity explicit, for example the different phenomenological approaches of the area, the mathematical principles underpinning the properties, the arguments supporting procedures, or the semiotic representations used.

Freudenthal (1983) highlighted three phenomena, with their respective procedures, that are involved in the learning of area as mental object: (1) fair sharing, associated with situations in which the given object must be distributed by exploiting regularities, by estimation, or by measurement; (2) comparing and reproducing, associated with situations in which two surfaces must be compared and where it is necessary to obtain a depiction of a surface with a different shape from the initial one, either by breaking and rebuilding transformations, by inclusion, estimation, measurement, or by transformations (congruencies, affinities, sharings); (3) measuring, associated situations in which a surface is linked to a measurement process, whether to distribute, measure, or value, or by additivity of composed figures, exhaustion of units or subunits, approximations from the inside to the outside, general geometric relations or general formulas.

The phenomena described by Freudenthal (1983) make it clear that area is not only related to the act of assigning a numerical value to a quantity of surface, but also to contexts that require reorganizing and distributing certain geometric shapes, not in order to quantify the area of such surfaces, but rather to establish relationships of inclusion or equivalence. Freudenthal's analysis (1983) considers qualitative and quantitative aspects of the figures and makes explicit the parts - or substructures - of the conceptual structure of the area that arise from the action of measuring fair sharing or transforming. Although the author's didactic approach is broad, we consider that it has some limitations. For example, it does not specify the different semiotic representations involved in the action of measuring, fair sharing or transforming. Similarly, it does not specify the arguments involved in the resolution of area tasks.

We consider it appropriate to attempt to unify the aspects that can allow a global understanding of the area concept, taking into account the existing complexity, which is associated with a broad knowledge of the variety of procedures that a task can admit, in addition to the study and analysis of the different concepts and properties involved in the measurement processes. Since the concepts, properties and procedures are manifested explicitly through the use of different representations, it is necessary to broaden and deepen the understanding of the type of relationships that can be established between the various forms of representation of the area, whether geometric, numerical or algebraic. Likewise, it is necessary to detail the different mathematical objects that make up the concept of area as a whole and how these can be related in the resolution of a mathematical task.

### *2.2 Registers of semiotic representations*

The registers of semiotic representation proposed by Duval (1995) allow an analysis of the different external representations used in certain mathematical task resolutions. External representations act as a stimulus in new constructions of mental structures and express the network of personal meanings used by subjects. According to Duval (2006), the representations of mathematical objects are produced by using representation systems that have a different nature. These can be semiotic, that is through signs, or of a non-semiotic nature related to neuronal networks, or through physical instruments. While in mathematics the objects of knowledge are not perceptually or instrumentally accessible, the use of semiotic registers is considered indispensable (Duval, 1999).

Duval (2006) points out that access to a represented object occurs as long as two conditions meet: (1) provide at least two different registers to represent the objects; (2) accomplish transformations from one register to another, even without being aware of the representations articulated. If this does not occur, the representation and the represented object might be confused, so it is necessary to appropriate a variety of semiotic representation systems, which can transform one semiotic representation into another. Thus, the cognitive complexity that underlies the thinking processes in mathematics is related to the existence of two specific forms of transformation: conversion and treatment (Duval, 2006), while conversions occur between different registers, and treatments take place within the same register.

The tendency towards using the area formulas to calculate it, mentioned above, allows us to infer that the most widely used representation register among students is the *base x height* formula, and the other formulas derived from it. This area-formula association may overlook important aspects of the measurement processes, such as the role played by the different treatment transformations between numerical registers, or the geometric representations that can be used. This may include the addition of units or subunits that make up a surface and how these units produce the multiplicative process that gives rise to the formula. There can also be geometric representations related to cutting, moving, and pasting, or to the decomposition of surfaces that makes it possible to simplify the use of formulas, or to obtain the area using additive processes, or by estimation in the case of irregular surfaces. These omissions may lead to confusion between the concept of area and its numeric representation based on the formula. Besides making conversion transformations between registers of a numeric and geometric nature more difficult, this also hampers the key aspects to promote comprehensive use of formulas and to account for the complexity of the area concept.

Godino, Wilhelmi, Blanco, Contreras, and Giacomone (2016) point out that the resolution of mathematical tasks can involve various registers of semiotic representation. These may include oral and written natural language; numerical depictions, whether integer, fractional or decimal; graphic depictions, whether linear-planar or spatial; and alphanumeric representations, such as algebraic representation treatments, and conversions may appear according to each registers' own rules. Depending on the nature of procedures used in an area task, in a school context that covers from primary school up to 13-14 year-old students, the representations can be geometrical (related to surface decompositions or actions of cutting, moving, and pasting), numerical (related to operations in the set of naturals), algebraic (related to theorems) or in written language (to justify the procedure used). Thus, two types of analysis are necessary to account for the complexity of the area concept, one referring to the mathematical objects involved, highlighting the representations, and the other to the relationships between such objects and their contextual function in the mathematical activity.

### *2.3 Objects, meanings and area configurations*

Identifying the different mathematical objects that shift in task resolutions, it is worth using some tools from the Onto-semiotic approach (OSA). The OSA approach distinguishes a variety of mathematical objects depending on their nature, and also allows an analysis of which objects emerge from the mathematical practices. At the same time, it recognizes language as a primary object in its role as an ostensive facet of the mathematical objects (Godino, Batanero & Font, 2007). The OSA approach takes a pragmatic view in comprehending the mathematical object, that is understanding that a subject has a mathematical object defined by the ability to recognize the properties and characteristics of that object, to relate it to other mathematical objects, and to use it in a variety of problem situations (Font, Godino & D'Amore, 2007). Thus, the comprehension of the area concept is through the ability that a person has to resolve a mathematical task appropriately, recognizing the different objects that intervene in said resolution, besides the relation among them. The relationships between the different mathematical objects are defined by notions of semiotic functions, established between an antecedent (expression) and a consequent (content) dependency relationship of a representational, instrumental, or structural type (Godino *et al.*, 2007).

The definition of the meaning of a mathematical object is through the system of practices performed by a person, or shared within an institution in certain problem situations (Godino, Batanero & Font, 2019). Considering the meaning of a mathematical object, in terms of practices, it is possible to distinguish between the sense and meaning of mathematical objects. The sense corresponds to the partial meaning of the object. That is to say that the meaning of an object can be divided into different types of more specific practices or into a sub-set of practices that might be useful in certain contexts (Godino *et al*., 2007). For its part, the global meaning of an object is reconstructed by systematically exploring the contexts of use of the object and the systems of practices involved (Godino *et al*., 2019). Thus, the global area meaning is defined by specific practices performed in different contexts.

Godino *et al.* (2007) propose an explicit typology of objects which conditions the description and analysis of mathematical activity. There are six types of primary objects: (1) languages (terms, expressions, notations, graphics) in their various registers; (2) problem situations (intra or extra-mathematical applications, exercises); (3) concepts - definition (introduced employing definitions or descriptions: line, point, number, mean, function); (4) propositions (statements on concepts); (5) procedures (algorithms, operations, calculation techniques), and (6) arguments (statements used to validate or explain deductive or other propositions and procedures). The emergence of mathematical objects through systems of practices underlines the complexity of such mathematical objects and the need for the articulation of the components in which such complexity is framed. This articulation corresponds to an epistemic configuration (Borji, Erfani & Font, 2019) and is taken into consideration to detail the different partial area meanings.

## 3. Method

The study is situated in an interpretative paradigm with a qualitative approach (Bassey, 1990) and is part of a larger study presented in Author1, Author2, and Author3 (2020). To characterize the different partial area meanings, we focus on the contents studied in the mathematic school, covered from primary to the first part of secondary school (aged 13-14). We started by considering the different manifestations of the area proposed by Corberán (1996), which illustrate a progression from problem situations that can be solved only with geometrical procedures to others, where numerical and algebraic procedures are also useful. However, we consider that these manifestations proposed by Corberán (1996) do not account for essential aspects of the characterization of the partial meanings mobilized when solving area tasks. These aspects are the mathematical objects that emerge; the registers of representation used; the way these registers of representation condition the emergence from some object, and how transformations inside and between the register occur.

The OSA approach, and its primary object configuration, is used to characterize in detail the partial area meanings, as it allows us to identify specific mathematical objects in the analysis of student responses. Each partial area meaning is a semiotic system, characterized by a concrete configuration of operational and discursive practices. Thus, the global area meaning is shaped by the articulation of the different subsystems of practices that determine each partial meaning (Godino, Burgos & Gea, 2021). The characterization of the partial meanings considers theoretical and empirical elements in its construction, which it incorporates following a top-down and bottom-up approach (Niss, 2006). In the former, top-down, the theoretical framework is given before and outside the actual research in which it is put into practice (Niss, 2006). This means that the specific literature review on the area (Douady & Perrin-Glorian, 1989; Huang & Witz, 2013; Kamii & Kysh, 2006; Kospentaris *et al*., 2011; Outhred & Mitchelmore, 2000; Sarama & Clements, 2009) allows us to identify certain mathematical objects that act as analytical categories, and that can shape the partial area meanings (Pm). These Pms range from a geometric approach (Pm1 -area as a portion of closed space- and Pm2 -area as magnitude/attribute) to a numerical (Pm3 -area with the number of two-dimensional units covering a surface) and algebraic approach (Pm4 -area as a product of two linear dimensions). Each of the four Pms has a specific classification of representations, which are made explicit by the procedures used.

In the bottom-up approach no specific theory is imposed on the data, and it did not collect according to a theory-based design (Niss, 2006), so some mathematical objects emerge as analytical categories from the analysis of 83 student responses to a set of activities (semistructured questionnaire) involving area measurement, allowing the first partial area meanings to be reinterpreted. The students answered the questionnaire individually, in a single 90-minute session, and were allowed to use a ruler to measure lengths. Each questionnaire had seven questions following an increasing cognitive demand. The design of the questionnaire progressively included tasks that allowed only the use of geometrical procedures, others that combined geometrical and numerical procedures, and others that accepted geometrical, numerical, and algebraic procedures. The students that participated formed part of four regular groups from two different schools in Barcelona (two groups from each school), and it was a random selection. Students had received training concerning the area measurement.

The results of this analysis revealed the requirement to consider the instrumental value that representations have (Font, Godino & D`Amore, 2007), as they allow certain practices to be carried out that would not be possible with other representations. For example, a geometrical representation of area allows the decomposition of flat shapes. From here, specific procedures used by students, for certain representations, reveal the need to identify the objects that are exclusive to each Pm, and those that are common to all or more than one Pm. Thus, it can be seen that Pm1 and Pm2 share objects that overlap in student resolutions (geometric representations, concepts, properties), and it is thus possible to unify them in a single Pm. In this way, the four initial partial meanings (Pm) are restructured and refined in three partial meanings (PM) as a result of the analysis of the students' answers to a questionnaire (Figure 1). These three PMs are (1) area as space delimited by a closed line - the beginning of the perception of the attribute; (2) area as the number of two-dimensional units that cover a surface; and (3) area as the product of two linear dimensions.

***[Figure 1 here]***

The bottom-up approach allowed us to define the PMs as systems of practices built from students' answers. Thus, actions that are part of the systems of practices implemented in task resolutions, the representations used, and relationships between these representations, are identified. The analysis of the representations is performed on two levels: on the one hand, differentiating types of semiotic representations used, and on the other, identifying the transformations within and among these representations. Subsequently, textualized operational and discursive practices used by students are identified, and from these practices, procedures that are made explicit through representation, and the explicit, or implicit, properties that validate these procedures. The properties and procedures allow us to infer which concepts and arguments are involved in the students' practices. As a whole, their practices make it possible to identify how the partial area meanings, and their primary objects, are involved in a resolution process through a network of semiotic functions. These semiotic functions made it possible to relate the partial area meanings in the same resolution. Similarly, the interpretation of these semiotic functions allowed us to define the global area meaning as a coordination of the three partial area meanings. To illustrate this, we considered the process followed by the advanced solver profile identified in Author 1, Author 2, and Author 3 (2020).

Table 1 presents the task proposed to the group of students. Figures 2 and 3 represent the sequence of actions identified in the resolution of the proposed tasks.

***[Table 1 here]***

- *Large square area (question a)*

(1) Direct calculation of the area through the square-length formula.

(2) Product between the number of rows and columns that structure the square. This action allows us to guess that the formula for the area of the square is used indirectly.

***[Figure 2 here]***

Since procedures and properties are made explicit through representations, the analysis of the sequence of actions focuses on the same. Two types of representations are identified: symbolic by multiplying the number of rows by the number of columns (Figure 2b), and by using the base x height formula (Figure 2c); geometric by breaking down the square into 12 rows and 12 columns (Figure 2a). Both representations allow a treatment transformation within a symbolic register and a conversion between a geometric and a symbolic register. In the first case, the numerical area representation (12 rows by 12 columns) is also represented algebraically through the base x height formula. It is a symbolic treatment to move from a one-dimensional to a two-dimensional magnitude. In the second case, the decomposition of the square into 12 rows and 12 columns is converted into a numerical representation by multiplying the number of rows and columns, and then into an algebraic representation by using the formula of *base x height*.

- *Black square area (question b)*

(1) Decomposition of the surface of the large and small squares into congruent triangles. This action implicitly uses the property of accumulation and additivity, understanding that figures can be composed and recomposed in regions of equal area.

***[Figure 3 here]***

(2) Direct calculation of the area of the square, as an action after using the Pythagorean theorem to obtain the measurement of the side of the square.

(3) Calculation of a fraction; by decomposing surfaces into regions of an equal area using the property of accumulation and additivity.

The above sequence of actions allows two treatment transformations to be identified in each register of representation, one based on the *base x height* formula (Figures 3b and 3c), and the other on the calculation of the fraction occupied by the black square, with respect to the large square (Figures 3a and 3d). This also makes it possible to convert a geometric register (Figure 3a) into a symbolic one (Figures 3d and 3e).

## 5. Results

### *5.1 Mathematical objects involved in a resolution process*

From the sequence of actions (or particular practices), the analysis of the system of practices is carried out by identifying the mathematical objects that intervene in the resolution of the task, employing an interpretative process. Table 2 shows the detail of the objects involved in the resolution of question a. Table 3 shows the details of the objects involved in the resolution of question b.

***[Table 2 here]***

***[Table 3 here]***

The analysis of the system of practices makes it possible to identify how the objects described in the OSA approach are mobilized in a resolution process. In turn, this analysis allows an extended and refined description of the Pms presented in Author 1, Author 2, and Author 3 (2020), which emerge only from a process of reviewing the specialized literature. The analysis of the system of practices that students execute provides information on the description of the objects that determine each of the three PMs. It is evident that certain objects, such as properties and procedures, are materialized through semiotic representations. Simultaneously, the representations conditioned the actions and practices used and, therefore, the configuration of objects of each PM. The concepts and arguments are identified indirectly, from their implicit use in the procedures. By contrast, procedures are identified from the explicit representations used by students. Below we show the object configurations that define each PM, and how these PMs interacted in the competent area resolution tasks.

The methodology of top-down and bottom-up analysis (Niss, 2006) allowed a more detailed characterization of the initial partial meanings, which had been constructed from the theoretical review. The students' analysis responses revealed that the instrumental value of the representations used involves the implicit mobilization of definitions and concepts. Moreover, the use of properties that underlies the practices carried out by the students revealed it as a common element for all the partial area meanings. Tables 4, 5, and 6 show the configuration of primary objects that determine each partial meaning.

***[Table 4 here]***

***[Table 5 here]***

***[Table 6 here]***

### *5.2 Relations between PMs*

The analysis of the system of practices allows us to identify how the different PMs relate to each other through a network of semiotic functions. The semiotic functions are detailed below according to Figures 3 and 4 presented in the above section.

(1) Fragment C of Figure 2 shows a semiotic function between the numerical values of the length of the sides of the square (antecedent) and the formula of the area of the square (consequent). The numerical values have an operative meaning suggesting that the area should be calculated through the formula. The purpose, use or meaning of such an action is to obtain the numerical value of the area using PM3. Thus, the numerical value of the lengths of the square is put into correspondence with the multiplication of *side x side*, to produce the formula of the area of the square.

(2) Fragments A and B, in Figure 2, show a semiotic function between the numerical values of the length of the sides of the square (antecedent) and the process of spatial structuring of the square into 12 rows and 12 columns (consequent). The numerical values have a conceptual and operational value since they implicitly involve the concept of spatial structuring, which leads to organizing the surface into rows and columns through PM2. The purpose, use or operational meaning of such an action is to obtain the numerical value of the area by putting PM3 into play. Thus, PM2 and PM3 are a related conversion between a geometric and a symbolic register.

(3) Fragment C of Figure 3 shows a semiotic function between the numerical values of the length of the sides of the black square (antecedent) and the formula of the area of the same square (consequent). The numerical values have an operative meaning, suggesting that area should be calculated using the formula. The purpose, use or meaning of the measuring action is an auxiliary step, before the numerical treatment of the area using

PM3. Thus, the numerical value of the lengths of the square is put in correspondence with the multiplication of base x height, to produce the formula of the area of the square.

(4) Fragment B of Figure 3 shows a semiotic function between the numerical values of the length of the sides of the large square (antecedent) and Pythagoras's theorem (consequent). Furthermore, it shows a semiotic function between Pythagoras's theorem (antecedent) and the formula of the area for the black square (consequent). As in the previous practice, the numerical values have an operative meaning. The purpose, use or meaning of the use of Pythagoras's theorem is an auxiliary step, before the numerical treatment of the area using PM3. Thus, the numerical value of the square lengths is put in correspondence with the multiplication of base x height, to produce the formula of the area's square.

(5) Fragment A of Figure 3 shows a semiotic function between the graphic representation of the Task (antecedent) and the procedure of decomposition of surfaces into congruent units (consequent). It also shows a semiotic function between the procedure of decomposition of surfaces into congruent units (antecedent), and an additive and fractional process (consequent). The lines drawn in the resolution have an operative value, as they implicitly suggest that the large square can be decomposed into smaller units and sub-units, implementing PM1. A visual treatment guided by the decomposition of the large square into right triangles is applied, and a symbolic treatment by calculations involving addition (1 triangle + 1 triangle...) and fractions (2/9 of the total area of the large square). The purpose, use, or meaning of the above actions is to obtain the proportion that the black square occupies of the large square, putting PM1 and PM2 into relation.

(6) Fragment E of Figure 3 shows a semiotic function between the angular lines of the graphical representation of the Task (antecedent) and the numerical values of the length of the sides of the large square (consequent). Also, it shows a semiotic function between the numerical values of the length of the sides of the large square (antecedent) and the formula for the area of triangles (base x height /2). Finally, it shows a semiotic function between the triangle area formula (antecedent) and a multiplicative process (consequent). The lines drawn have an operative value, as they implicitly suggest that the large square can be decomposed into smaller units and sub-units, implementing PM1. A visual treatment guided by the decomposition of the large square into right triangles is applied, in addition to a symbolic treatment using calculations involving the multiplication of the area of triangles. The purpose, use, or meaning of the above actions is to calculate the area of the black square by adding up the areas of the triangles that make it up. Thus, the units of measurement are put into correspondence with a multiplicative process that allows the area of the black square to be obtained. This involves two conversions between a geometrical and a symbolic register, putting PM1, PM2, and PM3 into relation.

(7) Fragment D of Figure 3 shows a semiotic function between the lines of the graphical representation of the task (background) and the procedure of decomposition of surfaces into congruent units (consequent). Also, it shows a semiotic function between the decomposition procedure of surfaces into congruent units (antecedent) and a multiplicative process (consequent). The lines have an operative value, as in the previous case. A visual treatment guided by the decomposition of the large square into right triangles is applied, and a symbolic treatment through calculations involving addition (1 triangle + 1 triangle...) and multiplication and division (2/9 of 144). The purpose, use, or meaning of the above actions is to calculate the area of the black square through the fraction it occupies concerning the total area. Thus, the units of measurement are put into

correspondence with a process of multiplication and division that allows the area of the black square to be obtained. This is a conversion between a geometric and a symbolic register, putting PM1, PM2, and PM3 into relation.

The strategic coordination of the different PMs, employing semiotic functions, is associated with coordination between registers of a different nature and a wide variety of procedures. Figure 4 shows the procedures connected to each PM according to those found in the students' responses to the tasks. The concentric nested circle structure allows us to visualize how geometrical procedures underpin the justification of numerical and algebraic procedures. Therefore, students who evidenced some difficulties when solving the task shown in Table 1 used procedures from the exterior annulus in Figure 4.

***[Figure 4 here]***

In Figure 4 the PMs are separated by dotted lines, indicating that the procedures of each PM are not exclusive, but rather that, as a whole, they contribute to shaping the global area meaning. As the repertoire of procedures acquired by students is wide, it is possible to convey arithmetic competence in a comprehensive way, based on the justified use of the different objects that make up the global area meaning.

**6. Discussion and conclusions**

The conversion transformations between geometric and symbolic registers are supported by the different primary objects involved in the resolution presented, which demonstrates the need for integration between the manipulation of ostensive registers and the interpretation that is made of this manipulation (Godino *et al.*, 2016). In this respect, the representations are presented as key objects since they allow us to make explicit the procedures and, from here, concepts, properties, and the arguments used. The relationships between the different PMs show that geometric representations and their associated transformations allow for the comprehensive use of formulas, which is evidenced by the students' ability to obtain better justifications of the procedures. Thus, we agree with Huang (2017) that the use of 2D geometry, understood as geometric representations, allows a comprehensive approach, in conceptual terms, to the use of formulas and the concept of area.

The articulation between the specialized literature and student resolutions, through OSA, allowed the phenomenology of Freudenthal (1983) and the concepts, properties and procedures described by Sarama and Clements (2009) to be integrated and unified. In turn, it integrates and coordinates key elements for the process of solving area tasks, which have not been evidenced in other studies in this way. Examples include the importance of using a geometric approach (Douady & Perrin-Glorian, 1989; Kordaki & Potari 1998; Kospentaris *et al*., 2011; Mamona-Downs & Papadopoulos, 2006; Zacharos, 2006), the use of semiotic registers of representation that are made explicit through the use of procedures of a different nature, the value of the units of measurement (Huang & Witz, 2013; Kamii & Kysh, 2006; Outhred & Mitchelmore, 2000; Sarama & Clements, 2009), the properties and principles involved (Freudenthal, 1983; Sarama & Clements, 2009), as well as the arguments that support the different mathematical objects. The detailed characterization of the PMs and the mathematical objects that make them up allow us to identify those elements necessary for the analysis of student resolutions, making it possible to search for evidence of different mathematical objects that form part of the global area meaning and its complexity. The examples in the analysis showed that the use of PMs makes it possible to characterize the mathematical mobilized objects and how they are coordinated, in the context of resolution of area tasks. This detailed characterization of objects and relations among them allows an approximation to relate the personal meaning that students give to the concept of area and the institutional meaning that is to be built in school mathematics.

Each of the PMs constitutes a system of practices, defined by a specific configuration of primary objects. The concepts, procedures, representations, principles, and arguments are unique to each configuration. However, the properties are a common element to all PMs, since the conservation of the area, the transitivity, and the accumulation and additivity, are independent of the contexts of use that area may have. The PMs also incorporate a detailed description of these primary objects involved in mathematical practices and contemplate, in an integrated manner, the instrumental value of the representations and their treatment and conversion transformations, as well as the procedures derived from each representation. As a whole, the PMs' characterization allows us to search for evidence of the knowledge that is mobilized when solving an area task and of the understanding in terms of competence (Font *et al*., 2007).

We consider that the use of the characterization of PMs could be useful for the didactic design of area tasks, both throughout compulsory education and in the initial training of teachers. This characterization allows the design of such tasks to be oriented towards the comprehensive use of geometric properties, procedures, and concepts involved in the processes of area measurement, which are fundamental aspects to understand the origin and functioning of formulas. However, because no classroom experience using the PMs has been implemented, it is not possible to assume that teaching using the different PMs will guarantee the development of robust understanding by students.

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**Conflicts of interest**

The authors claim to have no conflict of interest

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**Table 1**. Task solved by students in Author 1, Author 2 and Author 3 (2020). [Compiled by authors]

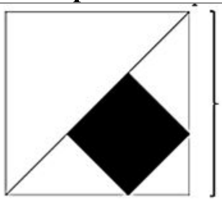
| **Formulation** | **Graphic representation of the Task** |
|---|---|
| **a- What is the area of the 12 cm square? Why?**<br><br>**b- How many possible ways** can you calculate the area of the black square? Please **explain** |  |

**Table 2**. Objects involved in the resolution of question *a*. [Compiled by authors]

| **Textualized operational and discursive practices** | **Primary objects** |
|---|---|
| **Practice 1:** to square the number indicating the length of the side of the square. | *Representations:* written (justification); symbolic (formula).<br>*Procedures:* use linear dimensions and apply formulas.<br>*Arguments:* the area of a square surface is determined by the product of the two linear dimensions of the square. |
| **Practice 2:** to structure the square in rows and columns with squares of 1 cm side. Count the number of two-dimensional units or multiply the number of rows by the number of columns. | *Representations:* written (justification); geometric (rows and columns); symbolic (addition; multiplication).<br>• *Treatments:* within the symbolic register of representation (RR) using the formula of the area's square, multiplying number of rows by the number of columns and counting squares to obtain the area's square. Within the geometric RR by breaking down the square area into squares of 1cm side and visualizing rows and columns.<br>• *Conversions:* between geometric and symbolic RR by breaking down the square surface into squares of 1cm and obtaining the area by counting these squares or multiplying rows and columns. Between written RR and geometric RR; and between written RR and symbolic RR.<br>*Definitions and concepts:* units of measurement; spatial structuring.<br>*Procedures:* decomposition of surfaces into congruent units; structuring of surfaces into rows and columns; counting of two-dimensional units; multiplication of rows and columns.<br>*Arguments:* the units of the area covering a column in a square are given by a dimension of the square. The number of columns is given by the other dimension. Multiplying the length measurements is a shortcut to finding the number of area units which covers the square.<br>*Properties:* accumulation and additivity. |

**Table 3**. Objects involved in the resolution of question *b*. [Compiled by authors]

| **Textualized operational and discursive practices** | **Primary objects** |
|---|---|
| **Practice 3:** to square the number that comes from measuring the side of the square with a ruler. | *Representations:* written (justification); symbolic (formula).<br>*Procedures:* use linear dimensions and apply formulas.<br>*Arguments:* the area of a square is determined by the product of the two linear dimensions of the square. |

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| **Practice 4:** use Pythagoras's theorem to obtain the measurement of the diagonal of the large square. Then, divide by three to obtain the measurement of the side of the square. This last value must be squared to obtain the area of the square. | *Representations:* written (justification); symbolic (formula; Pythagoras's theorem).<br>• *Treatments:* within symbolic RR using numerical and algebraic procedures employing the Pythagorean theorem and the formula of the area of squares, respectively<br>• *Conversions:* between written RR and symbolic RR<br>*Procedures:* to calculate the length of the hypotenuse of a right triangle; calculate lengths by visual estimates (one third of the entire diagonal); use linear dimensions and apply formulas.<br>*Arguments:* the square's area is determined by the product of the two linear dimensions of the square. |
| **Practice 5:** to decompose the large square into 18 small triangles and the black square into 4 small triangles. Count the number of triangles that make up the black square and establish the relationship to the total number of triangles that make up the large square. Identify that the area of the black square is 2/9 of the total. Calculate 2/9 of 144 to obtain the area of the black square. | *Representations:* written (justification); geometric (breakdown into triangles; symbolic (addition; fraction of a quantity).<br>• *Treatments:* within RR geometry by breaking down the square surface into congruent triangles and visualizing the number of triangles that make up the black square. Within symbolic RR, counting the number of triangles that make up both squares and establishing the proportion that the black square occupies out of the total in a fractional numerical RR. Also calculating the fraction of a quantity.<br>• *Conversions:* between geometric and symbolic RR, breaking down the surface into congruent units and obtaining the area by counting these units and calculating the fraction of a quantity.<br>*Procedures:* decompose surfaces into congruent units and the units into sub-units to facilitate the measurement of the area of a surface. Measure areas as an additive process, counting units and/or subunits that cover the two-dimensional space. Calculate areas of squares, rectangles, and triangles identifying the relationship between the respective formulas.<br>*Arguments:* the measurement of the area of a surface can have different numbers associated with it, coming from the measurement made with different two-dimensional and/or standard units of measurement. The mental act of cutting the two-dimensional space into parts of equal area and then counting them serves as a basis for comparing areas, as it allows relationships to be established according to the parts that make up the surface.<br>*Properties and propositions:* accumulation and additivity; transitivity. Any regular polygon can be broken down into congruent triangles. |
| **Practice 6:** to visualize that the side of the square can be broken down into three segments of the same length. Each of the three segments corresponds to one leg of the right triangles into which the large square can be broken down. The black square is composed of four of these right triangles. Use the formula for the area of triangles to calculate the area of one of the four | *Representations:* written (justification); geometric (breakdown into triangles; symbolic (multiplication; formula).<br>• *Treatments:* within RR geometry by breaking down the surface of the black square into congruent triangles and displaying the number of triangles that make up the black square.<br>• *Conversions:* between geometric RR and symbolic RR by breaking down the surface into congruent triangles and obtaining the area of the black square through the product between the area of one triangle and four.<br>*Procedures:* decompose segments into sub-segments of the same length; decompose surfaces into congruent units and the units into subunits to facilitate the measurement of the area of a surface; calculate areas of squares and triangles, identifying the relationship between their formulas. |

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| triangles that make up the black square. Multiply the value of the area of the triangle by four, since the square is composed of four of these triangles. | *Arguments:* the area of the triangle is half the square of equal base and height that contains it, so the formula for the area of the triangle is base per height divided by two. The mental act of cutting the two-dimensional spaces into parts of equal area and then counting them serves as a basis for comparing areas, as it allows relationships to be established according to the parts that make up the surface.<br>*Properties and propositions:* accumulation and additivity; transitivity. Every polygon can be broken down into triangles. |

**Table 4.** Extended characterization PM1. *Area as a space delimited by a closed line.* [Compiled by authors]

| **Mathematical objects** | **Description** |
|---|---|
| **Problem situations (contexts of use)** | (S2) To establish equivalence or inclusion relationships between different surfaces.<br>(S3) Distribute an area equally. |
| **Representations** | (R1) *Oral-written:* using adjectives such as "equal", "thinner" "wider", "double", "half", "a quarter" related to surfaces. It is used in the concepts, terms, propositions and properties of the area. In the formulation of the tasks, in the justification of the processes and in the explanation of the results.<br>(R2) *Gestural:* actively, using parts of the body to compare and/or estimate amounts of surface area.<br>(R3) *Manipulative:* using physical objects. It is shown through physical elements such as Tangram, which can be cut out; or through technological means such as Geogebra, Cabri or other.<br>(R4) *Geometric:* using convenient decompositions to compare and/or estimate quantities of surfaces. It allows us to visualize the purpose process carried out, for example the decomposition of surfaces, the drawing of lines and auxiliary elements, and the possible reconfigurations.<br>(R5) *Symbolic:* the $R^+$ set to compare two or more surfaces.<br>(R6) *Transformations:* treatments and conversions. |
| **Concepts** | (C1) Surface: the quality that allows objects to be measured, which has length and width. The comparison between the qualities of the objects is the beginning of the measurement activity. The area can be measured if it makes sense to say that the objects to be measured are wide or narrow.<br>(C2) Magnitude: the area is considered a magnitude, since it is a property that can be measured. Magnitude allows a quantitative meaning to be assigned to a quantity of area. The measurement of the quantity of area corresponds to a number that is related to a unit of measurement. Thus, the comparison is converted into measurement by establishing a unit of measurement.<br>(C3) Amount of surface: corresponds to the extension occupied by an enclosed space. In this way, by establishing an equivalence relationship between a set of surfaces (having the same extension), the equivalence would correspond to the amount of surface area. |
| **Properties (Pp) and Propositions (Pr)** | (Pp1) Conservation: by cutting/decomposing a given region and reorganizing its parts another figure can be formed; however, the area remains the same.<br>(Pp2) Transitivity: to understand that if the amount of "area A" is equal to (less or more than) the amount of "area B" and the amount of "area B" |

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| | is equal to (less or more than) the amount of "area C", then "area A" is equal to (less or more than) "area C".<br>(Pp3) Accumulation and additivity: figures can be composed and recomposed in regions of equal area.<br>(Pr1) The parallelograms that are on the same base and between the same parallels are congruent or equal.<br>(Pr2) A parallelogram that has the same base as a triangle, both placed between the same parallels, has twice the area of the triangle.<br>(Pr3) The triangles placed on equal bases and between the same parallel are equal.<br>(Pr4) Two polygons are congruent if their sides and angles are respectively equal or congruent.<br>(Pr5) Every polygon can be broken down into triangles.<br>(Pr6) Two polygons are finely equivalent if each of them can be broke down into the same finite number of congruent triangles.<br>(Pr7) Every triangle is equidecomposable from a parallelogram.<br>(Pr8) Two equivalent figures have equal areas. |
| **Procedures** | (P1) Compare two or more surfaces directly by total and/or partial overlapping.<br>(P2) Compare two or more surfaces indirectly by cutting and pasting.<br>(P3) Decompose two or more surfaces conveniently, graphically, or mentally.<br>(P4) Carry out movements of rotation, translation, and superimposition of figures. |
| **Arguments** | (A1) Comparing two or more surfaces by placing one shape over another is useful for establishing equivalence and/or to include relationships.<br>(A2) The mental act of cutting the two-dimensional space into parts of the equal area serves as a basis for comparing areas, as it allows relationships to be established according to the parts that make up the surface, taking into account their lengths.<br>(A3) By changing the shape of a surface there is no change in the area of the surface, as the figures can be decomposed and reorganized while keeping the same "parts." |

**Table 5.** Extended characterization PM2. *Area as the number of two-dimensional units that cover a surface.* [Compiled by authors]

| **Mathematical objects** | **Description** |
|---|---|
| **Problem situations (contexts of use)** | (S4) Situations in which an area appears to be linked to a measurement process (with non-standard units of measurement) for comparison, distribution or valuation. |
| **Representations** | (R7) Oral-written: using adjectives such as "smaller"; "bigger"; "double"; "half", etc. about two-dimensional units of measurement and a quantity of surface area. It is used in the concepts, terms, propositions, and properties of the area. In the formulation of tasks, in the justification of processes, and the explanation of results.<br>(R8) Gestural: using parts of the body in an active way (for example, hands and feet). It is used to cover surfaces and compare and/or estimate their areas.<br>(R9) Manipulative: using physical objects to cover surfaces. It is shown through physical elements too, for example iterate units; or through technological means such as Geogebra, Cabri, among others. |

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| | (R10) Geometric: using grids and partitions in congruent figures (squares and/or triangles) of the units of measurement and surfaces. It allows the visualization of the process carried out, for example, for the decomposition of the units of measurement, the drawing of lines and auxiliary elements, and possible reconfigurations.<br>(R11) Symbolic: the R+ set for the unit count or sum of areas; using calculations to obtain the area. |
| **Concepts** | (C4) Unit of measurement: to fix quantity of a physical magnitude. The unit of area is related to the unit of distance for convenience. Thus, if the distance is in centimeters, the area will be measured in square centimeters. In general, for any unit (U) of distance, the area will be measured in the corresponding square unit (U).<br>(C5) Spatial structuring: requiring a surface to be structured with squares aligned in rows and columns to understand the area as truly two-dimensional.<br>(C6) Non-standard units of measurement: corresponds to a region of the plane delimited by a closed polygonal line (U). The properties that non-standard units of measurement must comply with are those of being easily reproducible, and easily divisible without leaving gaps when the surface is covered with units or their fractions. |
| **Properties (Pp) and Propositions (Pr)** | (Pp1) Conservation: by cutting/decomposing a given region and reorganizing its parts, another figure can be formed. However, the area remains the same.<br>(Pp2) Transitivity: to understand that if the amount of "area A" is equal to (less or more than) the amount of "area B" and the amount of "area B" is equal to (less or more than) the amount of "area C", then "area A" is equal to (less or more than) "area C."<br>(Pp3) Accumulation and additivity: figures can be composed and recomposed in regions of equal area.<br>(Pr9) The unit of measurement can be divided into parts (be fractionated) to facilitate the process of measuring areas. |
| **Procedures** | (P5) Decompose surfaces into congruent units and/or sub-units to facilitate the process of measuring areas.<br>(P6) Iterate two-dimensional units of measurement to compare and measure areas.<br>(P7) Approach surfaces from the inside.<br>(P8) Structure surfaces in rows and columns considering length measurements.<br>(P9) Measure areas as an additive process by counting units and/or sub-units that cover the surface. |
| **Arguments** | (A4) The surface's area can have different numerical values associated with it, coming from measurements made with different units. Thus, the total value results from the count of the units (or fraction thereof) that cover the surface.<br>(A5) There is an inverse relationship between the size of the unit of measurement and the number of units that cover the surface, since the larger the size of the unit of measurement, the fewer units cover the surface, and vice versa.<br>(A6) The square is presented as the best option to measure areas of polygonal surfaces, due to the ease of iterating and to cover rectangles and squares accurately, without overlaps or gaps. Therefore, the final value of the area of a surface is expressed in square units.<br>(A7) The units of area covering a column in a square and/or rectangle are given by a dimension of the square and/or rectangle. The number of columns is given by the other dimension. Multiplying the length |

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| | measurements is a shortcut to finding the number of area units that cover the square and/or rectangle. |

**Table 6.** Extended characterization PM3. *Area as a two-dimensional linear product*. [Compiled by authors]

| Mathematical objects | Description |
|---|---|
| **Problem situations (contexts of use)** | (S5) Situations in which an area seems to be linked to a measurement process (with standard units of measurement) for comparison, distribution, or valuation. |
| **Representations** | (R13) *Oral-written:* using adjectives such as "minor", "major", "double", "half", etc., related to the numerical values of the area. Used in concepts, terms, propositions, and properties about the area. In the task statements, process justifications, and the explanation of the results.<br>(R14) *Manipulative:* using physical elements such as Geogebra, Cabri, among others.<br>(R15) *Geometric:* using partitions at known figures to calculate the area of unknown figures, which allows us to visualize the process carried out, for example the decomposition of different surfaces, the drawing of lines and auxiliary elements, and the possible reconfigurations.<br>(R16) *Symbolic:* the R+ set for the indirect calculation of the area. Using other algebraic expressions such as the relation between the area of a square and its diagonal $d^2 = l^2 + l^2$; or using Pythagoras's theorem to obtain the length measurements of a right triangle $h^2 = l^2 + l^2$.<br>(R17) *Transformations:* treatments and conversions. |
| **Concepts** | (C7) Area formula: using formulas allows us to quantify an area through standardized measurement units in an abbreviated form. The standard way of expressing the measurement facilitates the communication of a measurement precisely so that another person can understand the result of the measurement. In this sense, "calculating the surface area" would be a suitable expression.<br>(C8) Square units: in the two-dimensional in area treatments, this magnitude can be expressed as the product of two lengths (l x l), which is why it is measured with units derived from the length (m2, cm2...)<br>(C9) Standard measurement units: area units such as square regions (cm2, m2, km2, etc.)<br>(C10) Length: characteristic of an object found by quantifying the distance between the end points of that object. |
| **Properties (Pp) and Propositions (Pr)** | (Pp1) Conservation: by cutting/decomposing a given region and reorganizing its parts, another figure can be formed; however, the area remains the same.<br>(Pp2) Transitivity: to understand that if the amount of "area A" is equal to (less or more than) the amount of "area B" and the amount of "area B" is equal to (less or more than) the amount of "area C", then "area A" is equal to (less or more than) "area C".<br>(Pp3) Accumulation and additivity: figures can be composed and recomposed in regions of equal area.<br>(Pr10) To calculate the area of a figure, it is a matter of decomposing the figure into a finite number of parts so that these parts can be put back together to form a simpler figure. |
| **Procedures** | (P10) Measure linear dimensions and use formulas. |

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| | (P11) Calculate areas of known figures to obtain areas of the unknown figures.<br>(P12) Calculate areas of squares, rectangles, and triangles by identifying the relationship between their formulas. |
| **Arguments** | (A8) The area of a square and/or rectangular surface is determined by the product of the two linear dimensions of a rectangle and/or square. Thus, the formula of base x height allows us to find the area of square and/or rectangular surfaces.<br>(A9) The area of the triangle is half of a rectangle with the same base and height that contains it. Therefore, the formula for the area of the triangle is base per height divided by two.<br>(A10) The surface area can have different numerical values associated with it, coming from the measurement made with different standards of measurement units. There is an inverse relationship between these standards and the surface area, since the larger the standard unit of measurement, the smaller the number assigned to the area, and vice versa. |

**Figure 1**. Process followed for the differentiation and reinterpretation from Pm to PM.

[Compiled by authors]

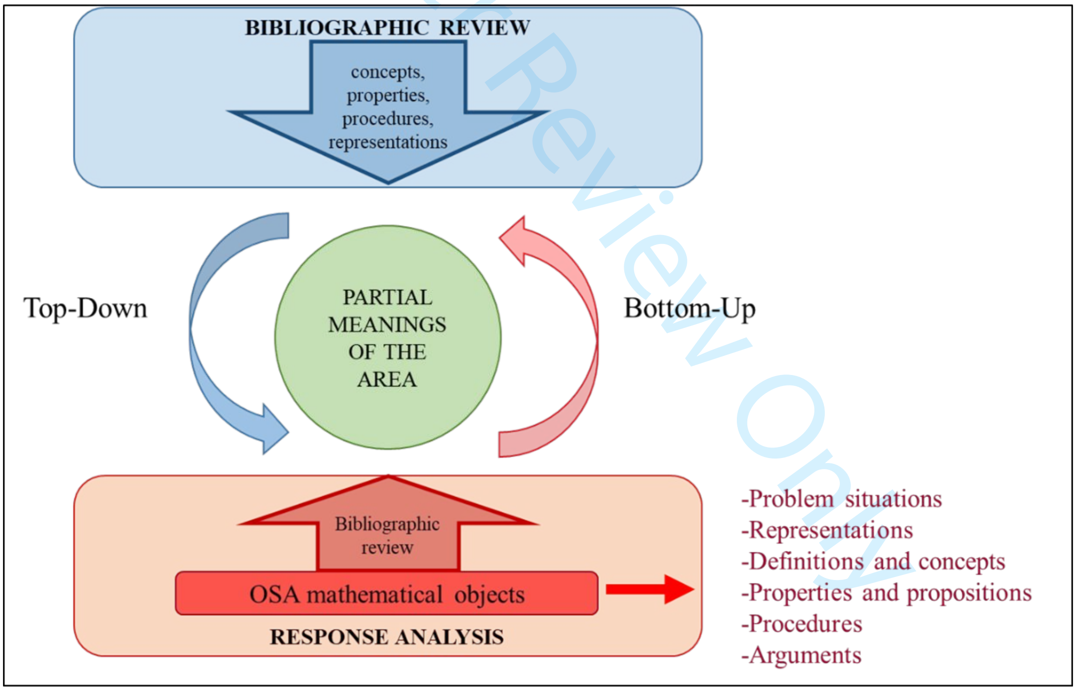



**Figure 2**. Sequence of actions for the calculation of area of the large square (question a).

[Compiled by authors]

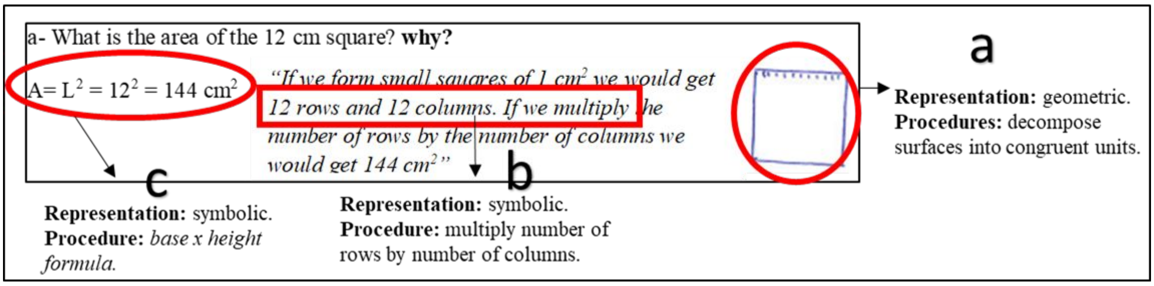



**Figure 3.** Sequence of actions for the calculation of area of the black square (question b). [Compiled by authors]

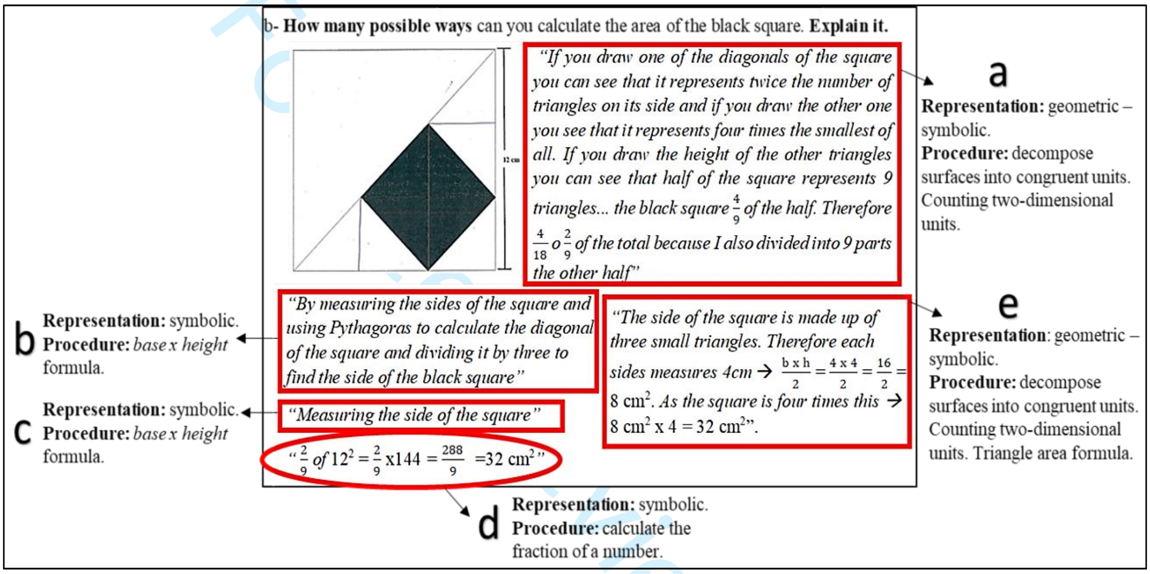



**Figure 4**. Procedures of the PMs. [Compiled by authors]

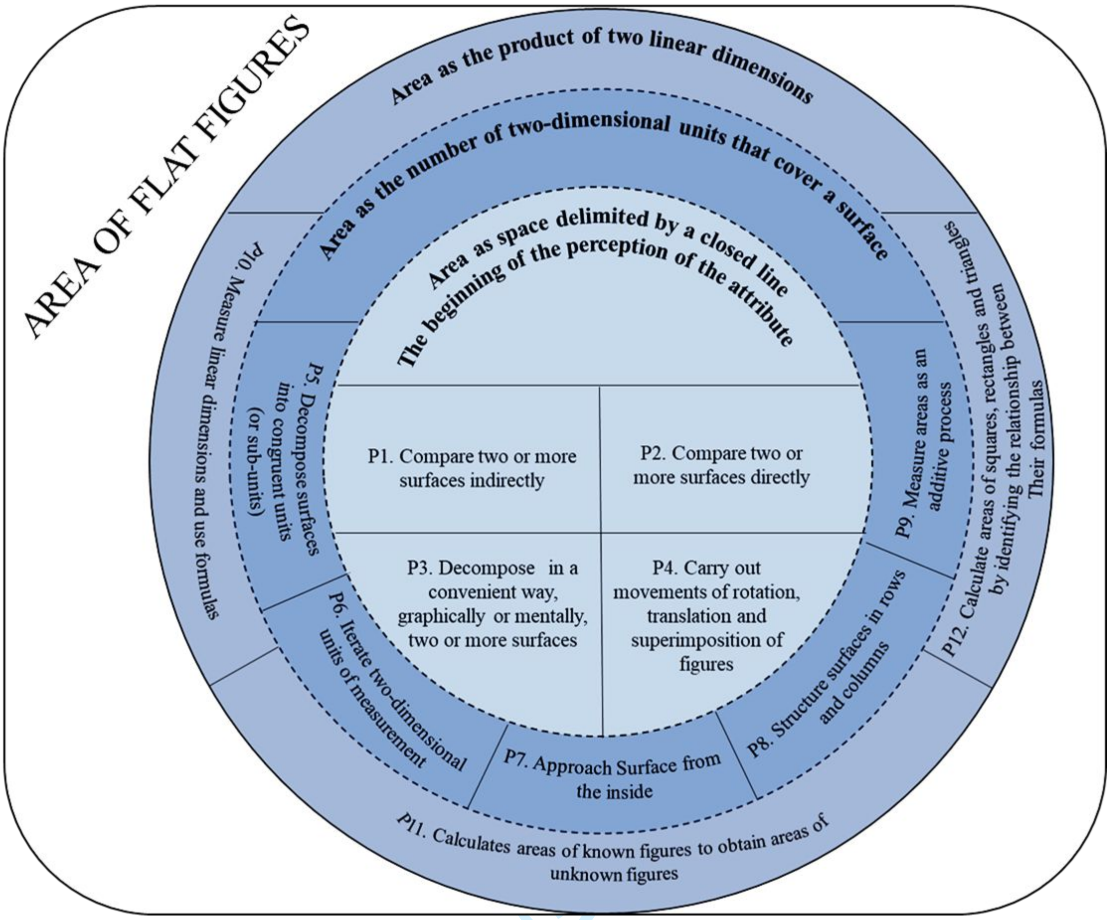

AREA OF FLAT FIGURES
Area as the product of two linear dimensions
Area as the number of two-dimensional units that cover a surface
Area as space delimited by a closed line
The beginning of the perception of the attribute
P1. Compare two or more surfaces indirectly
P2. Compare two or more surfaces directly
P3. Decompose in a convenient way, graphically or mentally, two or more surfaces
P4. Carry out movements of rotation, translation and superimposition of figures
P5. Decompose surfaces into congruent units (or sub-units)
P6. Iterate two-dimensional units of measurement
P7. Approach Surface from the inside
P8. Structure surfaces in rows and columns
P9. Measure areas as an additive process
P10. Measure linear dimensions and use formulas
P11. Calculates areas of known figures to obtain areas of unknown figures
P12. Calculate areas of squares, rectangles and triangles by identifying the relationship between Their formulas

**Figure 1**. Process followed for the differentiation and reinterpretation from Pm to PM. [Compiled by authors]

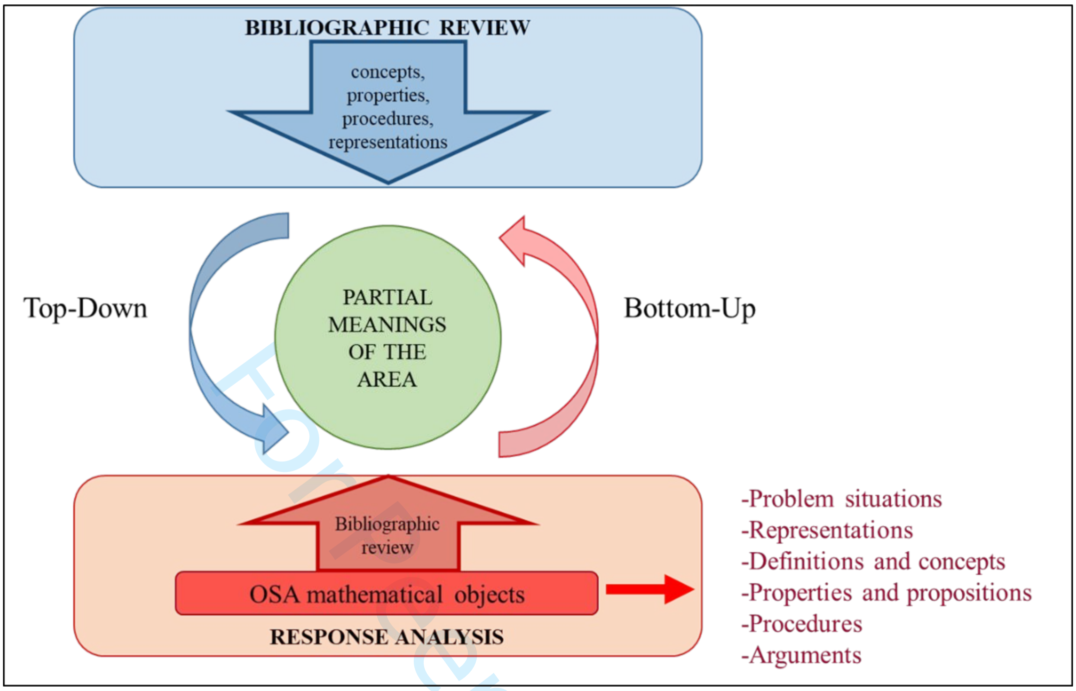



**Figure 2**. Sequence of actions for the calculation of area of the large square (question a). [Compiled by authors]

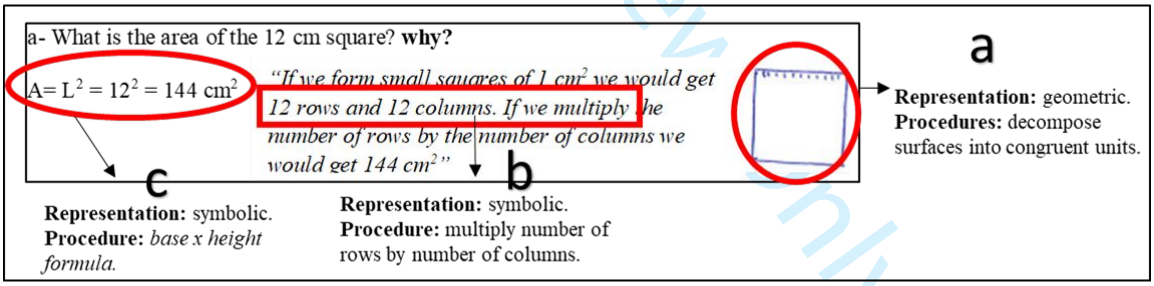
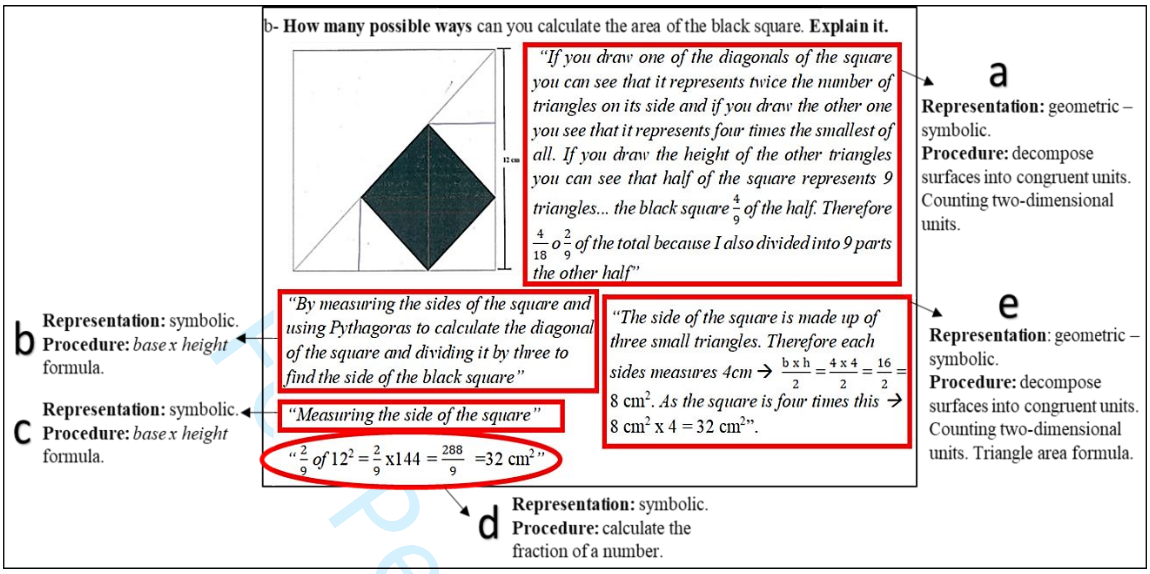


**Figure 3.** Sequence of actions for the calculation of area of the black square (question b). [Compiled by authors]

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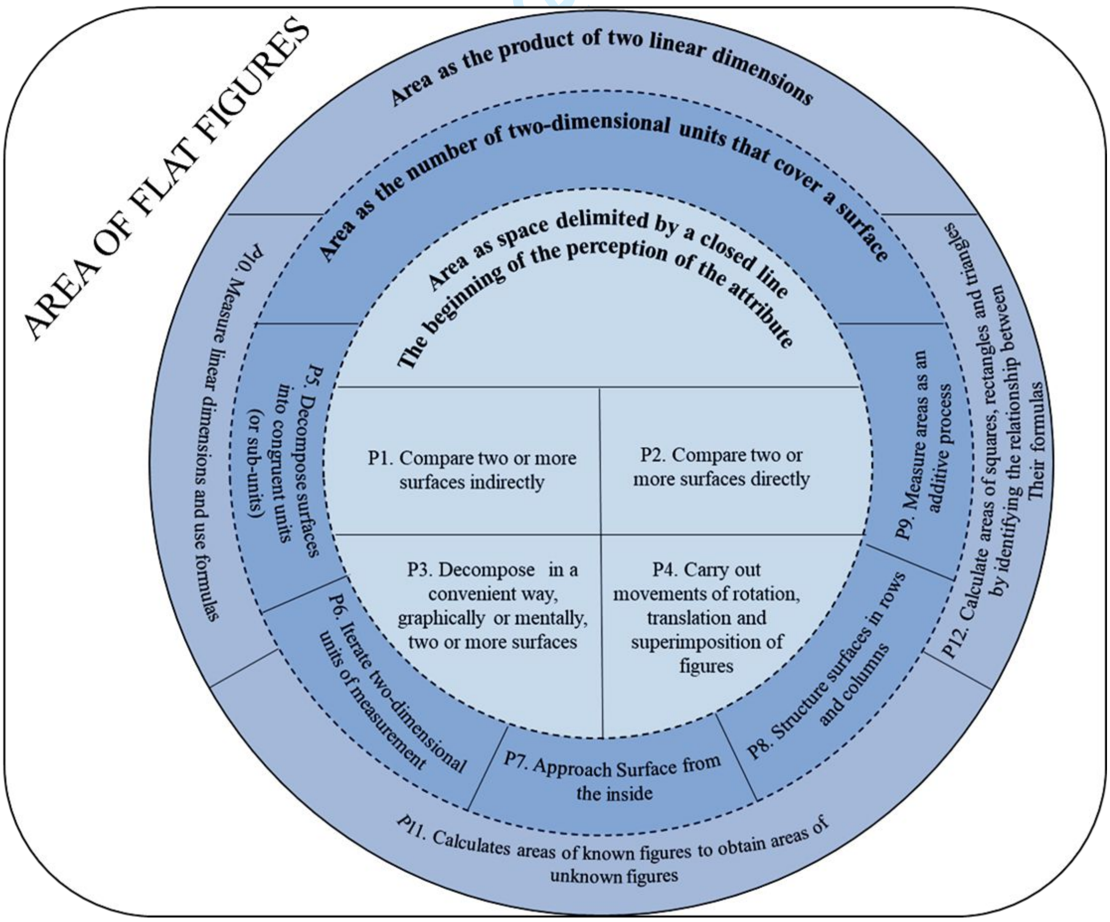


**Table 1**. Task solved by students in Author 1, Author 2 and Author 3 (2020). [Compiled by authors]

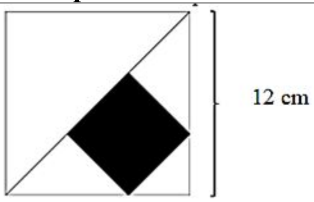
| **Formulation** | **Graphic representation of the Task** |
|---|---|
| **a- What is the area of the 12 cm square? Why?**<br><br>**b- How many possible ways** can you calculate the area of the black square? Please **explain** |   |

**Table 2**. Objects involved in the resolution of question *a*. [Compiled by authors]

| **Textualized operational and discursive practices** | **Primary objects** |
|---|---|
| **Practice 1:** to square the number indicating the length of the side of the square. | *Representations:* written (justification); symbolic (formula).<br>*Procedures:* use linear dimensions and apply formulas.<br>*Arguments:* the area of a square surface is determined by the product of the two linear dimensions of the square. |
| **Practice 2:** to structure the square in rows and columns with squares of 1 cm side. Count the number of two-dimensional units or multiply the number of rows by the number of columns. | *Representations:* written (justification); geometric (rows and columns); symbolic (addition; multiplication).<br>• *Treatments:* within the symbolic register of representation (RR) using the formula of the area's square, multiplying number of rows by the number of columns and counting squares to obtain the area's square. Within the geometric RR by breaking down the square area into squares of 1cm side and visualizing rows and columns.<br>• *Conversions:* between geometric and symbolic RR by breaking down the square surface into squares of 1cm and obtaining the area by counting these squares or multiplying rows and columns. Between written RR and geometric RR; and between written RR and symbolic RR.<br>*Definitions and concepts:* units of measurement; spatial structuring.<br>*Procedures:* decomposition of surfaces into congruent units; structuring of surfaces into rows and columns; counting of two-dimensional units; multiplication of rows and columns.<br>*Arguments:* the units of the area covering a column in a square are given by a dimension of the square. The number of columns is given by the other dimension. Multiplying the length measurements is a shortcut to finding the number of area units which covers the square.<br>*Properties:* accumulation and additivity. |

**Table 3**. Objects involved in the resolution of question *b*. [Compiled by authors]

| **Textualized operational and discursive practices** | **Primary objects** |
|---|---|
| **Practice 3:** to square the number that comes from measuring the side of the square with a ruler. | *Representations:* written (justification); symbolic (formula).<br>*Procedures:* use linear dimensions and apply formulas.<br>*Arguments:* the area of a square is determined by the product of the two linear dimensions of the square. |

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| **Practice 4:** use Pythagoras's theorem to obtain the measurement of the diagonal of the large square. Then, divide by three to obtain the measurement of the side of the square. This last value must be squared to obtain the area of the square. | *Representations:* written (justification); symbolic (formula; Pythagoras's theorem).<br>• *Treatments:* within symbolic RR using numerical and algebraic procedures employing the Pythagorean theorem and the formula of the area of squares, respectively<br>• *Conversions:* between written RR and symbolic RR<br>*Procedures:* to calculate the length of the hypotenuse of a right triangle; calculate lengths by visual estimates (one third of the entire diagonal); use linear dimensions and apply formulas.<br>*Arguments:* the square's area is determined by the product of the two linear dimensions of the square. |
| **Practice 5:** to decompose the large square into 18 small triangles and the black square into 4 small triangles. Count the number of triangles that make up the black square and establish the relationship to the total number of triangles that make up the large square. Identify that the area of the black square is 2/9 of the total. Calculate 2/9 of 144 to obtain the area of the black square. | *Representations:* written (justification); geometric (breakdown into triangles; symbolic (addition; fraction of a quantity).<br>• *Treatments:* within RR geometry by breaking down the square surface into congruent triangles and visualizing the number of triangles that make up the black square. Within symbolic RR, counting the number of triangles that make up both squares and establishing the proportion that the black square occupies out of the total in a fractional numerical RR. Also calculating the fraction of a quantity.<br>• *Conversions:* between geometric and symbolic RR, breaking down the surface into congruent units and obtaining the area by counting these units and calculating the fraction of a quantity.<br>*Procedures:* decompose surfaces into congruent units and the units into sub-units to facilitate the measurement of the area of a surface. Measure areas as an additive process, counting units and/or subunits that cover the two-dimensional space. Calculate areas of squares, rectangles, and triangles identifying the relationship between the respective formulas.<br>*Arguments:* the measurement of the area of a surface can have different numbers associated with it, coming from the measurement made with different two-dimensional and/or standard units of measurement. The mental act of cutting the two-dimensional space into parts of equal area and then counting them serves as a basis for comparing areas, as it allows relationships to be established according to the parts that make up the surface.<br>*Properties and propositions:* accumulation and additivity; transitivity. Any regular polygon can be broken down into congruent triangles. |
| **Practice 6:** to visualize that the side of the square can be broken down into three segments of the same length. Each of the three segments corresponds to one leg of the right triangles into which the large square can be broken down. The black square is composed of four of these right triangles. Use the formula for the area of triangles to calculate the area of one of the four | *Representations:* written (justification); geometric (breakdown into triangles; symbolic (multiplication; formula).<br>• *Treatments:* within RR geometry by breaking down the surface of the black square into congruent triangles and displaying the number of triangles that make up the black square.<br>• *Conversions:* between geometric RR and symbolic RR by breaking down the surface into congruent triangles and obtaining the area of the black square through the product between the area of one triangle and four.<br>*Procedures:* decompose segments into sub-segments of the same length; decompose surfaces into congruent units and the units into subunits to facilitate the measurement of the area of a surface; calculate areas of squares and triangles, identifying the relationship between their formulas. |

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| triangles that make up the black square. Multiply the value of the area of the triangle by four, since the square is composed of four of these triangles. | *Arguments:* the area of the triangle is half the square of equal base and height that contains it, so the formula for the area of the triangle is base per height divided by two. The mental act of cutting the two-dimensional spaces into parts of equal area and then counting them serves as a basis for comparing areas, as it allows relationships to be established according to the parts that make up the surface.<br>*Properties and propositions:* accumulation and additivity; transitivity. Every polygon can be broken down into triangles. |

**Table 4.** Extended characterization PM1. *Area as a space delimited by a closed line.* [Compiled by authors]

| **Mathematical objects** | **Description** |
|---|---|
| **Problem situations (contexts of use)** | (S2) To establish equivalence or inclusion relationships between different surfaces.<br>(S3) Distribute an area equally. |
| **Representations** | (R1) *Oral-written:* using adjectives such as "equal", "thinner" "wider", "double", "half", "a quarter" related to surfaces. It is used in the concepts, terms, propositions and properties of the area. In the formulation of the tasks, in the justification of the processes and in the explanation of the results.<br>(R2) *Gestural:* actively, using parts of the body to compare and/or estimate amounts of surface area.<br>(R3) *Manipulative:* using physical objects. It is shown through physical elements such as Tangram, which can be cut out; or through technological means such as Geogebra, Cabri or other.<br>(R4) *Geometric:* using convenient decompositions to compare and/or estimate quantities of surfaces. It allows us to visualize the purpose process carried out, for example the decomposition of surfaces, the drawing of lines and auxiliary elements, and the possible reconfigurations.<br>(R5) *Symbolic:* the $R^+$ set to compare two or more surfaces.<br>(R6) *Transformations:* treatments and conversions. |
| **Concepts** | (C1) Surface: the quality that allows objects to be measured, which has length and width. The comparison between the qualities of the objects is the beginning of the measurement activity. The area can be measured if it makes sense to say that the objects to be measured are wide or narrow.<br>(C2) Magnitude: the area is considered a magnitude, since it is a property that can be measured. Magnitude allows a quantitative meaning to be assigned to a quantity of area. The measurement of the quantity of area corresponds to a number that is related to a unit of measurement. Thus, the comparison is converted into measurement by establishing a unit of measurement.<br>(C3) Amount of surface: corresponds to the extension occupied by an enclosed space. In this way, by establishing an equivalence relationship between a set of surfaces (having the same extension), the equivalence would correspond to the amount of surface area. |
| **Properties (Pp) and Propositions (Pr)** | (Pp1) Conservation: by cutting/decomposing a given region and reorganizing its parts another figure can be formed; however, the area remains the same.<br>(Pp2) Transitivity: to understand that if the amount of "area A" is equal to (less or more than) the amount of "area B" and the amount of "area B" |

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| | is equal to (less or more than) the amount of "area C", then "area A" is equal to (less or more than) "area C". (Pp3) Accumulation and additivity: figures can be composed and recomposed in regions of equal area. (Pr1) The parallelograms that are on the same base and between the same parallels are congruent or equal. (Pr2) A parallelogram that has the same base as a triangle, both placed between the same parallels, has twice the area of the triangle. (Pr3) The triangles placed on equal bases and between the same parallel are equal. (Pr4) Two polygons are congruent if their sides and angles are respectively equal or congruent. (Pr5) Every polygon can be broken down into triangles. (Pr6) Two polygons are finely equivalent if each of them can be broke down into the same finite number of congruent triangles. (Pr7) Every triangle is equidecomposable from a parallelogram. (Pr8) Two equivalent figures have equal areas. |
| **Procedures** | (P1) Compare two or more surfaces directly by total and/or partial overlapping. (P2) Compare two or more surfaces indirectly by cutting and pasting. (P3) Decompose two or more surfaces conveniently, graphically, or mentally. (P4) Carry out movements of rotation, translation, and superimposition of figures. |
| **Arguments** | (A1) Comparing two or more surfaces by placing one shape over another is useful for establishing equivalence and/or to include relationships. (A2) The mental act of cutting the two-dimensional space into parts of the equal area serves as a basis for comparing areas, as it allows relationships to be established according to the parts that make up the surface, taking into account their lengths. (A3) By changing the shape of a surface there is no change in the area of the surface, as the figures can be decomposed and reorganized while keeping the same "parts." |

**Table 5.** Extended characterization PM2. *Area as the number of two-dimensional units that cover a surface.* [Compiled by authors]

| **Mathematical objects** | **Description** |
|---|---|
| **Problem situations (contexts of use)** | (S4) Situations in which an area appears to be linked to a measurement process (with non-standard units of measurement) for comparison, distribution or valuation. |
| **Representations** | (R7) Oral-written: using adjectives such as "smaller"; "bigger"; "double"; "half", etc. about two-dimensional units of measurement and a quantity of surface area. It is used in the concepts, terms, propositions, and properties of the area. In the formulation of tasks, in the justification of processes, and the explanation of results. (R8) Gestural: using parts of the body in an active way (for example, hands and feet). It is used to cover surfaces and compare and/or estimate their areas. (R9) Manipulative: using physical objects to cover surfaces. It is shown through physical elements too, for example iterate units; or through technological means such as Geogebra, Cabri, among others. |

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| | (R10) Geometric: using grids and partitions in congruent figures (squares and/or triangles) of the units of measurement and surfaces. It allows the visualization of the process carried out, for example, for the decomposition of the units of measurement, the drawing of lines and auxiliary elements, and possible reconfigurations.<br>(R11) Symbolic: the R+ set for the unit count or sum of areas; using calculations to obtain the area. |
| **Concepts** | (C4) Unit of measurement: to fix quantity of a physical magnitude. The unit of area is related to the unit of distance for convenience. Thus, if the distance is in centimeters, the area will be measured in square centimeters. In general, for any unit (U) of distance, the area will be measured in the corresponding square unit (U).<br>(C5) Spatial structuring: requiring a surface to be structured with squares aligned in rows and columns to understand the area as truly two-dimensional.<br>(C6) Non-standard units of measurement: corresponds to a region of the plane delimited by a closed polygonal line (U). The properties that non-standard units of measurement must comply with are those of being easily reproducible, and easily divisible without leaving gaps when the surface is covered with units or their fractions. |
| **Properties (Pp) and Propositions (Pr)** | (Pp1) Conservation: by cutting/decomposing a given region and reorganizing its parts, another figure can be formed. However, the area remains the same.<br>(Pp2) Transitivity: to understand that if the amount of "area A" is equal to (less or more than) the amount of "area B" and the amount of "area B" is equal to (less or more than) the amount of "area C", then "area A" is equal to (less or more than) "area C."<br>(Pp3) Accumulation and additivity: figures can be composed and recomposed in regions of equal area.<br>(Pr9) The unit of measurement can be divided into parts (be fractionated) to facilitate the process of measuring areas. |
| **Procedures** | (P5) Decompose surfaces into congruent units and/or sub-units to facilitate the process of measuring areas.<br>(P6) Iterate two-dimensional units of measurement to compare and measure areas.<br>(P7) Approach surfaces from the inside.<br>(P8) Structure surfaces in rows and columns considering length measurements.<br>(P9) Measure areas as an additive process by counting units and/or sub-units that cover the surface. |
| **Arguments** | (A4) The surface's area can have different numerical values associated with it, coming from measurements made with different units. Thus, the total value results from the count of the units (or fraction thereof) that cover the surface.<br>(A5) There is an inverse relationship between the size of the unit of measurement and the number of units that cover the surface, since the larger the size of the unit of measurement, the fewer units cover the surface, and vice versa.<br>(A6) The square is presented as the best option to measure areas of polygonal surfaces, due to the ease of iterating and to cover rectangles and squares accurately, without overlaps or gaps. Therefore, the final value of the area of a surface is expressed in square units.<br>(A7) The units of area covering a column in a square and/or rectangle are given by a dimension of the square and/or rectangle. The number of columns is given by the other dimension. Multiplying the length |

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| | measurements is a shortcut to finding the number of area units that cover the square and/or rectangle. |

**Table 6.** Extended characterization PM3. *Area as a two-dimensional linear product*. [Compiled by authors]

| **Mathematical objects** | **Description** |
|---|---|
| **Problem situations (contexts of use)** | (S5) Situations in which an area seems to be linked to a measurement process (with standard units of measurement) for comparison, distribution, or valuation. |
| **Representations** | (R13) *Oral-written:* using adjectives such as "minor", "major", "double", "half", etc., related to the numerical values of the area. Used in concepts, terms, propositions, and properties about the area. In the task statements, process justifications, and the explanation of the results.<br>(R14) *Manipulative:* using physical elements such as Geogebra, Cabri, among others.<br>(R15) *Geometric:* using partitions at known figures to calculate the area of unknown figures, which allows us to visualize the process carried out, for example the decomposition of different surfaces, the drawing of lines and auxiliary elements, and the possible reconfigurations.<br>(R16) *Symbolic:* the R+ set for the indirect calculation of the area. Using other algebraic expressions such as the relation between the area of a square and its diagonal $d^2 = l^2 + l^2$; or using Pythagoras's theorem to obtain the length measurements of a right triangle $h^2 = l^2 + l^2$.<br>(R17) *Transformations:* treatments and conversions. |
| **Concepts** | (C7) Area formula: using formulas allows us to quantify an area through standardized measurement units in an abbreviated form. The standard way of expressing the measurement facilitates the communication of a measurement precisely so that another person can understand the result of the measurement. In this sense, "calculating the surface area" would be a suitable expression.<br>(C8) Square units: in the two-dimensional in area treatments, this magnitude can be expressed as the product of two lengths (l x l), which is why it is measured with units derived from the length (m2, cm2...)<br>(C9) Standard measurement units: area units such as square regions (cm2, m2, km2, etc.)<br>(C10) Length: characteristic of an object found by quantifying the distance between the end points of that object. |
| **Properties (Pp) and Propositions (Pr)** | (Pp1) Conservation: by cutting/decomposing a given region and reorganizing its parts, another figure can be formed; however, the area remains the same.<br>(Pp2) Transitivity: to understand that if the amount of "area A" is equal to (less or more than) the amount of "area B" and the amount of "area B" is equal to (less or more than) the amount of "area C", then "area A" is equal to (less or more than) "area C".<br>(Pp3) Accumulation and additivity: figures can be composed and recomposed in regions of equal area.<br>(Pr10) To calculate the area of a figure, it is a matter of decomposing the figure into a finite number of parts so that these parts can be put back together to form a simpler figure. |
| **Procedures** | (P10) Measure linear dimensions and use formulas. |

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| | (P11) Calculate areas of known figures to obtain areas of the unknown figures.<br>(P12) Calculate areas of squares, rectangles, and triangles by identifying the relationship between their formulas. |
| **Arguments** | (A8) The area of a square and/or rectangular surface is determined by the product of the two linear dimensions of a rectangle and/or square. Thus, the formula of base x height allows us to find the area of square and/or rectangular surfaces.<br>(A9) The area of the triangle is half of a rectangle with the same base and height that contains it. Therefore, the formula for the area of the triangle is base per height divided by two.<br>(A10) The surface area can have different numerical values associated with it, coming from the measurement made with different standards of measurement units. There is an inverse relationship between these standards and the surface area, since the larger the standard unit of measurement, the smaller the number assigned to the area, and vice versa. |