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SUBGROUP COLLECTIONS CONTROLLING THE HOMOTOPY TYPE OF A p -LOCAL COMPACT GROUP

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ABSTRACT. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local compact group. We prove that the (un-completed) homotopy type of the nerve of the linking system $\mathcal{L}$ is determined by the collection of subgroups of S that are $\mathcal{F}$ -centric and $\mathcal{F}$ -radical. This generalizes work of Broto, Grodal, Levi, Oliver, and the second author, who prove an analogous result for p -local finite groups.

1. INTRODUCTION

The structure of a “ p -local compact group” was introduced by Broto, Levi, and Oliver in [BLO07] and provides a common framework for the study of the mod p homotopy type of various types of classifying spaces. Examples include classical objects such as p -completed classifying spaces of finite groups and compact Lie groups. More broadly, one can use the framework to study classifying spaces of homotopy-theoretic generalizations of groups, such as p -compact groups [DW94]. For example, the p -completed classifying space of a finite loop space is the classifying space of p -local compact group [BLO14]. Other examples constructed from exotic p -local finite groups are described in [GLR19].

To describe a p -local compact group, one begins with a “discrete p -toral” group S (Definition 2.1). A “fusion system” over S (Definition 2.2) is a subcategory of the category of groups, with objects given by all subgroups of S . “Saturated” fusion systems (Definition 2.4) satisfy additional axioms requiring S to behave like a Sylow p -subgroup of the hypothetical supergroup G , and require the morphism sets of the fusion system to behave as though they were homomorphisms induced by conjugation in G (even though such a supergroup may not exist).

Associated to a fusion system $\mathcal{F}$ is a “centric linking system” $\mathcal{L}$ (Definition 2.6). While morphism sets in $\mathcal{F}$ mimic *homomorphisms* induced by conjugation, an associated linking system $\mathcal{L}$ for $\mathcal{F}$ has morphism sets that mimic *group elements* of a hypothetical supergroup G that induce the homomorphisms in $\mathcal{F}$ via conjugation. The nerve $|\mathcal{L}|$ is analogous to the classifying space of a group, BG . And indeed, if a p -local compact group arises from a compact Lie group, then the p -completion of the nerve of $\mathcal{L}$ is a model for the p -completion of BG [BLO07, Thm. 9.10], and the same is true if $\mathcal{L}$ arises from a p -compact group [BLO07, Thm. 10.7].

Classifying spaces of compact Lie groups and finite groups admit mod p homology decompositions in terms of orbit categories with respect to certain families of subgroups. Such decompositions are a key tool in the study of homotopy uniqueness

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and of maps between classifying spaces [Dwy97, JMO92, JM92]. Similar decompositions exist for p -local compact groups [BLO07, Prop. 4.6]. The two taken together are key ingredients in the proofs that the p -local compact groups associated to Lie groups and finite groups model the p -completion of the groups' classifying spaces.

Centric and radical subgroups of a group G play a key part in the homotopy type of BG . For example, Dwyer [Dwy97] showed that the collection of p -radical, p -centric subgroups of a finite group G are enough to recover the p -completed homotopy type of BG for any of the classical homology decompositions, and later the same was established for compact Lie groups ([JMO92], [Lib11]).

In fusion systems over *finite* p -groups, the result that centric and radical subgroups determine the homotopy type of the classifying space was proved in [BCG⁺05, Thm. 3.5], using an induction via “pruning subgroups.” The corresponding result for p -local compact groups is not in the literature, and that is the gap that we fill with this paper.

We make use of the “bullet construction” of [BLO07]: $\mathcal{L}^\bullet \subseteq \mathcal{L}$ is a full subcategory of the centric linking system such that the inclusion $\mathcal{L}^\bullet \subseteq \mathcal{L}$ induces an homotopy equivalence on nerves [BLO07, Prop. 4.5]. Attractively for computation, $\text{Obj}(\mathcal{L}^\bullet)$ contains finitely many S -conjugacy classes of subgroups ([BLO07, Lemma 3.2]) and contains all subgroups that are both centric and radical.

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p -toral group S , and let $\mathcal{L}$ be a centric linking system associated to $\mathcal{F}$. Let $\mathcal{H}$ be a collection of $\mathcal{F}$ -centric subgroups of S that is closed under $\mathcal{F}$ -conjugacy and contains all subgroups of S that are both $\mathcal{F}$ -centric and $\mathcal{F}$ -radical. Let $\mathcal{H}^\bullet = \{P^\bullet \mid P \in \mathcal{H}\}$ and assume that $\mathcal{H}^\bullet \subseteq \mathcal{H}$. Let $\mathcal{L}^\mathcal{H} \subseteq \mathcal{L}$ denote the full subcategory of $\mathcal{L}$ whose objects are in $\mathcal{H}$. Then the inclusion of $\mathcal{L}^\mathcal{H}$ in $\mathcal{L}$ induces a homotopy equivalence of nerves $|\mathcal{L}^\mathcal{H}| \simeq |\mathcal{L}|$.*

Our proof follows the same general strategy as [BCG⁺05] but we clarify and streamline their argument, and handle some extra obstacles that occur because the linking system is not finite. Theorem 1.1 is also closely related to results in Appendix A of [BLO14], in which the authors study the mod p homotopy type of transporter systems, another type of category used to describe the classifying space of a p -local compact group. Corollary A.10 of [BLO14] is similar to Theorem 1.1, but assumes that the collection of subgroups being considered is closed under supergroups, which is not the case when dealing with $\mathcal{F}$ -centric $\mathcal{F}$ -radical subgroups.

Another advantage of our approach is that Theorem 1.1 gives a genuine homotopy equivalence of nerves, whereas the techniques of [BLO14, Cor. A.10] necessarily can only give equivalences after p -completion because there is a mod p homology argument involved. It is true that results such as [BLO03, Prop. 1.1] for the p -local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ associated to a finite group G , along with the similar result [BLO07, Thm 9.10] for compact Lie groups, only tell us that $|\mathcal{L}_S(G)|$ agrees with BG after p -completing both sides. However, more precise versions of this statement have been obtained in some cases without p -completion. For example, Libman and Viruel [LV09] give conditions on a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ such that $|\mathcal{L}| \simeq B\Gamma$ for some discrete group Γ . See also [COS08] for a different example identifying the uncompleted nerve, this time in the simply-connected case.

Work of Ślomińska [Ślo91] allows one to describe the homotopy type of $|\mathcal{L}|$ as a homotopy colimit indexed on a poset. This approach was taken by Libman

[Lib06] to describe a “normalizer decomposition” for p -local finite groups, which in particular gives a decomposition of the uncompleted nerve of the linking system. In forthcoming work [BCG⁺], the current authors, together with Grbić and Strumila, prove an analogous theorem for p -local compact groups. In this context, Theorem 1.1 reduces the size of the indexing category for the decomposition and allows for explicit computations, in some cases giving homotopy pushout descriptions for $|\mathcal{L}|$. For example, the general normalizer decomposition recovers the homotopy pushout descriptions for $\mathrm{BSU}(2)$ and $\mathrm{BSO}(3)$ originally due to Dwyer, Miller, and Wilkerson [DMW87].

Organization. Section 2 gathers background material on p -local compact groups. Section 3 discusses normalizer fusion and linking systems, adapting results from [BCG⁺05]. In Section 4, for an arbitrary fully normalized P in $\mathcal{F}$, we construct $\tilde{P}$, the largest supergroup of P over which all $\mathcal{F}$ -automorphisms of P extend, and we show that $\tilde{P}$ coincides with the group $\hat{P}$ used in [BCG⁺05]. In Section 5 we prove the main theorem.

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2. BACKGROUND

In this section, we gather definitions and preparatory results. We review the definition of a p -local compact group, and we also review some lemmas on linking systems.

Definition 2.1. A *discrete p -toral group* is a group P given by an extension

$$1 \longrightarrow (\mathbb{Z}/p^\infty)^k \longrightarrow P \longrightarrow \pi_0 P \longrightarrow 1,$$

where k is a nonnegative integer which we call the *rank*, and $\pi_0 P$ is a finite p -group.

- The group $(\mathbb{Z}/p^\infty)^k$ is the *identity component* of P , denoted P_0 .
- The *size* of a discrete p -toral group P is an ordered pair $\mathrm{size}(P) := (k, c)$, where k is the rank of P and $c = |\pi_0 P|$. The pairs (k, c) are given the lexicographic order (see [CLN07, A.5]).

Subgroup inclusions respect size as in the finite group case: if $P_1 \subseteq P_2$ are discrete p -toral groups, then $\mathrm{size}(P_1) \leq \mathrm{size}(P_2)$, with equality if and only if $P_1 = P_2$ (see [BLO07, Sec. 1]).

Definition 2.2. [BLO07, Defn. 2.1] A *fusion system* $\mathcal{F}$ over a discrete p -toral group S is a subcategory of the category of groups, defined as follows. The objects of $\mathcal{F}$ are all of the subgroups of S . The morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ contain only group monomorphisms, and satisfy the following conditions.

- (a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for all $P, Q \subseteq S$. In particular, all subgroup inclusions and conjugations by elements of S are in $\mathcal{F}$.

- (b) Every morphism in $\mathcal{F}$ factors as the composite of an isomorphism in $\mathcal{F}$ followed by a subgroup inclusion.

We think of the homomorphisms of a fusion system $\mathcal{F}$ as mimicking the idea of conjugation in a supergroup G of S . Accordingly, two groups P and P' that are objects of $\mathcal{F}$ are called $\mathcal{F}$ -conjugate if they are isomorphic as objects of $\mathcal{F}$.

Definition 2.3. Let $\mathcal{F}$ be a fusion system over the discrete p -toral subgroup S .

- (1) We say that $P \subseteq S$ is *fully centralized* if for all $Q \subseteq S$ that are $\mathcal{F}$ -conjugate to P , we have $\text{size}(C_S P) \geq \text{size}(C_S Q)$.
- (2) We say that $P \subseteq S$ is *fully normalized* if for all $Q \subseteq S$ that are $\mathcal{F}$ -conjugate to P , we have $\text{size}(N_S P) \geq \text{size}(N_S Q)$.

The following definition, of “saturation,” is intended to axiomatize the consequences of the group S being a Sylow p -subgroup of G , together with morphisms coming from conjugation by elements of G . This is a technical condition that is assumed in order to guarantee good group-like properties, as in Definition 2.7.

Definition 2.4. [BLO07, Defn. 2.2] A fusion system $\mathcal{F}$ is *saturated* if the following three conditions hold:

- (I) If $P \subseteq S$ is fully normalized in $\mathcal{F}$, then P is fully centralized in $\mathcal{F}$, the group $\text{Out}_{\mathcal{F}}(P) := \text{Aut}_{\mathcal{F}}(P) / \text{Aut}_P(P)$ is finite, and the group $\text{Out}_S(P) := \text{Aut}_S(P) / \text{Aut}_P(P)$ is a Sylow p -subgroup of $\text{Out}_{\mathcal{F}}(P)$.
- (II) If $P \subseteq S$ and $\phi \in \text{Hom}_{\mathcal{F}}(P, S)$ are such that $\phi(P)$ is fully centralized, and if we set

$$N_{\phi} := \{g \in N_S \mid \phi c_g \phi^{-1} \in \text{Aut}_S(\phi(P))\},$$

then there exists $\bar{\phi} \in \text{Hom}_{\mathcal{F}}(N_{\phi}, S)$ such that $\bar{\phi}|_P = \phi$.

- (III) If $P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots$ is an increasing sequence of subgroups of S with the property that $P_{\infty} = \bigcup_{n=1}^{\infty} P_n$, and if $\phi \in \text{Hom}(P_{\infty}, S)$ is any homomorphism such that $\phi|_{P_n} \in \text{Hom}_{\mathcal{F}}(P_n, S)$ for all n , then $\phi \in \text{Hom}_{\mathcal{F}}(P_{\infty}, S)$.

The goal of this paper is to show one can safely restrict to a sub-collection of subgroups of S ; the objects we consider are exactly those that satisfy both of the following two conditions.

Definition 2.5. Let $\mathcal{F}$ be a fusion system over a discrete p -toral group S .

- (1) A subgroup $P \subseteq S$ is called $\mathcal{F}$ -centric if P contains all elements of S that centralize it, and likewise all $\mathcal{F}$ -conjugates of P contain their S -centralizers.
- (2) A subgroup $P \subseteq S$ is called $\mathcal{F}$ -radical if $\text{Out}_{\mathcal{F}}(P)$ contains no nontrivial normal p -subgroup.

A linking system, whose definition we recall next, has more morphisms than the fusion system. The motivating example satisfies $|\mathcal{L}|_p^{\wedge} \simeq |BG|_p^{\wedge}$ for a compact Lie group G , though linking systems are more general than this.

Definition 2.6. [BLO07, Defn. 4.1] Let $\mathcal{F}$ be a fusion system over a discrete p -toral group S . A *centric linking system associated to $\mathcal{F}$* is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$ -centric subgroups of S , together with a functor $\mathcal{L} \xrightarrow{\pi} \mathcal{F}$ and “distinguished” monomorphisms $P \xrightarrow{\delta_P} \text{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$ -centric subgroup $P \subseteq S$ satisfying the following conditions.

- (A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, the center $Z(P)$ acts freely on $\text{Hom}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \subseteq \text{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\text{Hom}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

- (B) For each $\mathcal{F}$ -centric subgroup $P \subseteq S$ and each $g \in P$, the functor π sends $\delta_P(g) \in \text{Aut}_{\mathcal{L}}(P)$ to $c_g \in \text{Aut}_{\mathcal{F}}(P)$.
- (C) For each $f \in \text{Hom}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \delta_P(g) \downarrow & & \downarrow \delta_Q(\pi(f)(g)) \\ P & \xrightarrow{f} & Q. \end{array}$$

With these definitions in place, we arrive at the object of study.

Definition 2.7. A p -local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over the discrete p -toral group S , and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of $(S, \mathcal{F}, \mathcal{L})$ is defined as $B\mathcal{F} := |\mathcal{L}|_p^{\wedge}$.

A priori, computing the classifying space of a p -local compact group requires handling an infinite number of isomorphism classes of objects of $\mathcal{L}$. However, Broto, Levi, and Oliver constructed a functorial retraction $(-)^{\bullet}: \mathcal{F} \rightarrow \mathcal{F}$ that lifts to the associated linking system, and whose image contains a finite number of conjugacy classes.

Proposition 2.8. [BLO07, Defn. 3.1, Lemma 3.2, Prop. 3.3, Prop. 4.5] *Let $\mathcal{F}$ be a saturated fusion system over a discrete p -toral group S . There is an idempotent endofunctor $(-)^{\bullet}: \mathcal{F} \rightarrow \mathcal{F}$, the bullet functor, such that the full subcategory $\mathcal{F}^{\bullet} \subseteq \mathcal{F}$ with $\text{Obj}(\mathcal{F}^{\bullet}) := \{P^{\bullet} \mid P \subseteq S\}$ is closed under $\mathcal{F}$ -conjugacy, contains finitely many S -conjugacy classes, and contains all subgroups $P \subseteq S$ that are both $\mathcal{F}$ -centric and $\mathcal{F}$ -radical. If $\mathcal{L}$ is a linking system associated to $\mathcal{F}$, then $(-)^{\bullet}$ lifts to an idempotent endofunctor of $\mathcal{L}$, and the inclusion $\mathcal{L}^{\bullet} \subseteq \mathcal{L}$ induces a homotopy equivalence of nerves.*

Proposition 2.8 says that, for computational purposes, we can restrict to the category $\mathcal{L}^{\bullet}$, which has a finite number of isomorphism classes of objects. Our goal in this paper is to show that one can restrict to a yet smaller collection of objects, namely those that are both $\mathcal{F}$ -centric and $\mathcal{F}$ -radical, without changing the nerve of the associated linking system.

The remainder of this section gathers lemmas related to lifting morphisms from a fusion system to the associated linking system. In a fusion system, all of the morphisms between subgroups are actual group homomorphisms, but morphisms in a linking system cannot be viewed in this way. Given a morphism φ in $\mathcal{L}$, there is an *associated* homomorphism of groups, namely the homomorphism $\pi(\varphi)$ in $\mathcal{F}$. But the morphisms in $\mathcal{F}$ are analogous to group homomorphisms induced by conjugation in a supergroup $G \supseteq S$, while the morphisms in $\mathcal{L}$ itself are analogous to the group elements that induce the homomorphism.

Nevertheless, the last three lemmas of this section establish that several common features of group homomorphisms also exist for morphisms in $\mathcal{L}$. First, we can uniquely complete liftings from $\mathcal{F}$ to $\mathcal{L}$.

Lemma 2.9. [BLO07, Lemma 4.3] *Given morphisms $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ and $\psi \in \text{Hom}_{\mathcal{F}}(Q, R)$, and lifts $\tilde{\psi}$ and $\tilde{\psi\varphi}$ of ψ and $\psi \circ \varphi$, respectively, to $\mathcal{L}$, there is a unique compatible lift of φ to $\mathcal{L}$ making the diagram on the right a commuting lift to $\mathcal{L}$ of the diagram in $\mathcal{F}$ on the left.*

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \text{id} \downarrow & & \downarrow \psi \\ P & \xrightarrow{\psi \circ \varphi} & R \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\exists! \tilde{\varphi}} & Q \\ \text{id} \downarrow & & \downarrow \tilde{\psi} \\ P & \xrightarrow{\tilde{\psi\varphi}} & R \end{array}$$

Next we need an analogue in $\mathcal{L}$ of inclusions. Given $P \subseteq Q$, there is a preferred morphism $i : P \rightarrow Q$ in $\mathcal{F}$, namely the subset inclusion. In $\mathcal{L}$ there is no natural notion of subgroup inclusion, but the next lemma says that we can make a coherent choice of lifts of the subgroup inclusions in $\mathcal{F}$ to morphisms in $\mathcal{L}$.

Lemma 2.10. [JLL12, Prop. 1.5, Rem. 1.6] *The poset of inclusions of subgroups in $\mathcal{F}$ lifts to a compatible sub-poset $\{\iota_P^Q : P \rightarrow Q\}$ of $\mathcal{L}$. In particular, ι_P^P is the identity morphism of P , and given inclusions $P \subseteq Q \subseteq R$ we have $\iota_Q^R \circ \iota_P^Q = \iota_P^R$.*

Lastly, we need restriction and corestriction of morphisms in fusion and linking systems. In a fusion system, a morphism $\varphi : P \rightarrow Q$ can be restricted to a subgroup $A \subseteq P$ because the subgroup inclusion $A \hookrightarrow P$ is necessarily a morphism of $\mathcal{F}$ (Definition 2.2(a)). Similarly, if $P_\varphi := \text{im}(\varphi : P \rightarrow Q)$, then the isomorphism $P \xrightarrow{\varphi} P_\varphi$ is a morphism of $\mathcal{F}$ (Definition 2.2(b)), and we call it the corestriction of φ to P_φ .

Once we have fixed a compatible subposet of inclusions in $\mathcal{L}$ as in Lemma 2.10, we can also define restrictions and corestrictions in the linking system. For $A \subseteq P$, a morphism $\varphi \in \text{Hom}_{\mathcal{L}}(P, Q)$ has a restriction $\varphi|_A := \varphi \circ \iota_A^P$ in $\text{Hom}_{\mathcal{L}}(A, Q)$. The first part of the next lemma says that there is also a unique corestriction φ to $P_\varphi := \text{im}(\pi(\varphi) : P \rightarrow Q)$. The same reference that we cite shows that morphisms in $\mathcal{L}$ can be corestricted to any subgroup containing P_φ , but we do not need this generality.

Lemma 2.11.

- (1) [JLL12, Lemma 1.7(i)] *Let $\varphi \in \text{Hom}_{\mathcal{L}}(P, Q)$ and let $P_\varphi := \pi(\varphi)(P)$. There is a unique map $\varphi \in \text{Hom}_{\mathcal{L}}(P, P_\varphi)$ such that $\varphi = \iota_{P_\varphi}^Q \circ \varphi$, giving a commuting lift to $\mathcal{L}$ (on the right) of the commuting diagram in $\mathcal{F}$ (on the left).*

$$\begin{array}{ccc} P & \xrightarrow{\pi(\varphi)} & P_\varphi \\ \text{id} \downarrow & & \downarrow \subseteq \\ P & \xrightarrow{\pi(\varphi)} & Q \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\varphi} & P_\varphi \\ \text{id} \downarrow & & \downarrow \iota_{P_\varphi}^Q \\ P & \xrightarrow{\varphi} & Q \end{array}$$

- (2) [BLO14, Prop. A.2, Cor. A.5] *Given an isomorphism $\varphi : P \rightarrow P'$ in $\mathcal{F}$, every lift of φ to $\mathcal{L}$ is an isomorphism.*

- (3) Given a diagram in $\mathcal{F}$ on the left, and a lift $\tilde{\varphi}$ of φ to $\mathcal{L}$, there is a unique lift $\tilde{\varphi}|_P$ of $\varphi|_P$ making the diagram on the right commute in $\mathcal{L}$, and if $\varphi|_P$ is an isomorphism, so is $\tilde{\varphi}|_P$.

$$\begin{array}{ccc}
 P & \xrightarrow{\varphi|_P} & Q \\
 \subseteq \downarrow & & \downarrow \subseteq \\
 P' & \xrightarrow{\varphi} & Q'
 \end{array}
 \qquad
 \begin{array}{ccc}
 P & \xrightarrow{\tilde{\varphi}|_P} & Q \\
 \iota \downarrow & & \downarrow \iota \\
 P' & \xrightarrow{\tilde{\varphi}} & Q'
 \end{array}$$

Proof. For the first statement in (3), apply (1) to $\tilde{\varphi} \circ \iota_P^{P'}$ (see also [Lib06, Prop. 2.11]). The second statement in (3) follows from (2). $\square$

3. NORMALIZER FUSION SUBSYSTEMS

Quillen's Theorem A allows the establishment of a homotopy equivalence between the nerve of a category and the nerve of a subcategory by studying the nerves of overcategories or undercategories for the inclusion. The proof of Theorem 1.1 relies on undercategories for its inductive strategy. In this section, we establish the first of a sequence of equivalences necessary for the proof.

Suppose that $(S, \mathcal{F}, \mathcal{L})$ is a p -local compact group, and $P \subseteq S$. Let $P \downarrow \mathcal{L}$ be the undercategory of P : objects are morphisms $P \rightarrow Q$ in $\mathcal{L}$, and morphisms are commuting triangles. Let $P \downarrow_{\neq} \mathcal{L}$ denote the full subcategory of $P \downarrow \mathcal{L}$ consisting of objects $P \rightarrow Q$ that are non-isomorphisms of $\mathcal{L}$. To analyze $P \downarrow_{\neq} \mathcal{L}$, whose nerve will be one stage in a sequence of equivalences, we use the “normalizer fusion subsystem” $P \downarrow_{N_{\mathcal{L}}P}$ described in [BLO14, Sec. 2] for p -local compact groups. We follow [BCG⁺05], where the analogue of Theorem 1.1 is established for p -local finite groups. The goal of the section is to prove the following.

Proposition 3.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local compact group, and let P be a fully $\mathcal{F}$ -normalized subgroup of S . There is a retraction $r: (P \downarrow_{\neq} \mathcal{L}) \rightarrow (P \downarrow_{\neq} N_{\mathcal{L}}P)$ that induces a homotopy equivalences of nerves.*

We give the definition of a normalizer subsystem and basic lemmas, and then follow [BCG⁺05] in defining a retraction functor. The proof of Proposition 3.1 concludes the section. Most of the section consists of suitable specialization or generalization of results of [BLO14] and [BCG⁺05].

Definition 3.2. Let $\mathcal{F}$ be a saturated fusion system over a discrete p -toral group S , and let $P \subseteq S$ a fully $\mathcal{F}$ -normalized subgroup. The *normalizer of P in $\mathcal{F}$* , denoted $N_{\mathcal{F}}P$, is a fusion system over the discrete p -toral group $N_S P$. If $Q, R \subseteq N_S P$, then $\text{Hom}_{N_{\mathcal{F}}P}(Q, R)$ is given by

$$\left\{ \varphi \in \text{Hom}_{\mathcal{F}}(Q, R) \mid \exists \varphi' \in \text{Hom}_{\mathcal{F}}(Q \cdot P, R \cdot P) \text{ with } \varphi'|_Q = \varphi \text{ and } \varphi'(P) = P \right\}.$$

Note that the objects Q and R in Definition 3.2 are subgroups of $N_S P$, but do not have to contain P itself. If they happen to contain P , then $\text{Hom}_{N_{\mathcal{F}}P}(Q, R)$ is just the subset of $\text{Hom}_{\mathcal{F}}(Q, R)$ consisting of morphisms that take P to P . Otherwise, the definition is requiring that morphisms extend to the smallest subgroups that do contain P , namely $Q \cdot P$ and $R \cdot P$, in a way that takes P to P .

Saturation is a key technical requirement, and fortunately is inherited by the normalizer fusion system.

Lemma 3.3. [BLO14, Thm. 2.3] *If $\mathcal{F}$ is a saturated fusion system, then so is $N_{\mathcal{F}}P$.*

We would like an associated centric linking system. If $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, there is a candidate linking system $N_{\mathcal{L}}P$ associated to $N_{\mathcal{F}}P$ that is given by a subcategory of $\mathcal{L}$. In the following definition, note that subgroups of $N_S P$ that are $\mathcal{F}$ -centric are necessarily $N_{\mathcal{F}}P$ -centric as well.

Definition 3.4. The category $N_{\mathcal{L}}P$, the *normalizer in $\mathcal{L}$ of P* , is defined as a subcategory of $\mathcal{L}$. The object set of $N_{\mathcal{L}}P$ is given by $N_{\mathcal{F}}P$ -centric subgroups. The morphism sets $\text{Hom}_{N_{\mathcal{L}}P}(Q, R)$ are given by

$$\left\{ \varphi \in \text{Hom}_{\mathcal{L}}(Q \cdot P, R \cdot P) \mid \pi(\varphi)|_Q \in \text{Hom}_{N_{\mathcal{F}}P}(Q, R) \text{ and } \pi(\varphi)(P) = P \right\},$$

where $\pi: \mathcal{L} \rightarrow \mathcal{F}$ is the projection from the linking system to the fusion system.

Lemma 3.5. [Gon16, Lemma 1.21] *If P is a fully normalized subgroup in $\mathcal{F}$, then the category $N_{\mathcal{L}}P$ is a centric linking system associated to $N_{\mathcal{F}}P$.*

Before we go on to undercategories, we pause to note easy properties of $N_{\mathcal{F}}P$.

Lemma 3.6. *If $\varphi: P \rightarrow Q$ is a morphism in $N_{\mathcal{F}}P$, then $P \triangleleft Q$ and $\varphi(P) = P$.*

Proof. If $\varphi: P \rightarrow Q$ is a morphism in $N_{\mathcal{F}}P$, then the definition says that there is a morphism $\varphi': P \rightarrow Q \cdot P$ such that $\varphi'|_P = \varphi$ and $\varphi'(P) = P$. Hence $\varphi(P) = P$. Since $Q \subseteq N_S P$, we know P is normal in Q . $\square$

Next we define the categories used for the proof of the main result.

Definition 3.7.

- (1) We write $P \downarrow N_{\mathcal{L}}P$ to denote the undercategory (or “coslice category”) of P in $N_{\mathcal{L}}P$, i.e. the category whose objects are morphisms $P \rightarrow Q$ of $N_{\mathcal{L}}P$ and whose morphisms are commuting triangles under P .
- (2) We write $P \downarrow_{\neq} N_{\mathcal{L}}P$ (resp., $P \downarrow_{\neq} \mathcal{L}$) to denote the full subcategory of $P \downarrow N_{\mathcal{L}}P$ (resp. $P \downarrow \mathcal{L}$) whose objects are maps $P \rightarrow Q$ that are not isomorphisms in $\mathcal{L}$.

The bulk of the work of this section is to construct a retraction functor r from $P \downarrow \mathcal{L}$ to $P \downarrow N_{\mathcal{L}}P$ and show that the retraction restricts to a functor from $P \downarrow_{\neq} \mathcal{L}$ to $P \downarrow_{\neq} N_{\mathcal{L}}P$ (following the model of [BCG⁺05]). Let $P \xrightarrow{\varphi} Q$ be an object in $P \downarrow \mathcal{L}$. In an ideal world, we would like to construct an object $P \rightarrow Q_{\varphi}$ of $P \downarrow N_{\mathcal{L}}P$ that depends only on φ . Sometimes this works: if $\varphi(P) = P$, then we will indeed be able to take $Q_{\varphi} = Q \cap N_S P$. In general, however, we only have $\varphi(P) \cong P$; we would like to twist everything by this isomorphism, but it turns out that such a twist requires making a choice, and unfortunately, the group Q_{φ} is not uniquely defined by φ . However there will still be coherent maps between the outcomes for all of the objects φ in $P \downarrow \mathcal{L}$, making the retraction into a functor.

Construction 3.8. For a fully normalized subgroup P and a morphism $P \xrightarrow{\varphi} Q$ in $\mathcal{L}$, let $P_{\varphi} := \text{im}(\pi(\varphi): P \rightarrow Q)$. By Lemma 2.11, there exists a unique morphism $\underline{\varphi}$ in $\mathcal{L}$ such that $\varphi = \iota_{P_{\varphi}}^Q \circ \underline{\varphi}$ (the “corestriction” of φ to P_{φ}). Then P_{φ} is necessarily $\mathcal{F}$ -isomorphic to P , so there exists a morphism $f_{\varphi}: N_S(P_{\varphi}) \rightarrow N_S P$ in $\mathcal{F}$ whose restriction to P_{φ} corestricts to an isomorphism to $P \subseteq N_S P$ (see [BLO14,

Lemma 2.2(b)] with $K = \text{Aut } P$). If φ happens to be a morphism in $N_{\mathcal{L}}P$, i.e. if $P_\varphi = P$, then we choose f_φ to be the identity map. Then define

$$Q_{f_\varphi} := \text{im} [f_\varphi: N_S(P_\varphi) \cap Q \rightarrow N_S P].$$

In $\mathcal{F}$ we have a diagram with restriction/corestriction morphisms of f_φ labelled with underlines:

$$(3.9) \quad \begin{array}{ccccccc} P & \xrightarrow{\cong} & P & \xrightarrow{\subseteq} & Q_{f_\varphi} & \xrightarrow{\subseteq} & N_S(P) \\ & \searrow \pi(\underline{\varphi}) & \uparrow \underline{f_\varphi} & & \uparrow \underline{f_\varphi} & \searrow \subseteq \circ (\underline{f_\varphi})^{-1} & \uparrow f_\varphi \\ & & P_\varphi & \xrightarrow{\subseteq} & N_S(P_\varphi) \cap Q & \xrightarrow{\subseteq} & N_S(P_\varphi) \end{array}$$

We want to lift diagram (3.9) to $\mathcal{L}$, as shown in (3.10) below. We first choose a lift F_φ for f_φ (using the identity for F_φ if $P = P_\varphi$), and then we lift the subset containments of (3.9) as the preferred inclusion maps of $\mathcal{L}$, all marked simply as ι to declutter the diagram. By Lemma 2.11, there are unique isomorphisms $\underline{F_\varphi}$ and $\underline{F_\varphi}$ to fill in the other vertical arrows. We define $\eta(\varphi) = \iota \circ \underline{F_\varphi}^{-1}$ to fill in the dotted arrow. Further, we already have a lift of the diagonal arrow $P \rightarrow P_\varphi$, and we fill in $P \rightarrow P$ with the composite of isomorphisms $\underline{F_\varphi} \circ \underline{\varphi}$:

$$(3.10) \quad \begin{array}{ccccccc} P & \xrightarrow{\cong} & P & \xrightarrow{\iota} & Q_{f_\varphi} & \xrightarrow{\iota} & N_S(P) \\ & \searrow \underline{\varphi} & \uparrow \underline{F_\varphi} & & \uparrow \underline{F_\varphi} & \searrow \eta(\varphi) = \iota \circ \underline{F_\varphi}^{-1} & \uparrow F_\varphi \\ & & P_\varphi & \xrightarrow{\iota} & N_S(P_\varphi) \cap Q & \xrightarrow{\iota} & N_S(P_\varphi) \end{array}$$

Given φ and choices of f_φ and F_φ , the rest of the diagram is uniquely determined. We define

$$(3.11) \quad r(\varphi) = \iota_P^{Q_{f_\varphi}} \circ (\underline{F_\varphi} \circ \underline{\varphi}) \in \text{Obj}(P \downarrow N_{\mathcal{L}}P)$$

and we observe that

$$(3.12) \quad \eta(\varphi) = \iota_{N_Q(P_\varphi)}^{Q_{f_\varphi}} \circ \underline{F_\varphi}^{-1}$$

is a morphism in $P \downarrow \mathcal{L}$ from $r(\varphi)$ to φ .

Remark 3.13.

- (1) If $\varphi: P \rightarrow Q$ is in $\mathcal{L}$ and $\pi(\varphi)(P) = P$, then $Q_{f_\varphi} = N_Q P$.
- (2) $r(P \xrightarrow{\iota} Q) = (P \xrightarrow{\iota} N_Q P)$

We extract some further details from the construction of r for later use. One result we need is that the retraction r preserves the property of not being $\mathcal{F}$ -isomorphic to P .

Lemma 3.14. *If $\varphi: P \rightarrow Q$ is not an isomorphism, then Q_{f_φ} properly contains P .*

Proof. In the bottom row of diagram (3.9), the middle entry is actually $N_Q(P_\varphi)$. If φ is not an isomorphism, then Q properly contains P_φ , so $N_Q(P_\varphi)$ properly contains P_φ as well ([BLO07, Lemma 1.8]). It follows that in the top row, Q_{f_φ} properly contains P , as required. $\square$

We also need to know that r actually is a retraction.

Lemma 3.15. *If $\varphi: P \rightarrow Q$ is a morphism in $N_{\mathcal{L}}P$, then $r(\varphi) = \varphi$, i.e., r is a retraction.*

Proof. By Lemma 3.6, we know that $P \subseteq Q \subseteq N_S P$ and $\pi(\varphi)(P) = P$. Hence we have $P_\varphi = P$ and $N_S(P_\varphi) \cap Q = Q$, since $Q \subseteq N_S P$. Because F_φ is the identity, we find $Q_{f_\varphi} = Q$ and $\iota_P^{Q_{f_\varphi}} \circ (\underline{F_\varphi} \circ \underline{\varphi}) = \iota_P^Q \circ \underline{\varphi} = \varphi$. $\square$

Lastly, we must still establish that r is actually a functor, despite the choices that were made during its construction.

Lemma 3.16.

- (1) *The retraction r is a functor from $P \downarrow \mathcal{L}$ to $P \downarrow N_{\mathcal{L}}P$.*
- (2) *Let $i: (P \downarrow N_{\mathcal{L}}P) \rightarrow (P \downarrow \mathcal{L})$ denote the inclusion functor. Then there is a natural transformation η from $i \circ r$ to the identity on $P \downarrow \mathcal{L}$.*

Proof. To prove that r is a functor, we must establish that, despite having to make choices in defining $r(\varphi)$ for each object φ of $P \downarrow \mathcal{L}$, we can choose compatible morphisms in $P \downarrow N_{\mathcal{L}}P$ between the objects $P \rightarrow Q_{f_\varphi}$ for different φ , making r into a functor.

Our strategy is to prove that a morphism β in $P \downarrow \mathcal{L}$ (which is a commuting diagram in $\mathcal{L}$ under P as on the left), together with choices of f_φ and $f_{\varphi'}$, gives rise to a unique $\hat{\beta}$ in $N_{\mathcal{L}}P$ making a commutative ladder in $\mathcal{L}$ (on the right):

$$(3.17) \quad \begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \downarrow = & & \downarrow \beta \\ P & \xrightarrow{\varphi'} & Q' \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} P & \xrightarrow{r(\varphi)} & Q_{f_\varphi} & \xrightarrow{\eta(\varphi)} & Q \\ \downarrow = & & \downarrow \exists! \hat{\beta} & & \downarrow \beta \\ P & \xrightarrow{r(\varphi')} & Q'_{f_{\varphi'}} & \xrightarrow{\eta(\varphi')} & Q' \end{array}$$

While $\hat{\beta}$ depends not only on β but also on the choices of f_φ and $f_{\varphi'}$ (and their lifts to the linking system), we omit that dependence from the notation.

Our setup allows us to construct the commuting diagram (3.18) in $\mathcal{L}$ below, working from right to left. (The notation is as in diagram (3.10).) Once again we denote the preferred “inclusions” in $\mathcal{L}$ simply by ι to reduce clutter. We are given β and the commutativity of the outermost rectangle by hypothesis. The composites $P \rightarrow Q_{f_\varphi}$ and $Q_{f_\varphi} \rightarrow Q$ across the top row are $r(\varphi)$ and $\eta(\varphi)$, respectively, and

similarly are $r(\varphi')$ and $\eta(\varphi')$ for the second row.

$$(3.18) \quad \begin{array}{ccccccc} & & & \varphi & & & \\ & & & \curvearrowright & & & \\ P & \xrightarrow{\varphi} & P_\varphi & \xrightarrow{F_\varphi} & P & \xrightarrow{\iota} & Q_{f_\varphi} \xrightarrow{(F_\varphi)^{-1}} N_S(P_\varphi) \cap Q \xrightarrow{\iota} Q \\ & \searrow & \downarrow \hat{\beta} & & \downarrow \underline{\beta} & & \downarrow \beta \\ P & \xrightarrow{\varphi'} & P_{\varphi'} & \xrightarrow{F_{\varphi'}} & P & \xrightarrow{\iota} & Q_{f_{\varphi'}} \xrightarrow{(F_{\varphi'})^{-1}} N_S(P_{\varphi'}) \cap Q' \xrightarrow{\iota} Q' \\ & \swarrow & \uparrow & & \uparrow & & \uparrow \\ & & & \varphi' & & & \end{array}$$

First, $\beta \circ \iota$ corestricts uniquely to $\underline{\beta}$ so that the rightmost square commutes. The map $\hat{\beta}$ is then uniquely defined by requiring commutativity of the next square to the left.

To test if the left rectangle commutes, we use the result that every morphism in $\mathcal{L}$ is a categorical monomorphism ([BLO14, Prop. A.2(d), Cor. A.5]). It is sufficient to check that composing both ways around the left square with the composite $\iota \circ F_{\varphi'}: Q_{f_{\varphi'}} \rightarrow Q'$ are the same. The two compositions are the same by commutativity of the other two squares and the outer rectangle.

Lastly, uniqueness of $\hat{\beta}$ guarantees functoriality of r despite the choices made in the construction.

For (2), observe that (3.17), thought of as a diagram in $P \downarrow \mathcal{L}$, is exactly the diagram required to show that η is a natural transformation from $i \circ r$ to the identity. $\square$

We can now put together the proof of Proposition 3.1, whose statement we reproduce for convenience.

Proposition 3.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local compact group, and let P be a fully $\mathcal{F}$ -normalized subgroup of S . There is a retraction $r: (P \downarrow_{\mathcal{L}} \mathcal{L}) \rightarrow (P \downarrow_{\mathcal{L}} N_{\mathcal{L}}P)$ that induces a homotopy equivalence of nerves.*

Proof. The function $r: (P \downarrow \mathcal{L}) \rightarrow (P \downarrow N_{\mathcal{L}}P)$ of Construction 3.8 is a retraction functor by Lemmas 3.15 and 3.16, and it preserves non-isomorphisms by Lemma 3.14. The natural transformation η from Lemma 3.16 restricts to a natural transformation from $i \circ r$ to the identity on $P \downarrow_{\mathcal{L}} \mathcal{L}$. Hence r induces a homotopy equivalence between nerves. $\square$

4. THE SUBGROUP $\tilde{P}$

The overall strategy for proving Theorem 1.1 is to study a sequence of undercategories in order to apply Quillen's Theorem A. In Section 3, we established a homotopy equivalence between the nerves of $P \downarrow_{\mathcal{L}} \mathcal{L}$ and $P \downarrow_{\mathcal{L}} N_{\mathcal{L}}P$. In this section, we construct a supergroup $\tilde{P}$ of P inside of $N_S P$, and a functor from $P \downarrow_{\mathcal{L}} N_{\mathcal{L}}P$ to $\tilde{P} \downarrow N_{\mathcal{L}}P$. Our goal is the following proposition.

Proposition 4.1. *If P is fully normalized, $\mathcal{F}$ -centric, and not $\mathcal{F}$ -radical, then $|P \downarrow_{\mathcal{L}} N_{\mathcal{L}}P| \simeq |\tilde{P} \downarrow N_{\mathcal{L}}P|$.*

The first part of the section leads up to the definition of $\tilde{P}$ and its elementary properties (equation (4.6) and Lemma 4.7). The second part of the section sets up and proves the key extension property of $\tilde{P}$ (Lemma 4.11), and the section

concludes with the proof of Proposition 4.1. We begin with an observation about conjugation by images of an extension of a morphism.

Lemma 4.2. *Let $\alpha \in \text{Aut}_{\mathcal{F}}(P)$, and assume that α extends to $\tilde{\alpha}: Q \rightarrow S$ in $\mathcal{F}$, where $P \subseteq Q \subseteq N_S P$ and $\tilde{\alpha}|_P = \alpha$. If $y \in Q$, then as automorphisms of P ,*

$$\alpha \circ c_y \circ \alpha^{-1} = c_{\tilde{\alpha}(y)}.$$

Proof. Remembering that α itself cannot be applied to y , we compute

$$\begin{aligned} \alpha \circ c_y \circ \alpha^{-1}(p) &= \alpha(y \cdot (\alpha^{-1}p) \cdot y^{-1}) \\ &= \tilde{\alpha}(y \cdot (\alpha^{-1}p) \cdot y^{-1}) \\ &= \tilde{\alpha}(y) \cdot p \cdot \tilde{\alpha}(y^{-1}). \end{aligned}$$

□

Motivated by Lemma 4.2, one makes the following definition.

Definition 4.3. Let $\alpha \in \text{Aut}_{\mathcal{F}}(P)$. We define

$$N_{\alpha} := \{y \in N_S P \mid \alpha \circ c_y \circ \alpha^{-1} \in \text{Aut}_S(P)\}.$$

As a corollary of Lemma 4.2 and Definition 4.3, we find that N_{α} is the largest subgroup of $N_S P$ over which α could possibly extend.

Corollary 4.4. *If $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ extends to $\tilde{\alpha}: Q \rightarrow S$ where $P \subseteq Q \subseteq N_S P$, then $Q \subseteq N_{\alpha}$.*

There is also a uniqueness property for extensions of elements of $\text{Aut}_{\mathcal{F}}(P)$, as described by the following lemma. The lemma considerably strengthens what one could conclude just from Lemma 4.2.

Lemma 4.5. [BLO07, Prop. 2.8] *Let P be $\mathcal{F}$ -centric, and suppose $P \subseteq Q$. If $\alpha \in \text{Aut}_{\mathcal{F}}(P)$, and $\tilde{\alpha}$ and $\tilde{\alpha}'$ are extensions of α to morphisms $Q \rightarrow S$, then there exists $z \in Z(P) \subseteq Q$ such that $\tilde{\alpha}' = \tilde{\alpha} \circ c_z$. In particular, $\tilde{\alpha}(Q) = \tilde{\alpha}'(Q)$.*

We are interested in the largest supergroup of P over which *all* $\mathcal{F}$ -automorphisms of P must extend. If P is a fully normalized subgroup, let $\tilde{P}$ be defined by

$$(4.6) \quad \tilde{P} := \bigcap_{\alpha \in \text{Aut}_{\mathcal{F}}(P)} N_{\alpha}.$$

Lemma 4.7. *If P is fully normalized, then $P \triangleleft \tilde{P} \triangleleft N_S P$, and $C_S(P) \subseteq \tilde{P}$.*

Proof. The centralizer of P is contained in every N_{α} , and therefore in $\tilde{P}$.

Let $x \in N_S P$. We claim that

$$x N_{\alpha} x^{-1} = N_{\alpha \circ c_{x^{-1}}}.$$

To see the inclusion from left to right, suppose that $\tilde{\alpha}: N_{\alpha} \rightarrow S$ is an extension of α . Then $\tilde{\alpha} \circ c_{x^{-1}}: x N_{\alpha} x^{-1} \rightarrow S$ is an extension of $\tilde{\alpha} \circ c_{x^{-1}} \in \text{Aut}_{\mathcal{F}}(P)$, and so $x N_{\alpha} x^{-1} \subseteq N_{\alpha \circ c_{x^{-1}}}$ by Corollary 4.4. The reverse inclusion is the same argument.

The preceding paragraph proves that conjugation by elements of $N_S P$ permutes the groups N_{α} for various $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ and therefore stabilizes their intersection, namely $\tilde{P}$. □

With the basic properties of $\tilde{P}$ in place, we consider its extension properties. The goal is Lemma 4.11, which establishes the existence and uniqueness of certain extensions of automorphisms of P over subgroups of $N_S P$ containing $\tilde{P}$.

Lemma 4.8. *Let P be fully normalized, let $\alpha \in \text{Aut}_{\mathcal{F}}(P)$, and let $\tilde{\alpha}: N_{\alpha} \rightarrow S$ be an extension of α . Then $\text{im}(\tilde{P} \xrightarrow{\tilde{\alpha}} S) = \tilde{P}$.*

Proof. It suffices to show that $\tilde{\alpha}(\tilde{P}) \subseteq \tilde{P}$ because we then have a subgroup containment between groups of equal size (in the sense of Definition 2.1), which are therefore equal. Let $y \in \tilde{P}$. First we check that $\tilde{\alpha}(y) \in N_S P$. By Lemma 4.2, we know that

$$c_{\tilde{\alpha}(y)} = \alpha \circ c_y \circ \alpha^{-1},$$

which stabilizes P because $y \in N_S P$, and therefore $\tilde{\alpha}(y)$ is in $N_S P$.

To finish the proof, we must show that $\tilde{\alpha}(y) \in N_{\beta}$ for all $\beta \in \text{Aut}_{\mathcal{F}}(P)$. Applying Lemma 4.2 again, we find

$$\begin{aligned} \beta \circ c_{\tilde{\alpha}(y)} \circ \beta^{-1} &= \beta \circ (\alpha \circ c_y \circ \alpha^{-1}) \circ \beta^{-1} \\ &= (\beta \circ \alpha) \circ c_y \circ (\beta \circ \alpha)^{-1}. \end{aligned}$$

Since $y \in \tilde{P}$, we know $y \in N_{\beta \circ \alpha}$, so the last automorphism is in $\text{Aut}_S(P)$ as required. $\square$

Recall that for a finite group H , we write $O_p(H)$ for the largest normal p -subgroup of H , i.e. the intersection of all Sylow p -subgroups of H . Only the first half of the proof of the following proposition is necessary for Corollary 4.10, but the equality statement shows that our $\tilde{P}$ agrees with the group $\hat{P}$ that plays a similar role in the proof of [BCG⁺05, Prop. 3.11].

Proposition 4.9. *Let P be fully normalized. The image of $\tilde{P}$ under the natural map $c: N_S P \rightarrow \text{Out}_{\mathcal{F}}(P)$ equals $O_p(\text{Out}_{\mathcal{F}}(P))$.*

Proof. First we prove that $O_p(\text{Out}_{\mathcal{F}}(P)) \subseteq c(\tilde{P})$. Because P is fully normalized, we know that $O_p(\text{Out}_{\mathcal{F}}(P)) \subseteq \text{Out}_S(P)$, so any element of $O_p(\text{Out}_{\mathcal{F}}(P))$ can be represented by c_w for some $w \in N_S P$. We would like to show that for any $\beta \in \text{Aut}_{\mathcal{F}}(P)$, we have $w \in N_{\beta}$, so as to conclude that $w \in \tilde{P}$.

Because $O_p(\text{Out}_{\mathcal{F}}(P))$ is contained in $\text{Out}_S(P)$ and is normal in $\text{Out}_{\mathcal{F}}(P)$, for any $[\beta] \in \text{Out}_{\mathcal{F}}(P)$, there exists $s \in N_S P$ such that

$$[\beta] \cdot [c_w] \cdot [\beta^{-1}] = [c_s],$$

and by adjusting the choice of s using an element of P if necessary, we can assume that $\beta \cdot c_w \cdot \beta^{-1} = c_s$. Therefore $w \in N_{\beta}$. Since β was arbitrary, we find that $w \in \tilde{P}$, as required.

Next we show that $c(\tilde{P}) \subseteq O_p(\text{Out}_{\mathcal{F}}(P))$. It suffices to show that $c(\tilde{P})$ is a normal subgroup of $\text{Out}_{\mathcal{F}}(P)$, and since the quotient map $\text{Aut}_{\mathcal{F}}(P) \rightarrow \text{Out}_{\mathcal{F}}(P)$ is surjective, it suffices to show that the image of $\tilde{P}$ in $\text{Aut}_{\mathcal{F}}(P)$ is normal in $\text{Aut}_{\mathcal{F}}(P)$. Let $y \in \tilde{P}$ and $\beta \in \text{Aut}_{\mathcal{F}}(P)$; we must show that $\beta \circ c_y \circ \beta^{-1} = c_z$ for some $z \in \tilde{P}$. By Lemma 4.2 we know that $\beta \circ c_y \circ \beta^{-1} = c_{\tilde{\beta}(y)}$ as automorphisms of P , where $\tilde{\beta}$ is the extension of β to N_{β} guaranteed by Axiom (II) of saturation. Moreover, $\tilde{\beta}(y)$ is in $\tilde{P}$ by Lemma 4.8. $\square$

The following corollary is the first of two critical ingredients in the proof of Proposition 4.1, the other being the extension property proved in Lemma 4.11.

Corollary 4.10. *Let P be a fully normalized subgroup. If P is not $\mathcal{F}$ -radical, then $\tilde{P}$ properly contains P .*

Proof. If P is not $\mathcal{F}$ -radical, then by definition $O_p(\text{Out}_{\mathcal{F}}(P))$ is nontrivial. By Proposition 4.9, the inverse image of $O_p(\text{Out}_{\mathcal{F}}(P))$ along $N_S P \rightarrow \text{Out}_{\mathcal{F}}(P)$ is $\tilde{P}$, which therefore properly contains P . $\square$

Finally, we establish that for $\mathcal{F}$ -centric subgroups, we can extend maps in the $\mathcal{L}$ -normalizer of P .

Lemma 4.11. *Let P be $\mathcal{F}$ -centric, and let $Q, Q' \subseteq N_S P$ with $P \subseteq Q \cap Q'$. Given $\varphi \in \text{Hom}_{N_{\mathcal{L}}(P)}(Q, Q')$, there exists a unique $\hat{\varphi} \in \text{Hom}_{N_{\mathcal{L}}P}(\tilde{P} \cdot Q, \tilde{P} \cdot Q')$ such that $\hat{\varphi}|_Q = \varphi$.*

Proof. Observe first that $Q, Q' \subseteq N_S P$, so they both normalize $\tilde{P}$ by Lemma 4.7; thus $\tilde{P} \cdot Q$ and $\tilde{P} \cdot Q'$ are groups, contained in $N_S P$. Let $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ be the restriction to P of $\pi(\varphi) \in \text{Hom}_{N_{\mathcal{F}}P}(Q, Q')$. Then $Q \subseteq N_{\alpha}$ and also $\tilde{P} \subseteq N_{\alpha}$ by Corollary 4.4. Hence $\tilde{P} \cdot Q \subseteq N_{\alpha}$, and by axiom (II) of saturation, there exists $f_{\alpha}: \tilde{P} \cdot Q \rightarrow S$ in $\mathcal{F}$ with $f_{\alpha}|_P = \alpha$. Further, by Lemma 4.8 we know that $f_{\alpha}(\tilde{P}) \subseteq \tilde{P}$.

By definition, $\pi(\varphi): Q \rightarrow Q'$, like f_{α} , is an extension of α . By Lemma 4.5, there exists $z \in Z(P)$ such that $f_{\alpha}|_Q = \pi(\varphi) \circ c_z$, so by replacing f_{α} with $f_{\alpha} \circ c_{z^{-1}}$ if necessary, we can assume that $f_{\alpha}|_Q = \pi(\varphi)$ and still satisfies $f_{\alpha}(\tilde{P}) = \tilde{P}$. Hence $f_{\alpha} \in \text{Hom}_{N_{\mathcal{F}}P}(\tilde{P} \cdot Q, \tilde{P} \cdot Q')$.

Choose a lift $\tilde{\varphi}$ of f_{α} to $N_{\mathcal{L}}(P)$. We now have the commutative diagram below in $N_{\mathcal{F}}P$ (on the left) and a proposed lifting of that diagram to $N_{\mathcal{L}}P$ on the right that may or may not commute:

$$\begin{array}{ccc} \tilde{P} \cdot Q & \xrightarrow{f_{\alpha}} & \tilde{P} \cdot Q' \\ \uparrow \subseteq & & \uparrow \subseteq \\ Q & \xrightarrow{\pi(\varphi)} & Q' \end{array} \qquad \begin{array}{ccc} \tilde{P} \cdot Q & \xrightarrow{\tilde{\varphi}} & \tilde{P} \cdot Q' \\ \uparrow \iota & & \uparrow \iota \\ Q & \xrightarrow{\varphi} & Q' \end{array}$$

Both compositions around the right-hand square project to the same map in $\mathcal{F}$, because the left-hand square commutes in $\mathcal{F}$. Since $P \subseteq Q$, we know that Q is $\mathcal{F}$ -centric, so by the axioms of a linking system there exists $x \in C_S(Q) = Z(Q)$ such that

$$\begin{aligned} \iota_{Q'}^{\tilde{P} \cdot Q'} \circ \varphi &= \tilde{\varphi} \circ \iota_Q^{\tilde{P} \cdot Q} \circ \delta_Q(x) \\ &= \left(\tilde{\varphi} \circ \delta_{\tilde{P} \cdot Q}(x) \right) \circ \iota_Q^{\tilde{P} \cdot Q} \end{aligned}$$

where the second line uses property (C) in the definition of a linking system. Since $\hat{\varphi} := \tilde{\varphi} \circ \delta_{\tilde{P} \cdot Q}(x)$ restricts to φ and φ preserves P , so does $\hat{\varphi}$, and hence $\hat{\varphi}$ is in $N_{\mathcal{L}}P$. Lastly, $\hat{\varphi}$ is unique because $\iota_Q^{\tilde{P} \cdot Q}$ is an epimorphism in a categorical sense ([BLO14, Prop. A.2]). $\square$

The proof of Proposition 4.1 is now a routine matter of checking diagrams.

Proof of Proposition 4.1. We exhibit functors in both directions between $P \downarrow_{\neq} N_{\mathcal{L}}P$ and $\tilde{P} \downarrow_{\neq} N_{\mathcal{L}}P$, with appropriate natural transformations. We define a functor $G: (\tilde{P} \downarrow_{\neq} N_{\mathcal{L}}P) \rightarrow (P \downarrow_{\neq} N_{\mathcal{L}}P)$ by precomposing with the distinguished inclusion $P \xrightarrow{\iota} \tilde{P}$, which is a morphism of $N_{\mathcal{L}}P$. The image of G is $P \downarrow_{\neq} N_{\mathcal{L}}P$ by Corollary 4.10.

In the other direction, to define $F : (P \downarrow_{\neq} N_{\mathcal{L}}P) \rightarrow (\tilde{P} \downarrow N_{\mathcal{L}}P)$, suppose that $P \xrightarrow{\varphi} Q$ is an object of $P \downarrow_{\neq} N_{\mathcal{L}}P$. Then P is a subgroup of Q , so by Lemma 4.11 there exists a unique morphism $\hat{\varphi} : \tilde{P} \rightarrow \tilde{P} \cdot Q$ of $P \downarrow N_{\mathcal{L}}P$ that extends φ , and we define $F(\varphi) = \hat{\varphi}$. We define F on a morphism β by

$$(4.12) \quad F \left(\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ & \searrow \varphi' & \downarrow \beta \\ & & Q' \end{array} \right) = \left(\begin{array}{ccc} \tilde{P} & \xrightarrow{\hat{\varphi}} & \tilde{P} \cdot Q \\ & \searrow \hat{\varphi}' & \downarrow \hat{\beta} \\ & & \tilde{P} \cdot Q' \end{array} \right)$$

where $\hat{\beta}$ is the (unique) extension of β guaranteed by Lemma 4.11.

We want a natural transformation ν from the identity functor on $P \downarrow_{\neq} N_{\mathcal{L}}P$ to the composite GF . To define $\nu(P \xrightarrow{\varphi} Q)$, we need a morphism $Q \rightarrow \tilde{P} \cdot Q$ of $\mathcal{L}$ under P , and we use

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ & \searrow & \downarrow \nu(\varphi) := \iota_Q^{\tilde{P} \cdot Q} \\ & & \tilde{P} \cdot Q. \end{array}$$

$GF(\varphi) = \hat{\varphi} \circ \iota_P^{\tilde{P}}$

where the diagram commutes because $\hat{\varphi}|_P = \varphi$.

To check naturality of ν for the morphism β shown in (4.12), consider the diagram below, where the front triangle is $\nu(\varphi) : \varphi \rightarrow GF(\varphi)$, the back triangle is $\nu(\varphi') : \varphi' \rightarrow GF(\varphi')$, and $\hat{\beta} = GF(\beta)$. The unlabelled map is $GF(\beta \circ \varphi) = \widehat{\beta \circ \varphi} \circ \iota_P^{\tilde{P}} = \iota_{Q'}^{\tilde{P} \cdot Q'} \circ \beta \circ \varphi$.

$$(4.13) \quad \begin{array}{ccccc} & & P & \xrightarrow{\varphi'} & Q' \\ & & \parallel & \searrow & \downarrow \iota_{Q'}^{\tilde{P} \cdot Q'} \\ & & P & \xrightarrow{\varphi} & Q \\ & & \searrow \hat{\varphi} \circ \iota_P^{\tilde{P}} & \downarrow \iota_Q^{\tilde{P} \cdot Q} & \nearrow \hat{\beta} \\ & & & \tilde{P} \cdot Q & \end{array}$$

The rectangle on the right commutes by the construction of $\hat{\beta}$ (Lemma 4.11). The back rectangle commutes because the other faces commute, establishing naturality of ν .

The natural transformation $\text{Id} \Rightarrow FG : (\tilde{P} \downarrow N_{\mathcal{L}}P) \rightarrow (\tilde{P} \downarrow N_{\mathcal{L}}P)$ applied to $\varphi : \tilde{P} \rightarrow Q$ (where $Q \subseteq N_S P$) is the morphism

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\varphi} & Q \\ & \searrow \widehat{\varphi \circ \iota_P^{\tilde{P}}} & \downarrow \iota_Q^{\tilde{P} \cdot Q} \\ & & \tilde{P} \cdot Q. \end{array}$$

The diagram commutes by the uniqueness in Lemma 4.11, since both ways around the diagram are extensions of the composite $P \xrightarrow{\iota_P^{\tilde{P}}} \tilde{P} \xrightarrow{\varphi} Q$ to a map $\tilde{P} \rightarrow \tilde{P} \cdot Q$.

Naturality follows from a diagram similar to (4.13) with P replaced by $\tilde{P}$ and the slanted maps adjusted accordingly. $\square$

5. PROOF OF THEOREM 1.1

In this section, we prove the main theorem of the paper, which we reproduce for the reader's convenience.

Theorem 1.1. Let $\mathcal{F}$ be a saturated fusion system over a discrete p -toral group S , and let $\mathcal{L}$ be a centric linking system associated to $\mathcal{F}$. Let $\mathcal{H}$ be a collection of $\mathcal{F}$ -centric subgroups of S that is closed under $\mathcal{F}$ -conjugacy and contains all subgroups of S that are both $\mathcal{F}$ -centric and $\mathcal{F}$ -radical. Let $\mathcal{H}^\bullet = \{P^\bullet \mid P \in \mathcal{H}\}$ and assume that $\mathcal{H}^\bullet \subseteq \mathcal{H}$. Let $\mathcal{L}^\mathcal{H} \subseteq \mathcal{L}$ denote the full subcategory of $\mathcal{L}$ whose objects are in $\mathcal{H}$. Then the inclusion of $\mathcal{L}^\mathcal{H}$ in $\mathcal{L}$ induces a homotopy equivalence of nerves $|\mathcal{L}^\mathcal{H}| \simeq |\mathcal{L}|$.

First we note that we can work within the subcategory $\mathcal{L}^\bullet$ of $\mathcal{L}$ (see Proposition 2.8).

Lemma 5.1. *With the notation of Theorem 1.1, $\mathcal{L}^\mathcal{H} \hookrightarrow \mathcal{L}$ induces a homotopy equivalence of nerves if and only if $\mathcal{L}^{\mathcal{H}^\bullet} \hookrightarrow \mathcal{L}^\bullet$ induces a homotopy equivalence of nerves.*

Proof. Because $\mathcal{H}^\bullet \subseteq \mathcal{H}$, there is a commuting diagram

$$\begin{array}{ccc} \mathcal{L}^{\mathcal{H}^\bullet} & \longrightarrow & \mathcal{L}^\bullet \\ \uparrow \downarrow & & \uparrow \downarrow \\ \mathcal{L}^\mathcal{H} & \longrightarrow & \mathcal{L} \end{array}$$

where the downward vertical arrows are given by inclusion of subcategories and the upward arrows are given by the functor $P \mapsto P^\bullet$. The vertical arrows induce homotopy equivalences of categories because the down-and-up composite is the identity on the top row, and the distinguished inclusions $\iota_P^{P^\bullet}$ provide a natural transformation from the identity functor on the bottom row to the up-and-down composite [BLO07, Prop. 4.5 (a)]. $\square$

The proof of Theorem 1.1 follows the general argument of [BCG⁺05, Thm. 3.5], while dealing with the changes needed for the infinite situation. By Lemma 5.1, it is sufficient to assume that $\mathcal{H} = \mathcal{H}^\bullet$, i.e. that $\mathcal{L}^\mathcal{H}$ is a (full) subcategory of $\mathcal{L}^\bullet$. Hence we can start with $\mathcal{L}^\bullet$, and get to $\mathcal{L}^\mathcal{H}$ by inductively pruning conjugacy classes of subgroups that are not in $\mathcal{H}$, starting with the smallest subgroups. Since $\mathcal{L}^\bullet$ has a finite number of conjugacy classes, this process terminates in $\mathcal{L}^\mathcal{H}$.

Proof of Theorem 1.1. By Lemma 5.1, it is sufficient to assume that $\mathcal{L}^\mathcal{H} \subseteq \mathcal{L}^\bullet$. Since $\mathcal{L}^\bullet$ has a finite number of conjugacy classes of objects, we can make a finite list of conjugacy classes $\langle P_1 \rangle, \dots, \langle P_n \rangle$ of $\mathcal{L}^\bullet$ such that

- $P_1, \dots, P_n$ are fully normalized representatives of distinct conjugacy classes of objects of $\mathcal{L}^\bullet$, and represent all conjugacy classes that are in $\mathcal{L}^\bullet$ but not in $\mathcal{H}$.
- The list of sizes is non-increasing: $\text{size}(P_i) \geq \text{size}(P_{i+1})$ for $i = 1, \dots, n-1$.

Let $\mathcal{L}_0 = \mathcal{L}^{\mathcal{H}}$, and for each $i = 0, \dots, n-1$, let $\mathcal{L}_{i+1}$ be the full subcategory of $\mathcal{L}^\bullet$ whose objects are $\text{Obj}(\mathcal{L}_i) \cup \langle P_{i+1} \rangle$. Thus

$$(5.2) \quad \mathcal{L}^{\mathcal{H}} = \mathcal{L}_0 \subsetneq \mathcal{L}_1 \subsetneq \dots \subsetneq \mathcal{L}_n = \mathcal{L}^\bullet,$$

and each subcategory in the sequence contains one more isomorphism class of $\text{Obj}(\mathcal{L})$ than the one before it.

We seek to prove that for all i , the inclusion $\mathcal{L}_i \subsetneq \mathcal{L}_{i+1}$ induces a homotopy equivalence of nerves, and we wish to apply Quillen's Theorem A. We must show that for all $Q \in \text{Obj}(\mathcal{L}_{i+1})$, the nerve of $Q \downarrow \mathcal{L}_i$ is contractible. If Q is actually an object of $\mathcal{L}_i$, then the desired statement is true because the identity map of Q is an initial object. Thus, we need only consider $Q \in \langle P_{i+1} \rangle$. For all such Q , the undercategories $Q \downarrow \mathcal{L}_i$ are isomorphic, so we need only prove that the nerve of $P_{i+1} \downarrow \mathcal{L}_i$ is contractible (where we have assumed that P_{i+1} is fully normalized). To simplify notation, let $P := P_{i+1}$, which remains fixed for the remainder of the proof.

First we assert that we have an isomorphism of categories

$$(P \downarrow \mathcal{L}_i) \cong (P \downarrow_{\neq} \mathcal{L}^\bullet).$$

To see this, note that if $P \rightarrow R$ is a morphism of $\mathcal{L}^\bullet$ that is not an isomorphism, then $R \in \text{Obj}(\mathcal{L}_i)$, because $\text{size}(R) > \text{size}(P)$. Hence it is sufficient to prove that for all fully normalized subgroups P , the nerve of $P \downarrow_{\neq} \mathcal{L}^\bullet$ is contractible.

Consider the sequence of categories

$$\begin{array}{ccccc} & \xleftarrow{(-)^\bullet} & & \xleftarrow{\supseteq} & & \xleftarrow{(-) \circ \iota_P^\bullet} \\ P \downarrow_{\neq} \mathcal{L}^\bullet & & P \downarrow_{\neq} \mathcal{L} & & P \downarrow_{\neq} N_{\mathcal{L}} P & & \tilde{P} \downarrow_{\neq} N_{\mathcal{L}} P \\ & \xrightarrow{\subseteq} & & \xrightarrow{r} & & \xrightarrow{\varphi \mapsto \hat{\varphi}} \end{array}$$

We assert that each adjacent pair has homotopy equivalent nerves. For the first pair, the map from right to left takes a non-isomorphism $P \rightarrow Q$ to the composite $P \rightarrow Q \rightarrow Q^\bullet$, which is likewise a non-isomorphism, since $\text{size}(Q^\bullet) \geq \text{size}(Q) > \text{size}(P)$. Hence the composite $(-)^\bullet \circ (\subseteq)$ is the identity on $P \downarrow_{\neq} \mathcal{L}^\bullet$, and the natural map $Q \rightarrow Q^\bullet$ is a natural transformation from the identity on $P \downarrow_{\neq} \mathcal{L}$ to the composite $(\subseteq) \circ (-)^\bullet$.

The middle pair has homotopy equivalent nerves by Proposition 3.1. Lastly, because P is fully normalized, $\mathcal{F}$ -centric, and not $\mathcal{F}$ -radical (because P is not in $\mathcal{H}$), the third pair also has homotopy equivalent nerves by Proposition 4.1. However, the identity map $\tilde{P} \rightarrow \tilde{P}$ is an initial object of $\tilde{P} \downarrow_{\neq} N_{\mathcal{L}} P$, which therefore has contractible nerve, so we conclude that

$$|P \downarrow \mathcal{L}_i| \cong |P \downarrow_{\neq} \mathcal{L}^\bullet| \simeq |\tilde{P} \downarrow_{\neq} N_{\mathcal{L}} P| \simeq *.$$

We have proved that in the sequence (5.2), each successive pair of categories has homotopy equivalent nerves by Quillen's Theorem A. Since the inclusion $\mathcal{L}^\bullet \subseteq \mathcal{L}$ induces a homotopy equivalence of nerves [BLO07, Prop. 4.5], the proof is finished. $\square$

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