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Dynamics of a Two-Level Laser Model with Delay

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In 1982, Arecchi *et al.* proposed a simple two level laser model to interpret the first evidence of chaos and generalized multistability in a Q-switched CO2 laser. In this framework, laser dynamics is described by means of a set of two ordinary differential equations for the photon number and the population inversion between the two resonant levels. A sinusoidal function accounted for cavity loss modulation. In this work, we first prove the existence of a periodic orbit for the original two level non-autonomous laser. Then, we transform this model into a four-dimensional autonomous dynamical system in order to provide a mathematical analysis which confirms the seminal results already obtained. Finally, by replacing the sinusoidal loss modulation with a delayed function of photon number we confirm the occurrence of chaos and multistability for such a delayed model with delay times of the order of reciprocal of the modulation frequencies.

I. INTRODUCTION

In the beginning of the eighties, a pioneer experiment on chaotic behavior in a CO2 laser with periodic modulation applied to the cavity loss parameter, was presented [1]. A simple two-level laser model was used for modeling the observed dynamics including generalized multistability [5]. The paper is organized as follows. In the next section, after having briefly recalled the set of two differential ordinary differential equations of the two-level laser model of 1982 and its original parameters, we provide its corresponding nonlinear second order non-autonomous ordinary differential equation (jerk form) and also its corresponding three-dimensional autonomous dynamical system. Then, we show that the main dynamics features (phase portraits, bifurcation diagrams) of these two models are perfectly identical to the original one. In Sec. 3, we transform this two-level non-autonomous model into a four-dimensional autonomous dynamical system and we provide a mathematical analysis that confirms the seminal results already obtained by Arecchi *et al.* [1]. In Sec. 5, by replacing the sinusoidal loss modulation by a delayed variable, we show that such two-level laser model with delay is perfectly analogous to the one analyzed by Grigorieva *et al.* [3] and for which they demonstrated multistability in a local vicinity of the stationary state.

II. TWO-LEVEL NON-AUTONOMOUS LASER MODEL

A. Original model

According to Arecchi *et al.* [1], the laser dynamics is modeled with a set of two differential equations for the photon number n and the population inversion Δ between the two resonant levels as follows (see Meucci and Ginoux [4] for more details):

$$\begin{aligned} \frac{dn}{dt} &= -k_1\left[1 + m\cos(\Omega t)\right]n + Gn\Delta, \\ \frac{d\Delta}{dt} &= -\gamma\Delta + R - 2Gn\Delta, \end{aligned} \tag{1}$$

where k_1 is the decay rate of photon number, γ is the population inversion decay rate and R is the pump rate, G is the field-matter coupling constant and $m\cos(\Omega t)$ is a sinusoidal signal with amplitude m and modulation frequency $f_{mod} = \Omega/2\pi$ applied to the optical cavity.

B. Rescaled form

Introducing the normalized photon number $x = n/(\gamma/2G)$, the normalized population inversion $y = \Delta/(k_1/G)$ and the non-dimensional time $t' = \gamma t$ the dynamical system becomes

$$\begin{aligned}\frac{dx}{dt} &= -\varepsilon x \left(1 + m\cos(\omega t) - y\right), \\ \frac{dy}{dt} &= -y - xy + p_0,\end{aligned} \tag{2}$$

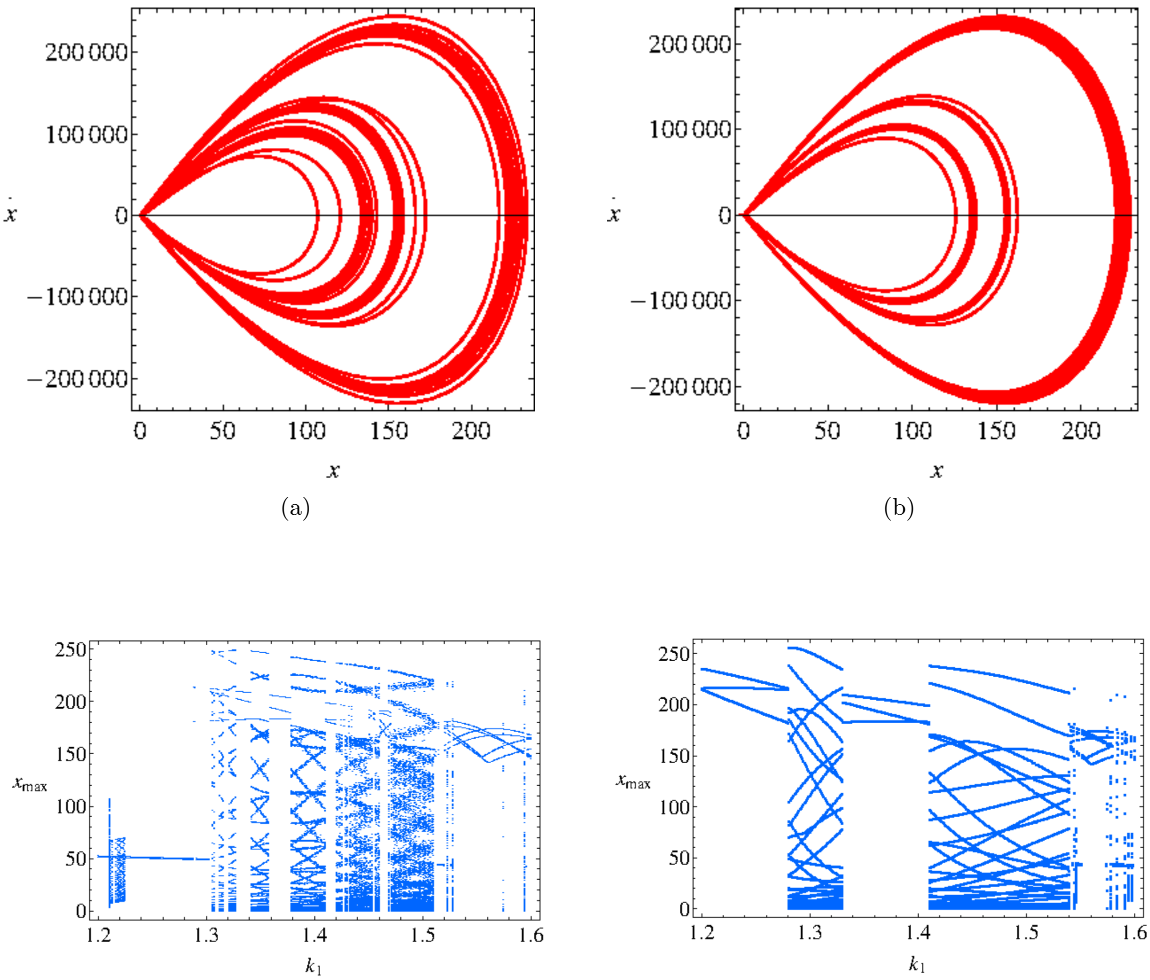
where $\varepsilon = k_1/\gamma$, $\omega = \Omega/\gamma$, $p_0 = RG/\gamma k_1$ is the normalized pump parameter and the rescaled modulation frequency is now equal to f_{mod}/γ. For sake of simplicity, we use the notation $t' = t$. In their original work, Arecchi *et al.* [1] used the following parameters: $k_1 = 1.2 - 1.6 \times 10^7 s^{-1}$, $\gamma = 10^3 s^{-1}$, $RG = 2.0 \times 10^{11} s^{-2}$, $m = 0.03$ and $f_{mod} = 64.33 - 78.8 kHz$. So, the rescaled modulation frequency will have now for values $64.33 - 78.8$.

C. Jerk form

Now, by posing $k(t) = 1 + m\cos(\omega t)$ and by using linear combinations, Eqs. (2) can be written as the following nonlinear second order non-autonomous ordinary differential equation (jerk form):

$$\frac{d}{dt}\left(\frac{\dot{x}}{x}\right) + (1+x)\left(\frac{\dot{x}}{x}\right) + (1+x)\,\varepsilon k\,(t) = \varepsilon\left(p_0 - \dot{k}\,(t)\right) \tag{3}$$

Then, a comparison between the two-level non-autonomous laser model (2) and its corresponding jerk form (3) highlights a perfect identity between both phase portraits and bifurcation diagrams for which the chosen bifurcation parameter is k_1. In Figures 1, parameters are $k_1 = 1.5 \times 10^7 s^{-1}$ and $f_{mod} = 78.8$ for all. Initial conditions are $x(0) = 0.3$ and $y(0) = 1.01$.



(a) (b)

(c) (d)

FIG. 1: Phase portraits and bifurcation diagrams of the two-level laser model (2) (a & c) and its jerk form (3) (b & d).

D. Existence of a periodic orbit

In 1982, Arecchi *et al.* [1] had highlighted the existence of a periodic orbit for the two-level non-autonomous laser model (1-2). In this subsection, we provide an analytic proof of the existence of such periodic orbit for ε sufficiently small. Then, we show, through an example, that this result can be extended to some values of $\varepsilon \approx 15\%(k_1/\gamma)$. First, let's recall the basic results from the averaging theory that we need for studying the periodic orbits of the differential system (2). Let Δ be the open subset of $\mathbf{R}^n$ where the differential system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{4}$$

is defined, here the $\mathcal{C}^2$ functions F_k for $k = 0, 1, 2$ are T-periodic. We want to study the T-periodic orbits of system (4) with $\varepsilon = 0$ which persist for $\varepsilon > 0$ sufficiently small. We assume that the unperturbed system (4) with $\varepsilon = 0$ has a submanifold $\mathcal{U}$ filled with periodic solutions.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (4) with $\varepsilon = 0$ such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. The linear part of system (4) with $\varepsilon = 0$ along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ is

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y} \tag{5}$$

Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of this linear differential system (5), and let $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ be the projection $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Suppose that there is a k–dimensional submanifold $\mathcal{U}$ of Δ filled with T–periodic solutions of system (4) with $\varepsilon = 0$. Then the T–periodic solutions of system (4) with $\varepsilon = 0$ which persist in the system (4) with $\varepsilon > 0$ enough small can be found using the next result.

Theorem 1. *Let W be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{Cl}(W) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. Suppose that*

(i) $\mathcal{U} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \quad \alpha \in \mathrm{Cl}(W)\} \subset \Delta$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (4) with $\varepsilon = 0$ is T–periodic;

(ii) for each $\mathbf{z}_\alpha \in \mathcal{U}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (5) verifying that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the lower right corner a $(n-k) \times (n-k)$ matrix A_α with $\det(A_\alpha) \neq 0$, and in the upper right corner the $k \times (n-k)$ zero matrix.

Let $\mathcal{F} : \mathrm{Cl}(W) \to \mathbb{R}^k$ be the function

$$\mathcal{F}(\alpha) = \xi \left(\frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right). \tag{6}$$

If there is $a \in W$ such that $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system (4) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 1 was proved by Malkin in [6] and by Roseau in [7], for a clear proof see [8].

Using the averaging theory described above we can now prove the following result on the periodic orbits of the differential system (2).

Theorem 2. *For $\varepsilon > 0$ sufficiently small and $p_0 \neq 1$ the differential system (2) has a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that $(x(0, \varepsilon), y(0, \varepsilon)) = (O(\varepsilon^2), p_0 + O(\varepsilon))$.*

Proof. We do the change of variables $(x, y) = (\varepsilon X, y)$, then the differential system (2) becomes

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon X \left(1 + m\cos(\omega t) - y\right), \\ \frac{dy}{dt} &= -y - \varepsilon X y + p_0, \end{aligned} \tag{7}$$

This differential system for $\varepsilon = 0$ has the equilibrium point $(X, y) = (X_0, p_0)$ for all $X_0 \in \mathbf{R}$. This equilibrium point can be considered a periodic orbit of the differential system (7) with $\varepsilon = 0$ of period $T = 2\pi/\omega$.

We shall apply Theorem 1 to system (7). Using the notation of the previous section $\mathbf{x} = (X, y)$, $F_0(t, \mathbf{x}) = (0, -y + p_0)$, $F_1(t, \mathbf{x}) = (X(1 + m\cos(\omega t) - y), -Xy)$, $F_2(t, \mathbf{x}, \varepsilon) = 0$, $\mathbf{z}(t) = (X_0, p_0)$ and $\alpha = X_0$. Then the fundamental matrix of the linear differential system (6) for the differential system (7) is

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Consequently we have

$$M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}(T) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

and all the assumptions of Theorem 1 are satisfied.

Since $\xi(X, y) = X$ the function (6) for the differential system (7) is

$$\mathcal{F}(X_0) = \frac{2\pi(p_0 - 1)X_0}{\omega}.$$

So, the zero a of this function with respect to the variable X_0 is $a = 0$. Hence from Theorem 1 we obtain that the differential system (7) for $\varepsilon > 0$ sufficiently small has a periodic solution $(X(t), y(t)) = (O(\varepsilon), p_0 + O(\varepsilon))$ because

$$\mathcal{F}'(X_0) = \frac{2\pi(p_0 - 1)}{\omega} \neq 0.$$

Hence going back to the variables (x, y) we obtain that the differential system (2) has a periodic solution $(x(t), y(t)) = (O(\varepsilon^2), p_0 + O(\varepsilon))$. This completes the proof of the theorem.

□

In order to emphasize Theorem 2, we plot on Figure 2, the periodic orbit of the differential system (7). In Figure 2, parameters are $k_1 = 1.5 \times 10^7 s^{-1}$, $f_{mod} = 78.8$, $p_0 = 15$, $\varepsilon = 2100 \approx 15\%(k_1/\gamma)$.

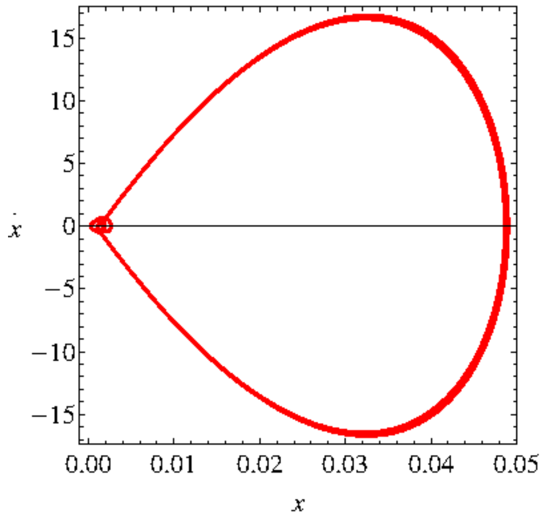


FIG. 2: Periodic orbit of the differential system (7).

III. TWO-LEVEL AUTONOMOUS LASER MODEL

By posing $z = \omega t$, the two-level non-autonomous laser model (2) can be also transformed into the following three-dimensional autonomous dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= -\varepsilon x \left(1 + m\cos(z) - y\right), \\ \frac{dy}{dt} &= -y - xy + p_0, \\ \frac{dz}{dt} &= \omega, \;\; z(0) = 0. \end{aligned} \tag{8}$$

Of course, system (8) exhibits exactly the same dynamics features (phase portrait and bifurcation diagram) as the two previous ones (2, 3). However, although system (8) has been transformed into an autonomous dynamical system, the presence of the sinusoidal function in the right hand side of its first equation still precludes any mathematical stability analysis. Thus, by recalling that a sinusoidal function is nothing else but the solution of a harmonic oscillator, the two-level non-autonomous laser model (2) can be transformed into a four-dimensional autonomous dynamical system which reads as follows:

$$\frac{dx}{dt} = -\varepsilon x\left(1 + mz - y\right),$$
$$\frac{dy}{dt} = -y - xy + p_0,$$
$$\frac{dz}{dt} = -\omega^2 u, \;\; z(0) = 1, \tag{9}$$
$$\frac{du}{dt} = z, \;\; u(0) = 0.$$

A. Fixed points

By using the classical nullclines method, it can be shown that the dynamical system (9) admits two fixed points.

$$I_1\left(0, p_0, 0, 0\right) \;\; ; \; I_2\left(p_0 - 1, 1, 0, 0\right) \tag{10}$$

B. Jacobian matrix

The Jacobian matrix of the dynamical system (9) reads:

$$J = \begin{pmatrix} -\varepsilon(1 + mz - y) & \varepsilon x & -\varepsilon m x & 0 \\ -y & -1 - x & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{11}$$

By replacing the coordinate of the fixed points I_1 (10) in the Jacobian matrix (11), one obtains the four following eigenvalues:

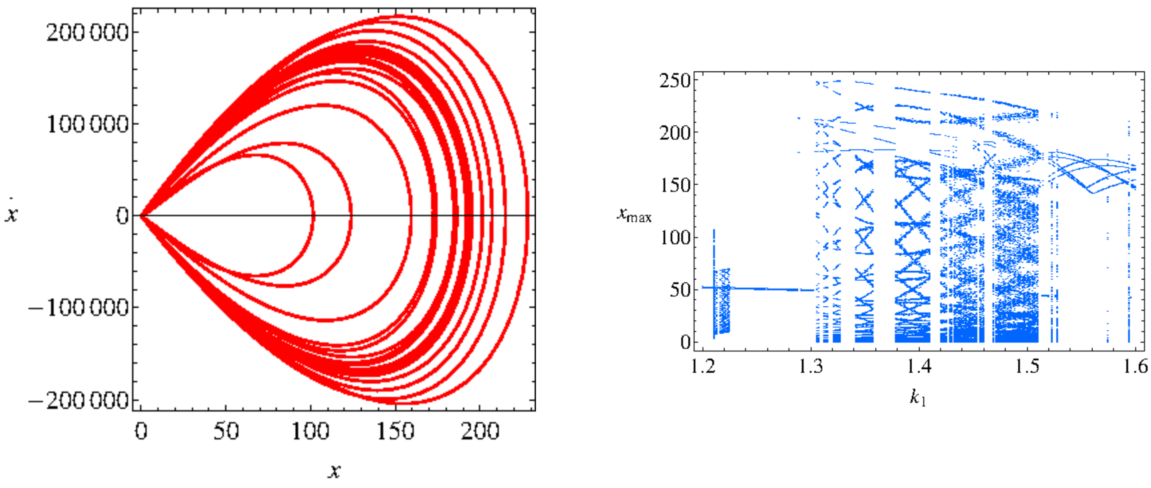
$$\lambda_1 = -1, \; \lambda_2 = \varepsilon\left(p_0 - 1\right), \; \lambda_{3,4} = \pm i\omega. \tag{12}$$

With this parameters set $p_0 - 1 > 0$, λ_2 is real and positive while λ_1 is real and negative. So, according to the Lyapunov theorem, the fixed point I_1 is *unstable*.

By replacing the coordinate of the fixed points I_2 (10) in the Jacobian matrix (11), one obtains the four following eigenvalues:

$$\lambda_{1,2} = -\frac{p_0}{2} \pm i\frac{1}{2}\sqrt{4\varepsilon\left(p_0 - 1\right) - p_0^2} \quad \text{and} \quad \lambda_{3,4} = \pm i\omega. \tag{13}$$

With this parameters set, $p_0 > 0$ and both real parts of $\lambda_{1,2}$ are negative. Hence, according to the Lyapunov theorem, the fixed point I_2 is *stable*. Notice that no Hopf bifurcation can occur in system (9). It is important to note that the emergence of chaos is related to the interplay between the two stationary points. The unstable fixed point I_1 provides the re-injection to I_2, allowing time evolution of the trajectory in phase space. Here again the phase portrait and bifurcation diagram presented in Figs. 3 are perfectly analogous to the previous ones (see Figs. 1). Initial conditions are still $x(0) = 0.3$ and $y(0) = 1.01$ and the parameters set is the same as above except $k_1 = 1.4 \times 10^7 s^{-1}$.



(a) (b)

FIG. 3: Phase portrait and bifurcation diagram of the two-level autonomous laser model (9).

Then, by using the Lyapunov Exponents Toolbox (LET) developed by Steve Siu for MatLab® and involving the two algorithms proposed by Wolf *et al.* [9] and Eckmann and Ruelle [10], we have obtained for the original parameter set the same Lyapunov Characteristic Exponents for all models (2,3,8,9). Now, let's highlight the multistability regions of the two-level non-autonomous laser model (2) and so to all models (3,8,9). To this aim, we have introduced in a previous publication [5] what we called *multistability bifurcation diagram* and which consists in plotting (as usual) the maxima of the variable $x(t)$ as a function of the initial condition $y(0)$ instead of the control parameter k_1 (see Fig. 4).

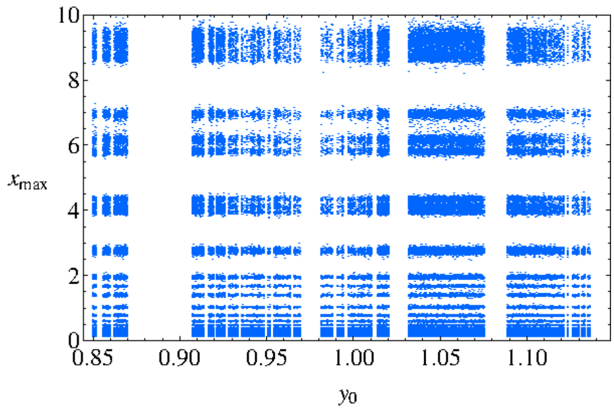
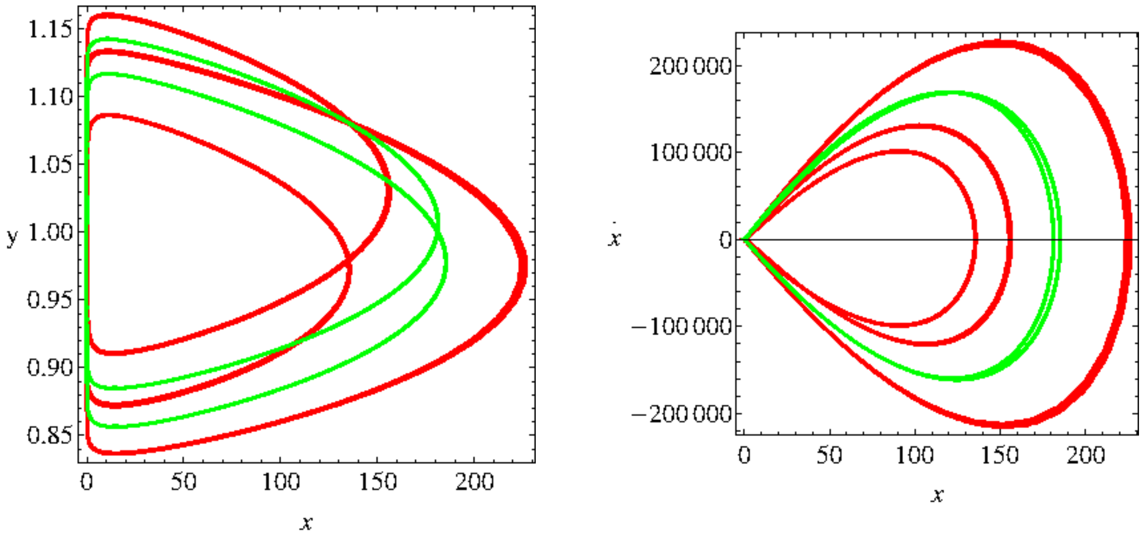


FIG. 4: Multistability bifurcation diagram of the two-level non-autonomous laser model (2).

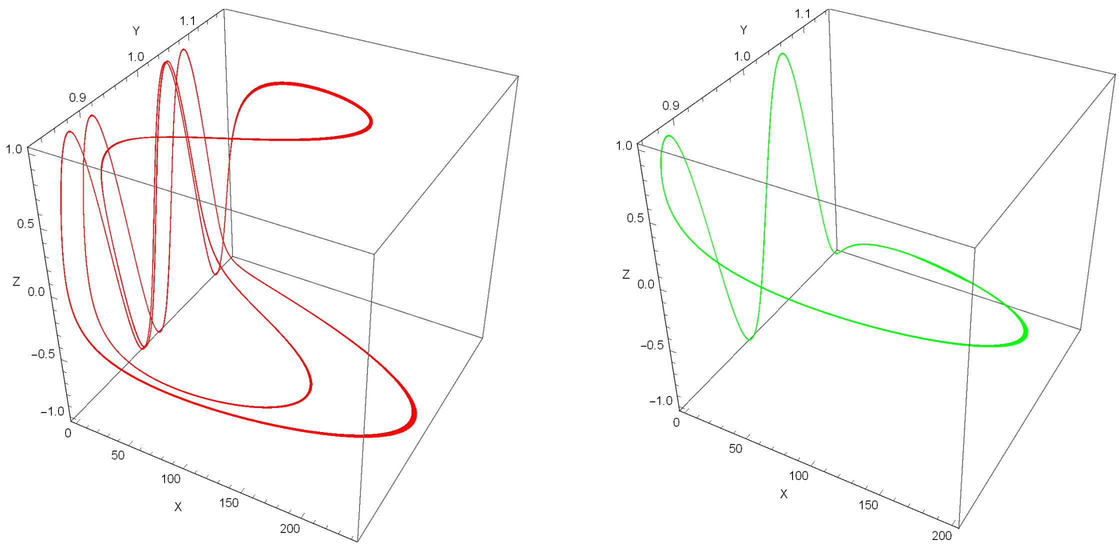
Figure 4 represents the maxima of the solution $x(t)$ as a function of the initial condition $y(0) = y_0$ for $k_1 = 1.4 \times 10^7 s^{-1}$. As an example, we deduce from Fig. 4 that for $y(0) = 0.89$ and $y(0) = 1.01$, two attractors coexist (see Figs. 5a & 5b).



(a) (b)

FIG. 5: Phase portrait of the two-level non-autonomous laser model (2).

Figures 5a & 5b represent the phase portraits of the two-level non-autonomous laser model (2) in the $(x,\, y)$ plane and the $(x,\, \dot{x})$ plane for $x(0) = 0.3$, $y(0) = 1.01$ in red and $y(0) = 0.89$ in green respectively. Of course, we found exactly the same phase portraits for the two previous models (3,8,9).



(a) (b)

FIG. 6: Phase portrait of the four-dimensional autonomous dynamical system (9).

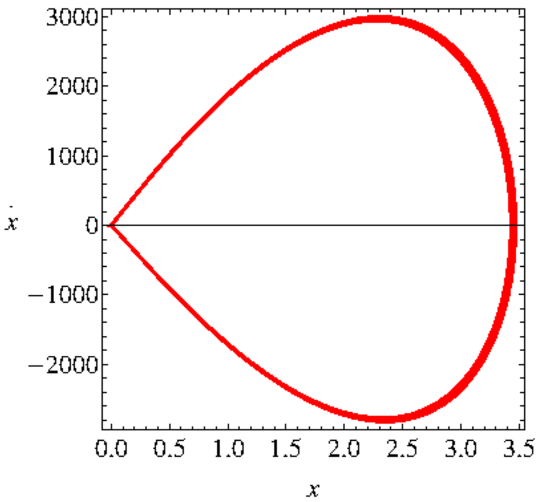
Figures 6a & 6b represent the phase portraits of four-dimensional autonomous dynamical system (9) in the $(x,\, y,\, z)$ space phase for $x(0) = 0.3$, $y(0) = 1.01$ in red and $y(0) = 0.89$ in green respectively. Here again two attractors coexist.

IV. TWO-LEVEL LASER MODEL WITH DELAY

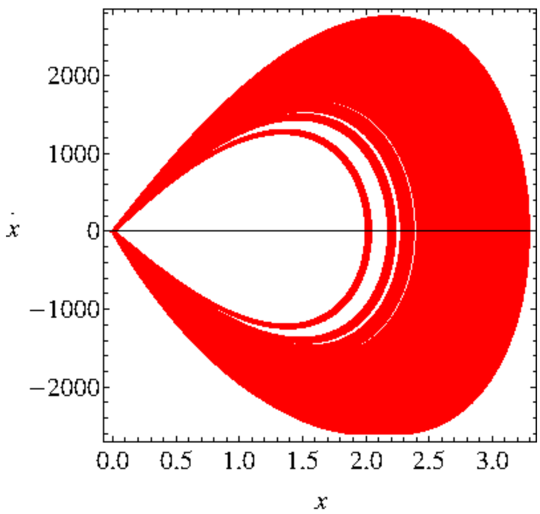
Starting from the seminal works of Arecchi *et al.* [2], we propose now to replace in Eqs. (2) the sinusoidal loss modulation $\cos(\omega t)$ by the delayed function $x(t - \tau)$. The two-level laser model with delay reads thus:

$$\frac{dx}{dt} = -\varepsilon x\left(1 + mx(t-\tau) - y\right), \\ \frac{dy}{dt} = -y - xy + p_0, \tag{14}$$

where $\varepsilon = k_1/\gamma$, $p_0 = RG/\gamma k_1$ and τ is the period of propagation and transformation an optical signal in feedback circuit. In order to make a comparison with the previous models (2,3,8,9) we propose to choose initially for τ the reciprocal of the rescaled modulation frequency, i.e., $\tau = 1/f_{mod} = 1/78.8 \approx 0.012$. Thus, for this two-level laser model with delay, all parameters are as follows: $k_1 = 1.5 \times 10^7 s^{-1}$, $\gamma = 10^3 s^{-1}$, $RG = 2.0 \times 10^{11} s^{-2}$, $m = 0.03$ and τ which is now the *bifurcation parameter* will vary with the range of values 0.012 and 20. In figures 7a & 7b, we have plotted both phase portrait for $\tau = 0.012$ and $\tau = 0.2$ of the two-level laser model with delay (14). Parameter $k_1 = 1.5 \times 10^7 s^{-1}$ in Figures 7, initial conditions are still $x(0) = 0.3$ and $y(0) = 1.01$.



(a) $\tau = 0.012s$

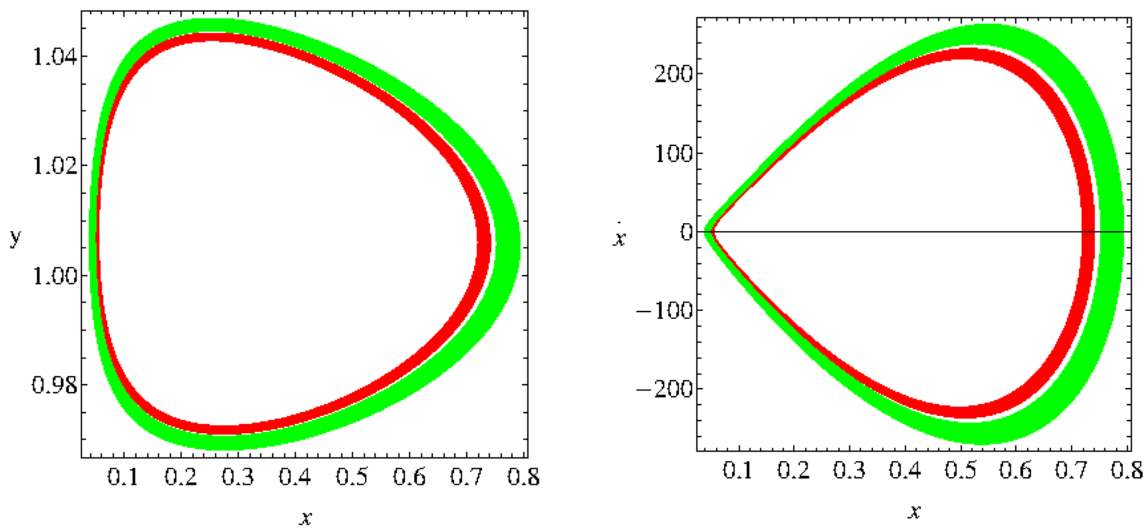


(b) $\tau = 0.2s$

FIG. 7: Phase portrait of the two-level laser model with delay (14).

From Fig. 7a we observe again a perfect analogy between the phase portrait of the two-level laser model with delay (14) and the two-level non-autonomous laser model (2, 3) as well as the three and four-dimensional autonomous laser models (8,9). Fig. 7a also confirms the existence of a periodic orbit according to Theorem 2 while Fig. 7b highlights the chaotic solution of the two-level laser model with delay (14).

In order to highlight the multistability of the two-level laser model with delay (14), we have plotted in Figs. 8a & 8b its phase portraits in the (x, y) plane and the $(x, \dot{x})$ plane for $x(0) = 0.3$, $y(0) = 1.01$ in red and $y(0) = 1.02$ in green respectively. All parameters are the same as above except $k_1 = 1.4 \times 10^7 s^{-1}$ and $\tau = 18$.



(a) (b)

FIG. 8: Phase portrait of the two-level laser model with delay (14).

Thus, figures 8 confirm the multistability already established by Grigorieva *et al.* [3]. In order to specify the windows of multistability we have plotted as previously the maxima of the variable $y(t)$ as a function of the initial condition $y(0)$ in a bifurcation diagram presented in figure 9. All parameters are the same as above except $k_1 = 1.4 \times 10^7 s^{-1}$ and $\tau = 18$.

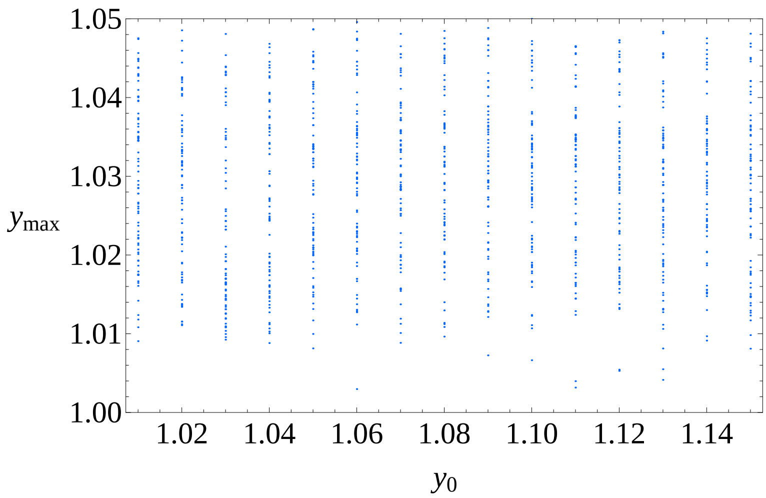


FIG. 9: Multistability bifurcation diagram of the two-level laser model with delay (14).

We observe in Fig. 9 vertical continuous series of points from the bottom to the top which corresponds to a multistability regions. Thus, as exemplified in Figs. 8, we verify that for $y(0) = 1.02$, $y(0) = 1.03$, ... we observe the coexistence of two *periodic orbits* and so, multistability.

V. DISCUSSION

Tentative discussion to be verified by Riccardo.
More than forty years ago, Arecchi *et al.* [1] disclosed a first occurrence of chaos as well as multistability in a CO2 laser with periodic modulation applied to the cavity loss parameter modeled by a set of two nonlinear differential equations, the so-called *two-level non-autonomous laser model* (1). Using a classical normalization, we provided the rescaled *two-level non-autonomous laser model* equations (2) and recalled the expression of its *jerk form* (3). Then, we have stated an analytic proof of the existence of a *periodic orbit* for this original model for ε sufficiently small. We have also shown that this result can be extended to some greater values of $\varepsilon \approx 15\%(k1/\gamma)$ which are realistic since they can be simply obtained by increasing γ, i.e. the population inversion decay, of a factor 10. Secondly, we have transformed the original *two-level non-autonomous laser model* (2) into a three-dimensional autonomous dynamical system (8) and four-dimensional autonomous dynamical system (9) the *topological equivalence* of which has been established by the comparison of both *phase portraits* and classical *bifurcation diagrams* and *Lyapunov Characteristic Exponents*. Such a transformation enabled on the one hand, to perform a mathematical stability analysis and on the other hand, to confirm the *multistabilty*, i.e. the co-existence of *periodic orbits* for various initial conditions thanks to the use of a *multistability bifurcation diagram* we have introduced in a previous publication. At last, following the works of Grigorieva *et al.* [3], we have replaced in the original *two-level non-autonomous laser model* (2 the sinusoidal loss modulation $\cos(\omega t)$ by the delayed function $x(t - \tau)$ which led us to the *two-level laser model with delay* (14). After having also shown the *topological equivalence* between models (2) and (14), we confirmed the *multistability* for this *two-level laser model with delay* (14) still using our *multistability bifurcation diagram.* From this analysis it follows, on the one hand, that the *two-level non-autonomous laser models* (1, 2, 3), the *two-level autonomous laser models* (8-9) and the textittwo-level laser model with delay (14) are *topologically equivalent.* On the other hand, this work proves the existence and co-existence of *periodic orbits* for all these models. Finally, this work has also enabled to characterize the role and importance of the *bifurcation parameters* in each model: k_1 for (1, 2, 3, 8, 9) and τ for (14).

[1] Arecchi, F. T., Meucci, R., Puccioni, G. P. & Tredicce, J. R. [1982] "Experimental evidence of subharmonic bifurcations, multistability, and turbulence in a q-switched gas laser," *Phys. Rev. Lett.* 49, 1217-1220.

[2] Arecchi, F. T., Giacomelli, G., Lapucci, A. & Meucci, R. [1991] "Dynamics of a CO_2 laser with delayed feedback: The short-delay regime," *Phys. Rev. A.* 43(9), 4997-5004.

[3] Grigorieva, E. V., Kaschenkob, I. S. & Kaschenkob, S. A. [2014] "Multistability in a Laser Model with Large Delay," *Automatic Control and Computer Sciences*, 48, 623-629.

[4] Meucci, R. & Ginoux, J. M. [2023] *Nonlinear Dynamics of Laser*, Topics in Systems Engineering: Volume 4, World Scientific Publishing.

[5] Meucci, R., Ginoux, J.M., Mehrabbeik, M., Jafari, S. & Sprott, J.C. [2022] "Generalized multistability and its control in a laser," *Chaos* 32, 083111.

[6] Malkin, I.G. [1956] *Some problems of the theory of nonlinear oscillations*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow.

[7] Roseau, M. [1966] *ibrations non linéaires et théorie de la stabilité*, (French) Springer Tracts in Natural Philosophy, Vol. 8, New York: Springer.

[8] Buică, A., Françoise, J.P. & Llibre, J. [2007] "Periodic solutions of nonlinear periodic differential systems with a small parameter," Comm. on Pure and Appl. Anal. 6, 103–111.

[9] Wolf, A., Swift, J.B., Swinney, H.L. & Vastano, J. A. [1985] "Determining Lyapunov exponents from a time series," *Physica D*, 16, 285-317.

[10] Eckmann, J. P. & Ruelle, D. [1985] "Ergodic theory of chaos and strange attractors," *Rev. Mod. Phys.*, 57, 617-656.