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THE SCHEME OF CHARACTERS IN SL_2

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ABSTRACT. The aim of this article is to study the $\mathrm{SL}_2(\mathbb{C})$–character scheme of a finitely generated group. Given a presentation of a finitely generated group Γ, we give equations defining the coordinate ring of the scheme of $\mathrm{SL}_2(\mathbb{C})$–characters of Γ (finitely many equations when Γ is finitely presented). We also study the scheme of abelian and non-simple representations and characters. Finally we apply our results to study the $\mathrm{SL}_2(\mathbb{C})$–character scheme of the Borromean rings.

1. Introduction

For a finitely generated group Γ, we fix a presentation

$$\Gamma = \langle \gamma_1, \dots, \gamma_n \mid r_l,\ l \in L \rangle \qquad (1)$$

with L possible infinite. Let

$$\pi \colon \mathbb{F}_n \twoheadrightarrow \Gamma$$

denote the natural surjection from the free group $\mathbb{F}_n$ on n generators. Hence the kernel of π is the subgroup of $\mathbb{F}_n$ normally generated by the relations r_l, $l \in L$. Our first goal is to describe the scheme of characters of Γ in $\mathrm{SL}_2(\mathbb{C})$ from the scheme of $\mathbb{F}_n$ and the presentation (1).

The scheme of representations and characters in $\mathrm{SL}_2(\mathbb{C})$ are denoted respectively by $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. The corresponding algebras of functions are the universal algebra of $\mathrm{SL}_2(\mathbb{C})$-representations

$$A(\Gamma) = \mathbb{C}[R(\Gamma, \mathrm{SL}_2(\mathbb{C}))]$$

and the universal algebra of $\mathrm{SL}_2(\mathbb{C})$-characters

$$B(\Gamma) = \mathbb{C}[X(\Gamma, \mathrm{SL}_2(\mathbb{C}))] \cong A(\Gamma)^{\mathrm{SL}_2(\mathbb{C})}.$$

We give the definitions in Section 2. The algebras $A(\Gamma)$ and $B(\Gamma)$ may be *non-reduced*, i.e. they may contain non-zero nilpotent elements (see the examples in Section 8, for instance, or the more general result of [18] that provides the existence of arbitrary singularities). If instead of the scheme we consider the underlying variety, then the algebra of functions of the variety is the quotient $B(\Gamma)_{red} \coloneqq B(\Gamma)/N(\Gamma)$, where $N(\Gamma)$ is the nilradical of $B(\Gamma)$.

For $\gamma \in \mathbb{F}_n$, let $t_\gamma \in B(\mathbb{F}_n)$ denote the the evaluation function of characters at γ.

Theorem 1. *Let $\Gamma = \langle \gamma_1, \dots, \gamma_n \mid r_l,\ l \in L \rangle$ be a finitely generated group. Then*

$$B(\Gamma) \cong B(\mathbb{F}_n) / \big(t_{r_l} - 2,\ t_{\gamma_i r_l} - t_{\gamma_i},\ t_{\gamma_j \gamma_k r_l} - t_{\gamma_j \gamma_k} \mid l \in L,\ 1 \le i \le n,\ 1 \le j < k \le n \big)$$

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We compare this result to Theorem 3.2 in [14] by González-Acuña and Montesinos-Amilibia, who give a presentation for the variety instead of the scheme. More precisely, they deal with the reduced algebra and prove that the reduction $B(\Gamma)_{red}$ of $B(\Gamma)$ is given by

$$B(\Gamma)_{red} \cong B(\mathbb{F}_n)/\operatorname{rad}\left(t_{r_l}-2,\ t_{\gamma_i r_l}-t_{\gamma_i} \mid l\in L,\ 1\le i\le n\right).$$

Our motivation for Theorem 1 is that $B(\Gamma)$ may be non-reduced and that, even when $B(\Gamma)$ is reduced, the radical in the ideal of González-Acuña and Montesinos-Amilibia is needed, see Example 28).

In order to work with shorter words and to simplify computations, we may write each relation as a product of two words

$$r_l = u_l^{-1} v_l, \qquad \forall l\in L. \tag{2}$$

Addendum to Theorem 1. *Let $\Gamma = \langle \gamma_1,\dots,\gamma_n \mid u_l = v_l,\ l\in L\rangle$ be a finitely generated group. Then*

$$B(\Gamma)\cong B(\mathbb{F}_n)/\big(t_{v_l}-t_{u_l},\ t_{\gamma_i v_l}-t_{\gamma_i u_l},\ t_{\gamma_j\gamma_k u_l}-t_{\gamma_j\gamma_k v_l} \mid l\in L,\ 1\le i\le n,\ 1\le j<k\le n\big).$$

The first step of the proof uses the isomorphism between $B(\Gamma)$ and the skein algebra of Γ, proved by Przytycki and Sikora in [33], but that follows also from a result of Procesi [31]. The proof uses trace identities of Vogt and Magnus to find the explicit presentation.

In this paper we also define the scheme of *non-simple* representations and characters (some times called reducible representations, but we already use the word reduced with another meaning). In Proposition 37 we prove that the scheme of non-simple characters is isomorphic to the scheme of characters of its abelianization. We also describe the scheme of characters for abelian groups in Theorem 38.

In this article we are mainly interested in the field of complex numbers. Nevertheless, all results (in particular Theorem 1) are valid over any algebraically closed field of characteristic 0.

Remark 2. Recently G. Miura and S. Suzuki have given a similar description of $B(\Gamma)$ when Γ has three generators [25]. We discuss the differences between both results in Remark 27.

The paper is organized as follows. Preliminaries on affine schemes are provided in Section 2, which can be skipped by readers familiar with schemes and used as a reference. Then in Section 3 we discuss the scheme of representations, and the local properties of these are given in Section 4. Section 5 is devoted to the proof of Theorem 1. The schemes of non-simple and of abelian characters are studied respectively in Sections 6 and 7. In Section 8 we provide examples of non-reduced character schemes of three-manifolds, and in the final section we compute the scheme of characters of the Borromean rings exterior.

2. Preliminaries on affine algebraic schemes

In this section we review the basic tools in affine algebraic schemes required. For more details see for instance [9, 23, 27].

2.1. Affine algebraic schemes.

Let A be a finitely generated, commutative $\mathbb{C}$-algebra. Recall that A is the quotient of a polynomial algebra by an ideal, i.e. there exists $n\in\mathbb{N}$ and an ideal $I\subset\mathbb{C}[x_1,\dots,x_n]$ such that $A\cong\mathbb{C}[x_1,\dots,x_n]/I$. It follows from Zariski's lemma that there is a one-to-one correspondence between the maximal ideals $\mathfrak{m}\subset A$ and algebra morphisms $A\to\mathbb{C}$. In what follows we let $\operatorname{Spm}(A)$ denote the maximal spectrum of A i.e.

$$\operatorname{Spm}(A) = \{\mathfrak{m}\subset A \mid \mathfrak{m} \text{ is a maximal ideal.}\}$$

We can think of elements of A as complex-valued *functions* on $\mathrm{Spm}(A)$ in the following way: if $f \in A$ and $\mathfrak{m} \in \mathrm{Spm}(A)$, then $A/\mathfrak{m} \cong \mathbb{C}$ and $f(\mathfrak{m}) := f \bmod \mathfrak{m}$. Therefore, we have $f(\mathfrak{m}) = 0$ if and only if $f \in \mathfrak{m} \subset A$. Notice that we might have that $f(\mathfrak{m}) = 0$ for all $\mathfrak{m} \in \mathrm{Spm}(A)$, but $f \neq 0$; if his happens then $f \in A$ is nilpotent. Also each $\mathfrak{m} \in \mathrm{Spm}(A)$ determines, by evaluation, an algebra morphism $\phi_{\mathfrak{m}}: A \to \mathbb{C}$ given by $\phi_{\mathfrak{m}}(f) = f(\mathfrak{m})$ for $f \in A$. The two important properties are the following:

- The elements of A separate points. Indeed, if $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Spm}(A)$ then $\mathfrak{m} \neq \mathfrak{m}'$ implies that there exists $f \in \mathfrak{m}$ and $f \notin \mathfrak{m}'$, and therefore $0 = f(\mathfrak{m}) \neq f(\mathfrak{m}') \neq 0$.
- Every algebra morphism $\phi: A \to \mathbb{C}$ is the evaluation at a unique point i.e. there exists a unique point $\mathfrak{m} = \mathrm{Ker}(\phi) \in \mathrm{Spm}(A)$ such that for all $f \in A$ we have $\phi(f) = f(\mathfrak{m})$.

For any subset M of A we define

$$V(M) := \{\mathfrak{m} \in \mathrm{Spm}(A) \mid \mathfrak{m} \supset M\} = \{\mathfrak{m} \in \mathrm{Spm}(A) \mid \forall f \in M, f(\mathfrak{m}) = 0\}.$$

It is clear that $V(M) = V(I)$ if I is the ideal generated by M. We will endow $\mathrm{Spm}(A)$ with the Zariski topology, where the closed sets are of the form $V(I)$ for I and ideal of A. Notice that $V(0) = \mathrm{Spm}(A)$ and $V(1) = \varnothing$. For more details see Appendix A in [23].

We will call a pair $X = (\mathrm{Spm}(A), A)$ an *affine algebraic scheme*. Given an affine algebraic scheme X, we define the *coordinate algebra* of X as $\mathbb{C}[X] := A$, and the *underlying space* $|X| = \mathrm{Spm}(A)$. In what follows it will be convenient to write $x \in |X|$ for an element in $|X| = \mathrm{Spm}(A)$. In this case the corresponding maximal ideal in A will be denoted by $\mathfrak{m}_x$.

Remark 3. It is worth noticing the word "algebraic" in the definition of *affine algebraic scheme*. Affine schemes in general are constructed from the prime spectrum of a commutative ring with unity, though here we work with the maximal spectrum of a finitely generated commutative $\mathbb{C}$-algebra. See [27] or [9].

Example 4. The n-dimensional affine space has the structure on an affine algebraic scheme $\mathbb{A}^n := (\mathbb{C}^n, \mathbb{C}[x_1, \ldots, x_n])$. A point $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$ corresponds to the maximal ideal $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$, and each maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is of this form. The value of $f \in \mathbb{C}[x_1, \ldots, x_n]$ at $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$ is simply $f(p)$.

Example 5. We consider the *dual numbers* $A_2 := \mathbb{C}[\epsilon]/(\epsilon^2)$. We have that $\mathrm{Spm}(A_2) = \{*\}$ has only one point, the maximal ideal $* := (\epsilon)$. The value of $f = a + b\epsilon \in A_2$ at the point $*$ is $f(*) = a$. Hence for $\epsilon \in A_2$ we have $\epsilon(*) = 0$, but $0 \neq \epsilon$ and $\epsilon^2 = 0$ in A_2. The scheme $(\{*\}, A_2)$ is called the *double point*, and similarly $A_n := \mathbb{C}[\epsilon]/(\epsilon^n)$ gives us a point of multiplicity n.

A morphism between two affine algebraic schemes $X = (\mathrm{Spm}(A), A)$ and $Y = (\mathrm{Spm}(B), B)$ is a pair (α, α^*) such that $\alpha: \mathrm{Spm}(A) \to \mathrm{Spm}(B)$ is a map, $\alpha^*: B \to A$ is an algebra homomorphism satisfying

$$g(\alpha(\mathfrak{m})) = \alpha^*(g)(\mathfrak{m}) \quad \text{for all } \mathfrak{m} \in \mathrm{Spm}(A) \text{ and for all } g \in B \tag{3}$$

i.e. for all $g \in B$ the two maps $g \circ \alpha$ and $\alpha^*(g)$ are equal as maps on $|X| = \mathrm{Spm}(A)$. Notice that α is continuous in the Zariski-topology, for if $J \subset B$ is an ideal we have

$$\begin{aligned} \alpha^{-1}\big(V(J)\big) &= \{\mathfrak{m} \in \mathrm{Spm}(A) \mid \alpha(\mathfrak{m}) \supset J\} \\ &= \{\mathfrak{m} \in \mathrm{Spm}(A) \mid \forall g \in J,\ \alpha^*(g)(\mathfrak{m}) = g(\alpha(\mathfrak{m})) = 0\} \\ &= V\big(\alpha^*(J)\big). \end{aligned}$$

In what follows we will write $\alpha\colon X \to Y$ for a morphism between the schemes X and Y. It is understood that $\alpha\colon |X| \to |Y|$, and $\alpha^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$ satisfy equation (3) i.e. for all $g \in \mathbb{C}[Y]$ and for all $x \in |X|$ we have $\alpha^*(g)(x) = g(\alpha(x))$.

Remark 6.
- An algebra homomorphism $\alpha^*\colon B \to A$ determines a unique map $\alpha\colon \mathrm{Spm}(A) \to \mathrm{Spm}(B)$ in the following way: for $\mathfrak{m} \in \mathrm{Spm}(A)$ we consider the map $\phi_{\mathfrak{m}}\colon B \to \mathbb{C}$ given by $\phi_{\mathfrak{m}}(g) = \alpha^*(g)(\mathfrak{m})$. This map is an algebra homomorphism and we can put $\alpha(\mathfrak{m}) := \mathrm{Ker}(\phi_{\mathfrak{m}})$, and we obtain $g(\alpha(\mathfrak{m})) = \alpha^*(g)(\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Spm}(A)$. Notice that $\alpha(\mathfrak{m}) = (\alpha^*)^{-1}(\mathfrak{m})$ since $\mathrm{Ker}(\phi_{\mathfrak{m}}) = (\alpha^*)^{-1}(\mathfrak{m})$.
- The map $\alpha\colon \mathrm{Spm}(A) \to \mathrm{Spm}(B)$ does not determine α^* uniquely. This only happens if A has no nilpotent elements.

A scheme morphism $\alpha\colon X \to Y$ is called a *closed immersion* if the underlying continuous map $\alpha\colon |X| \to |Y|$ is a homeomorphism between $|X|$ and a closed subset of $|Y|$, and the algebra homomorphism $\alpha^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$ is surjective.

2.2. **Tangent space.** Let X be an affine algebraic scheme. For every $x \in |X|$, the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a finite dimensional $\mathbb{C}$-vector space. By definition its dual space is the tangent space of X at x which will be denoted by

$$T_x(X) := \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^* .$$

Lemma 7. *Let $\mathfrak{m} \in \mathrm{Spm}(A)$. Then there is a one-to-one correspondence between linear forms $\ell \in (\mathfrak{m}/\mathfrak{m}^2)^*$ and scheme morphisms $(\alpha, \alpha^*)\colon (\mathrm{Spm}(A_2), A_2) \to (\mathrm{Spm}(A), A)$ such that $\alpha(*) = \mathfrak{m}$.*

Proof. Recall that $\mathrm{Spm}(A_2) = \{*\}$ has only one point which corresponds to the maximal ideal (ϵ). We consider the map $\alpha\colon \mathrm{Spm}(A_2) \to \mathrm{Spm}(A)$ given by $\alpha(*) = \mathfrak{m}$. In order to get a morphism of schemes, we are looking for an algebra homomorphism $\alpha^*\colon A \to A_2$ which satisfies equation (3). That is

$$\alpha^*\colon A \to A_2 \text{ given by } \alpha^*(f) = \alpha_0^*(f) + \epsilon\,\alpha_1^*(f)$$

where $\alpha_i^*\colon A \to \mathbb{C}$ are $\mathbb{C}$-linear maps, such that

$$\alpha^*(f)(*) = \alpha_0^*(f) = f(\alpha(*)) = f(\mathfrak{m})\,,$$

and

$$\forall\ f_1, f_2 \in A, \quad \alpha_1^*(f_1 f_2) = f_1(\mathfrak{m})\alpha_1^*(f_2) + \alpha_1^*(f_1) f_2(\mathfrak{m})\,. \tag{4}$$

Notice that equation (4) implies that $\alpha_1^*(1) = 0$, and hence $\alpha_1^*|_{\mathbb{C}} \equiv 0$.

Let $\ell \in (\mathfrak{m}/\mathfrak{m}^2)^*$ be given. We define a linear map $\alpha_1^*\colon A \to \mathbb{C}$ satisfying equation (4) by putting $\alpha_1^*(f) := \ell(f - f(\mathfrak{m}))$. Indeed,

$$\begin{aligned}\alpha_1^*(f_1 f_2) &= \ell(f_1 f_2 - f_1(\mathfrak{m}) f_2(\mathfrak{m})) \\ &= \ell((f_1 - f_1(\mathfrak{m}))(f_2 - f_2(\mathfrak{m}))) + f_2(\mathfrak{m})\ell(f_1 - f_1(\mathfrak{m})) + f_1(\mathfrak{m})\ell(f_2 - f_2(\mathfrak{m}))\,,\end{aligned}$$

and equation (4) follows since $(f_1 - f_1(\mathfrak{m}))(f_2 - f_2(\mathfrak{m})) \in \mathfrak{m}^2$ and $\ell|_{\mathfrak{m}^2} \equiv 0$.

Let $(\alpha, \alpha^*)\colon (\mathrm{Spm}(A_2), A_2) \to (\mathrm{Spm}(A), A)$ be a scheme morphisms such that $\alpha(*) = \mathfrak{m}$ i.e. for all $f \in A$ $\alpha^*(f) = f(\mathfrak{m}) + \epsilon\alpha_1^*(f)$ where $\alpha_1^*\colon A \to \mathbb{C}$ is linear and satisfies equation (4). The restriction $\alpha_1^*|_{\mathfrak{m}}$ is linear, and by equation (4) α_1^* vanishes on $\mathfrak{m}^2$. Hence α_1^* defines a linear form on $\mathfrak{m}/\mathfrak{m}^2$. □

2.3. **Closed subschemes.** Let $I \subset A$ be an ideal, and we let $\pi: A \to A/I$ denote the projection. We obtain the affine algebraic scheme $(\mathrm{Spm}(A/I), A/I)$. Notice that there is a one-to-one correspondence between the ideals in A/I and the ideals in A which contain I. Hence, there is a natural scheme morphism

$$(\alpha, \alpha^*): (\mathrm{Spm}(A/I), A/I) \to (\mathrm{Spm}(A), A) \quad \text{given by } \alpha(\overline{\mathfrak{m}}) = \pi^{-1}(\overline{\mathfrak{m}}) \text{ and } \alpha^* = \pi.$$

We have $\mathrm{Im}(\alpha) = V(I)$, and $\alpha: \mathrm{Spm}(A/I) \to V(I)$ a homeomorphism since it is closed; that is for every ideal $\bar{J} \subset A/I$ we have

$$\alpha(V(\bar{J})) = V(I) \cap V\big(\pi^{-1}(\bar{J})\big).$$

We endow the closed set $V(I)$ with the ring A/I, and we call the pair $(V(I), A/I)$ a *closed subscheme* of $(\mathrm{Spm}(A), A)$. In contrast to classical algebraic geometry, the closed subschemes of $(\mathrm{Spm}(A), A)$ and the ideals of A are in one-to-one correspondence. It follows that every affine algebraic scheme is isomorphic to a closed subscheme of some affine space $\mathbb{A}^n$.

For two closed subschemes $(V(I_k), A/I_k)$, $k = 1, 2$, the subscheme $(V(I_1 \cap I_2), A/(I_1 \cap I_2))$ is called their *union* and $(V(I_1 + I_2), A/(I_1 + I_2))$ is called their *intersection.* Since and these are actually set-theoretic unions and intersections of the underlying spaces. See Section I.2.1 in [9].

We call an affine algebraic scheme X *reduced* if $\mathbb{C}[X]$ is reduced. If $N \subset \mathbb{C}[X]$ denotes the nilradical of $\mathbb{C}[X]$, then the algebra $\mathbb{C}[X]_{red} := \mathbb{C}[X]/N$ is reduced. Moreover, N is the intersection of all maximal ideals of A. Hence $V(N) = V(0) = |X|$, and it follows that $X_{red} := (|X|, \mathbb{C}[X]_{red})$ is a reduced, closed subscheme of X.

2.4. **Local to global properties.** For a finitely generated algebra A we let $N(A)$ denote its nilradical. For an ideal $I \subset A$ we let $\mathrm{Ann}(I) := \{a \in A \mid aI = 0\}$ denote the *annihilator* of I. The point of $\mathrm{Spm}(A)$ represented by a maximal ideal $\mathfrak{m}$ is called *reduced* if the local ring $A_\mathfrak{m}$ is reduced. The non-reduced points of $\mathrm{Spm}(A)$ form a Zariski closed subset (which might be empty). This follows from the the following lemma.

Lemma 8. *Let A be a finitely generated, commutative $\mathbb{C}$-algebra A, $I \subset A$ an ideal, and $\mathfrak{m} \subset A$ a maximal ideal. Then*

(1) $N(A_\mathfrak{m}) = N(A)_\mathfrak{m}$;
(2) $V\big(\mathrm{Ann}(I)\big) = \{\mathfrak{m} \in \mathrm{Spm}(A) \mid I_\mathfrak{m} \neq 0\}$;
(3) The set of non-reduced points in $\mathrm{Spm}(A)$ is Zariski-closed.

Proof. (1) If $\frac{a}{s} \in N(A_\mathfrak{m})$ then there exists $n \in \mathbb{N}$ such that $\left(\frac{a}{s}\right)^n = \frac{a^n}{s^n} = \frac{0}{1}$. Hence there exists $t \in A \setminus \mathfrak{m}$ such that $a^n t = 0$ in A. Hence $(ta)^n = 0$ in A, and $\frac{a}{s} = \frac{at}{st} \in N(A)_\mathfrak{m}$. The other inclusion is obvious.

(2) We have

$$\mathfrak{m} \notin V\big(\mathrm{Ann}(I)\big) \Leftrightarrow \mathfrak{m} \not\supset \mathrm{Ann}(I) \Leftrightarrow \exists\, s \in A \setminus \mathfrak{m},\ \forall\, b \in I\ s\, b = 0$$

and

$$I_\mathfrak{m} \neq 0 \Leftrightarrow \forall\, b \in I,\ \exists s \in A \setminus \mathfrak{m}\ s\, b = 0$$

Clearly the first statement implies the second. But also the second implies the first since A is noetherian. For if $I = (b_1, \ldots, b_k)$ and $s_i \in A \setminus \mathfrak{m}$ such that $s_i b_i = 0$ then for the product $s = s_1 \cdots s_k$ we have $sb = 0$ for all $b \in I$.

(3) We have $N(A_\mathfrak{m}) = N(A)_\mathfrak{m} \neq 0$ if and only $\mathfrak{m} \in V\big(\mathrm{Ann}(N(A))\big)$. Hence the non-reduced points $\mathfrak{m} \in \mathrm{Spm}(A)$ form the Zariski-closed set $V\big(\mathrm{Ann}(N(A))\big)$. □

The first local to global property is the following (see [24, Corollary 5.19]):

Lemma 9. *Let A be a ring. Then A is reduced if and only if for every maximal ideal $\mathfrak{m} \subset A$ the local ring $A_\mathfrak{m}$ is reduced.*

Let A be a subring of B. An element $b \in B$ is called *integral* over A if it is a root of a monic polynomial with coefficients in A. An integral domain A is called *integrally closed* or *normal* if it is integrally closed in its field of fractions F i.e. $\alpha \in F$, and α integral over A implies $\alpha \in A$. There is an other local to global property (see [24, Prop. 6.16]:

Lemma 10. *Let A be an integral domain. Then A is normal if and only if for every maximal ideal $\mathfrak{m} \subset A$ the localization $A_\mathfrak{m}$ is normal.*

2.4.1. *Tangent cone.* Let B be a noetherian, local ring with maximal ideal $\mathfrak{m}$. In this situation we have by Krull's intersection theorem that

$$\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = \{0\}, \tag{5}$$

and we will define the associated *associated graded ring of B* as

$$\mathrm{gr}(B) = B/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots .$$

The multiplication of two *homogeneous* elements $\bar{a} \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and $\bar{b} \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$ is defined as follows:

$$\bar{a} \cdot \bar{b} := a\, b \mod \mathfrak{m}^{n+k+1}$$

where a and b are elements of B representing $\bar{a}$ and $\bar{b}$ respectively.

For any finitely generated, commutative $\mathbb{C}$-algebra A the *tangent cone* of the schema $X = (\mathrm{Spm}(A), A)$ at $x \in X$ is the schema

$$\big(\mathrm{Spm}(\mathrm{gr}(A_\mathfrak{m})), \mathrm{gr}(A_\mathfrak{m})\big) \text{ where } \mathfrak{m} = \mathfrak{m}_x .$$

A local noetherian ring B with maximal ideal $\mathfrak{m}$ inherits some properties of its associated graded algebra $\mathrm{gr}(B)$. For example, if $\mathrm{gr}(B)$ is reduced, normal or regular, then so is the local ring B. More precisely, it follows from (5) that for $a \in B$, $a \neq 0$, there is a minimal n such that $a \in \mathfrak{m}^n$, $a \notin \mathfrak{m}^{n+1}$, and we define the *initial term* $\mathrm{in}(a) \in \mathrm{gr}(B)$ of a as

$$\mathrm{in}(a) := a \mod \mathfrak{m}^{n+1} \in \mathfrak{m}^n/\mathfrak{m}^{n+1} \subset \mathrm{gr}(B) .$$

It is easy to see that $\mathrm{in}(ab) = \mathrm{in}(a) \cdot \mathrm{in}(b)$. This implies

Proposition 11. *Let B be a local noetherian ring with maximal ideal $\mathfrak{m}$.*

(1) If the associated graded algebra $\mathrm{gr}(B)$ is reduced, then B is reduced.
(2) If the associated graded algebra $\mathrm{gr}(B)$ is normal, then B is normal.

See [22, Section 17] and [8, Chapter 5] for more details.

3. The scheme of representations and characters

The general reference for this section is Lubotzky's and Magid's book [19]. In what follows we let Γ denote a finitely generated group with presentation (1).

3.1. **The universal algebra $A(\Gamma)$.** The $\mathbb{C}$-algebra $A(\Gamma)$ comes with a representation $\rho_u \colon \Gamma \to \mathrm{SL}_2(A(\Gamma))$, and the pair $(A(\Gamma), \rho_u)$ has the following universal property: for every commutative $\mathbb{C}$-algebra A and every representation $\rho \colon \Gamma \to \mathrm{SL}_2(A)$ there exists a unique algebra morphism $f \colon A(\Gamma) \to A$ such that $\rho = f_* \circ \rho_u$ where $f_* \colon \mathrm{SL}_2(A(\Gamma)) \to \mathrm{SL}_2(A)$ is the induced map.

The algebra $A(\Gamma)$ can be constructed from the presentation of Γ in (1), see [19, Prop. 1.2]. In more detail, the algebra $A(\Gamma)$ is a quotient of the finitely generated polynomial algebra, and the construction goes at follows: put

$$X^{(k)} = \begin{pmatrix} x_{11}^{(k)} & x_{12}^{(k)} \\ x_{21}^{(k)} & x_{22}^{(k)} \end{pmatrix} \text{ for } 1 \le k \le n.$$

and consider the polynomial ring on $4k$ variables $\mathbb{C}[x_{ij}^{(k)}]$, where $1 \le i, j \le 2$ and $1 \le k \le n$. Then $A(\Gamma)$ is the quotient of $\mathbb{C}[x_{ij}^{(k)}]$ by the ideal J generated by

$$\det(X^{(k)}) - 1,\ 1 \le k \le n, \text{ and } (m_l)_{ij} - \delta_{ij},\ 1 \le i, j \le 2,\ l \in L$$

where $m_l = r_l(X^{(1)}, \dots, X^{(n)})$ is a matrix in $\mathrm{GL}(\mathbb{C}[x_{ij}^{(k)}])$, $(m_l)_{ij}$ are the entries of m_l, and δ_{ij} is the Kronecker delta.

3.2. **The representation scheme and the representation variety.** For a finitely generated group Γ the $\mathrm{SL}_2(\mathbb{C})$-*scheme of representations* $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is defined as pair

$$R(\Gamma, \mathrm{SL}_2(\mathbb{C})) := \big(\mathrm{Spm}(A(\Gamma)), A(\Gamma)\big)$$

where $A(\Gamma)$ is the universal algebra of Γ (see Section 3.1).

A single representation $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ corresponds to $\mathbb{C}$-algebra morphism $A(\Gamma) \to \mathbb{C}$ i.e. to a maximal ideal $\mathfrak{m}_\rho \subset A(\Gamma)$, and each maximal ideal $\mathfrak{m} \subset A(\Gamma)$ determines a representation $\rho_{\mathfrak{m}}\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$. Hence maximal ideals in $A(\Gamma)$ correspond exactly to representations $\Gamma \to \mathrm{SL}_2(\mathbb{C})$, and we will use this identification in what follows.

Regular functions on $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ are exactly the elements of $A(\Gamma)$. More precisely, if $f \in A(\Gamma)$ and $\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then

$$f(\rho) = f \bmod \mathfrak{m}_\rho \in A(\Gamma)/\mathfrak{m}_\rho \cong \mathbb{C}\,.$$

If N is the nil-radical of $A(\Gamma)$, then the reduced algebra $A(\Gamma)_{\mathrm{red}} = A(\Gamma)/N$ is isomorphic to the coordinate ring of the representation variety. This justifies the notation $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ for the representation variety. The surjection $A(\Gamma) \twoheadrightarrow A(\Gamma)_{\mathrm{red}}$ induces a *closed immersion* $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}} \to R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, the underlying continuous map is a homeomorphism in the Zariski topology. In particular the scheme $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and the variety $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ have the same dimension. The scheme $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ might support more regular functions than $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$. More precisely, different regular functions on the scheme can have the same values on maximal ideals.

If the nil-radical of $A(\Gamma)$ is trivial then we call the algebra $A(\Gamma)$ and the representation scheme $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ *reduced*.

3.3. **The universal algebra $B(\Gamma)$, the character scheme, and trace functions.** The group $\mathrm{SL}_2(\mathbb{C})$ acts on the 2×2 matrices by conjugation. This induces an action of $\mathrm{SL}_2(\mathbb{C})$ on the algebra $A(\Gamma)$, and the algebra $B(\Gamma) = A(\Gamma)^{\mathrm{SL}_2(\mathbb{C})} \subset A(\Gamma)$ of invariants is called the *universal algebra of* $\mathrm{SL}_2(\mathbb{C})$-*characters*. The algebra $B(\Gamma)$ is also a quotient of a finitely generated polynomial algebra, and the affine GIT quotient is naturally defined as the *scheme of characters*

$$X(\Gamma, \mathrm{SL}_2(\mathbb{C})) := (\mathrm{Spm}(B(\Gamma)), B(\Gamma))\,,$$

see [19]. A single character $\chi\colon \Gamma \to \mathbb{C}$ corresponds to $\mathbb{C}$-algebra morphism $B(\Gamma) \to \mathbb{C}$ i.e. to a maximal ideal in $B(\Gamma)$. Also the algebra $B(\Gamma)$ might be non-reduced.

Remark 12. Notice that both $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ can be reducible.

Given an element $\gamma \in \Gamma$ there exists a word $w_\gamma(\gamma_1,,\dots,\gamma_n) \in \mathbb{F}_n$ which represents γ. The matrix $w_\gamma(X^{(1)},\dots,X^{(n)}) \in \mathrm{GL}_2(\mathbb{C}[x_{ij}^{(k)}])$, and

$$t_\gamma \coloneqq \mathrm{tr}\left(w_\gamma(X^{(1)},\dots,X^{(n)})\right)$$

represents an element in $B(\Gamma)$. We call t_γ the *trace function* associated to $\gamma \in \Gamma$.

The *first fundamental theorem* of invariant theory asserts that the trace functions t_γ, $\gamma \in \Gamma$, generate the algebra $B(\Gamma)$ (see [19] for instance). Besides finding an explicit finite set of generators, we aim to describe the relations satisfied by the t_γ. For the moment we just state some basic relations that follow from properties of the trace. Namely for all $a, b \in \Gamma$ and the neutral element $e \in \Gamma$ we have:

$$t_e = 2, \quad t_a = t_{a^{-1}}, \quad t_{ab} = t_{ba}, \quad \text{and} \quad t_{ab} + t_{ab^{-1}} = t_a t_b. \tag{6}$$

The last equality follows from the Cayley–Hamilton theorem.

3.4. **Free groups.** The general reference for this section is Goldman's Handbook article [12] and the references therein.

Let $\mathbb{F}_n$ be a free group of rank n. It is a classical result that the algebra $B(\mathbb{F}_n)$ is reduced. By works of Cartier [6], the coordinate ring $\mathbb{C}[G]$ of an algebraic group G in characteristic zero is reduced (see Section 3.h of [23] for a modern approach). But for $\mathrm{SL}_2(\mathbb{C})$ a more elementary argument is sufficient.

Theorem 13. *The algebra $B(\mathbb{F}_n)$ is reduced.*

Proof. The polynomial $x_1x_4 - x_2x_3 - 1 \in \mathbb{C}[x_1,x_2,x_3,x_4]$ is irreducible, and an irreducible element in a factorial ring is prime. Therefore the coordinate algebra

$$\mathbb{C}[\mathrm{SL}_2(\mathbb{C})] = \mathbb{C}[x_1,x_2,x_3,x_4]/(x_1x_4 - x_2x_3 - 1)$$

is a domain, and hence reduced. Moreover, we obtain that

$$A(\mathbb{F}_n) \cong \bigotimes_{k=1}^{n} \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]$$

is reduced [3, V, §15, Theorem 3]. It follows that $B(\mathbb{F}_n) = A(\mathbb{F}_n)^{\mathrm{SL}_2(\mathbb{C})} \subset A(\mathbb{F}_n)$ is also reduced. □

An explicit presentation of $B(\mathbb{F}_n)$ is given by Ashley, Burelle, and Lawton in [1].

For a free group $\mathbb{F}_n = \langle \gamma_1,\dots,\gamma_n \mid \varnothing \rangle$ of rank n the algebra $B(\mathbb{F}_n)$ is generated by the following $n(n^2+5)/6$ elements

$$t_{\gamma_i},\ 1 \le i \le n, \quad t_{\gamma_i\gamma_j},\ 1 \le i < j \le n, \quad \text{and} \quad t_{\gamma_i\gamma_j\gamma_k},\ 1 \le i < j < k \le n$$

(see [14]). In the following examples we give the explicit presentation $B(\mathbb{F}_n)$ in the rank two and three.

Example 14. The rank two case goes back to Fricke and Klein; they proved that $B(\mathbb{F}_2)$ is isomorphic to free polynomial ring $\mathbb{C}[t_{\gamma_1}, t_{\gamma_2}, t_{\gamma_1\gamma_2}]$ (see Section 2.2 in [12]).

Example 15. The case of rank three is not isomorphic to a polynomial algebra. More precisely, for a free group on three generators $\mathbb{F}_3 = \langle a,b,c \mid \varnothing \rangle$, the coordinate ring of $X(\mathbb{F}_3, \mathrm{SL}_2(\mathbb{C}))$ has been computed for instance in [14, 20]:

$$B(\mathbb{F}_3) = \mathbb{C}[t_a,t_b,t_c,t_{ab},t_{ac},t_{bc},t_{abc}]/(F) \tag{7}$$

where

$$F = F(t_a,t_b,t_c,t_{ab},t_{ac},t_{bc},t_{abc}) = t_{abc}^2 - p\,t_{abc} + q \tag{8}$$

with $p, q \in \mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}]$ given by

$$p = t_a t_{bc} + t_b t_{ac} + t_c t_{ab} - t_a t_b t_c \tag{9}$$

and

$$q = t_a^2 + t_b^2 + t_c^2 + t_{ab}^2 + t_{ac}^2 + t_{bc}^2 + t_{ab} t_{ac} t_{bc} - t_a t_b t_{ab} - t_a t_c t_{ac} - t_b t_c t_{bc} - 4\,. \tag{10}$$

The trace functions t_{abc}, t_{acb} are solutions of the quadratic equation over $\mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}]$

$$X^2 - p\,X + q = 0\,. \tag{11}$$

We have

$$t_{abc} + t_{acb} = p, \quad t_{abc} t_{acb} = q \quad \text{and} \quad (t_{abc} - t_{acb})^2 = \Delta \tag{12}$$

where

$$\Delta = p^2 - 4q\,. \tag{13}$$

(see Section 5.1 in [12]).

4. Local properties

In this section we keep on reviewing properties of the scheme of representations and the scheme of characters, now focusing on local properties. For instance, in order to check that the scheme is reduced, it is sufficient to prove that it is locally reduced.

4.1. **Zariski tangent space.** For a representation $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$, $Z^1(\Gamma, \mathrm{Ad}\,\rho)$ denotes the space of 1-cocycles or crossed morphisms twisted by $\mathrm{Ad}\,\rho$, namely linear maps

$$d\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})$$

that satisfy

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)}\, d(\gamma_2), \qquad \forall \gamma_1, \gamma_2 \in \Gamma.$$

Weil's construction [40] gives the following theorem (see also [19]):

Theorem 16. *The Zariski tangent space to the scheme $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ at ρ is naturally isomorphic to $Z^1(\Gamma, \mathrm{Ad}\,\rho)$.*

Proof. By Lemma 7, we have that a tangent vector of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ at ρ corresponds to a scheme morphism

$$(\alpha, \alpha^*)\colon (\{*\}, A_2) \to \big(\mathrm{Spm}\big(A(\Gamma)\big), A(\Gamma)\big)$$

such that $\alpha(*) = \rho$, and $\alpha^*\colon A(\Gamma) \to A_2$ is an algebra homomorphism. In turn, this corresponds to a representation $\rho_\alpha\colon \Gamma \to \mathrm{SL}_2(A_2)$ such that for all $\gamma \in \Gamma$

$$\rho_\alpha(\gamma) = (\mathrm{id} + \epsilon\, d(\gamma))\rho(\gamma) \text{ such that } d\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C}).$$

Notice that $\mathrm{SL}_2(A_2) = \{(\mathrm{id} + \epsilon X)A \mid A \in \mathrm{SL}_2(\mathbb{C}) \text{ and } \mathrm{tr}(X) = 0\}$.

Now, it is easy to check that ρ_α is a homomorphism if and only if $d\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})$ is a cocycle. □

The cohomology of Γ with coefficients in the Γ-module $\mathfrak{sl}_2(\mathbb{C})$ twisted by $\mathrm{Ad}\,\rho$ is isomorphic to

$$H^1(\Gamma, \mathfrak{sl}_2(\mathbb{C})) \cong Z^1(\Gamma, \mathrm{Ad}\,\rho)/B^1(\Gamma, \mathrm{Ad}\,\rho),$$

where $B^1(\Gamma, \mathrm{Ad}\,\rho)$ denotes the subspace of inner crossed morphisms, namely crossed morphisms $d\colon \Gamma \to \mathfrak{sl}_2(\mathbb{C})$ for which there exists $a \in \mathfrak{sl}_2(\mathbb{C})$ so that

$$d(\gamma) = a - \mathrm{Ad}_{\rho(\gamma)}(a), \qquad \forall \gamma \in \Gamma.$$

Theorem 17. *If ρ is simple, then the Zariski tangent space to $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ at χ_ρ is naturally isomorphic to $H^1(\Gamma, \mathrm{Ad}\,\rho)$.*

This is [19, Thm 2.13].

Recall from the previous section that the variety of representations $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ is the union of affine varieties. We denote by $\dim_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ the maximal dimension of all components of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ containing ρ.

Lemma 18. *For any representation $\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, we have*

$$\dim_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}} \leq \dim Z^1(\Gamma; \mathrm{Ad}\,\rho),$$

with equality if, and only if, ρ is reduced and ρ is a smooth point of the representation variety.

Proof. The proof is an easy consequence of the following inequalities:

$$\begin{aligned}\dim_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}} &\leq \dim T_\rho \left(R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}\right) \\ &\leq \dim T_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \dim Z^1(\Gamma, \mathrm{Ad}\,\rho),\end{aligned}$$

where T_ρ denotes the Zariski tangent space at ρ of both the variety or the scheme. □

Definition 19. We call $\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ *scheme smooth* if

$$\dim_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}} = \dim Z^1(\Gamma; \mathrm{Ad}\,\rho),$$

Example 20. If Γ is a finite group, then every $\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is scheme smooth, because $H^1(\Gamma, \mathrm{Ad}\,\rho) = 0$ and the following lemma.

Lemma 21. *If $H^1(\Gamma, \mathrm{Ad}\,\rho) = 0$, then ρ is scheme smooth.*

Proof. We use that every $d \in Z^1(\Gamma, \mathrm{Ad}\,\rho)$ is inner, namely there is $a \in \mathfrak{sl}_2(\mathbb{C})$ such that $d(\gamma) = a - \mathrm{Ad}_\rho(\gamma) a$ for every $\gamma \in \Gamma$. It can be checked that d is a vector tangent to the orbit by conjugation, to the path of conjugation by $\exp(ta)$. This yields $\dim_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}} = \dim Z^1(\Gamma, \mathrm{Ad}\,\rho)$. □

In what follows we call a representation $\rho \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ *reduced* and *normal* if the local ring $A(\Gamma)_{\mathfrak{m}_\rho}$ is reduced and normal, respectively. Here $\mathfrak{m}_\rho$ denotes the maximal ideal associated to ρ.

The tangent cone $TC_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is the spectrum of the graded $\mathbb{C}$-algebra associated to the local ring $A(\Gamma)_{\mathfrak{m}_\rho}$. Moreover, the tangent space $T_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is the smallest affine subspace which contains the tangent cone (see [27, III.§3]). From Proposition 11 we obtain:

Lemma 22. *Let $\rho \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be a representation.*

Then ρ is reduced and normal if the tangent cone $TC_\rho R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is reduced and normal, respectively.

5. Computing the character scheme of a finitely presented group

In this section we prove Theorem 1, for that purpose we recall the skein algebra. Let $\mathbb{C}\Gamma$ denote the group ring. The tensor algebra over $\mathbb{C}\Gamma$ is denoted by $\mathrm{T}(\mathbb{C}\Gamma)$.

Definition 23. The *skein algebra* is

$$\mathcal{S}_\Gamma = \mathrm{T}(\mathbb{C}\Gamma) / \left(e - 2,\ \alpha \otimes \beta - \beta \otimes \alpha,\ \alpha \otimes \beta - \alpha\beta - \alpha\beta^{-1} \mid \alpha, \beta \in \Gamma\right).$$

The definition of the skein algebra is motivated by the trace functions identities in Equation (6). Based on work of Brumfiel and Hilden [4], Przytycki and Sikora proved in [33] the following:

Theorem 24 ([33])**.** *There is an isomorphism of $\mathbb{C}$-algebras $\Phi\colon \mathcal{S}_\Gamma \xrightarrow{\cong} B(\Gamma)$ determined by $\Phi([\gamma]) = t_\gamma$.*

As explained by Marché in [21], the theorem also follows from a more general result of Procesi [31].

Recall we have a presentation $\Gamma = \langle \gamma_1, \dots, \gamma_n \mid r_l,\ l \in L\rangle$ as in (1). The natural projection

$$\pi\colon \mathbb{F}_n \to \Gamma$$

induces a surjection $\pi_*\colon \mathcal{S}_{\mathbb{F}_n} \twoheadrightarrow \mathcal{S}_\Gamma$ that factors to an epimorphism

$$\bar\pi\colon \mathcal{S}_{\mathbb{F}_n}/\mathcal{K} \twoheadrightarrow \mathcal{S}_\Gamma,$$

where $\mathcal{K} \subset \mathcal{S}_{\mathbb{F}_n} \cong B(\mathbb{F}_n)$ is the ideal

$$\mathcal{K} = ([\alpha] - [\alpha'] \mid \alpha \in \mathbb{F}_n,\ \pi(\alpha) = \pi(\alpha'))\ .$$

Lemma 25. *The map $\bar\pi\colon \mathcal{S}_{\mathbb{F}_n}/\mathcal{K} \twoheadrightarrow \mathcal{S}_\Gamma$ is an isomorphism of $\mathbb{C}$-algebras.*

Proof. To construct the inverse, start with a set-theoretic section $s\colon \Gamma \to \mathbb{F}_n$ of the projection $\pi\colon \mathbb{F}_n \to \Gamma$, extend it linearly to $\mathrm{T}(\mathbb{C}\Gamma) \to \mathrm{T}(\mathbb{C}\mathbb{F}_n)$ and compose the extension with the projection to $\mathcal{S}_{\mathbb{F}_n}/\mathcal{K}$.

By construction of $\mathcal{K}$, we have for all $\alpha, \beta \in \Gamma$, $s(\alpha\beta) - s(\alpha)s(\beta) \in \mathcal{K}$, therefore s induces a morphism of algebras $\bar s\colon \mathcal{S}_\Gamma \to \mathcal{S}_{\mathbb{F}_n}/\mathcal{K}$, that satisfies $[\alpha] = \bar s(\bar\pi([\alpha]))$ for all $\alpha \in \mathbb{F}_n$ and $[\beta] = \bar\pi(\bar s([\beta]))$ for all $\beta \in \Gamma$. Since classes of the elements in the group span linearly the skein algebra, $\bar s$ and $\bar\pi$ are inverse from one another. □

Corollary 26. *Let $I = (t_\alpha - t_\beta \mid \alpha, \beta \in \mathbb{F}_n,\ \pi(\alpha) = \pi(\beta))$. Then*

$$B(\Gamma) \cong B(\mathbb{F}_n)/I.$$

Proof of Theorem 1. Using Corollary 26, the proof amounts to find the suitable generating set for the ideal I. Given $\alpha, \beta \in \mathbb{F}_n$, we introduce the element

$$\Theta(\alpha, \beta) = t_{\alpha\beta} - t_\alpha,$$

so that

$$I = (\Theta(\alpha, \beta) \mid \alpha, \beta \in \mathbb{F}_n,\ \beta \in \ker(\pi))\,.$$

The proof consists in finding a generating set for the ideal I according to the statement of the theorem. Firstly, as $\ker(\pi)$ is normally generated by the relations r_l, $l \in L$, by using the equalities

$$\begin{aligned}
\Theta(\alpha, \beta_1\beta_2) &= \Theta(\alpha\beta_1, \beta_2) + \Theta(\alpha, \beta_1), && \forall \alpha, \beta_1, \beta_2 \in \mathbb{F}_n,\\
\Theta(\alpha, \gamma\beta\gamma^{-1}) &= \Theta(\gamma^{-1}\alpha\gamma, \beta), && \forall \alpha, \beta, \gamma \in \mathbb{F}_n,
\end{aligned}$$

we get:

$$I = (\Theta(\alpha, r_l) \mid \alpha \in \mathbb{F}_n,\ l \in L)\,.$$

For an element $\alpha \in \mathbb{F}_n$, let $|\alpha|$ denote its word length in the canonical generators. Define

$$I_k = (\Theta(\alpha, r_l) \mid \alpha \in \mathbb{F}_n,\ |\alpha| \le k,\ l \in L)\,. \tag{14}$$

so that $I = \bigcup_k I_k$. We claim that

$$I_{k+1} = I_k, \quad \text{for } k \ge 2. \tag{15}$$

For that purpose we use an equality due to Vogt [14, Lemma 4.1.1]:

$$\begin{aligned} 2t_{abcd} &= t_a t_b t_c t_d - t_c t_d t_{ab} - t_b t_c t_{ad} - t_a t_d t_{bc} - t_a t_b t_{cd} \\ &\quad + t_{ad} t_{bc} - t_{ac} t_{bd} + t_{ab} t_{cd} + t_d t_{abc} + t_c t_{abd} + t_b t_{acd} + t_a t_{bcd} \,. \end{aligned} \tag{16}$$

From this equation, we easily deduce

$$\begin{aligned} 2\Theta(abc, d) &= (t_{abc} - t_a t_{bc} - t_c t_{ab} + t_a t_b t_c)\Theta(1, d) \\ &\quad + (t_{ab} - t_a t_b)\Theta(c, d) - t_{ac}\Theta(b, d) + (t_{bc} - t_b t_c)\Theta(a, d) \\ &\quad + t_a\Theta(bc, d) + t_b\Theta(ac, d) + t_c\Theta(ab, d), \end{aligned} \tag{17}$$

which implies (15).

As $I = I_2$, I is generated by $\Theta(1, r)$, $\Theta(\gamma_i^{\pm 1}, r)$, and $\Theta(\gamma_i^{\pm 1}\gamma_j^{\pm 1}, r)$, so we just need to get rid of the powers -1 and to reduce to $i < j$. This is proved by using the equalities

$$\Theta(\alpha, r) = t_\alpha\Theta(1, r) - \Theta(\alpha^{-1}, r), \tag{18}$$

$$\Theta(\alpha\beta, r) = t_\alpha\Theta(\beta, r) - \Theta(\alpha^{-1}\beta, r), \tag{19}$$

(the first one is obviously a particular case of the second one), which in its turn follow from the trace identity $t_a t_b = t_{ab} + t_{a^{-1}b}$. □

Proof of the Addendum. To simplify notation, assume there is only one relation r and decompose it as $r = u^{-1}v$. We prove the equalities

$$(t_{ar} - t_a \mid a \in \mathbb{F}_n) = (t_{bv} - t_{bu} \mid b \in \mathbb{F}_n) = (t_{bv} - t_{bu} \mid b \in \{1, \gamma_i, \gamma_i\gamma_j\}) \,. \tag{20}$$

The first equality is elementary by writing $a = bu$. The second equality follows from

$$t_{bv} - t_{bu} = \Theta(b, v) - \Theta(b, u)$$

by applying the same arguments as in the proof of Theorem 1, in particular equalities (17), (18), and (19), allow to reduce the word length of b. Then the addendum follows from (20). □

Remark 27. The ideal I_1 defined in (14) is the ideal considered by González-Acuña and Montesinos-Amilibia [14]. They proved that the coordinate ring of the representation variety (not the scheme) is isomorphic to

$$\big(B(\mathbb{F}_n)/I_1\big)_{red} \cong B(\mathbb{F}_n)/\operatorname{rad}(I_1) \,.$$

In Section 8 we see that the ideal I_1 may be non-radical even if the coordinate ring $B(\Gamma)$ is reduced (whether the ideal I_1 is a radical ideal or not may depend on the presentation of Γ). Thus, when we compute a scheme, I_1 is not sufficient and we need to consider I_2. The following example illustrates that the ideals I_2 and I_1 may be different, even when $B(\Gamma)$ is reduced.

Example 28. Let $\Gamma = \langle a, b \mid r\rangle$, $r = baba^{-1}b^{-1}a^{-1}$, be the trefoil group. The algebra $B(\Gamma)$ is a quotient of $B(\mathbb{F}_2) \cong \mathbb{C}[t_a, t_b, t_{ab}]$. As Γ is a one-relator group, we obtain:

$$I_1 = (t_r - 2,\ t_{ar} - t_a,\ t_{br} - t_b) \quad \text{and} \quad I_2 = (t_r - 2,\ t_{ar} - t_a,\ t_{br} - t_b,\ t_{abr} - t_{ab}) \,.$$

Let's compute I_1. By using the trace relations we obtain $t_{ar} = t_b$ and hence $t_{ar} - t_a = t_b - t_a$. Moreover, we have

$$t_{br} = t_b t_r - t_{rb^{-1}} \quad \text{and} \quad t_{rb^{-1}} = t_a \,.$$

Therefore,

$$t_{br} - t_b = t_b t_r - t_a - t_b = t_b(t_r - 2) + (t_b - t_a),$$

and we obtain $I_1 = (t_r - 2, t_a - t_b)$. The trace relations (6) give:

$$\begin{aligned} t_{baba^{-1}b^{-1}a^{-1}} &= t_{aba}t_{bab} - t_{(ba)^3} \\ t_{aba} &= t_{ab}t_a - t_b \\ t_{bab} &= t_{ab}t_b - t_a \\ t_{(ba)^3} &= t_{ab}^3 - 3\,t_{ab}\,, \end{aligned}$$

and therefore

$$t_r - 2 = (t_{ab}t_a - t_b)(t_{ab}t_b - t_a) - t_{ab}^3 + 3\,t_{ab}\,.$$

We obtain the following primary decomposition of I_1:

$$I_1 = (t_a - t_b, t_b^2 - t_{ab} - 2) \cap (t_a - t_b, (t_{ab} - 1)^2)$$

and I_1 is clearly not radical (see also [16]). Notice that the first ideal corresponds to the characters of non-simple representations $t_{ab^{-1}} - 2 = t_a t_b - t_{ab} - 2$.

On the other hand, we have that $ab\,r = (ab)baba^{-1}(ab)^{-1}$, and hence $t_{ab} = t_{baba^{-1}}$ modulo I_2. Now,

$$\begin{aligned} t_{ab\,r} - t_{ab} &= t_{baba^{-1}} - t_{ab} \\ &= t_{bab}t_a - t_{(ba)^2} - t_{ab} \\ &= (t_{ab}t_b - t_a)t_a - (t_{ab}^2 - 2) - t_{ab} \end{aligned}$$

We obtain the following primary decomposition of I_2:

$$I_2 = (t_a - t_b, t_b^2 - t_{ab} - 2) \cap (t_a - t_b, t_{ab} - 1)$$

and I_2 is radical (see also the notebook [16]).

Remark 29. We observe also that $\mathbb{C}[t_{\gamma_1}, t_{\gamma_2}, t_{\gamma_1\gamma_2}]/I_1$ depends on the presentation and not only on the group. An explicit example is given by an other presentation of the trefoil group (the group of Example 28): the presentation $\langle x, y \mid x^2y^{-3}\rangle$ produces a radical ideal I_1 (see [16] for the computations).

Remark 30. The ideal I_3 is the ideal considered by Miura and Suzuki in [25]. They proved that the skein module of a group with three generators and two relations (and hence the coordinate ring of the representation scheme) is isomorphic to $B(\mathbb{F}_3)/I_3$. As pointed out in the proof of Theorem 1, we have in general that $I_2 = I_3$, see Equation (15), and therefore it is sufficient to consider less generators of the ideal.

Remark 31. Our result was mainly motivated by Corollary 10.1.7 in [11]. This corollary states that any product of matrices (and their inverses) constructed from a given set $\{A_1, \ldots, A_n\} \subset \mathrm{SL}_2(\mathbb{C})$ can be written as a linear combination of id, A_i, $1 \le i \le n$, and A_iA_j, $1 \le i < j \le n$.

6. The scheme of non-simple representations

In concordance with [19] we use the term *simple* representation instead of the more common term *irreducible* representation in order to avoid confusion with irreducible algebraic varieties or schemes, as well as reduced or non-reduced schemes. Thus the term *non-simple* in the title of this section would be *reducible* in other papers.

Definition 32 ([19])**.** Let $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be a representation. We say that ρ is *simple* if the image $\rho(\Gamma)$ spans $M_2(\mathbb{C})$ as a $\mathbb{C}$-vector space.

Lemma 33. *Let $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be a representation. Then ρ is simple if and only if there exist γ_1, γ_2, γ_3 in Γ such that*

$$\mathrm{tr}\left(\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3)\right) - \mathrm{tr}\left(\rho(\gamma_1)\rho(\gamma_3)\rho(\gamma_2)\right) \neq 0\,.$$

Proof. Let $\mathrm{id} \in \mathrm{SL}_2(\mathbb{C})$ denote the identity matrix. For a matrix $M \in M_2(\mathbb{C})$ we put $M_0 = M - \frac{\mathrm{tr}(M)}{2}\mathrm{id}$. Then id, A, B and C generate $M_2(\mathbb{C})$ as a $\mathbb{C}$-vector space if and only if (A_0, B_0, C_0) is a $\mathbb{C}$-basis for the subspace $\mathfrak{sl}_2(\mathbb{C})$ of trace free matrices of $M_2(\mathbb{C})$. Now observe that the map $\mathfrak{sl}_2(\mathbb{C})^3 \to \mathbb{C}$ given by

$$(X_0, Y_0, Z_0) \mapsto \mathrm{tr}(X_0 Y_0 Z_0) \tag{21}$$

is a determinant map, e.g. multilinear and alternating (see Section V in [10]).

A direct calculation gives

$$\begin{aligned}\mathrm{tr}(ABC) = \mathrm{tr}(A_0B_0C_0) - \frac{1}{2}\big(&\mathrm{tr}(A)\,\mathrm{tr}(BC)\\ &+ \mathrm{tr}(B)\,\mathrm{tr}(AC) + \mathrm{tr}(C)\,\mathrm{tr}(AB) - \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(C)\big),\end{aligned} \tag{22}$$

and hence

$$\mathrm{tr}(ABC) - \mathrm{tr}(ACB) = 2\ \mathrm{tr}(A_0B_0C_0)\,.$$

Thus $\mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$ if and only if A_0, B_0, C_0 are linearly independent (equivalently, they span $\mathfrak{sl}_2(\mathbb{C})$). As $\mathrm{id} \in \rho(\Gamma)$, the lemma follows. □

Remark 34. Notice that Equation (22) and the fact that (21) is a determinant map imply the equality $t_{abc} + t_{acb} = p$ in (12).

Motivated by Lemma 33 we define:

Definition 35. Let Γ be a group, and $B(\Gamma)$ the universal algebra of $\mathrm{SL}_2(\mathbb{C})$-characters. The ideal $J_{ns} \subset B(\Gamma)$ is defined by

$$J_{ns} = J_{ns}(\Gamma) := \left(t_{abc} - t_{acb} \mid a, b, c \in \Gamma\right),$$

and the quotient

$$B_{ns}(\Gamma) = B(\Gamma)/J_{ns}$$

is called the universal algebra of non-simple $\mathrm{SL}_2(\mathbb{C})$-characters.

Remark 36. We have

$$J_{ns} \subset B(\Gamma) = A(\Gamma)^{\mathrm{SL}_2(\mathbb{C})} \subset A(\Gamma)\,,$$

and Lemmata 3.4.1, 3.4.2 and Remark 3.4.3 in [28] give that

$$J_{ns}A(\Gamma) \bigcap B(\Gamma) = J_{ns}$$

and

$$\left(A(\Gamma)/J_{ns}A(\Gamma)\right)^{\mathrm{SL}_2(\mathbb{C})} \cong A(\Gamma)^{\mathrm{SL}_2(\mathbb{C})}/J_{ns} = B_{ns}(\Gamma)\,.$$

For an abelian group Γ_0 we have $J_{ns}(\Gamma_0) = (0)$ since $abc = acb$ in Γ_0, and hence $B_{ns}(\Gamma_0) = B(\Gamma_0)$.

Proposition 37. *Let Γ be a finitely generated group. The abelianization morphism $\Gamma \to \Gamma_{ab}$ induces an isomorphism of algebras $B(\Gamma_{ab}) \xrightarrow{\cong} B_{ns}(\Gamma)$.*

Proof. By writing $c = b^{-1}a^{-1}d$, we have an equality of sets

$$\{t_{abc} - t_{bac} \mid a, b, c \in \Gamma\} = \{t_d - t_{[b,a]d} \mid a, b, d \in \Gamma\}$$

So both sets generate the same ideal in $B(\Gamma)$. The left hand side set spans J_{ns}, the right hand side set spans the ideal of the abelianization, by Theorem 1. □

7. Abelian groups

Throughout this section Γ denotes a finitely generated abelian group. Przytycki and Sikora proved that the skein algebra of an abelian group is reduced [32, 33], hence its character scheme is also reduced. In [36] Sikora described the variety of characters when Γ is torsion free. Here we discuss the case with torsion, and we show in particular that $B(\Gamma)$ is reduced with completely different methods.

Let β denote the first Betti number of Γ and T its torsion subgroup, so that we have a short exact sequence

$$1 \to T \to \Gamma \to \mathbb{Z}^\beta \to 1$$

with T finite. The scheme of characters of T is a variety with finite cardinality. For each $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$, set

$$X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi = \mathrm{res}^{-1}(\chi)$$

where $\mathrm{res}\colon X(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to X(T, \mathrm{SL}_2(\mathbb{C}))$ is the map induced by restriction.

Theorem 38. *For Γ an abelian group as above, $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is reduced and*

$$X(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \bigcup_{\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))} X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$$

is its decomposition into irreducible components. Furthermore

$$X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi \cong \begin{cases} (\mathbb{C}^*)^\beta & \text{if } \chi \text{ is not central} \\ (\mathbb{C}^*)^\beta / \sim & \text{if } \chi \text{ is central} \end{cases}$$

where $(\lambda_1, \dots, \lambda_\beta) \sim (\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_\beta})$, for $\lambda_1, \dots, \lambda_\beta \in \mathbb{C}^*$.

When $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$ is central, a point in $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ is the character of a representation that maps the generators of $\mathbb{Z}^\beta$ to diagonal matrices with eigenvalues $\lambda_i^{\pm 1}$. Thus the action of the involution is the action of the Weyl group. In particular:

Corollary 39. *The singular locus of $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is the set of central characters of Γ.*

To prepare the proof of Theorem 38, we need a lemma on the Zariski tangent space.

Lemma 40. *If $\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is non-central, then $\dim Z^1(\Gamma, \mathrm{Ad}\rho) \leq \beta + 2$.*

Proof of Lemma 40. We may assume $\beta \geq 1$, as the finite case has been considered in Example 20. Chose elements $\gamma_1, \dots, \gamma_\beta \in \Gamma$ so that they project to a generating set of $\Gamma/T \cong \mathbb{Z}^\beta$. This choice yields a splitting $\Gamma \cong \mathbb{Z}^\beta \times T$. To bound the dimension of the space of 1-cocycles, consider the monomorphism of vector spaces:

$$\begin{array}{rcl} Z^1(\Gamma, \mathrm{Ad}\rho) & \to & \mathfrak{sl}_2(\mathbb{C}) \times \overset{(\beta)}{\cdots} \times \mathfrak{sl}_2(\mathbb{C}) \times Z^1(T, \mathrm{Ad}\rho|_T) \\ d & \mapsto & (d(\gamma_1), \dots, d(\gamma_\beta), d|_T) \end{array} \tag{23}$$

where $\rho|_T$ and $d|_T \in Z^1(T, \mathrm{Ad}\rho|_T)$ denote the respective restrictions to T of ρ and d. Since T is finite, $H^1(T, \mathrm{Ad}\rho|_T) = 0$ and therefore

$$\dim Z^1(T, \mathrm{Ad}\rho|_T) = B^1(T, \mathrm{Ad}\rho|_T) = \begin{cases} 0 & \text{if } \rho|_T \text{ is central} \\ 2 & \text{otherwise} \end{cases} \tag{24}$$

because $\dim B^1(T, \mathrm{Ad}\rho|_T) = \dim \mathfrak{sl}_2(\mathbb{C}) - \dim \mathfrak{sl}_2(\mathbb{C})^{\mathrm{Ad}\rho(T)}$.

Next we distinguish cases, according to whether the restriction $\rho|_T$ is central or not.

When the restriction $\rho|_T$ is central, we may assume that $\rho(\gamma_1)$ is non-central. Then for every $d \in Z^1(\Gamma, \mathrm{Ad}\rho)$, from $\gamma_1\gamma_i = \gamma_i\gamma_1$ we get (following the rules of crossed homomorphism or Fox calculus):

$$(\mathrm{Ad}_{\rho(\gamma_1)} - \mathrm{id}) d(\gamma_i) = (\mathrm{Ad}_{\rho(\gamma_i)} - \mathrm{id}) d(\gamma_1) \tag{25}$$

As $\rho(\gamma_1)$ is non-central, $\operatorname{rank}(\mathrm{Ad}_{\rho(\gamma_1)} - \mathrm{id}) = 2$ (because the kernel of $(\mathrm{Ad}_{\rho(\gamma_1)} - \mathrm{id})$ is the tangent line to the 1-parameter group containing $\rho(\gamma_1)$). Hence, from (25) applied to $i = 2, \ldots, \beta$ and using (24), the image of (23) is contained in the kernel of a linear map $\mathfrak{sl}_2(\mathbb{C})^\beta \to \mathfrak{sl}_2(\mathbb{C})^{\beta-1}$ of rank $2(\beta-1)$, and we get the bound $\dim Z^1(\Gamma, \mathrm{Ad}\rho) \leq 3\beta - 2(\beta-1)$.

When the restriction $\rho|_T$ is non-central chose $\gamma_0 \in T$ such that $\rho(\gamma_0)$ is non-central. Then apply the same argument as above to the equalities $\gamma_0\gamma_i = \gamma_i\gamma_0$, for $i = 1, \ldots, \beta$. Now the image of (23) is contained in the kernel of a linear map $\mathfrak{sl}_2(\mathbb{C})^\beta \oplus \mathbb{C}^2 \to \mathfrak{sl}_2(\mathbb{C})^\beta$ of rank 2β, hence $\dim Z^1(\Gamma, \mathrm{Ad}\rho) \leq 3\beta + 2 - 2\beta$. □

For a morphism $\tau: T \to \mathbb{Q}/\mathbb{Z}$ and a splitting $\Gamma \cong \mathbb{Z}^\beta \times T$, define the subset of representations $V_\tau \subset (R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ as:

$$V_\tau = \{\exp(\theta a)\exp(2\pi\tau a) \mid \theta \in \hom(\Gamma, \mathbb{C}),\ a \in \mathcal{Q}\}, \tag{26}$$

where $\mathcal{Q}$ denotes the quadric

$$\mathcal{Q} = \{a \in \mathfrak{sl}_2(\mathbb{C}) \mid \det(a) = 1\} = \mathfrak{sl}_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{C}).$$

To see that the set V_τ consists of representations, notice that $\exp(2\pi a) = \mathrm{id}$ for every $a \in \mathcal{Q}$. The splitting $\Gamma \cong \mathbb{Z}^\beta \times T$ is used to extend $\tau: T \to \mathbb{Q}/\mathbb{Z}$ to the whole Γ, but the set V_τ does not depend on this extension or splitting.

We shall use V_τ to describe the preimage of $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ by the natural projection $R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, that we denote by

$$R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi = \{\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \mid \chi_{\rho|_T} = \chi\}.$$

Consider $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$ non-central. Every representation of T with character χ is of the form $\exp(2\pi\tau a)$, for some $a \in \mathcal{Q}$ and some $\tau: T \to \mathbb{Q}/\mathbb{Z}$. Notice that for $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$ non-central, we can chose either τ or $-\tau$, but this is the only ambiguity in the choice of τ. In fact $V_\tau = V_{\tau'}$ if and only if $\tau = \pm\tau'$.

Lemma 41. *For $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$ non-central, and τ as above, $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ is reduced and*

$$R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi = V_\tau \cong (\mathbb{C}^*)^\beta \times \mathcal{Q}.$$

In particular $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ is irreducible and smooth.

Proof. As all abelian subgroups of $\mathrm{SL}_2(\mathbb{C})$ are contained in a one-parameter group, we have equality of sets $(R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi)_{\mathrm{red}} = V_\tau$. Using Lemmas 18 and 40, since $\dim V_\tau = \beta + 2$ every point in $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is scheme smooth (hence reduced), and $(R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi)_{\mathrm{red}} = R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$.

Finally, notice that the natural map $(\mathbb{C}^*)^\beta \times \mathcal{Q} \to V_\tau$ is a bijection because $\exp(2\pi\tau a)$ is non-central. By construction this map is regular, and by smoothness and Zariski's main theorem [26, 3.20] it is biregular. □

When χ is central, $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ contains representations with parabolic elements, hence we need other tools to study it.

Definition 42. The *cone of group homomorphisms of rank at most one* from Γ to $\mathfrak{sl}_2(\mathbb{C})$ is the determinantal variety:

$$\hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C})) \cong \hom_1(\Gamma/T, \mathfrak{sl}_2(\mathbb{C})) \cong \hom_1(\mathbb{C}^\beta, \mathfrak{sl}_2(\mathbb{C})).$$

Namely, $\hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ consists of all group homomorphisms in $\hom(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ whose image is contained in a linear space of dimension at most 1. This is the $\mathbb{C}$-cone on the Segre variety $\mathbb{P}^{\beta-1} \times \mathbb{P}^2$.

Now consider a central character $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$. We chose $\rho_0 \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ a central representation of Γ whose restriction to T has character χ.

Lemma 43. *For $\chi \in X(T, \mathrm{SL}_2(\mathbb{C}))$ central, and given $\rho_0 \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ also central,*

$$(R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi)_{red} = \{\exp(\theta)\rho_0 \mid \theta \in \hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))\}.$$

Furthermore $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ is scheme-smooth at non-central representations.

Proof. As in Lemma 41, equality $(R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi)_{\mathrm{red}} = \{\exp(\theta)\rho \mid \theta \in \hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ as sets is a consequence of the fact that every abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$ is contained in a one-parameter group, which in its turn is the image by the exponential of a line in $\mathfrak{sl}_2(\mathbb{C})$. Scheme smoothness at non-central representations follows again from Lemmas 18 and 40 and the dimension count. □

Lemma 44. *The cone* $\hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ *is isomorphic to the quadratic cone to* $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ *at any central representation and it is the whole tangent cone.*

Proof. Let ρ be a central representation. In particular $\mathrm{Ad}\,\rho$ is trivial and

$$Z^1(\Gamma, \mathrm{Ad}\rho) = \hom(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$$

(namely, group homomorphisms with no restriction on the dimension of the image). To compute the quadratic cone, notice that, by Baker-Campbell-Hausdorff formula, for any $\theta \in \hom(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$:

$$e^{t\theta(\gamma_1)} e^{t\theta(\gamma_2)} e^{-t\theta(\gamma_1)} e^{-t\theta(\gamma_2)} = e^{t^2[\theta(\gamma_1), \theta(\gamma_2)] + O(t^3)}, \qquad \forall \gamma_1, \gamma_2 \in \Gamma.$$

This yields that $\hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ is the quadratic cone at ρ. (This can be viewed equivalently with the obstruction theory of Goldman.) Not only $\hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ is the quadratic cone, by construction it is also the whole tangent cone, because all higher order terms in Baker-Campbell-Hausdorff formula vanish when $[\theta(\gamma_1), \theta(\gamma_2)] = 0$. □

Related to Lemma 44, notice that a theorem of Goldman and Millson [13, Thm. 9.3] guarantees that the singularities of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ are at most quadratic, as Γ is virtually a Bieberbach group.

Since $\hom_1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ is a determinantal scheme it is reduced and normal [5], and by Lemma 22, we get:

Corollary 45. *A central representation is a reduced and normal point of the scheme $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$.*

From the previous results in this section we deduce:

Corollary 46. *The scheme $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is reduced and normal. Its singular locus is precisely the set of central representations, that have quadratic singularities modeled in the $\mathbb{C}$-cone on $\mathbb{P}^{\beta-1} \times \mathbb{P}^2$.*

Proof of Theorem 38. As $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is reduced and normal, and its irreducible components are $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$, $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is also reduced and normal, and is components are the $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$.

Fix $a \in \mathcal{Q}$ and consider

$$\begin{array}{rcl} \tilde{\varphi}_a \colon \hom(\Gamma, \mathbb{C}) \cong \mathbb{C}^\beta & \to & R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi \\ \theta & \mapsto & \exp(\theta a)\rho_0 \end{array}$$

where $\rho_0 \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ is a central representation when χ is central as in Lemma 43, or $\rho_0 = \exp(2\pi\tau a)$ for some nontrivial $\tau \colon T \to \mathbb{Z}/\mathbb{Q}$ as in (26) when χ is not central. The map $\tilde{\varphi}_a$ factors through the exponential map on each factor $\mathbb{C}$ to

$$\varphi_a \colon (\mathbb{C}^*)^\beta \to R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi.$$

We compose it with the projection $\pi: R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi \to X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ and we claim that the composition

$$\pi \circ \varphi_a : (\mathbb{C}^*)^\beta \to X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi \tag{27}$$

is a surjection. When χ is not central, the claim follows from Lemma 41 and the description of V_τ in Equation (26), as $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathcal{Q}$ (hence every orbit by conjugation in $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ meets the image of φ_a). When χ is central, we use the description of Lemma 43 and in this case we similarly prove that every orbit by conjugation in $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ of a semi-simple representation meets the image of φ_a. We conclude the surjectivity by recalling that every character is the character of a semi-simple representation.

Next we discuss the inverse images of the surjection $\pi \circ \varphi_a$ in (27). We use two elementary properties:

- If two semisimple representations have the same character, then they are conjugate. Notice that the image of φ_a consists of only semisimple representations.
- We write a diagonal representation of Γ as $\mathrm{diag}(\theta, \theta^{-1})$ for $\theta: \Gamma \to \mathbb{C}^*$ a homomorphism. If $\mathrm{diag}(\theta_1, \theta_2^{-1})$ is conjugate to $\mathrm{diag}(\theta_2, \theta_2^{-1})$, then $\theta_1 = \theta_2^{\pm 1}$.

To apply these remarks to our situation, we may assume that $a \in \mathcal{Q}$ is diagonal. Now, if τ is central, then it follows that $\pi \circ \varphi_a$ in (27) factors to the quotient as in the statement of the theorem

$$(\mathbb{C}^*)^\beta / \sim \to X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi \tag{28}$$

which is a bijection. This does not hold anymore when τ is non central, as the restriction to T of a representation in the image of φ_a is fixed, it is a representation $\rho_0 = \exp(2\pi\tau a)$ with $\tau: T \to \mathbb{Z}/\mathbb{Q}$ nontrivial. We conclude that when τ is non central (27) is already a bijection.

The maps (27) and (28) are regular bijections, and by Zariski's main theorem [26, 3.20] they are both biregular, since $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))_\chi$ is normal. □

Next we describe the algebra $B(\mathbb{Z}^\beta) = \mathbb{C}[X(\mathbb{Z}^\beta, \mathrm{SL}_2(\mathbb{C}))]$ in terms of traces. Start with the presentation

$$\mathbb{Z}^\beta = \langle a_1, \dots, a_\beta \mid [a_i, a_j] = 1 \rangle$$

Proposition 47. *With the previous presentation,*

$$B(\mathbb{Z}^\beta) = \mathbb{C}[t_{a_i}, t_{a_i a_j}] / (t_{a_i a_j} - t_{a_j a_i}, f_{a_i,a_j}, g_{a_i,a_j,a_k}, h_{a_i,a_j,a_k,a_l})_{i,j,k,l \in \{1,\dots,\beta\}}$$

where the subindex are different on each $f_{a_i,a_j}, g_{a_i,a_j,a_k}, h_{a_i,a_j,a_k,a_l}$*, and*

$$f_{a,b} = t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 4, \tag{29}$$

$$g_{a,b,c} = t_a(t_a t_{bc} + t_b t_{ac} + t_c t_{ab} - t_a t_b t_c) - 2t_{ab} t_{ac} - 4t_{bc} + 2t_b t_c, \tag{30}$$

$$h_{a,b,c,d} = (2t_{ab} - t_a t_b)(2t_{cd} - t_c t_d) - (2t_{ac} - t_a t_c)(2t_{bd} - t_b t_d). \tag{31}$$

Furthermore, for computing trace functions of elements of $\mathbb{Z}^\beta$*, we use the standard trace identities* (6) *and* (16) *and add*

$$2t_{abc} = t_a t_{bc} + t_b t_{ac} + t_c t_{ab} - t_a t_b t_c. \tag{32}$$

Remark 48. (a) Equation (32) does not need to be included in the presentation of the algebra $B(\mathbb{Z}^\beta)$, but it is needed to describe the trace functions of group elements. It can be read as $t_{abc} = \frac{1}{2} p$, where p is as in (9). Notice that from the relations (29), (30) and (31) it follows that $p^2 - 4q = 0$, so the polynomial F in (8) reads as $F = (t_{abc} - \frac{1}{2} p)^2$. Vanishing of F is a standard trace identity, but we include (32) to get rid of the square.

(b) Using (32), (30) reads as

$$g_{a,b,c} = 2(t_a t_{abc} - t_{ab} t_{ac} - 2t_{bc} + t_b t_c).$$

Notice also that, for F as in (8):

$$\frac{\partial F}{\partial t_{bc}} = -t_a t_{abc} + t_{ab} t_{ac} + 2t_{bc} - t_b t_c = -\tfrac{1}{2} g_{a,b,c}.$$

(c) If we allow that some of the subindex are equal, then we can reduce to a single family of equations, because:

$$2f_{a,b} = g_{a,b,b} \qquad \text{and} \qquad 2g_{a,b,c} = h_{a,a,b,c}.$$

Proof of Proposition 47. By Theorem 38,

$$B(\mathbb{Z}^\beta) = (\mathbb{C}[\lambda_1, \dots, \lambda_\beta, \mu_1, \dots, \mu_\beta]/(\lambda_i\mu_i - 1)_{i=1,\dots,\beta})^\sigma$$

where $\sigma(\lambda_1, \dots, \lambda_\beta, \mu_1, \dots, \mu_\beta) = (\mu_1, \dots, \mu_\beta, \lambda_1, \dots, \lambda_\beta)$. A point in $X(\mathbb{Z}^\beta, \mathrm{SL}_2(\mathbb{C}))$ with coordinates $(\lambda_1, \dots, \lambda_\beta, \mu_1, \dots, \mu_\beta)$ is the character of the representation that maps each generator $a_i \in \mathbb{Z}^\beta$ to the diagonal matrix $\mathrm{diag}(\lambda_i, \frac{1}{\lambda_i})$. To compute the invariant subalgebra, we change coordinates

$$t_{a_i} = \lambda_i + \mu_i = \lambda_i + \tfrac{1}{\lambda_i}, \qquad x_i = \lambda_i - \mu_i = \lambda_i - \tfrac{1}{\lambda_i},$$

so

$$B(\mathbb{Z}^\beta) = \left(\mathbb{C}[t_{a_1}, \dots, t_{a_\beta}, x_1, \dots, x_\beta]/(t_{a_i}^2 - x_i^2 - 4)_{i=1,\dots,\beta}\right)^\sigma$$

and $\sigma(t_{a_1}, \dots, t_{a_\beta}, x_1, \dots, x_\beta) = (t_{a_1}, \dots, t_{a_\beta}, -x_1, \dots, -x_\beta)$. The algebra of invariants by σ is generated by the t_{a_i} and the quadratic monomials

$$z_{ij} = x_i x_j, \qquad i, j = 1, \dots, \beta.$$

Thus $B(\mathbb{Z}^\beta) = \mathbb{C}[t_{a_i}, z_{ij}]/I$ where I is the ideal generated by:

$$t_{a_i}^2 - z_{ii} - 4, \qquad z_{ij} - z_{ji}, \qquad z_{ij} z_{kl} - z_{il} z_{kj},$$

for $i, j, k, l \in \{1, \dots, \beta\}$, possibly equal. The z_{ij} can be written in terms of traces as follows:

$$z_{ij} = x_i x_j = \lambda_i \lambda_j + \frac{1}{\lambda_i \lambda_j} - \frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} = t_{a_i a_j} - t_{a_i a_j^{-1}} = 2t_{a_i a_j} - t_{a_i} t_{a_j}$$

and the presentation of $B(\mathbb{Z}^\beta)$ as a quotient of $\mathbb{C}[t_{a_i}, t_{a_i a_j}]$ follows by writing the z_{ij} in terms of traces.

Finally, notice that the unique relation we need to add to compute the trace of any element from these variables is (32), see the discussion in Remark 48, because the trace of elements of length larger that three is deduced from Vogt relation (16). □

We show a couple of examples that illustrate Theorem 38 when Γ has torsion.

Example 49. We compute the scheme of characters of

$$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} = \langle a, b \mid [a, b] = b^4 = 1 \rangle.$$

As $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is a quotient of $\mathbb{Z}^2$, by Proposition 47 the coordinates are (t_a, t_b, t_{ab}) and they satisfy

$$t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 4 = 0 \tag{33}$$

Furthermore, as $b^4 = 1$, we get

$$t_b(t_b + 2)(t_b - 2) = 0.$$

Thus there are three components:

- When $t_b = 0$, the character restricted to $\mathbb{Z}/4\mathbb{Z}$ is non-central. By replacing $t_b = 0$ in (33) we get $t_a^2 + t_{ab}^2 - 4 = 0$, which can be rewritten as
$$(t_a + it_{ab})(t_a - it_{ab}) = 4,$$
which is isomorphic to $\mathbb{C}^*$.
- When $t_b = \pm 2$, (33) becomes $(t_a \mp t_{ab})^2 = 0$, and since we know it is reduced, we can get rid of the square and so $t_a = \pm t_{ab}$. Namely, two lines, because $\mathbb{C}^*/\sim \cong \mathbb{C}$.

Example 50. Next consider
$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} = \langle a, b, c \mid [a,b] = [a,c] = [b,c] = c^4 = 1 \rangle.$$
The same considerations as in Example 49 yield that the coordinates for the scheme of characters are $(t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc})$ and it has three components:

- One component has equations:
$$\left.\begin{aligned} t_c &= 0 \\ t_a^2 + t_{ac}^2 - 4 &= 0 \\ t_b^2 + t_{bc}^2 - 4 &= 0 \\ -2t_{ab} + t_a t_b - t_{ac} t_{bc} &= 0 \end{aligned}\right\}$$
So it is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.
- The remaining two components have equations
$$\left.\begin{aligned} t_c \pm 2 &= 0 \\ t_{ac} \pm t_a &= 0 \\ t_{bc} \pm t_b &= 0 \\ t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 4 &= 0 \end{aligned}\right\}$$
and are isomorphic to $X(\mathbb{Z}^2, \mathrm{SL}_2(\mathbb{C})) \cong (\mathbb{C}^* \times \mathbb{C}^*)/\sim$.

8. Examples of non-reduced schemes of characters

Kapovich and Millson have proved in [18] that there are no restrictions on the local geometry of $\mathrm{SL}_2(\mathbb{C})$-character schemes of 3–manifold groups, in particular there are non-reduced character schemes of those groups. In this section we give explicit examples of non-reduced $\mathrm{SL}_2(\mathbb{C})$-character schemes of 3–manifold and orbifold groups.

Example 51. Let $S^3(K,3)$ denote the three-dimensional orbifold with underlying space S^3, branching locus the figure-eight knot K and branching index 3. It is a Euclidean orbifold [39, Ch. 13]. The Euclidean structure is relevant, because the translational part of the Euclidean holonomy is a non-integrable infinitesimal deformation of its rotational part. This is similar to the examples of Lubotzky and Magid in [19, pp. 40–43], for representations of Euclidean 2-orbifolds in $\mathrm{GL}_2(\mathbb{C})$ that are proved to be non-reduced. Furthermore in [13, §9.3] Goldman and Millson prove that those are double points, as the singularities are at most quadratic. We address triple points in Example 52 below, using Nil instead of Euclidean orbifolds.

The fundamental group of the figure eight knot exterior has a presentation:
$$\pi_1(S^3 \setminus K) = \langle a, b \mid ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b \rangle$$
where a and b are represented by meridian loops. Thus a presentation of the orbifold fundamental group can be obtained by adding the relation $a^3 = 1$:
$$\Gamma = \pi_1^{\mathrm{orb}}(S^3(K,3)) = \langle a, b \mid ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b,\ a^3 = 1 \rangle.$$

The algebra $B(\Gamma)$ is a quotient of $B(\mathbb{F}_2) \cong \mathbb{C}[t_a, t_b, t_{ab}]$ by and ideal I. A computer supported calculation yields [16]:

$$I = (t_a - 2, t_b - 2, t_{ab} - 2) \cap (t_a + 1, t_b + 1, t_{ab} + 1) \cap (t_a + 1, t_b + 1, (t_{ab} - 1)^2).$$

So $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ consists of three points, one of them non-reduced (double point). The point $t_a = t_b = t_{ab} = 2$ is the trivial character, $t_a = t_b = t_{ab} = -1$ is the non-trivial abelian character, and $(t_a, t_b, t_{ab}) = (-1, -1, 1)$ is the *double point* that corresponds to the character of a simple representation in $\mathrm{SU}(2)$. This representation is a lift of the rotational part of the holonomy of the Euclidean structure in $\mathrm{PSU}(2) \cong \mathrm{SO}(3)$.

We can also understand this double point as the result of a tangency. The variety of characters of the figure eight knot exterior is the plane curve given by

$$(x^2y - 2x^2 - y^2 + y + 1)(y - x^2 + 2) = 0,$$

where $x = t_a = t_b$ and $y = t_{ab}$. The group Γ is obtained from the figure eight knot by adding the relation $a^3 = 1$. A representation of an element of order three is either trivial (and has trace 2) or has trace -1. Therefore, the case where the image of a is non-trivial corresponds to $x = -1$. The line $x = -1$ intersects $y - x^2 - 2 = 0$ transversely (at the abelian representation) and $x^2y - 2x^2 - y^2 + y + 1 = 0$ tangentially, giving the double point.

The fact that the intersection between $x^2y - 2x^2 - y^2 + y + 1 = 0$ and $x = -1$ is not transverse corresponds to the fact that the trace of the meridian is not a local parameter at a Euclidean degeneration of hyperbolic cone manifolds [29].

Example 52. Let $S^3(\mathrm{Wh}, (m, n))$ be the orbifold with underlying space S^3, branching locus the Whitehead link Wh, and branching indexes m and n. The orbifold $S^3(\mathrm{Wh}, (4, 2))$ has Nil geometry [38, p. 112]. The isometry group of Nil surjects onto $\mathrm{Isom}(\mathbb{R}^2)$, which in its turn surjects onto $\mathrm{O}(2) \subset \mathrm{SO}(3)$, but the rotational part of the Nil holonomy of $S^3(\mathrm{Wh}, (4, 2))$ in $\mathrm{O}(2) \subset \mathrm{SO}(3) \cong \mathrm{PSU}(2)$ does not lift to $\mathrm{SU}(2)$, because of the elements of order two. Instead, it lifts to a representation of $S^3(\mathrm{Wh}, (8, 4))$ in $\mathrm{SU}(2)$, and this is the orbifold we consider.

As in the previous example, we notice that the trace of the meridian is not a local parameter at a Nil degeneration [30], it is in fact a singularity of order three. Hence this representation will be a non reduced point of the character scheme of $S^3(\mathrm{Wh}, (8, 4))$.

For the explicit computation, start with the fundamental group of the Whitehead link exterior

$$\pi_1(S^3 \setminus \mathrm{Wh}) = \langle a, b \mid aba^{-1}b^{-1}a^{-1}bab = baba^{-1}b^{-1}a^{-1}ba \rangle.$$

Its scheme of characters is the hypersurface of $\mathbb{C}^3$ with equation

$$(z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy)(x^2 + y^2 + z^2 - xyz - 4) = 0,$$

where $x = t_a$, $y = t_b$ and $z = t_{ab}$. The component $x^2 + y^2 + z^2 - xyz - 4 = 0$ consists of abelian characters. The intersection of $z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy = 0$ with the lines $x = \pm\sqrt{2}$ and $y = 0$ (corresponding to rotations of order 8 and 4 respectively) is $z^3 = 0$. Hence two triple points.

The whole scheme of characters $X(S^3(\mathrm{Wh}, (8, 4)))$ can be computed using the notebook [16]: besides the two triple points, it contains 21 simple points (3 simple characters, 4 central characters and 14 abelian non-central characters).

Example 53. In this example and the next one, we consider a 3-manifold M_n with boundary a torus obtained by attaching a $(2, 2n)$-torus link exterior C_n (a cable space) and K, the *orientable* I-bundle over the Klein bottle, along a boundary component.

The I-bundle K has the homotopy type of the Klein bottle, hence

$$\Gamma_K = \pi_1(K) \cong \langle \gamma, \mu \mid \mu\gamma\mu^{-1} = \gamma^{-1} \rangle,$$

and the peripheral subgroup is $\pi_1(\partial K) = \langle \gamma, \mu^2\rangle$.

Let C_n be the exterior of the $(2,2n)$-torus link, with $n \geq 2$. It is a Seifert-fibered space with orbifold surface an annulus with one cone point of order n. Hence

$$\Gamma_{C_n} = \pi_1(C_n) \cong \langle a, c \mid [a, c^n] = 1\rangle,$$

where c^n is a generator of the center. Up to conjugation $\pi_1(C_n)$ has two peripheral subgroups: $\langle ac, c^n\rangle$ and $\langle a, c^n\rangle$.

We obtain a 3-manifold $M_n = C_n \cup_h K$ by identifying the boundary component of K with a boundary component of C_n via a homeomorphism $h\colon \partial K \to \partial_2 C_n$, where $\partial_2 C_n$ is the component of ∂C_n with peripheral subgroup $\langle a, c^n\rangle$ (thus $\langle ac, c^n\rangle$ is the peripheral subgroup of M_n). The homeomorphism $h\colon \partial K \to \partial_2 C_n$ is chosen so that $h_*(\gamma) = a$, and $h_*(\gamma\mu^2) = c^n$. A presentation of the fundamental group of M_n is

$$\Gamma_n := \pi_1(M_n) \cong \Gamma_{C_n} *_{\mathbb{Z}\oplus\mathbb{Z}} \Gamma_K \cong \langle a, c, \gamma, \mu \mid \mu\gamma\mu^{-1}\gamma = [a, c^n] = 1,\ \gamma\mu^2 = c^n,\ a = \gamma\rangle .$$

We simplify the presentation:

$$\Gamma_n \cong \langle c, \mu, \gamma \mid \mu\gamma\mu^{-1} = \gamma^{-1},\ \gamma\mu^2 = c^n\rangle \cong \langle c, \mu \mid \mu c^n\mu^{-2}\mu^{-1}c^n\mu^{-2} = 1\rangle \cong \langle c, \mu \mid\ c^n\mu^{-3}c^n\mu^{-1} = 1\rangle .$$

The peripheral subgroup of $\pi_1(M_n)$ is $\langle \mu^{-2}c, c^n\rangle$.

Let us consider the case $n = 2$. The scheme $X(M_2, \mathrm{SL}_2(\mathbb{C}))$ can be computed from the notebook [16], we describe the components. Firstly, since the abelianization of $\pi_1(M_2)$ is $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, there are three components of non-simple (or abelian) characters, described in Example 49. In addition there are two components containing simple characters, given by the ideals

$$I = (t_c t_{c\mu} + 2t_\mu, t_\mu^2, t_c t_\mu, t_c^2) \quad \text{and} \quad J = (t_\mu, t_{c\mu}) .$$

The ideal J is radical but I is not: $\mathrm{rad}(I) = (t_c, t_\mu)$.

We check that $\mathbb{C}[t_c, t_\mu, t_{c\mu}]/I$ is the coordinate ring of a double line, following Section II.3.5 in [9], by considering new coordinates:

$$x := t_c\, t_{c\mu} + 2\, t_\mu\,, \quad y := t_c\,, \quad z := t_{c\mu}\,.$$

With these coordinates $I \cong (x, y^2)$ and therefore

$$\mathbb{C}[t_c, t_\mu, t_{c\mu}]/I \cong \mathbb{C}[x, y, z]/(x, y^2) \cong \mathbb{C}[y, z]/(y^2) .$$

The representation variety (not the scheme) of M_2 is studied by K. Baker and K. Petersen in [2]. In fact M_2 is a once-punctured torus bundle with tunnel number one. This torus bundle is also named M_2 in [2], and in [2, Section 2.1] the following presentation for $\pi_1(M_2)$ is given:

$$\pi_1(M_2) \cong \langle \alpha, \beta \mid \beta^{-2} = \alpha^{-1}\beta\alpha^2\beta\alpha^{-1}\rangle \quad \text{by putting } \mu = \alpha^{-1} \text{ and } c = \beta\alpha^{-1}.$$

Example 54. We continue Example 53: now we show that for general $n \geq 2$, the scheme $X(M_n, \mathrm{SL}_2(\mathbb{C}))$ is non-reduced, by generalizing the double line when $n = 2$. As $\pi_1(M_n)$ is generated by c and μ, $X(M_n, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}$ is a subvariety of $\mathbb{C}^3$ with coordinates $(t_c, t_\mu, t_{c\mu})$. Consider

$$Y_k = \{(t_c, t_\mu, t_{c\mu}) \in \mathbb{C}^3 \mid t_\mu = 0, t_c = 2\cos(\tfrac{\pi k}{n})\} \subset X(M_n, \mathrm{SL}_2(\mathbb{C}))_{\mathrm{red}}.$$

for $k = 1, \dots, n-1$.

Lemma 55. *For $k = 1, \dots, n-1$, Y_k is a component of $X(M_n, \mathrm{SL}_2(\mathbb{C}))_{red}$ containing simple characters.*

Lemma 56. *The scheme $X(M_n, \mathrm{SL}_2(\mathbb{C}))$ is non-reduced at any point of Y_k, for $k = 1, \dots, n-1$.*

Proof of Lemma 55. First we describe a one-parameter family of characters and next we prove that this family forms a component. Consider characters $\chi \in X(M_n, \mathrm{SL}_2(\mathbb{C}))$ such that:

- $\chi(\mu) = \chi(\mu\gamma) = 0$ and $\chi(\gamma) = 2$. Namely χ restricted to $\pi_1(K)$ is the character of a representation of $\pi_1(K)$ that maps μ to a rotation of angle π in $\mathbb{H}^3$ and γ to the identity.
- $\chi(c) = \chi(ac) = \cos(\frac{\pi k}{n})$ and $\chi(a) = 2$. Therefore χ restricted to $\pi_1(C_n)$ is the character of a representation of $\pi_1(C_n)$ that maps c to a rotation of angle $2\pi k/n$ in $\mathbb{H}^3$ and a to the identity.

Thus the restriction of χ to both $\pi_1(K)$ and $\pi_1(C_n)$ is the character of a representation with image a cyclic group, and the restriction to the attaching torus is central, i.e. contained in $\{\pm \mathrm{id}\}$. One can deform such a character by conjugating separately the image of $\pi_1(K)$ and the image of $\pi_1(C_n)$, yielding a one-parameter of conjugacy classes of representations of $\pi_1(M_n)$.

To prove that this family of characters is a component, we first look at representations of $\pi_1(K) = \langle \gamma, \mu \mid \mu\gamma\mu^{-1} = \gamma^{-1}\rangle$. They lie in two families:

(a) Representations that map γ to $\pm\mathrm{id}$, and μ to any value.
(b) Representations that preserve an unoriented hyperbolic geodesic $l \subset \mathbb{H}^3$, that map γ to an isometry preserving the orientation of l, and that map μ and $\mu\gamma$ to π-rotations with axis perpendicular to l.

Given ρ a representation of $\pi_1 M$, we make two assertions:

- If the restriction $\rho|_{\pi_1 K}$ is in case (a) and $\rho(\mu^2) \neq -\mathrm{id}$, then $\rho(\pi_1 M)$ is abelian.
- If the restriction $\rho|_{\pi_1 K}$ is in case (b) and $\rho(\gamma) \neq \pm\mathrm{id}$, then $\rho(\pi_1 M)$ preserves an unoriented geodesic in hyperbolic space.

Both assertions follow from the attaching relations $\gamma\mu^2 = c^n$ and $a = \gamma$ by elementary considerations, and they yield that Y_k is a component. □

Proof of Lemma 56. For any $\chi \in Y_k$ simple, χ restricted to C_n is constant (see the proof of Lemma 55). In particular its restriction to ∂M_n is also constant. By writing $\chi = \chi_\rho$, as we can construct deformations of $\chi = \chi_\rho$ that remain constant in ∂M_n,

$$\dim\big(\ker\big(H^1(M_n, \mathrm{Ad}\,\rho) \to H^1(\partial M_n, \mathrm{Ad}\,\rho)\big)\big) \geq 1.$$

On the other hand, by a standard argument on Poincaré duality [17, 35],

$$\operatorname{rank}\big(H^1(M_n, \mathrm{Ad}\,\rho) \to H^1(\partial M_n, \mathrm{Ad}\,\rho)\big) = 1$$

Thus $\dim H^1(M_n; \mathrm{Ad}\,\rho) \geq 2$. It follows that the Zariski tangent space to the scheme at a simple $\chi \in Y_k$ has dimension ≥ 2 by Theorem 17. As generic characters in Y_k are simple and $\dim Y_k = 1$, the scheme is non-reduced at generic points of Y_k. By Lemma 8, the set of non-reduced points is the Zariski-closed and it contains Y_k. □

Remark 57. The line Y_k restricts to a point in $X(\partial M_n, \mathrm{SL}_2(\mathbb{C}))$. The image of the restriction is always an isotropic subspace, and when the restriction is non-singular the image is a Lagrangian submanifold, cf. [17, 35]. This example proves that non-singularity is needed to have a Lagrangian submanifold.

Remark 58. The manifold M_n of Example 54 should be compared to an example of Schnauel and Zhang [34], who attach a cable space to a torus knot exterior. In this way they obtain a boundary slope not detected by valuations on the variety of characters.

9. The character scheme of the Borromean rings

Let us start with a presentation for the group Γ_{Bor} of the Borromean rings.

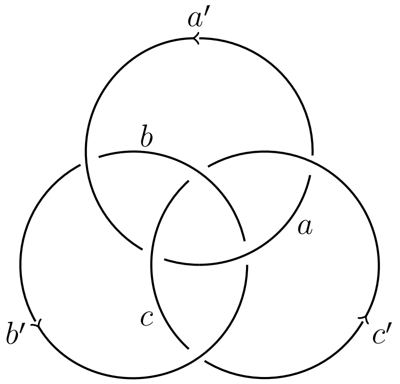

Figure 1. The Borromean rings

We take the over presentation and we obtain:

$$\begin{cases} a' &= c'a(c')^{-1}, \\ b' &= a'b(a')^{-1}, \\ c' &= b'c(b')^{-1}, \end{cases} \quad \text{and} \quad \begin{cases} a' &= cac^{-1}, \\ b' &= aba^{-1}, \\ c' &= bcb^{-1}. \end{cases}$$

This gives the presentation

$$\Gamma_{Bor} \cong \langle a,b,c \mid [a,[b,c^{-1}]],\ [b,[c,a^{-1}]],[c,[a,b^{-1}]] \rangle \tag{34}$$

and as usual one of the relation is a consequence of the other two relations. The generators a, b and c are meridians, and the corresponding longitudes are

$$\ell_a = [b,c^{-1}], \quad \ell_b = [c,a^{-1}] \quad \text{and} \quad \ell_c = [a,b^{-1}]\,.$$

In what follows we consider the free group $\mathbb{F}(a,b,c)$ and the surjection $\mathbb{F}(a,b,c) \twoheadrightarrow \Gamma_{Bor}$. Notice also that there is a surjective homomorphism $\Gamma_{Bor} \twoheadrightarrow \mathbb{Z}^3$ (the abelianization). Moreover, there are three obvious surjections $\Gamma_{Bor} \twoheadrightarrow \mathbb{F}_2$, one of the three meridians is mapped onto the trivial element the other meridians are mapped to a generating pair of $\mathbb{F}_2$.

From these surjections, we obtain an closed immersions

$$X(\mathbb{Z}^3, \mathrm{SL}_2(\mathbb{C})),\ X(\mathbb{F}_2, \mathrm{SL}_2(\mathbb{C})) \hookrightarrow X_{Bor} \hookrightarrow X(\mathbb{F}(a,b,c), \mathrm{SL}_2(\mathbb{C}))\,.$$

Here we write $X_{Bor} := X(\Gamma_{Bor}, \mathrm{SL}_2(\mathbb{C}))$ for short.

Recall that the coordinate ring of $X(F(a,b,c), \mathrm{SL}_2(\mathbb{C})) \subset \mathbb{C}^7$ is reduced and isomorphic to the quotient

$$\mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]/(t_{abc}^2 - p\,t_{abc} + q)$$

where p and q are the polynomials defined in equations (9) and (10) respectively.

The scheme X_{Bor} has several components. These components arise from surjections of Γ_{Bor} onto $\mathbb{Z}^3$ (the abelianization), and the free group $\mathbb{F}_2$ (one of the meridians is mapped onto the trivial element).

Let us first identify these components:

- First of all there is at least one the distinguished component X_0 which contains the character of a lift of the holonomy representation. We are looking for the ideal I_0.

- There are the characters of the non-simple representations $X_{ns} \subset X_{Bor}$. Since the Abelianization is a surjection $\Gamma_{Bor} \twoheadrightarrow \mathbb{Z}^3$ we obtain an inclusion $X(\mathbb{Z}^3, \mathrm{SL}_2(\mathbb{C})) \hookrightarrow X_{Bor}$. The ideal I_{ns} corresponding to this component was investigated in Section 7. According to Proposition 47, the ideal I_{ns} is generated by

$$
\begin{aligned}
(35)\quad & t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 4, \quad t_a^2 + t_c^2 + t_{ac}^2 - t_a t_c t_{ac} - 4, \quad t_b^2 + t_c^2 + t_{bc}^2 - t_b t_c t_{bc} - 4,\\
& t_a(t_a t_{bc} + t_b t_{ac} + t_c t_{ab} - t_a t_b t_c) - 2t_{ab}t_{ac} - 4t_{bc} + 2t_b t_c,\\
& t_b(t_a t_{bc} + t_b t_{ac} + t_c t_{ab} - t_a t_b t_c) - 2t_{ab}t_{bc} - 4t_{ac} + 2t_a t_c,\\
& t_c(t_a t_{bc} + t_b t_{ac} + t_c t_{ab} - t_a t_b t_c) - 2t_{ac}t_{bc} - 4t_{ab} + 2t_a t_b,\\
& 2t_{abc} - t_a t_{bc} - t_b t_{ac} - t_c t_{ab} + t_a t_b t_c.
\end{aligned}
$$

  The ideal I_{ns} is a prime, and hence radical, since $\mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]/I_{ns}$ the coordinate algebra of the irreducible variety $(\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)/\sim$ (see Theorem 38).
- Next there are representations which map one of the generators to a central element $\pm\mathrm{id} \in \mathrm{SL}_2(\mathbb{C})$. This gives rise to six 3-dimensional components $X_a^\pm$, $X_b^\pm$ and $X_c^\pm$ in X_{Bor}. Each of those components is isomorphic to $\mathbb{C}^3$ and has therefore no singular points. The ideal corresponding to $X_a^\pm$ is

$$I_a^\pm := \Big(t_a \mp 2, t_{ab} \mp t_b, t_{ac} \mp t_c, t_{abc} \mp t_{bc}\Big)$$

  and

$$\mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]/I_a^\pm \cong \mathbb{C}[t_b, t_c, t_{bc}]\,.$$

  Similar for the other generators. All these ideals are prime ideals, and hence they are radical.

Theorem 59. *The coordinate algebra $\mathbb{C}[X_{Bor}] = \mathbb{C}[X_{Bor}]_{red}$ is reduced. More precisely, we have*

$$\mathbb{C}[X_{Bor}] \cong \mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]/I_{Bor}$$

where $I_{Bor} = I_{ns} \cap I_a^+ \cap I_a^- \cap I_b^+ \cap I_b^- \cap I_c^+ \cap I_c^- \cap I_0$ is the intersection of prime ideals in $\mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]$. Therefore, the representation scheme has eight irreducible components

$$X_{Bor} = X_{ns} \cup X_a^+ \cup X_a^- \cup X_b^+ \cup X_b^- \cup X_c^+ \cup X_c^- \cup X_0\,.$$

Proof. Computer supported calculations [16] give us an ideal

$$I_7 = I_{ns} \cap I_a^+ \cap I_a^- \cap I_b^+ \cap I_b^- \cap I_c^+ \cap I_c^- \subset \mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}],$$

such that $I_{Bor} \subset I_7$.

If I, J are ideals in $\mathbb{C}[x_1, \ldots, x_n]$ then the *ideal quotient* is

$$(I : J) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid gf \in I \text{ for all } g \in J\}$$

(see Chapter 4 in [7]). It turns out that $(I : J)$ is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ containing I. Moreover we have that

$$V(I : J) \supset \overline{V(I) \smallsetminus V(J)}$$

where $V(I)$ denotes the vanishing set of the ideal in $\mathbb{C}^n$ and $\overline{S}$ the Zariski-closure of $S \subset \mathbb{C}^n$.

Now, we use ideal division to define the ideal $I_0 := (I_{Bor} : I_7)$. The notebook calculations give us that I_0 is radical in $\mathbb{Q}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]$, and $I_{Bor} = I_0 \cap I_7$. Hence

$$I_{Bor} = I_{ns} \cap I_a^+ \cap I_a^- \cap I_b^+ \cap I_b^- \cap I_c^+ \cap I_c^- \cap I_0\,.$$

The ideal I_0 is generated by

$$
\begin{aligned}
&t_{abc}^2 - t_{abc}\,p + q, \qquad && t_a t_{bc} - t_b t_{ac}, \qquad && t_b t_{ac} - t_c t_{ab},\\
&t_a t_{abc} - t_{ab} t_{ac}, \qquad && t_b t_{abc} - t_{ab} t_{bc}, \qquad && t_c t_{abc} - t_{ac} t_{bc}\,.
\end{aligned} \tag{36}
$$

Lemma 60. *The ideal I_0 is radical in $\mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]$.*

Proof. Computer supported calculations show that $I_0 \subset \mathbb{Q}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]$ is a radical ideal (see [16]). Now, Lemma 3.7 in [15] shows that

$$I_0 \cdot \mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}] \subset \mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]$$

is also radical. □

We let X_0 denote the corresponding subvariety of X_{Bor}. This equations had already be studied by Sparaco [37].

Lemma 61 (Sparaco)**.** *The variety X_0 is irreducible.*

Proof. The projection on to $\mathbb{C}^7 \to \mathbb{C}^4$ onto the subspace generated by the t_{ab}, t_{ac}, t_{bc} and t_{abc} coordinates induces a birational map between X_0 and the hypersurface with equation

$$
\begin{aligned}
t_{abc}^4 - t_{abc}^2(2t_{ab}t_{ac}t_{bc} - t_{ab}^2 - t_{ac}^2 - t_{bc}^2 + 4)\\
+ t_{ab}^2 t_{ac}^2 t_{bc}^2 - t_{ab}^3 t_{ac} t_{bc} - t_{ab} t_{ac}^3 t_{bc} - t_{ab} t_{ac} t_{bc}^3 + t_{ab}^2 t_{ac}^2 + t_{ab}^2 t_{bc}^2 + t_{ac}^2 t_{bc}^2
\end{aligned}
$$

It follows now that this hypersurface is irreducible and hence X_0 also (see [37]). □

It follows that the ideal I_0 is a prime ideal. □

9.1. **The distinguished component X_0.** Notice that X_0 is the unique component containing faithful representations. Hence X_0 is the canonical component of the character variety of the Borromean rings.

The singular locus X_0^{sing} is 1-dimensional. More precisely, the X_0^{sing} is the union of twelve lines and six points [16]. The lines are formed by characters of non-simple representations.

The characters of the central representations are part of X_0^{sing}. At each character of a central representation three of the lines intersect. So the lines form a *cube* with corners the characters of a central representations. Hence, the characters of the central representations are singular points of the singular set.

The characters of two central representations given by

$$
\begin{cases} a \mapsto \epsilon_a \mathrm{id} \\ b \mapsto \epsilon_b \mathrm{id} \\ c \mapsto \epsilon_c \mathrm{id} \end{cases} \quad \text{and} \quad \begin{cases} a \mapsto \epsilon'_a \mathrm{id} \\ b \mapsto \epsilon'_b \mathrm{id} \\ c \mapsto \epsilon'_c \mathrm{id} \end{cases}
$$

are connected by a line in X_0^{sing} if and only if two of the ϵ_x and ϵ'_x are equal. The line consists of characters of non-simple representations. For example the character of the trivial representation $(\mathrm{id}, \mathrm{id}, \mathrm{id})$, and the character of the central representation $(\mathrm{id}, \mathrm{id}, -\mathrm{id})$, are connected by the line

$$L = \{(2, 2, z, 2, z, z, z) \mid z \in \mathbb{C}\} \subset X_0\,.$$

The six isolated points in X_0^{sing} are characters of $\mathrm{SU}(2)$-representations. More precisely, the points are characters of binary-dihedral representations which map one of the generators to ±id and the other two to half-turns about two orthogonal lines. For example:

$$(\pm 2,0,0,0,0,0,0) \quad \text{is the character of} \quad \rho\colon \begin{cases} a \mapsto \pm\mathrm{id} \\ b \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ c \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{cases} .$$

These characters are the *midpoints* of the faces of the cube.

9.2. **Intersections between X_0 and the other components.** All components do intersect X_0. More precisely:

- $X_0 \cap X_a^{\pm}$: These intersection form the *faces* of the *cube*.
  The intersection $X_0 \cap X_a^{\pm}$ is two-dimensional and isomorphic to $\mathbb{C}^2$. In fact,
  $$I_0 + I_a^{\pm} = \left(t_a \mp 2, t_{ab} \mp t_b, t_{bc} \mp t_c, 2t_{bc} - t_b t_c, 2t_{abc} \mp t_b t_c\right)$$
  and hence for the coordinate ring
  $$\mathbb{C}[X_0 \cap X_a^{\pm}] \cong \mathbb{C}[t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}]/(I_0 + I_a^{\pm}) \cong \mathbb{C}[t_b, t_c]\,.$$
  Similar argument applies for $X_b^{\pm}$ and $X_c^{\pm}$. Notice that the lines in X_0^{sing} are contained in these intersection. For example: $X_0 \cap X_a^{+}$ contains the following four characters of central representations
  $$\begin{cases} a \mapsto \mathrm{id} \\ b \mapsto \epsilon_b \mathrm{id} \\ c \mapsto \epsilon_c \mathrm{id} \end{cases} \qquad \text{where } \epsilon_a, \epsilon_b \in \{\pm 1\},$$
  and the following four lines of characters of non-simple representations
  $$L_1 = \{(2,2,z,2,z,z,z) \mid z \in \mathbb{C}\}, \quad L_2 = \{(2,-2,z,-2,z,-z,-z) \mid z \in \mathbb{C}\},$$
  $$L_3 = \{(2,z,2,z,2,z,z) \mid z \in \mathbb{C}\}, \quad L_4 = \{(2,z,-2,z,-2,-z,-z) \mid z \in \mathbb{C}\}\,.$$
  form a *square*.
- $X_0 \cap X_{\mathrm{red}}$: This intersection is formed by the 12 lines of non-simple representations. It is contained in X_0^{sing}.

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