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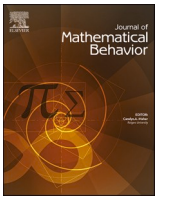

# Secondary-school teachers' noticing of aspects of mathematics teaching talk in the context of one-day workshops

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ABSTRACT

In a research project with one-day teacher education workshops for secondary-school mathematics teachers, our study explores the potential of tool-supported discussions in helping them to notice important and critical aspects of mathematics teaching talk. Mathematical practices of naming and explaining in teaching talk, students' content learning challenges, and noticing processes of identifying, interpreting and deciding are the components of our framework and the tools that guided the design and implementation of three workshops on linear equations, fractions and plane isometries. The data was collected during the discussions with the seven teachers and the teacher educator throughout these workshops. The coding of the discussions allowed us to see discourse moves that reveal the teachers' noticing of: (i) challenges in the identification of mathematical naming, (ii) mathematical explaining that voices the students' learning, (iii) classroom practice in relation to mathematical naming and explaining.

## 1. Introduction and research question

In many teacher education programs, mathematics teachers discuss students' mathematics learning and productions. This is also common in Catalonia-Spain, with rarely a complementary pedagogic focus on the place of mathematics teaching talk –i.e., the mathematical talk of the teacher in classroom teaching– in the students' learning and productions. Local programs tend to privilege attention to other relevant aspects of teaching, such as the choice of mathematical problems. This situation creates a contrast with programs in other countries' contexts, as well as with the rich research on mathematics teaching discourse including language (e.g., Ingram, 2020; Moschkovich, 2018; Wessel & Erath, 2018). In the US, for example, the enactment of productive discussions in mathematics classrooms (Stein, Engle, Smith, & Hughes, 2015) is at the center of several teacher education initiatives (e.g., Dunning, 2023; Jacobs, Koellner, Seago, Garnier, & Wang, 2020). The study reported in this paper was motivated by the situation in our context, by the importance of mathematics teaching talk in mathematics teaching and learning (Adler et al., 2023; Lampert, 1998; Pimm, 1987/2019), and by the question in Wilkinson (2018), "Do teachers model the mathematics register?" (p. 168) and what the teachers need to know and learn to do so.

Our study is part of a research project of one-day workshops with groups of secondary-school mathematics teachers and a pedagogic focus on aspects of mathematics teaching talk aimed at supporting the students' learning of the mathematical content at play in each workshop. We present instances of mathematics teaching talk to the teachers, and we draw on ideas from Mason (2002) and van

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Es & Sherin, (2002) to support teachers in what to tackle and how to interpret it. Our framework has three components: noticing processes of identifying, interpreting and deciding, mathematical naming and explaining in teaching talk, and students' content learning challenges. These components are mediational tools (Engeström, 2015), whose interaction in combination serves to select some aspects over others in the teacher education work. In this paper, we examine the tool-supported discussions with the seven teachers across three workshops on linear equations (WS1), fractions (WS2) and plane isometries (WS3), specifically the extent to which noticing practices of mathematical naming and explaining challenging content-meanings were facilitated. Our research question is as follows: *How can a focus on mathematical naming and explaining, in articulation with students' learning challenges, help teachers to notice important and critical uses of mathematics teaching talk?* We take WS1 to illustrate the detail of our tools and methods, and the data of the three workshops taken place in 2021 to report findings about the teachers' noticing. The emphasis in the study is thus upon the discussions and their mediation on the teachers' noticing, rather than upon the role and mediation of the teacher educator. Some uses of mathematics teaching talk are termed *important* because of their value for the enactment of students' content learning, while some others are termed *critical* because of their potential to reinforce or extend research-documented learning challenges.

## 2. Literature review and theoretical framework

Our theoretical and teacher education framework has three main components: teachers' noticing processes of identifying, interpreting and deciding, mathematical naming and explaining in teaching talk, and students' content learning challenges. The first component substantiates our social approach to teacher education work and research with secondary-school mathematics teachers. As a whole, the three components act as guiding tools in the design and practice of the workshops. The grounding of these components draws respectively on research in mathematics teacher education, classroom mathematics teaching and students' mathematics learning. In this section, we situate each component in relation to the research literature and to some operational decisions made in the study.

### 2.1. Teachers' noticing processes of identifying, interpreting and deciding

In his seminal book, Mason (2002) refers to the discipline of noticing as a fundamentally social practice, whose development enriches the reading of professional and educational practices, and, by doing this, it entails some potential for transformation in the working site. For teachers to develop a discipline of noticing aspects of teaching and further elucidating reasons for what they do and say in their teaching, they need to learn to give focused attention to these aspects and reasons so that they can reflect on them. The social stance in all this points to two central conditions of noticing development: participation in shared work in collaboration with others and their experiences, and learning specialized focused discourses (Morgan, 2014; van Es, 2011), for interpretation of and decisions about aspects and reasons that were already there and become progressively visible with professional and educational meaning. Noticing development thus requires a social site of participation in which some specialized focus of attention is made meaningful.

In the thinking of the workshops with the mathematics teachers as social sites of participation for the professional learning of a pedagogic discourse on mathematical naming and explaining in teaching talk aimed at enacting students' content learning, we take noticing as a three-layered process. Parallel to Mason (2002), van Es & Sherin, (2002) proposed the tracing of noticing in teacher education work by means of observing a continuum of three major processes and the evidence for them. For the purpose of our study, we adapt the three-layered approach to noticing as consisting of a continuum of processes that involve: (1) *identifying* instances of mathematical naming and explaining in teaching talk; (2) *interpreting* their importance and criticality in the creation of opportunities for overcoming students' mathematics learning challenges; (3) *deciding* how to change or elaborate instances of mathematical naming and explaining to better support students in their mathematics learning. It is a continuum because these processes do not remain static and there are many hybrids and nuances in between. For example, *interpreting* or analyzing how what has been identified in the teachers' noticing is related to something else can be built up with a diversity of reasons and types of evidence.

We are aware of the different approaches to noticing that Mason (2002) and van Es & Sherin, (2002) represent, with the former approach originally thought of as a model for learning to notice students' mathematical thinking. Mathematics teacher education research actually offers many differences in how noticing is conceptualized and examined (see the survey in Dindyal, Schack, Choy, & Sherin, 2021). van Es, Hand, Agarwal, & Sandoval, (2022) recently elaborated a framework of multidimensional noticing for equity in relation to how teachers attend to and reason about the enactment of teaching dimensions involved in creating opportunities for all students to own their mathematical reasoning. A focus on the teachers and their noticing (Mason, 2002) is here articulated with teaching teachers to notice aspects of the classroom practice decided in the research (van Es & Sherin, 2002). As in van Es, Hand, Agarwal, & Sandoval, (2022), we adopt a social approach to noticing that draws on stances from Mason (2002) and van Es & Sherin, (2002). We particularly see in the notion of multidimensional noticing for equity a place for a dimension regarding the access to and interaction with the mathematical talk, whose modelling cannot be taken for granted in all students' classroom interactions. The modelling in teaching talk of mathematical naming and explaining for all students is in this sense a way of creating opportunities of access to and interaction with the mathematical language and discourse (Moschkovich, 2018).

### 2.2. Mathematical naming and explaining in teaching talk

Some research and developmental studies have begun to address mathematical naming and explaining in teaching talk (e.g., Adler

et al., 2023; Planas et al., 2023), and how these relate to the creation of learning opportunities for all students. While mathematical naming and explaining are common practices in classroom teaching, even experienced teachers may not notice the importance and criticality of these practices, and it may be difficult for them to tackle and talk about their pedagogic function and, more specifically, about mathematics teaching talk and its pedagogic function. Professional talk about this talk is, of course, not the only way of working on the understanding of content teaching and learning that is language responsive. Teachers may be supported in attending to the management of classroom talk (Ingram, 2020; Moschkovich, 2018), to the kind of writing that is a tool of rich learning environments (Hebert & Powell, 2016), to the students' linguistic productions in their assessment of what happens in a lesson (Bailey & Durán, 2019), or to the scaffolding moves in teaching (Wessel & Erath, 2018). We claim that singling out mathematical naming and explaining in content teaching is also fundamental.

In our study, learning a specialized focused discourse for discussing aspects of mathematics teaching talk involves work on professional talk about mathematical naming and explaining. Our view of mathematical naming and explaining is rooted in the use of explicitness in language (Bailey, Maher, Wilkinson, & Nyakoojo, 2021), and in teaching that is explicit in the communication of content meaning intended as object of learning (Longwe, Fauskanger, & Kazima, 2022; Planas, 2021). Mathematical naming in teaching, by answering what-questions, such as 'What is it?', functions to make explicit what is being discussed so that students can focus on it. Linguistically, it consists of giving nouns or nominalized words and phrases from the mathematical register (Halliday, 1978), which for secondary-school linear equations includes *unknown*, *equal sign*, *equivalence*, or *algebraic expression*, but also *variable* and *value variation* for distinctions with linear functions (Ely & Adams, 2012). Mathematical naming in the form of phrases still includes the description of mathematical objects symbolically or visually represented, e.g., *y is triple the size of x* for $y = 3x$. Mathematical explaining is a step further, both linguistically and mathematically, in the sense of talk responding to why-questions, such as 'Why is this?'. It consists of giving more or less dense sentences that explain mathematical meanings and relationships. Planas (2021) reports instances of explaining equations proposed by the teachers in a piloting workshop, e.g.: "Get a sequence, which is to say, get a sequence of equivalent equations, or equations with the same solutions", "You have to use the transposition rules. That is, the rules for the generation of equivalent equations" (p. 282).

The interpretation of mathematical naming and explaining in Planas (2021) as the lexicalization of words, phrases, and sentences to communicate mathematical meaning in teaching talk is part of a broader notion of explanatory communication in mathematics teaching (Adler, 2021; Adler et al., 2023; Adler & Alshwaikh, 2019). Mathematical naming and explaining work together in teaching talk to open opportunities of content learning for all students. The learning of linear equations, for example, involves a revision of meanings for concepts like the equal sign, from calculation and relationship to structure. This learning can be hindered if students are not confronted with newer explicit meanings through mathematical talk in teaching. If the teacher writes $y = 3x$, then uses repeatedly "this equation" or ambiguous pronouns throughout the lesson, and does not refer to the structure by saying, e.g., *y is triple the size of x,* opportunities to communicate the function of the equal sign and the embedded algebra structure in the equation are missed. We keep the word "size" in our example, knowing that $x$ may represent something unmeasurable, to express that, in our attention to mathematics teaching talk, the goal is supporting mathematics learning rather than linguistic-mathematical correctness. We value mathematical naming and explaining in teaching talk, because we value the effects of interacting with and listening to certain forms of talk on students' learning (Hintz & Tyson, 2015; Moschkovich, 2018; Resnick, Michaels, & O'Connor, 2010).

### 2.3. Students' content-specific learning challenges

In the workshops with the secondary-school teachers, we aim to promote noticing of mathematics teaching talk that is responsive to students' content-specific learning. For this, we relate mathematical naming and explaining in teaching talk with mathematically content-specific learning challenges to guide the teachers' noticing in the discussion of the professional tasks. By learning challenges, we refer to a variety of stages that precede robust understanding (Schoenfeld, 2014), in line with our notion of learning as a continuous process of growing understanding (Engeström, 2001). Many of these challenges originate in the epistemic complexity of the mathematical contents of learning and are tied to the learning in development of the students for other contents. Mathematics education research also suggests links between students' learning challenges and classroom teaching practice (e.g., Hiebert & Grouws, 2007; William, 2007). We thus connect the teachers' pedagogic knowledge of these challenges –and knowledge on learning and learners– with noticing work on mathematics teaching talk, and on what to name and explain that can ideally support the learning of the specific content at play, regardless of cases of individual students and particular classrooms and lessons.

The design of each workshop required a literature review to identify and select students' content learning challenges to be presented to the teachers. In this paper, we do not detail the learning challenges selected for WS2 (e.g., understanding the fraction as one number) and WS3 (e.g., understanding the relationship between varying size and varying shape). Mathematics education research reports learning challenges regarding linear equations that are visible when students wrongly apply rules for transforming expressions and solving equations, with reasoning mostly based on notation manipulations and arithmetical properties. Some of these challenges are conceptually based, in the sense of a difficult seeing of mathematical structures (Mason et al., 2009) and of algebraic meanings of equality and equivalence. As Kieran (2013) and Rojano (2022) claimed, however, there is not a dichotomy between conceptual teaching and learning of structure sense and procedural teaching and learning of notation manipulation. In a similar vein, Zolkower et al. (2015) associated student conceptual learning of linear equations with their word use in talk about calculations and other procedures. Informed by our review, we finally selected the following two related students' challenges in the learning of linear equations: manipulation with understanding of literal-symbolic notation (e.g., Gay & Keith, 2002; Kieran, 1992), and attention to algebra structure sense (e.g., Rojano, 2022; Steinberg, Sleeman, & Ktorza, 1991). Structure sense in algebra here involves recognizing a familiar structure in its simplest form; dealing with a compound term as an object on its own and, through substitution, recognizing a

familiar structure within a more complex form; and choosing manipulations for a better use of structure.

A pedagogic focus on mathematical naming and explaining in teaching talk, in interaction with students' content learning challenges, can be argued as related to mathematical pedagogies that can benefit and are responsive to a focus on challenging mathematical meanings in classroom exploratory talk (Boyd & Kong, 2017; Lampert, 1990). In our local context, this is relevant because exploratory pedagogies of looking for ideas and new meanings in classroom practice are highly valued in mathematics teacher education on mathematics teaching. Gaining some research-documented knowledge of students' content learning challenges and practicing this knowledge in relation to mathematical naming and explaining can move professional talk and noticing in directions still aligned with mathematical pedagogies of classroom exploratory talk. The presentation of learning challenges and of teaching talk to the teachers can specifically reinforce knowledge for noticing mathematical naming and explaining aimed at supporting students' learning of linear equations, fractions and plane isometries through explorations of meaning.

## 3. Research design and analytical methods

The empirical site of this study were three one-day teacher education workshops on linear equations, fractions and plane isometries with seven mathematics teachers and the first author as the teacher educator. These workshops took place, one per month, in the school where three of them teach, the other four teachers being in the same educational district. The teachers had responded to a call sent by the project team, in collaboration with the educational district, with the invitation to participate in workshops on the teaching of secondary-school mathematics contents. They all had mathematics or science university degrees and their experience of mathematics teaching ranged from eight to twenty years. At the time of writing, a second round has begun with the second author as the teacher educator and the enrollment of new teachers. Both authors have experience of secondary-school mathematics teaching and of work with mathematics teachers. In subsection 3.1, we choose WS1 for illustrating design decisions and professional tasks.

### 3.1. Illustration of the workshops' design and tasks

The teachers' noticing during their participation in WS1 was supported by i) knowledge about students' challenges in manipulating with understanding literal-symbolic notation and in attending to algebra structure sense (first part), ii) knowledge about word use for mathematical naming and explaining (second part), iii) tasks with teaching instances of mathematical naming and explaining and prompts to enact processes of identifying, interpreting and deciding (third part). In the 30-minute first part, the teacher educator

**Table 1**
Summaries of PT1 and PT2 in WS1.

| First professional task (PT1) | | |
|---|---|---|
| Mathematical task | Naming $x = 5$ | Explaining $x = 5$ |
| (MT1)<br>Discuss whether $x = 5$ is a linear equation. | *-The unknown is five.*<br>*-x is five.*<br>*-x equals five.*<br>*-The value of x is five.*<br>*-An equation that is true for five and only five.*<br>*-A linear, or one-degree equation with number five.*<br>*-An equation that is five.* | *-It is linear because it is already solved and we can check the one solution, five.*<br>*-The power of x is one, so it is an equation of a linear type.*<br>*-It does not look like an equation, but it is and linear, because we do not read a power of two or more.*<br>*-It is an equation, and it is linear, but it is not written in standard form as usual.*<br>*-Check whether you can represent it like a times x plus b equals zero and how this gives you the quick answer.*<br>*-It is the simplest form of the equation because you cannot reduce it more.*<br>*-You can change the expressions in both sides, you can subtract five, and you see the constant and the variable.* |
| • PT1.1. How do these instances of teaching talk support the learning of linear equations?<br>• PT1.2. Which instances support algebra structure sense? Which ones rather promote unreasoned ways of manipulating notation and finding the solution?<br>• PT1.3. Propose your ways of naming and of explaining $x = 5$ to support algebra structure sense. | | |
| **Second professional task (PT2)** | | |
| **Mathematical task** | **Naming *same equation*** | **Explaining $x = 5$ in relation to $3x = 15$** |
| (MT2)<br>Discuss whether $x = 5$ and $3x = 15$ are the same equation. | *-Equal equation.*<br>*-Equivalent equation.*<br>*-Equation with everything identical or almost.*<br>*-Comparable in solution.*<br>*-Equation after applying some rules of transposition.*<br>*-Same number or expression adding or multiplying both sides.*<br>*-Same line graph.* | *-The value of x is not necessarily five in any equation, but it is for these two, so same equation.*<br>*-They are identical to each other in the numerical solution.*<br>*-Like five equals x same as x equals five. Different representation, same equation.*<br>*-Let us think of three times five and three times x, and how the three is important here.*<br>*-Fifteen divided by three is five, that is the key to start.*<br>*-The two equations are equal because you get the second one by simplifying the first.*<br>*-You can compare the equations by seeing that if you have the value of x in the first you can reason the value of x in the second.* |
| • PT2.1. How do these instances of teaching talk support the learning of linear equations?<br>• PT2.2. Which instances support algebra structure sense? Which ones rather promote unreasoned ways of manipulating notation and finding the solution?<br>• PT2.3. Propose your ways of naming *same equation* and of explaining $x = 5$ in relation to $3x = 15$ to support algebra structure sense. | | |

presented the learning challenges selected from the research literature. She approached algebra structure sense in terms of the mathematical practices that could be modelled in teaching talk to solve tasks of manipulation of algebraic equations (Kieran, 1992; Rojano, 2022). These practices were recognition of simpler structures such as notable products, switches between literal-symbolic forms such as factoring expressions, substitution such as replacing a variable by an expression, and application of algebraic identities. She then simulated research-informed situations where students struggled to capture algebra structure sense and attended to the numbers in the equations and to operational rules, considering each side or rule separately, with no evidence of use of strategies. She finally asked for experiences in the teachers' lessons of similar learning challenges.

In the 30-minute second part, there was one more presentation by the teacher educator regarding teaching talk to name and explain mathematical concepts and processes involved in algebra structure sense. Mathematical naming was presented as choices of words and phrases in teaching talk to make explicit, for all the students, responses to what-questions ('What is it?') about the mathematical contents being in focus and discussed in a lesson. Mathematical explaining was presented as choices of sentences in teaching talk to make explicit, for all the students, responses to why-questions ('Why is this?') about mathematical relationships, by possibly including linguistic marks such as "because", "because of this", "the reason is", "then", "which is to say" or "that is". The teacher educator then simulated practices of mathematical naming and explaining, informed by lesson data from a classroom research project of the team (Planas, Chico, García-Honrado, & Arnal, 2019). She specifically noted the equity issues involved in giving all students opportunities to listen to content-focused instances of mathematical talk in the classroom.

At the beginning of the 90-minute third part, the teachers were given paper sheets with two professional tasks, PT1 and PT2, to be approached individually and then discussed in group. PT1 and PT2 both contained: a) the description of a mathematical task, MT1 and MT2, b) examples of mathematical naming and explaining from secondary-school lessons of the former project in which MT1 and MT2 had been experimented (Planas, Chico, García-Honrado, & Arnal, 2019), and c) three prompts each, PT1.1, PT1.2 and PT1.3 and PT2.1, PT2.2 and PT2.3, to enact processes of identifying, interpreting and deciding on mathematical naming and explaining aimed at supporting the students' learning of linear equations. MT1 was to discuss whether $x = 5$ is a linear equation, and MT2 was to discuss whether $x = 5$ and $3x = 15$ are the same equation. Table 1 shows abbreviated versions of PT1 and PT2, translated from the original Catalan for the purpose of this paper. As it can be read, the real examples of mathematical naming and explaining in the tasks were not equally accurate or precise.

After a break, the last part of WS1 consisted of 90 min for discussion of mathematics teaching talk in mathematics teaching and learning, initially prompted by some general questions inspired by Wilkinson (2018), e.g., "Do we facilitate classroom talk to include our mathematical teaching talk?", "Do we model the mathematical talk sufficiently to our students for them to learn to use it?" The teacher educator also asked the teachers to report what they had noticed in the teaching talk exemplified in PT1 and PT2 so that some reflections could be shared and some proposals of mathematical naming and explaining could be further discussed. In this part, there was more talk of the teacher educator, compared to the third part for the teachers' discussion of the task prompts. The teachers' participation and their noticing throughout the three workshops, however, cannot be reduced to interactions with the teacher educator or even to expressions of individuality, because the professional tasks were mostly worked out through talk in collaboration.

### 3.2. *Methods for the analysis of the workshops' data*

WS1, WS2 and WS3 were audio-recorded, and the teachers' notes on the paper sheets were collected to double-check, when necessary, responses to the tasks' prompts. The first author listened to the audios and a research officer transcribed them. The audios and transcripts were uploaded to our university repository and access to the transcripts was permitted to three researchers in the project, including the second author, for familiarization and work with the data. Given our interest in examining the teachers' noticing in a particular direction, but also knowing that teachers' talk in teacher education can go beyond the interests of the researchers (Mason, 2002), we applied to the data a combined method of deductive and inductive thematic analysis (Proudfoot, 2023). The inductive analysis was important to value the teachers' voices in the workshops as much as possible. The deductive analysis followed a structured and theoretically informed method. The main components of our framework were used in combination to develop three codes: C1. (Identifying) *The teachers see mathematical names and explanations in the exemplified instances of teaching talk;* C2. (Interpreting) *The teachers justify relationships between some mathematical names and explanations and some students' learning challenges;* C3. (Deciding) *The teachers propose mathematical names and explanations and justify their value with respect to the students' learning.*

The initial stage of the analysis was for the coding. The first author applied C1, C2 and C3 to the WS1, WS2 and WS3 transcripts and the research officer prepared documents to group the pieces of data or episodes for each code. The data in which the teachers did not seem to attend to aspects prompted by the professional tasks were inductively coded with general themes such as *curricular regulations*, *time constraints* or *technological resources*, hence indicating some noticing around school mathematics teaching. The officer then prepared documents to group the pieces of data for each general theme. Although these themes lacked a focus on mathematical naming and explaining, this second set of documents helped to note and value directions in the teachers' noticing different to those intended in the research. For triangulation validity, the second author independently worked on the application of C1, C2 and C3 to the WS1 transcript and searched for general themes. He is also an experienced coder who knows well the theoretical framework of the study. Summaries of the results of the coding and of the general themes generated by the second author were done.

The second stage of the analysis consisted of meetings for peer review of the coding and the general themes generated. We discussed the WS1 episodes that we had coded differently, and we reconciled disagreements through discussion until final codes were assigned. A similar method was followed to refine the general themes. During these meetings, we came to see contents of the episodes coded as C1, C2 or C2, which provided more nuanced answers to our research question and could be thought of as discourse moves in the teachers' discussions. These moves indicated the teachers' noticing of: M1. *Challenges in the identification of mathematical naming*; M2.

*Mathematical explaining that voices the students' learning;* M3. *Classroom practice in relation to mathematical naming and explaining.* M1, M2 and M3 gave continuity to the teachers' noticing by responding to and accepting in the discussions specific nuances of the focus on mathematical naming and explaining in teaching talk. The beginning of an episode was the statement in which a teacher suggested an instance of mathematical naming and/or explaining –taken from the task or created– as an object of discussion or as challenging, in the sense of inhibiting or facilitating the learning of linear equations, fractions or plane isometries. The end of an episode was given by content moves in the discussion. Once M1, M2 and M3 were generated in the context of WS1, we independently worked on the WS2 and WS3 documents with the results of the coding to look for evidence of these same moves. There were more meetings for validity triangulation and a document was prepared with the nine episodes whose interpretation had not been agreed on.

For additional rigor and validity, the third stage of the analysis incorporated working seminars with two mathematics education researchers of the broader project, who are experts in interpretive analyses and know well the study and its framework. Prior to the seminars, they were given the descriptions of M1 to M3 and worked independently on the nine episodes whose interpretations had remained unclear or differently seen by the two authors. At this stage, the general themes were not further triangulated, and the analytical efforts concentrated on the specific nuances of the teachers' noticing in relation to mathematical naming and explaining. Interpretations of the nine episodes were shared in the seminars, two episodes were dismissed because we did not agree on a final assignation of any of the discourse moves, and the understanding of the other episodes was refined. Following the seminars, the two authors did one more review of the M1, M2 and M3 episodes. The final analysis of WS1, WS2 and WS3 produced a total of 73 episodes; 20, 21 and 32 respectively grouped into M1, M2 and M3. A summary of general themes was kept as a secondary result to back up and recognize the presence of spontaneous noticing processes. In subsections 4.1, 4.2 and 4.3, we illustrate the three moves with WS1 episodes. In subsection 4.4, we argue the progressive major focus of the teachers' noticing on mathematical naming and explaining in teaching talk.

## 4. Findings about the secondary-school teachers' noticing in the workshops

Our research question is: *How can a focus on mathematical naming and explaining, in articulation with students' learning challenges, help teachers to notice important and critical uses of mathematics teaching talk?* In this section, we approach an answer by illustrating episodes and discussing findings. In the episodes, the teachers respectively focus on identifying, interpreting and deciding on issues of mathematical naming and explaining. We list the findings separately and, at the same time, argue them being close to each other and as part of a continuum. There are not evident boundaries between every two processes within episodes either. The discourse moves indicating noticing processes did not take place independently from each other. The way in which the teachers interpreted some instances of mathematical explaining as critical, for example, prepared the elaboration of proposals. Pseudonyms have been assigned to all teachers.

### *4.1. Challenges in the identification of mathematical naming*

In this subsection, we reproduce and discuss an episode during the teachers' work on PT1 prompted by the question: *How do these instances of teaching talk support the students' learning of linear equations?* After some minutes of talk in which the teachers had discussed that the same numerical symbol, *5*, can represent properties accomplished by different mathematical structures if written *5* or $x = 5$, Toni noticed that the name *structure* was not mentioned in the PT1 instances. Carme and Isa further noticed that there were not explanations of $x = 5$ that addressed algebra structure sense. The following transcript is an English version of the conversation in the episode that was developed in Catalan:

| | |
|---|---|
| Toni: | I was surprised that none of the sentences on the sheet used the word structure. Did you see this? |
| Carme: | Now that you say it, the word is not even mentioned in any explanation of these teachers. |
| Isa: | Yes, just thinking that this is a kind of word never mentioned. We teach mathematical structures all the time, and do not say it by its name. |
| Toni: | So, we may not be teaching them, I mean, if we do not say it. |
| Isa: | I don't think it is an issue of saying the word. It is about what to do with the word and the students in the class after saying it. It is not only about vocabulary as we might first think. |
| Toni: | So, do we want to say why an equation is a mathematical structure? |
| Isa: | This word is tricky, and perhaps too much for the students. The curricular documents for algebra use it a thousand times and never say what it is. |
| Carme: | Yes, but do we want to say it and say what it is? What if not? A word for us when thinking about strong mathematics in our teaching? We don't need to define, just to explain the idea. It is like what happens with sets, we are all the time working with sets, just need to explain the idea. |
| Toni: | I may easily say what a set is about. Structure is right, but it is definitely too much for the students and for us. We may want to choose a better word, to talk about algebraic structures, and compare them with arithmetic structures and properties. That could work. |

During the processes of identifying mathematical names and explanations in WS1, but also in WS2 and WS3, the teachers' experienced challenges that were particularly focused on the mathematical naming and names in the tasks. They identified some mathematical names that were missing and several others that were unexplained in the teaching talk exemplified. This finding suggests the teachers' noticing of mathematical naming as more than linguistic and technical, or just referential. They overall noticed that the way mathematical contents are named in the mathematics classroom is an important part of their work of teaching. The episode above is singular in that it shows the challenging experiences of identifying missing names and unexplained names. On the one hand, the teachers identified names from the mathematical register and the local curricular documents (i.e., *sets, structure, algebraic structures*) that did not appear in the PT1 sentences and phrases. *Algebra structure sense*, nonetheless, had been named in the presentation of

students' learning challenges, and it was written in some tasks' prompts. On the other hand, still regarding the naming of *structure* in the context of linear equations and MT1, the teachers were challenged by "what to do with the word and the students in the class" to support mathematics learning once the word is said. They were specifically challenged by the mathematical explaining that comes after the naming, as Adler et al. (2023) also found, and by possible reasons for working on a mathematical concept without naming it in teaching talk. In this respect, the teachers noticed mathematical naming in articulation with mathematical explaining and how to teach the concept named.

In a few episodes, and similarly to what Longwe, Fauskanger, & Kazima, (2022) documented for a group of student teachers and the naming of concepts related to place value, the teachers identified a nuanced distinction between mathematical naming and naming as a pedagogical and teacher education practice. In the episode above, Toni, Isa and Carme noticed mathematical naming in teaching talk ("We may want to choose a better word, to talk about algebraic structures") as distinct from pedagogical naming in teacher education ("A word for us when thinking about strong mathematics in our teaching?"). Two other teachers raised this distinction in an episode on naming the equation as *x is five*. Mar and Jep noticed that, at some point in teaching, *x is five* requires talk that is algebraic, such as "An equation that is true when the number five is the value for x". Moreover, they noticed that the discussion of the naming *x is five* can contribute to "rethinking the teaching of algebra equations as different to the teaching of arithmetic operations".

### 4.2. Mathematical explaining that voices the students' learning

In this subsection, we reproduce and discuss an episode during the teachers' work on PT1 prompted by the questions: *Which instances support algebra structure sense? Which ones rather promote unreasoned ways of manipulating notation and finding the solution?* The episode initiates with Carme's talk about some equations not having "the equation looking", in regard to the instance, *it does not look like an equation, but it is and linear, because we do not read a power of two or more.* Jep, Toni and Mar draw on Carme's talk to link the exemplified mathematical explaining with the effect that they could imagine in their classrooms and with the thinking of equations limited to manipulation. An English version of the conversation in the episode, for the purpose of this paper, is as follows:

| | |
|---|---|
| Carme: | The teacher who says that it doesn't look like an equation, but it is …To me, this issue of the equation looking is important, and what the teacher says here is a bit confusing. |
| Jep: | What do you mean by confusing? |
| Carme: | If the students only practice putting equations in the standard form, it is difficult for them to see equations before or after that. More so if the teacher says that other forms don't look like equations. |
| Jep: | Yes, but I can imagine my students asking where the equation is if I just give them the x and the five. |
| Carme: | Exactly, and you can make it even worse for them if you say it doesn't look like an equation, you are a kind of confirming that equations should look differently. So, you choose a nice task for them to see other ways of looking, but don't really acknowledge all forms the same. |
| Toni: | I see it as putting myself in someone else's place. I can also imagine my students asking where the equation is, and so I would talk to them like this, saying it doesn't look like an equation, so that they know that I know what they may be thinking. Not too bad. |
| Carme: | Okay, not too bad, but what if you don't elaborate further on this? You could be supporting their thinking limited to the standard form. |
| Toni: | Yes, you need to put yourself in your place of teacher in the end. |
| Jep: | It would be interesting to ask my students to write examples of equations in their notebooks. Not sure what they would write, possibly many equations represented in the standard form, so zero on the right side. |
| Mar: | And the zero never on the left side. |
| Carme: | Yes, and if we say that others just don't look like equations, we are reinforcing this equation looking, and the equations prepared to start leaving this on one side and everything else on the other side… |
| Jep: | Some of my students might even answer that this one [$x=5$] has no solution, because you cannot follow the steps from the beginning. So, we need to be careful with what we say and how we explain it, because we cannot pretend that they will see an equation anyway. |

During the processes of interpreting mathematical naming and explaining in WS1, WS2 and WS3, the teachers' noticed the voicing of students' learning challenges in mathematical explanations as ways of making these challenges visible in teaching talk for productive mathematical discussion in the classroom. In the episode above, the voicing of what students might be thinking is viewed as strategic and "not too bad" as long as it becomes articulated with mathematical explaining in teaching and broader classroom practices (Wessel & Erath, 2018). For example, in order to better know what to voice and how in the mathematical talk of the teacher, Jep suggested asking students to write examples of equations in their notebooks. Whereas the teachers noticed some tensions in the deliberate voicing of some learning challenges, they agreed on strategic reasons for teaching talk occasionally not representing the teachers' mathematical thinking, and for justifying with caution talk saying that $x = 5$ or $5 = x$ do not look like equations. Carme first referred to literal-symbolic representations of equations limited to the standard form ($ax+b=0$) and to explaining that is detrimental to mathematical thinking, because it critically suggests the idea that equations only exist in this standard form. In the discussion, Toni interpreted the mathematical explaining that voices learning challenges as strategic and representing the teachers putting themselves in the place of some students to precede or interact with the mathematical talk that all students might need to listen to.

The discussion about voicing the students' learning challenges in teaching talk was the context for an episode on what comes to count as mathematical explaining compared to explaining that is "partially or not clearly mathematical". Isa said: "We can tell them this is an equation, and this is not, and then, this is an equation and looks like it, and this one looks like an equation, but it is not … and all combinations." The attention to these combinations was taken by the teacher educator to ask whether there might be some teaching talk that looked like mathematical explaining and was not actually explaining anything substantial regarding the content of teaching and learning. She posed to the teachers the example of *it is an equation, and it is linear, but it is not written in standard form as usual*, in PT1, and asked whether such a type of tautological talk was common in their teaching, to what they responded affirmatively. Isa made a

comparison with *the power of x is one, so it is an equation of a linear type*, where the description of the literal-symbolic representation critically functions as a reason for why $x = 5$ is a linear equation. She said that the explanation is mathematical, but it explains only the mathematical writing and does not support learning the concept. Quer saw other explanations of this kind in PT1, *If you want to know what a linear equation is, those examples do not explain that*, representing the complexity of accomplishing mathematical talk in teaching that is ideal or continuously supportive of the students' learning (Lampert, 1998).

### 4.3. Classroom practice in relation to mathematical naming and explaining

In this subsection, we reproduce and discuss an episode during the teachers' work on PT2 enacted by the prompt: *Propose your ways of naming same equation and of explaining $x = 5$ in relation to $3x = 15$ to support algebra structure sense.* The episode initiates with Quer's talk about reasons for $x = 5$ and $3x = 15$ being the same equation. While the teachers spent much of the time discussing the instances given on the paper sheet, and anticipating challenges that students might face in making meaning for linear equations from them, during their engagement in PT2, they also produced in collaboration ways of naming *same equation* and of explaining $x = 5$ in relation to $3x = 15$. This is an English version of the original conversation in this episode:

| | |
|---|---|
| Quer: | We can tell them that same here means same solution. We don't want them to run with the idea that same is exactly same when written. |
| Isa: | Yes, and I think it's a perfect moment for seeing the triple on both sides of the second one [$3x=15$]. |
| Quer: | Yes, we cannot let them run with the idea that same equation is exactly same when written. So, it's a good task for making sure that they learn different ways of representing the same equation. |
| Vera: | Listen to what I wrote. We tell them that this one [$3x=15$] is this one [$x=5$] but now supporting the triple weight on each side. So, if you make the equation support the same weight on each side, you don't change the relationship between the two sides. Is that a good explanation? |
| Quer: | Yes, talking about keeping weights. Good idea. |
| Isa: | But we don't want to make them think of measuring by naming weights. |
| Vera: | Yes, that's true. Anyway, they can still think of numbers as having a weight value, and five something weighing more than fifteen something. |
| Isa: | And we can then talk about number of weights. |
| Quer: | Not too bad to think of a model of weights, so that they see the structure and what is kept in the relationship. If we show them the weighing balance with equal weights on both sides, what's wrong with talking about weights? |
| Isa: | It's funny but I use visuals with the algebra tiles and the weighing balance in one lesson. I don't use this kind of talk later, when all the rules, tests, and manipulation stuff come in, whether we want it or not. |
| Vera: | That's a pity, isn't it? We are not really getting to the content. |

During the processes of deciding on mathematical naming and explaining in WS1, WS2 and WS3, the teachers' collaborated in the elaboration of improved instances of teaching talk with frequent attention to the interaction of these instances with the classroom practice (Wessel & Erath, 2018). The episode above illustrates this finding. Decisions on mathematical naming included naming "the triple weight on each side [of the equation]", which developed from "the triple on both sides" and turned into the discussion of naming *weights, weight values* or *number of weights* in the context of linear equations. Decisions on mathematical explaining included explaining that "same [in same equation] means same solution" so that students do not interpret that two equations are the same because they are written the same. One more proposal to explain the relationship between $x = 5$ and $3x = 15$, was as follows: "If you make the equation support the same weight on each side, you don't change the relationship between the two sides". This explanation was further discussed because of the suggestion of some measuring activity. More generally, this episode shows the teachers' noticing the adequacy of mathematical naming and explaining in teaching talk in relation to classroom practice, particularly the use of visuals with the algebra tiles and the weighing balance. The teachers specifically noticed that materials do not by themselves teach mathematics, although they can support classroom talk in which mathematical naming and explaining develop. For example, showing and using a weighing balance as a model of an equation in the classroom teaching could justify practices of mathematical naming and explaining including *weights* and *equal weights on both sides*. The adequacy of naming and explaining was therefore seen as relative to the model of linear equations chosen in the teaching. Unlike what happened with the *model of weights*, the adequacy of teaching talk for literal-symbolic work with equations reduced to mentions of rules and manipulation procedures was problematized without nuances.

A concern in this episode is the responsibility of teachers to use mathematical naming and explaining in teaching that is adequate or getting to the mathematical content and connected with the students' learning challenges and needs. The teachers drew on their own teaching practices and talk to exemplify how difficult this can be. The point that the teaching talk of Isa in different school lessons can be differently attentive to the students' learning challenges, because of some constraints, was taken up by the teacher educator so that the teachers could further discuss possibilities of improvement and change. They noticed mathematics teaching talk functioning differently –with respect to students' learning challenges– and, at the same time, that they were using talk differently, sometimes including unreasoned ways of talking about linear equations reduced to rules and manipulations. By sharing insights to propose instances of talk, they brought up their teaching and, by so doing, they noticed that changes in their teaching talk were about being more consistent or regular with talk that they already used in some of their lessons. Isa connected this reflection with the teachers whose talk was represented in the tasks: "These teachers are like us, talking in shortened and confusing ways in some lessons, to have everything done on time, and talking as they know is better in a few other lessons".

Subsection 4.4 presents some more evidence of the teachers' spontaneous noticing of concerns regarding school mathematics teaching, sometimes with the effect of postponing the focus on mathematical naming and explaining. Compared to WS1 and WS2, the teachers' noticing was more focused on mathematical naming and explaining in WS3. Although the point in this paper is not to argue the developmental nature of the teacher education workshops and of the study, the continuum and diversity of noticing processes with

an increasing focus on mathematical naming and explaining is an important finding.

### 4.4. Noticing mathematical naming and explaining in more focused and nuanced ways

In many moments of WS1 in October 2021, crossing episodes coded as M1, M2 or M3, the teachers introduced general themes such as technological resources, curricular regulations and time constraints. The initial coding of the WS1 data raised a variety of difficult questions to us when we saw the diversity of contents of the teachers' noticing. When the discussions of these other themes did not go back to or connect with the professional tasks, the teacher educator had reminded the teachers of specific prompts. This was done after minutes of discussion and not abruptly because the workshops were thought of as flexible sites for noticing the issues designed in the research (van Es & Sherin, 2002), but also those spontaneously introduced by the teachers (Mason, 2002). WS2 in November 2021 again provided data around general themes of school mathematics teaching in the teachers' talk (e.g., "The unit of fractions in the primary school is never finished and the consequences are for us in the secondary school"), and of the teacher educator's participation to redirect the discussions towards the prompts in the tasks. As the sessions went on, and we reached WS3 in December 2021, the teachers became more focused on the tasks and less involved in general discussions. They knew what to do because they had practiced similar tasks adapted for two other mathematical contents. In WS3, we found only a moment in which Carme talked about the lack of digital resources for the teaching of plane isometries, to which Vera and Quer responded by refocusing the discussion in the direction of naming and explaining as follows:

| | |
|---|---|
| Carme: | Same size, same shape. |
| Quer: | Yes, you don't change the size, you don't change the shape. So, it is an isometry because it keeps the size, and it keeps the shape. |
| Carme: | This is rather easy to teach because you don't need lots of examples, but if you don't have GeoGebra in the classroom, then it's too bad. Sometimes you don't have the resources, the technologies, and all your work and effort are like … like … |
| Vera: | Yes, but this is not about the resources we don't have, as it is not about teaching the words. It is about students' listening to our explanations connecting isometries with size, shape. |
| Quer: | Yes, and perhaps transformations that are distance-preserving … so that the students get the idea of what is key to isometries. |

Becoming more informed or trained could not have gone with becoming more focused on nuanced processes of identifying, interpreting and deciding on mathematical naming and explaining. The teachers could have known well what to do and still choose to spend time discussing general concerns about school mathematics teaching, or issues other than those intended in the workshops. To us, becoming more focused on the tasks and in more nuanced ways indicates some engagement in the pedagogic discourse of mathematical naming and explaining in teaching talk to supports students' learning. Table 2 shows the figures of some progress in the teachers' noticing of the pedagogic importance of mathematics teaching talk from the very basics of mathematical naming and explaining. The figures indicate that almost one half of the total of discourse moves analyzed were found in WS3 and only one fifth of the total were found in WS1. Lasting effects, however, cannot possibly be expected in the short-time context of three one-day workshops. Adler et al. (2023), for example, worked with secondary-school mathematics teachers on issues of word use in teaching during months in cycles of a lesson study.

In WS1, the teachers had first approached the tasks by noticing the technical vocabulary regardless of the concepts named, their explanation and learning, e.g., "Some of these instances have the vocabulary that the students need to learn, so they are good." This reminds us of what happened with a different group of six teachers who participated solely in a workshop on the mathematical naming and explaining of angles (Planas et al., 2023). In WS2, the teaching instances in the tasks were already more discussed from the perspective of the students' learning challenges around the mathematical content being named and explained, e.g., "This teacher called it [the fraction] a number, and that helps to overcome the idea that a fraction are two numbers". In WS3, as shown in the data with Carme, Quer and Vera above, the teachers clearly noticed the place of modelling mathematics teaching talk for the enactment of students' learning, and how mathematical naming and explaining relate to the mathematics that is made available to learn. Moreover, the teachers had decided that mathematical naming was necessary to avoid ambiguous pronouns that did not support the students' learning of mathematics.

## 5. Further discussion and conclusions

An assumption of our study was that the teachers' noticing of aspects of mathematics teaching talk can be developed in tool-supported discussions. In this paper, we have presented a model of one-day teacher education workshops on mathematical naming and explaining in teaching talk in association with students' content learning challenges. From our analysis, we conclude that the

**Table 2**
Distribution of M1, M2 and M3 across WS1, WS2 and WS3.

| *Discourse moves* | *WS1* | *WS2* | *WS3* | Total |
|---|---|---|---|---|
| M1. Challenges in the identification of mathematical naming | 5 | 4 | 11 | 20 |
| M2. Mathematical explaining that voices the students' learning | 3 | 9 | 9 | 21 |
| M3. Classroom practice in relation to mathematical naming and explaining | 6 | 12 | 14 | 32 |
| Total | 14 | 25 | 34 | 73 |

discussions in the three workshops guided the teachers on noticing that, by attending to and interpreting aspects of mathematics teaching talk, they can improve this talk and support students' learning. Here, an important finding of the study is that an interest in and comments about mathematical naming came out strongly in the teachers' noticing. In response to tasks' prompts around mathematical naming, the teachers largely discussed challenges regarding names that were missing or said but unexplained in the teaching talk exemplified (M1). More generally, the teachers remained attentive to links between mathematical naming and explaining and students' mathematics learning, not without reflection about the complexity of teaching talk that cannot be ideal because of the complexity of school mathematics teaching (Lampert, 1998). They developed ways of seeing mathematical naming and explaining as mathematical-linguistic choices with functions of modelling the mathematics register to the students (Wilkinson, 2018). The teachers particularly noticed the voicing of students' learning challenges as part of, in the sense of preparatory or transitional to, mathematical explaining in teaching talk (M2). Moreover, in their work with instances of mathematical naming and explaining from other teachers, they noticed talk that could have been their own and that was critical with respect to some of the learning challenges presented by the teacher educator. In relation to instances of naming and explaining that were noticed as critical and improvable, the teachers raised the relationship between the uses of teaching talk and the wider classroom practices in supporting the students' learning (M3).

Overall, the findings indicate a good take-up of the workshops by the teachers. Our tools in the design of the workshops and tasks mediated the teachers' noticing in a diversity of directions, including the noticing direction intended in the study. The teachers interacted with each other in the thinking of the professional tasks and, possibly because they all had experiences of mathematics teaching, they also introduced general contents of discussion such as the complexity of mathematics teaching at the level of classroom practices (e.g., "It is about what to do with the word and the students after saying it"), and at the curricular level (e.g., "when all the rules, tests, and manipulation stuff come in, whether we want it or not"). Despite the frequent switches in the contents of discussion, all the teachers moved back and forth between attention to mathematical naming and explaining and attention to their school teaching. In WS3, for example, Mar said: "Naming and explaining seem so obvious in our lessons that we forget about it, and so we plan other parts of our teaching … We can make better activities, better dynamics and also better talk". Based on this finding, we are thinking of designing future workshops and tasks in stronger connection with the participant teachers' own mathematics teaching talk and their seeing of the place of mathematical naming and explaining in it.

Our study provides some more ideas and options on how the design of the workshops could be improved. The presentations by the teacher educator, preceding the teachers' engagement in the professional tasks, could be reinforced by adding more research-based information. There are studies in sites of teacher education with secondary-school mathematics teachers that focus on the students' learning and use of particular words within algebra, for example, the relationships between these words and key concepts of linear equations, and the use of talk moves in the classroom teaching (e.g., Adler & Alshwaikh, 2019). Other studies have documented similar findings in the mathematical context of teaching and learning fractions (e.g., Wessel & Erath, 2018). As far as we know, plane isometries –a fundamental content of our secondary-school mathematics curriculum– have been less in focus in mathematics teacher education and mathematics education with a language perspective. Alongside findings from research on the students' learning of specific mathematical contents, the presentation of learning challenges to the teachers could thus be complemented by building on findings from language-based research on the learning and teaching of the content at play in each workshop.

In terms of the growing literature on mathematics teacher education and noticing as a social practice, the contribution of our study is twofold. First, it contributes to expanding van Es et al.'s (2022) notion of multidimensional noticing for equity by documenting teacher education work in which equity is approached in relation to the access, for all students, to the explicit modelling of mathematical talk. Second, the study provides a framework for researchers and teacher educators to design professional tasks with a focus on the mathematical-linguistic practices of naming and explaining, which in turn make possible the spontaneous emergence of some other foci on school mathematics teaching (Mason, 2002). By refining our current framework in stronger connection with the teachers' own mathematics teaching talk, we expect to build up a refined model of workshops that can be part of large teacher education programs with secondary-school mathematics teachers. We foresee the possibility of including workshops on mathematical naming and explaining, and on the importance of how school students are talked mathematics in the classroom, in programs that can have or keep on-the-job sessions with the teachers in the schools and classrooms. For local programs of mathematics teacher education, we plan to remain active in the initiated conversations with the educational and research authorities who are the funders of our wider project.

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**Declaration of Competing Interest**

We have no known conflict of interest to disclose.

**Data Availability**

Data will be made available on request.

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