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THE LIMIT CYCLES OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS FORMED BY CENTERS AND SEPARATED BY IRREDUCIBLE CUBIC CURVES II

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Abstract. In this paper we provide a lower bound for the maximum number of crossing limit cycles of some class of planar discontinuous piecewise linear differential systems formed by centers and separated by an irreducible algebraic cubic curve. First we prove that the systems constituted by three zones can exhibit 0, 1, 2, 3 or 4 crossing limit cycles having four intersection points with the cubic of separation. Second we prove that the systems constituted by two zones can exhibit 0, 1, or 2 crossing limit cycles having four intersection points with the cubic of separation.

1. Introduction

1.1. **Classification of the irreducible cubic polynomials.** A *cubic curve* is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $P(x, y) = 0$ for some polynomial $P(x, y)$ of degree three. This cubic is *irreducible* (respectively *reducible*) if the polynomial $P(x, y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x, y) = 0$ is *singular* if $P_x(x_0, y_0) = 0$ and $P_y(x_0, y_0) = 0$. A cubic curve is *singular* if it has some singular point, as usual here P_x and P_y denote the partial derivaties of P with respect to the variables x and y respectively.

A *flex* of an algebraic curve C is a point p of C such that C is nonsingular at p and the tangent at p intersects C at least three times. The next theorem characterizes all the irreducible cubic algebraic curves.

Theorem 1. *The following statements classify all the irreducible cubic algebraic curves.*

(a) *A cubic curve is nonsingular and irreducible if and only if it can be transformed with affine transformations into one of the following two curves;*

$$c_1(x, y) = y^2 - x(x^2 + bx + 1) = 0 \quad \text{with } b \in (-2, 2), \text{ or}$$
$$c_2(x, y) = y^2 - x(x - 1)(x - r) = 0 \quad \text{with } r > 1.$$

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(b) *A cubic curve is singular and irreducible if and only if it can be transformed with affine transformations into one of the following three curves:*

$$c_3(x,y) = y^2 - x^3 = 0, \qquad or$$
$$c_4(x,y) = y^2 - x^2(x-1) = 0, \quad or$$
$$c_5(x,y) = y^2 - x^2(x+1) = 0.$$

Statement (a) of Theorem 1 is proved in Theorem 8.3 of the book [4] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 1 follows directly from Theorem 8.4 of [4].

1.2. **Crossing limit cycles.** For $k = 1, \dots, 5$ let C_k be the five classes of planar discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curve $c_k(x,y) = 0$, or simply the irreducible cubic curve c_k.

Piecewise linear differential systems appear in the characterization of many real processes such as switches in electronic circuits, see for instance [2, 12, 13].

In the qualitative theory of piecewise linear differential systems one of the important and difficult problems is the determination of the existance and the number of limit cycles. We recall that a crossing periodic orbit, is a periodic orbit of a piecewise linear differential system with a cubic separation curve c_k with at least two points in c_k. If this periodic orbit is isolated we call it a *crossig limit cycle.* In recent years, much progress has been made in studying the existance and lower and upper bounds of crossing limit cycles of piecewise linear differential systems, see for example [1, 5, 6, 7, 8, 9, 15, 17, 18, 19].

There are previous results on the number of crossing limit cycles of discontinuous piecewise linear differential centers separated by a curve Σ. More precisely, if Σ is one straight line then such systems have no crossing limit cycles see [14, 16]. If Σ is a conic the number of crossing limit cycles have been studied in [10], and if Σ is a reducible cubic curve the number of crossing limit cycles have been analyzed in [11].

In [3] we started the study of crossing limit cycles of the discontinuous piecewise linear differential centers in $\mathbb{R}^2$ separated by an irreducible algebraic cubic curve. We proved that these differential systems only can exhibit at most three crossing limit cycles having two intersection points with the cubic of separation.

The objective of this paper is to provide lower bounds for the maximum number $\mathcal{N}$ of crossing limit cycles for the planar discontinuous piecewise linear differential centers which intersect the irreducible cubic curves c_i, with $i = 1 \dots 5$, and have four points of intersection with the cubic of separation. First, we study the number $\mathcal{N}$ when the crossing limit cycles intersect the curves c_2 and c_5, in four points in three different zones. Second, we give $\mathcal{N}$ when the crossing limit cycles intersect c_i with $i = 1 \dots 5$, in four points in two different zones.

Figures 1 and 2 show the different regions separated by the cubic curves c_i, with $i = 1 \dots 5$.

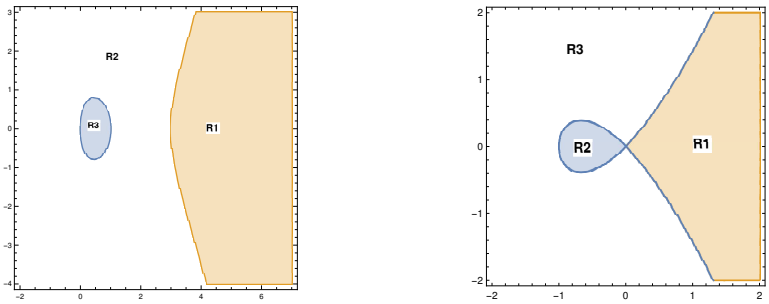


FIGURE 1. The three regions R_1, R_2 and R_3 of the plane separated by the curves c_2 on the left and c_5 on the right.

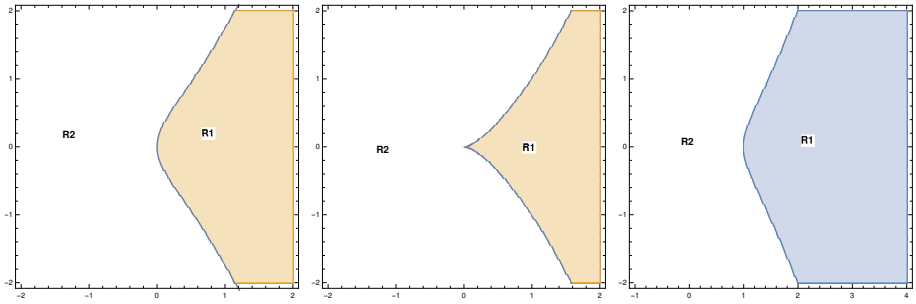


FIGURE 2. The two regions R_1 and R_2 of the plane separated by the curves c_1 on the left, c_3 on the middle and c_4 on the right.

1.3. **Crossing limit cycles contained in three regions intersecting the two cubic curves c_2 and c_5 in four points.** In this subsection we are interessted to provide lower bounds for the maximum number of *crossing limit cycles* of piecewise linear differential centers separated by the irreducible cubic curves c_2 or c_5, having four points of intersection with the cubic of separation. We note that such piecewise systems are formed by three pieces in each one there is a linear differential center.

Our first main result is the following.

Theorem 2. *We study the crossing limit cycles contained in three regions of the discontinuous piecewise linear centers in the class C_2 or C_5 and having four points of intersection with the cubic of separation.*

(a) *There are systems in C_2 and in C_5 exhibiting exactly one crossing limit cycle which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Figure* 3*, while the class C_5 has two possible configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 4*.*

(b) *There are systems in C_2 and in C_5 exhibiting exactly two crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Figure* 5*, while the class C_5 has two possible configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 6*.*

(c) *There are systems in C_2 and in C_5 exhibiting exactly three crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Figure* 7*, while the class C_5 has two possible configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 8*.*

(d) *There are systems in C_2 and in C_5 exhibiting exactly four crossing limit cycles which intersect c_2 or c_5 in four points see $(\mathcal{C}_2)$ and $(\mathcal{C}_5)$ of Figure* 9 *for the classes C_2 and C_5, respectively.*

Theorem 2 is proved in section 2.

1.4. **Crossing limit cycles contained in two regions intersecting the irreducible cubic curves c_i in four points.** Now we give our second main result which provides information on the number of crossing limit cycles of the discontinuous piecewise linear differential systems formed by two centers and intersect the cubic curves c_i, with $i = 1 \dots 5$ in four points.

Our second main result is the following.

Theorem 3. *We study the crosssing limit cycles contained only in two regions of the discontinuous piecewise linear centers in the class C_k for $k = 1 \dots 5$, and having four intersection points with the cubic of separation.*

(a) *There are systems in C_k exibiting exactly one crossing limit cycle intersecting the cubic curves c_i in four points. The classes C_1, C_3 and C_4 have one possible configuration, see $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ of Figure* 10 *and $(\mathcal{C}_4)$ of Figure* 12*, respectively. The classes C_2 and C_5 have two possible different configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure* 11*, and $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 13*.*

(b) *There are systems in C_k exibiting exactly two crossing limit cycles intersecting the cubic curves c_i in four points. The classes C_1, C_3 and C_4 have one possible configuration see $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ of Figure* 14*, respectively. The classes C_2 and C_5 have two possible different configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure* 15 *for the class C_2, and $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 16 *for the class C_5.*

Theorem 3 is proved in section 3.

2. Proof of Theorem 2

Proof of statement (a) of Theorem 2. First we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -\frac{x}{4} - \frac{5y}{16} + \frac{5}{4}, \quad \dot{y} = x + \frac{y}{4} + \frac{1}{2}, \tag{1}$$

this system has the first integral

$$H_1(x, y) = 4\left(x + \frac{y}{4}\right)^2 + 8\left(\frac{x}{2} - \frac{5y}{4}\right) + y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = \frac{1}{56}(14x - 7y - 79), \quad \dot{y} = x - \frac{y}{4} - \frac{211}{64}, \tag{2}$$

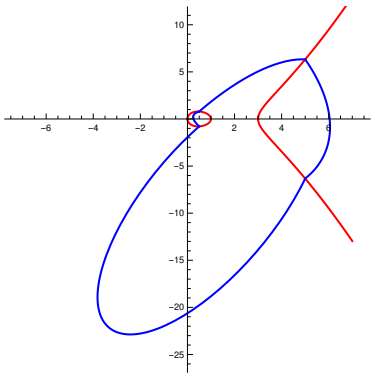
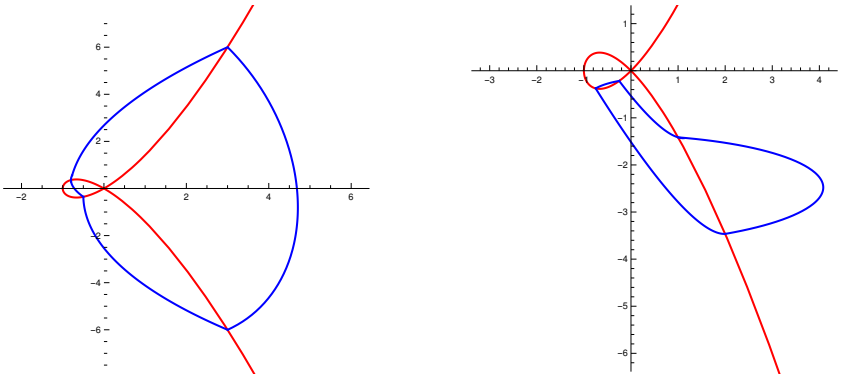


FIGURE 3. The unique limit cycle of the discontinuous piecewise linear differential system (1)–(3) contained in three zones.



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 4. The unique limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (5)–(7), and $(\mathcal{C}_5^2)$ for (9)–(11) contained in three zones.

this differential system has the first integral

$$H_2(x,y) = 4x^2 + x\left(-2y - \frac{211}{8}\right) + \frac{1}{14}y(7y + 158).$$

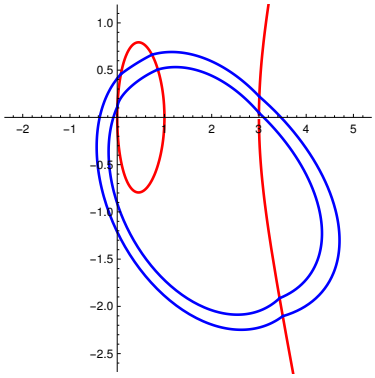
Now we consider the third linear differential center in the region R_3

$$\dot{x} = -x - \frac{5y}{4} + \frac{1}{2}, \quad \dot{y} = x + y - 2, \tag{3}$$

this differential system has the first integral

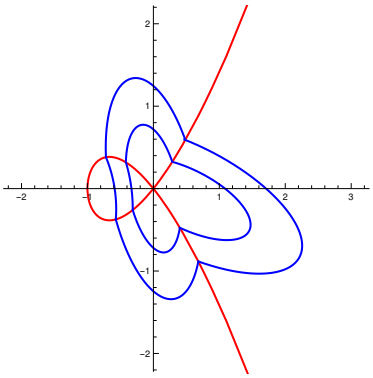
$$H_3(x,y) = 4(x+y)^2 + 8\left(-2x - \frac{y}{2}\right) + y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (1), (2) and (3) has exactly one crossing limit cycle, because the system

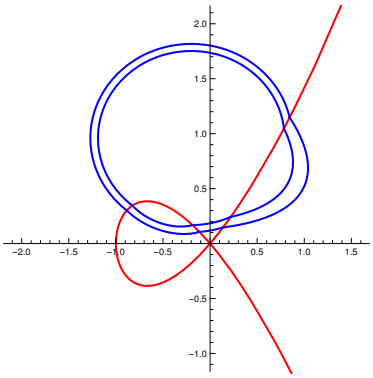


$(\mathcal{C}_2)$

FIGURE 5. The two limit cycles of the discontinuous piecewise linear differential system (12)–(14) contained in three zones.



$(\mathcal{C}_5^1)$



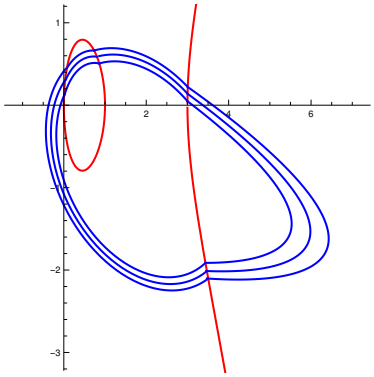
$(\mathcal{C}_5^2)$

FIGURE 6. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (15)–(17), and $(\mathcal{C}_5^2)$ for (18)–(20) contained un three zones.

of equations

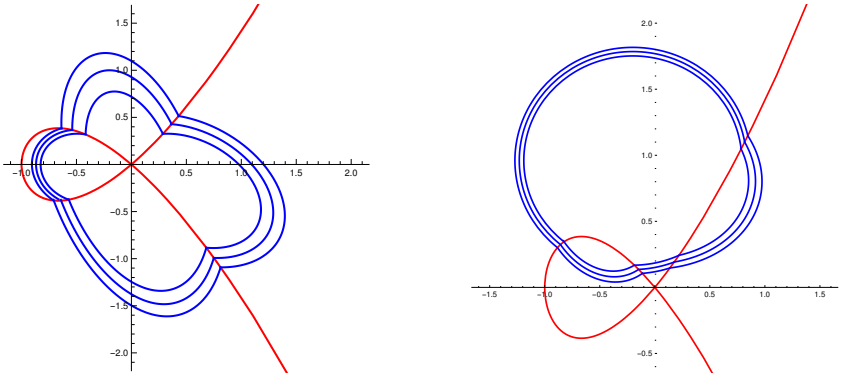
$$
\begin{aligned}
&H_1(\alpha,\beta)-H_1(\gamma,\delta)=0,\\
&H_2(\alpha,\beta)-H_2(f,g)=0,\\
&H_2(\gamma,\delta)-H_2(h,k)=0,\\
&H_3(f,g)-H_3(h,k)=0,\\
&\beta^2-\alpha(\alpha-1)(\alpha-3)=0,\\
&\delta^2-\gamma(\gamma-1)(\gamma-3)=0,\\
&g^2-f(f-1)(f-3)=0,\\
&k^2-h(h-1)(h-3)=0,
\end{aligned}
\tag{4}
$$

has a unique real solution $(\alpha,\beta,\gamma,\delta,f,g,h,k)=(5,-2\sqrt{10},5,2\sqrt{10},1/2,$ $-\sqrt{5}/2\sqrt{2},1/2,\sqrt{5}/2\sqrt{2})$. This completes the proof of statement (a) for the class C_2.



$(\mathcal{C}_2)$

FIGURE 7. The three limit cycles of the discontinuous piecewise linear differential system (21)–(23) contained in three zones.



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

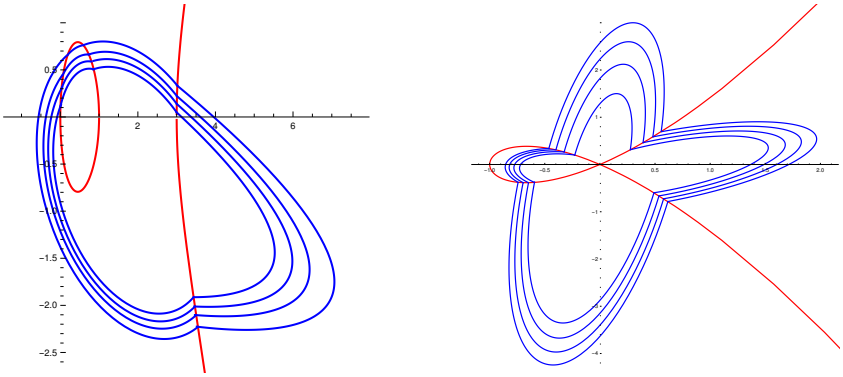
FIGURE 8. The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (24)–(26), and $(\mathcal{C}_5^2)$ for (27)–(29) contained in three zones.

Now we prove the existence of two different configurations of one crossing limit cycle for the class C_5. For the first possible configuration we consider the linear differential center

$$\dot{x} = -\frac{x}{8} - \frac{17y}{64} + \frac{3}{8}, \quad \dot{y} = x + \frac{y}{8} - 1, \tag{5}$$

in the region R_1, with its first integral

$$H_1(x,y) = 4\left(x + \frac{y}{8}\right)^2 + 8\left(-x - \frac{3y}{8}\right) + y^2.$$



$(\mathcal{C}_2)$ $(\mathcal{C}_5)$

FIGURE 9. The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2)$ for (30)–(32), and $(\mathcal{C}_5)$ for (33)–(35) contained in three zones.

In the region R_2 we consider the linear center

$$
\begin{aligned}
\dot{x} &= \frac{x}{4} - \frac{101y}{16} + \frac{3800\sqrt{2}+3584\sqrt{5}+2203113}{320\left(95\sqrt{2}+56\sqrt{5}-4380\right)},\\
\dot{y} &= x - \frac{y}{4} + \frac{-448604\sqrt{2}-315662\sqrt{5}+48\sqrt{10}+22560621}{160\left(95\sqrt{2}+56\sqrt{5}-4380\right)},
\end{aligned}
\tag{6}
$$

with its first integral

$$
\begin{aligned}
H_2(x,y) = \; & \frac{1}{40(95\sqrt{2}+56\sqrt{5}-4380)}\\
& (160(95\sqrt{2}+56\sqrt{5}-4380)x^2 + x(-80(95\sqrt{2}+56\sqrt{5}-4380)y\\
& -897208\sqrt{2}-631324\sqrt{5}+96\sqrt{10}+45121242) + y(1010(95\sqrt{2}\\
& +56\sqrt{5}-4380)y - 3800\sqrt{2}-3584\sqrt{5}-2203113).
\end{aligned}
$$

In the region R_3 we consider the following linear differential center

$$
\dot{x} = -x - 10y - \frac{1}{2}, \quad \dot{y} = x + y - 2, \tag{7}
$$

this differential system has the first integral

$$
H_3(x,y) = 4(x+y)^2 + 8\left(\frac{5}{12}\left(-3\sqrt{2}+\frac{81}{50}-\frac{288}{25\sqrt{5}}\right)x - y\right) + 16y^2.
$$

The discontinuous piecewise linear differential system formed by the linear differential centers (5), (6) and (7) has exactly one crossing limit cycle, because the system

of equations

$$
\begin{array}{l}
H_1(\alpha,\beta)-H_1(\gamma,\delta)=0,\\
H_2(\alpha,\beta)-H_2(f,g)=0,\\
H_2(\gamma,\delta)-H_2(h,k)=0,\\
H_3(f,g)-H_3(h,k)=0,\\
\beta^2-\alpha^2(\alpha+1)=0,\\
\delta^2-\gamma^2(\gamma+1)=0,\\
g^2-f^2(f+1)=0,\\
k^2-h^2(h+1)=0,
\end{array}
\tag{8}
$$

has a unique real solution $(\alpha,\beta,\gamma,\delta,f,g,h,k)=(3,-6,3,6,-1/2,-1/(2\sqrt{2}),-4/5,4/(5\sqrt{5}))$.

For the second configuration. In the region R_1 we consider the linear differential center

$$
\dot{x}=\frac{x}{7}-\frac{785y}{49}+\frac{70\sqrt{50-8\sqrt{6}}+40083}{490\left(\sqrt{2}-2\sqrt{3}\right)},\quad \dot{y}=x-\frac{y}{7}+\frac{1}{5},
\tag{9}
$$

which has the first integral

$$
H_1(x,y)=4\left(x-\frac{y}{7}\right)^2+8\left(\frac{x}{5}+\frac{\left(70\sqrt{2}-280\sqrt{3}-40083\right)y}{490\left(\sqrt{2}-2\sqrt{3}\right)}\right)+64y^2.
$$

In the region R_2 we consider the linear differential center

(10)

$$
\dot{x}=-\frac{3x}{7}-\frac{373y}{1764}+\frac{-23556784\sqrt{2}-26814823\sqrt{3}+14402472\sqrt{6}+83393133}{80960544},
$$

$$
\dot{y}=x+\frac{3y}{7}+\frac{-4861464\sqrt{2}+3092483\sqrt{3}+1548288\sqrt{6}-1763763}{56448\left(88\sqrt{2}-91\sqrt{3}+15\right)},
$$

with its first integral

$$
H_2(x,y)=4\left(x+\frac{3y}{7}\right)^2+\frac{y^2}{9}+\frac{A}{7056\left(88\sqrt{2}-91\sqrt{3}+15\right)\left(16\sqrt{3}-3\right)},
$$

where

$$
\begin{aligned}
A=\;& 88902216\sqrt{2}-37497657\sqrt{3}-82428288\sqrt{6}+153730473)x-8(7597532\sqrt{3}\\
&+266112\sqrt{6(259-32\sqrt{3})}-23982207)y.
\end{aligned}
$$

In the region R_3 we consider the linear differential system

$$
\dot{x}=\frac{x}{8}-\frac{577y}{64}+\frac{-10025\sqrt{3}-29787}{9216},\quad \dot{y}=x-\frac{y}{8}-\frac{1}{3},
\tag{11}
$$

which has the first integral

$$
H_3(x,y)=4\left(x-\frac{y}{8}\right)^2+8\left(\frac{\left(10025\sqrt{3}+29787\right)y}{9216}-\frac{x}{3}\right)+36y^2.
$$

The real solutions of the system of equations (8) with the values of $H_i(x,y)$ with $i=1,2,3$ given for this second configuration is $(\alpha,\beta,\gamma,\delta,f,g,h,k)=(1,-\sqrt{2},2,$

$-2\sqrt{3}, -1/4, -\sqrt{3}/8, -3/4, -3/8)$, then the discontinuous piecewise linear differential system (9), (10) and (11) has one crossing limit cycle ($\mathcal{C}_5^1$) of Figure 4. This completes the proof of statement (a) for the class C_5. □

Proof of statement (*b*) *of Theorem* 2. First we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -x - 5y - 1.8035, \quad \dot{y} = x + y - 0.664282, \tag{12}$$

this system has the first integral

$$H_1(x, y) = 4(x + y)^2 + 8(1.8035y - 0.664282x) + 16y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = -\frac{x}{2} - \frac{10y}{4} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \tag{13}$$

this differential system has the first integral

$$H_2(x, y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2.$$

The third linear differential center in the region R_3 is

$$\dot{x} = -2x - 8y + 0.837903, \quad \dot{y} = x - 2y - 0.169396, \tag{14}$$

this differential system has the first integral

$$H_3(x, y) = 4(x - 2y)^2 + 8(-0.169396x - 0.837903y) + 16y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (12), (13) and (14) have exactly two crossing limit cycles, because the system of equations (4) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (3.00039.., 0.0485802.., 3.43695.., -1.91305.., 0.860569.., 0.506666.., 0.00442503.., 0.114878..)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.00805.., 0.220494.., 3.50419.., -2.1034.., 0.735716.., 0.663523.., 0.0779996.., 0.458408..)$. This completes the proof of statement (b) for the class C_2.

Now we prove the existence of two different configurations of two crossing limit cycles for the class C_5. For the first configuration we consider the linear differential center

$$\dot{x} = -\frac{6x}{5} - \frac{369y}{100} + 0.152456, \quad \dot{y} = x + \frac{6y}{5} - 0.365572, \tag{15}$$

in the region R_1, with its first integral

$$H_1(x, y) = 4\left(x + \frac{6y}{5}\right)^2 + 8(-0.365572x - 0.152456y) + 9y^2.$$

In the region R_3 we consider the linear center

$$\dot{x} = -\frac{x}{5} - \frac{29y}{100}, \quad \dot{y} = x + \frac{y}{5}, \tag{16}$$

with its first integral

$$H_2(x, y) = 4\left(x + \frac{y}{5}\right)^2 + y^2.$$

In the region R_2 we consider the following linear differential center

$$\dot{x} = -y - 0.251587, \quad \dot{y} = x + 1.93017, \tag{17}$$

this differential system has the first integral

$$H_3(x,y) = 4x^2 + 8(1.93017x + 0.251587y) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (12), (13) and (14) have exactly two crossing limit cycles, because the system of equations (8) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.286549.., 0.325022.., 0.400999.., -0.474638.., -0.416749.., 0.318275.., -0.313035.., -0.259454..)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.484398.., 0.590171.., 0.681486.., -0.883698.., -0.719407.., 0.381077.., -0.569542.., -0.373673..)$.

For the second configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = -2x - \frac{25y}{4} + 6.3854, \quad \dot{y} = x + 2y + 0.134162, \tag{18}$$

which has the first integral

$$H_1(x,y) = 4(x+2y)^2 + 8(0.134162x - 6.3854y) + 9y^2.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{3}{2} - \frac{25y}{16}, \quad \dot{y} = x + \frac{1}{5}, \tag{19}$$

with its first integral

$$H_2(x,y) = 4x^2 + 8\left(\frac{x}{5} + \frac{3}{2}\right) + \frac{25y^2}{4}.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = -\frac{x}{2} - \frac{5y}{2} + 0.902952, \quad \dot{y} = x + \frac{y}{2} + 0.234228, \tag{20}$$

which has the first integral

$$H_3(x,y) = 4\left(x + \frac{y}{2}\right)^2 + 8(0.234228x - 0.902952y) + 9y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (18), (19) and (20) have exactly two crossing limit cycles, because the system of equations (8) has two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.844854.., 1.14753.., 0.137908.., 0.14711.., -0.884123.., 0.300962.., -0.112734.., 0.106189..)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.783948.., 1.04708.., 0.219822.., 0.242783.., -0.824415.., 0.345453.., -0.186379.., 0.168115..)$. This completes the proof of statement (b) for the class C_2. □

Proof of statement (c) of Theorem 2. First we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -3.95435x - 19.6368y - 6.39482, \quad \dot{y} = x + 3.95435y + 4.13635, \tag{21}$$

this system has the first integral

$$H_1(x,y) = 4(x + 3.95435y)^2 + 8(4.13635x + 6.39482y) + 16y^2.$$

The second linear differential center in the region R_2

$$\dot{x} = -\frac{x}{2} - \frac{5y}{2} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \tag{22}$$

this differential system has the first integral

$$H_2(x,y) = 4\left(x+\frac{y}{2}\right)^2 + 8\left(y-\frac{3x}{2}\right) + 9y^2.$$

Now we consider the third linear differential center in the region R_3

$$\dot{x} = -0.241343x - 1.00852y - 0.358353, \quad \dot{y} = x + 0.241343y - 0.869754, \tag{23}$$

this differential system has the first integral

$$H_3(x,y) = 4(x+0.241343y)^2 + 8(0.358353y - 0.869754x) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (21), (22) and (23) have exactly three crossing limit cycles, because the system of equations (4) has three real solutions $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (3.00039..,0.0485802..,3.43695..,-1.91305..,0.860569..,0.506666..,0.00442503..,0.114878..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (3.00318..,0.138419..,3.47181..,-2.01219..,0.803882..,0.588414..,0.0319264..,0.302877..)$ and $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (3.00805..,0.220494..,3.50419..,-2.1034..,0.735716..,0.663523..,0.0779996..,0.458408..)$.

Now we prove the existence of two different configurations of three crossing limit cycles for the class C_5. For the first configuration we consider the linear differential center

$$\dot{x} = -0.345578x - 1.11942y - 0.128163, \quad \dot{y} = x + 0.345578y - 0.440337, \tag{24}$$

in the region R_1, with its first integral

$$H_1(x,y) = 4(x+0.345578y)^2 + 8(0.128163y - 0.440337x) + 4y^2.$$

In the region R_3 we consider the linear center

$$\dot{x} = -\frac{x}{5} - \frac{29y}{100}, \quad \dot{y} = x + \frac{y}{5}, \tag{25}$$

with its first integral

$$H_2(x,y) = 4\left(x+\frac{y}{5}\right)^2 + y^2.$$

In the region R_2 we consider the following linear differential center

$$\dot{x} = --0.0923038x - 1.00852y - 0.0805185, \quad \dot{y} = x + 0.0923038y + 0.46371, \tag{26}$$

this differential system has the first integral

$$H_3(x,y) = 4(x+0.0923038y)^2 + 8(0.46371x + 0.0805185y) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (24), (25) and (26) has exactly three crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.286549..,0.325022..,0.681486..,-0.883698..,-0.416749..,0.318275..,-0.569542..,-0.373673..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.366248..,0.428095..,0.749792..,-0.991822..,-0.538507..,0.365825..,-0.639545..,-0.383969..)$ and $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (0.429988..,0.514188..,0.81153..,-1.09226..,-0.636378..,0.383743..,-0.706638..,-0.382736..)$.

For the second configuration. In the region R_1 we consider the linear differential center

$$\dot{x} = -0.121241x - 2.2647y + 1.93257, \quad \dot{y} = x + 0.121241y + 0.00611219, \tag{27}$$

which has the first integral

$$H_1(x,y) = 4(x+2y)^2 + 8(0.134162x - 6.3854y) + 9y^2.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{3}{2} - \frac{25y}{16}, \quad \dot{y} = x + \frac{1}{5}, \tag{28}$$

with its first integral

$$H_2(x,y) = 4x^2 + 8\left(\frac{x}{5} - \frac{3y}{2}\right) + \frac{25y^2}{4}.$$

In the region R_2 we consider the linear differential system

$$\dot{x} = -0.217737x - 1.04741y + 0.515416, \quad \dot{y} = x + 0.217737y + 0.350398, \tag{29}$$

which has the first integral

$$H_3(x,y) = 4(x+0.217737y)^2 + 8(0.350398x - 0.515416y) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (27), (28) and (29) have exactly three crossing limit cycles, because the system of equations (8) has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.844854.., 1.14753.., 0.137908.., 0.14711.., -0.884123.., 0.300962.., -0.112734.., 0.106189..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.816222.., 1.1.., 0.177162.., 0.192216.., -0.856934.., 0.324128.., -0.147429.., 0.136128..)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.783948.., 1.04708.., 0.219822.., 0.242783.., -0.824415.., 0.345453.., -0.186379.., 0.168115..)$. This completes the proof of statement (c) for the class C_5. □

Proof of statement (d) of Theorem 2. First we prove the statement for class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = -3.95435x - 19.6368y - 6.39482, \quad \dot{y} = x + 3.95435y + 4.13635, \tag{30}$$

this system has the first integral

$$H_1(x,y) = 4(x+3.95435y)^2 + 8(4.13635x - 6.39482y) + 16y^2.$$

The second linear differential center in the region R_2

$$\dot{x} = -\frac{x}{2} - \frac{5y}{2} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \tag{31}$$

this differential system has the first integral

$$H_2(x,y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2.$$

Now we consider the third linear differential center in the region R_3

$$\dot{x} = -0.241343x - 1.00852y - 0.358353, \quad \dot{y} = x + 0.241343y - 0.869754, \tag{32}$$

this differential system has the first integral

$$H_3(x,y) = 4(x+0.241343y)^2 + 8(0.358353y - 0.869754x) + 4y^2.$$

The discontinuous piecewise linear differential system formed by the linear differential centers (30), (31) and (32) have exactly four crossing limit cycles, because the system of equations (4) has four real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (3.00039.., 0.0485802.., 3.43695.., -1.91305.., 0.860569.., 0.506666.., 0.00442503.., 0.114878..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.00318.., 0.138419.., 3.47181.., -2.01219.., 0.803882.., 0.588414.., 0.0319264.., 0.302877..)$, $(\alpha_3, \beta_3, \gamma_3\delta_3, f_3, g_3, h_3, k_3) = (3.00805..,$

$0.220494.., 3.50419.., -2.1034.., 0.735716.., 0.663523.., 0.0779996.., 0.458408..$) and ($\alpha_4$, $\beta_4, \gamma_4, \delta_4, f_4, g_4, h_3 4, k_4$) = $(3.01815.., 0.332542.., 3.5491.., -2.22883.., 0.593338.., 0.762036.., 0.192725.., 0.66088..)$.

Now we prove the existence of two different configurations of four crossing limit cycles for the class C_5. For the first configuration we consider the linear differential center

$$\dot{x} = 0.897851x - 2.12174y - 0.62272, \quad \dot{y} = x - 0.897851y - 0.620434, \tag{33}$$

in the region R_1, with its first integral

$$H_3(x, y) = 4(x - 0.897851y)^2 + 8(0.62272y - 0.620434x) + 5.2624y^2.$$

In the region R_3 we consider the linear center

$$\dot{x} = \frac{x}{10} - \frac{17y}{450}, \quad \dot{y} = x - \frac{y}{10}, \tag{34}$$

with its first integral

$$H_2(x, y) = 4\left(x - \frac{y}{10}\right)^2 + \frac{y^2}{9}.$$

In the region R_2 we consider the following linear differential center

$$\dot{x} = -0.225709x - 1.26066y - 0.265909, \quad \dot{y} = x + 0.225709y + 0.334228, \tag{35}$$

this differential system has the first integral

$$H_3(x, y) = 4(x + 0.225709y)^2 + 8(0.334228x + 0.265909y) + 4.83887y^2.$$

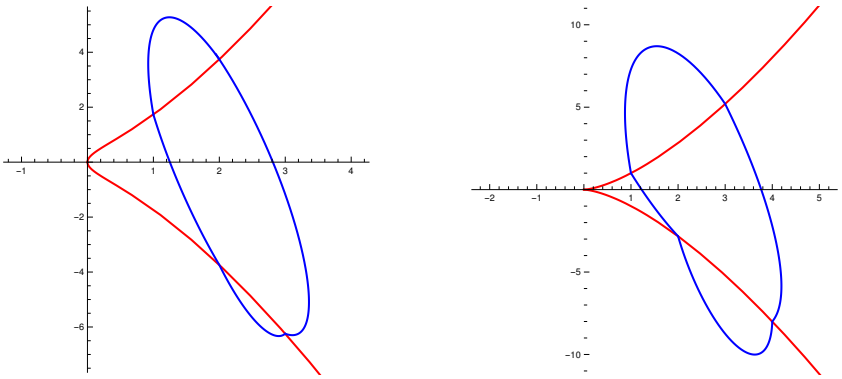
The discontinuous piecewise linear differential system formed by the linear differential centers (33), (34) and (35) have exactly four crossing limit cycles, because the system of equations (8) has four real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1)$ = $(0.275692.., 0.311385.., 0.490206.., -0.598415.., -0.22775.., 0.200142.., -0.593285.., -0.378363..)$, $(\alpha_2, \beta_2, \gamma_2\delta_2, f_2, g_2, h_2, k_2) = (0.391233.., 0.461461.., 0.535875.., -0.664112.., -0.324107.., 0.266457.., -0.647454.., -0.38443..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.480359.., 0.584453.., 0.577721.., -0.72566.., -0.399001.., 0.309322.., -0.696708.., -0.383691..)$ and $(\alpha_4, \beta_4, \gamma_4\delta_4, f_4, g_4, h_3 4, k_4) = (0.555792.., 0.693247.., 0.616543.., -0.783893.., -0.46287.., 0.339233.., -0.742.., -0.376889..)$. □

3. Proof of Theorem 3

Proof of statement (*a*) *of Theorem* 3. First we prove the statement for the class C_1. We consider the first linear differential center in the region R_2

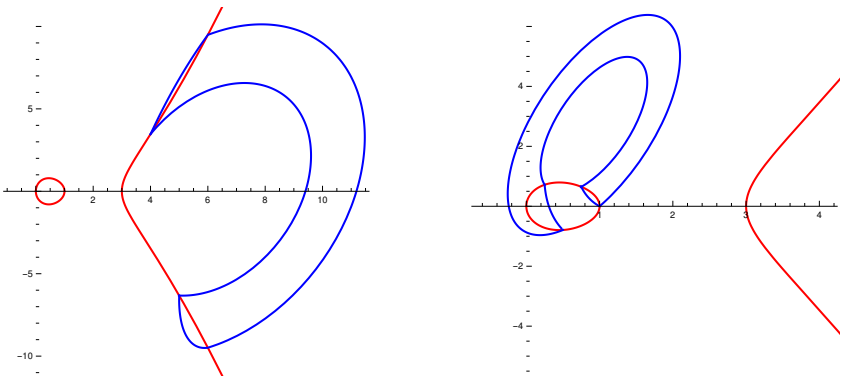
$$\begin{aligned} \dot{x} &= -\frac{x}{7} - \frac{113y}{3136} + \frac{\left(\sqrt{13}+1\right)\left(-187\sqrt{3}+192\sqrt{13}-64\right)}{5376}, \\ \dot{y} &= x + \frac{y}{7} + \frac{B}{75264}, \end{aligned} \tag{36}$$

with $B = \left(\sqrt{13}+1\right)\left(-10651\sqrt{13} + 896\sqrt{14} - 1792\sqrt{39} - 2618\sqrt{42} + 896\sqrt{182} + 18505\right)$. This system has the first integral



$(\mathcal{C}_1)$ $(\mathcal{C}_3)$

FIGURE 10. The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (36)–(37), and $(\mathcal{C}_3)$ for (43)–(44) contained in two zones.



$(\mathcal{C}_2^1)$ $(\mathcal{C}_2^2)$

FIGURE 11. The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (39)–(40), and $(\mathcal{C}_2^2)$ for (41)–(42) contained in two zones.

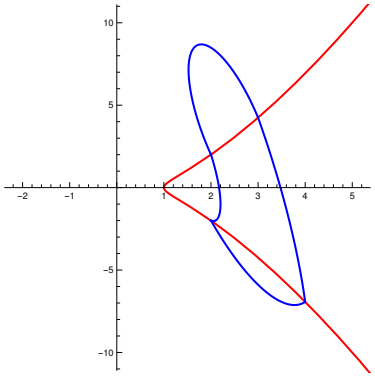
$$H_1(x,y) = \frac{1}{784(\sqrt{13}-1)}(3136(\sqrt{13}-1)x^2 + y(113(\sqrt{13}-1)y + 14(187\sqrt{3} - 192\sqrt{13}+64)) + x(896(\sqrt{13}-1)y - 10651\sqrt{13} + 896\sqrt{14} - 1792\sqrt{39} - 2618\sqrt{42} + 896\sqrt{182} + 18505)).$$

The second linear differential center in the region R_1 is

(37)

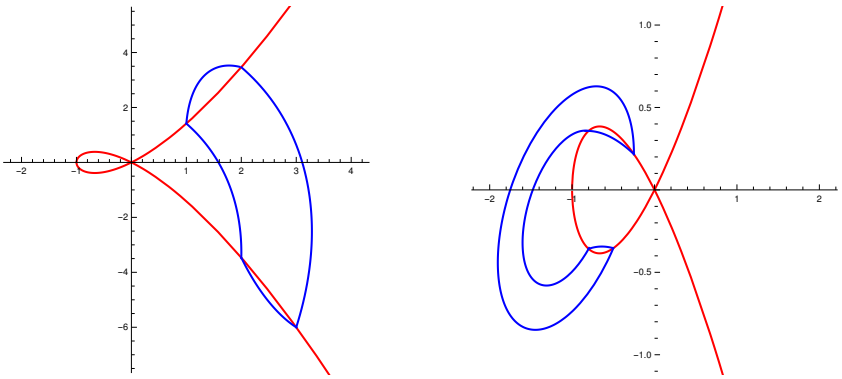
$$\dot{x} = -\frac{x}{6} - \frac{25y}{576} - \frac{-288\sqrt{13} + \frac{751}{\sqrt{3}} + 96}{576(\sqrt{13}-1)},$$

$$\dot{y} = x + \frac{y}{6} - \frac{6009\sqrt{13} + 576\sqrt{14} + 1152\sqrt{39} - 1502\sqrt{42} + 576\sqrt{182} - 10515}{3456(\sqrt{13}-1)},$$



$(\mathcal{C}_4)$

FIGURE 12. The unique crossing limit cycle of the discontinuous piecewise linear differential system (45)–(46) contained in two zones.



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 13. The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (47)–(48), and $(\mathcal{C}_5^2)$ for (49)–(50) contained in two zones.

this differential system has the first integral

$$H_2(x,y) = \frac{1}{432(\sqrt{13}-1)}(1728(\sqrt{13}-1)x^2 - x(-576(\sqrt{13}-1)y + 6009\sqrt{13} + 576\sqrt{14} + 1152\sqrt{39} - 1502\sqrt{42} + 576\sqrt{182} - 10515) + y(75(\sqrt{13}-1)y + 2(751\sqrt{3} - 864\sqrt{13} + 288))).$$

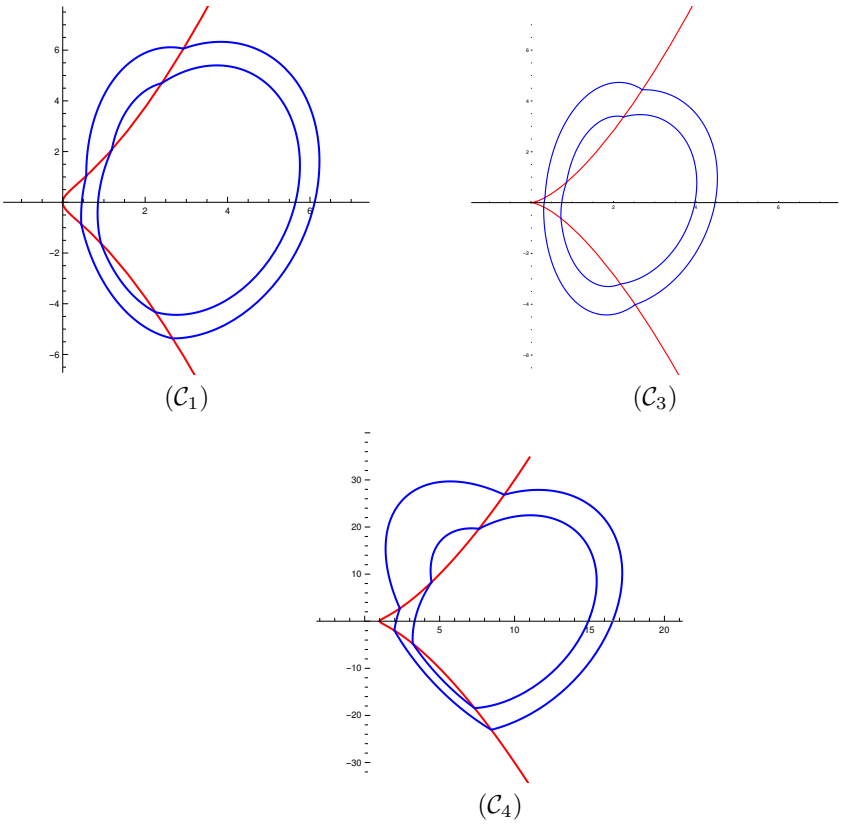


FIGURE 14. The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (51)–(52), $(\mathcal{C}_3)$ for (58)–(59), and $(\mathcal{C}_4)$ for (60)–(61) contained in two zones.

For the piecewise linear differential system (36)–(37) the unique real solution of the system of equations

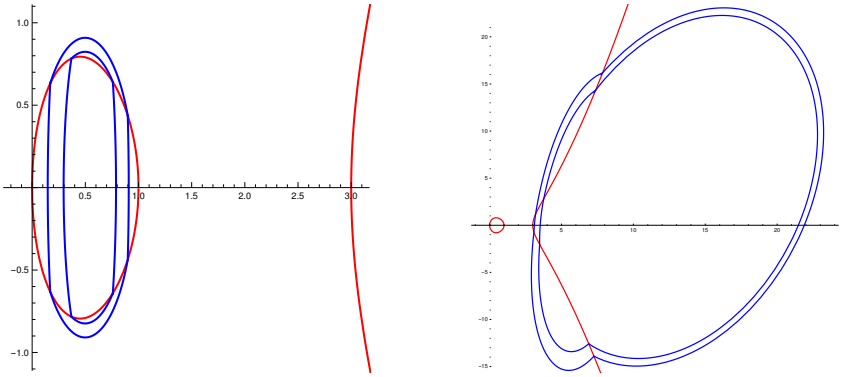
$$
\begin{array}{l}
H_1(\alpha_1,\beta_1)-H_1(\gamma_1,\delta_1)=0,\\
H_1(\alpha_2,\beta_2)-H_1(\gamma_2,\delta_2)=0,\\
H_2(\alpha_1,\beta_1)-H_2(\alpha_2,\beta_2)=0,\\
H_2((\gamma_1,\delta_1)-H_2(\gamma_2,\delta_2)=0,\\
c_i(\alpha_1,\beta_1)=0,\quad c_i(\alpha_2,\beta_2)=0,\\
c_i(\gamma_1,\delta_1)=0,\quad c_i(\gamma_2,\delta_2)=0,
\end{array}
\tag{38}
$$

when $i=1$, is $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(1,\sqrt{3},2,-\sqrt{14},2,\sqrt{14},3,-\sqrt{39})$.

Now we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

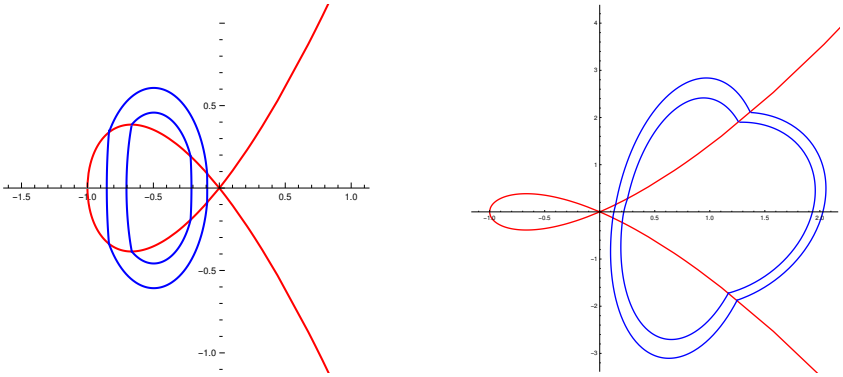
$$
\dot{x}=\frac{x}{6}-\frac{5y}{18}-1,\quad \dot{y}=x-\frac{y}{6}+\frac{1}{18}\left(12\sqrt{3}+6\sqrt{10}-151\right),
\tag{39}
$$

this differential system has the first integral



$(\mathcal{C}_2^1)$ $(\mathcal{C}_2^2)$

FIGURE 15. The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (54)–(55), and $(\mathcal{C}_2^2)$ for (56)–(57) contained in two zones.



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 16. The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (62)–(63), and $(\mathcal{C}_5^2)$ for (64)–(65) contained in two zones.

$$H_1(x,y)=\frac{2}{9}(18x^2+2x(-3y+12\sqrt{3}+6\sqrt{10}-151)+y(5y+36)).$$

The second linear differential center in the region R_2 is

$$\begin{aligned}\dot{x}&=\frac{x}{6}-\frac{13y}{144}+\frac{-82\sqrt{3}-205\sqrt{10}-96\sqrt{30}-5664}{4896},\\ \dot{y}&=x-\frac{y}{6}+\frac{-432\sqrt{10}+\sqrt{30\left(3936\sqrt{10}+24721\right)}-17964}{2448},\end{aligned}\tag{40}$$

this system has the first integral

$$H_2(x,y) = \frac{1}{36(2\sqrt{3}-5\sqrt{10})}(144(2\sqrt{3}-5\sqrt{10})x^2 - 4x(24\sqrt{3}y - 60\sqrt{10}y + 1117\sqrt{3} - 574 - 2649\sqrt{10} + 96\sqrt{30} - 720) + y(26\sqrt{3}y - 65\sqrt{10}y + 384\sqrt{3} - 1632\sqrt{10})).$$

The unique real solution of the system of equations (38) for $i = 2$, for the piecewise linear differential system (39)–(40) is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (4, 2\sqrt{3}, 5, -2\sqrt{10}, 6, 3\sqrt{10}, 6, -3\sqrt{10})$.

For the second configuration of the class C_2, we consider the linear system

$$\begin{aligned} \dot{x} &= \frac{x}{5} - \frac{41y}{400} + \frac{1440\sqrt{3} - 640\sqrt{10} - 160\sqrt{33} + 2503}{3200\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)}, \\ \dot{y} &= x - \frac{y}{5} + \frac{160\sqrt{10} + \dfrac{2503\sqrt{10} + 960\sqrt{30} - 3200}{-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}} - 4390}{6400}, \end{aligned} \tag{41}$$

this differential system has the first integral

$$H_1(x,y) = \frac{1}{800(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})}(3200(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})x^2 + x(13170\sqrt{3} - 6277\sqrt{10} + 480\sqrt{30} - 4390\sqrt{33} + 160\sqrt{330} - 1280(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})y) + 2y(164(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})y - 1440\sqrt{3} + 640\sqrt{10} + 160\sqrt{33} - 2503)).$$

The second linear differential center in the region R_2 is

$$\begin{aligned} \dot{x} &= \frac{x}{5} - \frac{13y}{100} + \frac{360\sqrt{3} - 160\sqrt{10} - 40\sqrt{33} + 579}{800\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)}, \\ \dot{y} &= x - \frac{y}{5} - \frac{-3873\sqrt{3} + 4898\sqrt{10} + 1440\sqrt{11} + 480\sqrt{30} + 2449\sqrt{33}}{3200\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)}, \end{aligned} \tag{42}$$

this system has the first integral

$$H_2(x,y) = \frac{1}{400(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})}1600(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})x^2 - x(640(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})y - 3873\sqrt{3} + 4898\sqrt{10} + 1440\sqrt{11} + 480\sqrt{30} + 2449\sqrt{33}) + 4y\left(52\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)y - 360\sqrt{3} + 160\sqrt{10} + 40\sqrt{33} - 579\right).$$

The unique real solution of the system of equations (38) for $i = 2$, for the piecewise linear differential system (41)–(42) is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (1/4, \sqrt{33}/8, 1/2, -\sqrt{5}/(2\sqrt{2}), 3/4, 3\sqrt{3}/8, 1, 0)$.

We prove the statement for the class C_3. We consider the first linear differential center in the region R_1

$$\begin{aligned} \dot{x} &= -\frac{x}{6} - \frac{25y}{576} + \frac{-128\sqrt{2} + 288\sqrt{3} + 483}{-384\sqrt{2} + 576\sqrt{3} + 1344}, \\ \dot{y} &= x + \frac{y}{6} + \frac{-96618\sqrt{2} + 9\sqrt{3(58745128\sqrt{2} + 89093337)} - 500192}{192384}, \end{aligned} \tag{43}$$

this differential system has the first integral

$$H_1(x,y) = \frac{1}{144(2\sqrt{2}-3\sqrt{3}-7)}(576(2\sqrt{2}-3\sqrt{3}-7)x^2 + x(192(2\sqrt{2}-3\sqrt{3}-7)y - 3770\sqrt{2}+6861\sqrt{3}+1152\sqrt{6}+14875) + y(25(2\sqrt{2}-3\sqrt{3}-7)y - 768\sqrt{2}+1728\sqrt{3}+2898).$$

The second linear differential center in the region R_2 is

$$\begin{aligned} \dot{x} &= -\frac{x}{9} - \frac{145y}{5184} + \frac{134932\sqrt{2}-69489\sqrt{3}-77682\sqrt{6}+250949}{577152}, \\ \dot{y} &= x + \frac{y}{9} + \frac{-23197\sqrt{2}+45834\sqrt{3}-1728\sqrt{6}+79814}{5184(2\sqrt{2}-3\sqrt{3}-7)}, \end{aligned} \tag{44}$$

this system has the first integral

$$H_2(x,y) = \frac{1}{648(-2\sqrt{2}-18\sqrt{3}+6\sqrt{6}-20)}(5184(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)x^2 + x(1152(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)y + 7645\sqrt{2}+193608\sqrt{3}+332692 - 67863\sqrt{6}) + y(145(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)y + 2304\sqrt{2}+10755\sqrt{3} - 6912\sqrt{6}+41343)).$$

This piecewise linear differential centers has a unique real solution of the system of equations (38) for $i = 3$, which is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (1, 1, 2, -2\sqrt{2}, 3, 3\sqrt{3}, 4, -8)$.

Now we prove the statement for the class C_4. We consider the linear differential center in the region R_1

$$\dot{x} = -\frac{x}{9} - \frac{181y}{8100} + \frac{2}{9}, \quad \dot{y} = x + \frac{y}{9} + \frac{\sqrt{2}}{3} - \frac{2071}{540} + \frac{8}{3\sqrt{3}}, \tag{45}$$

this differential system has the first integral

$$H_1(x,y) = 4x^2 + \frac{2}{135}x\left(60y + 180\sqrt{2} + 480\sqrt{3} - 2071\right) + \frac{y(181y-3600)}{2025}.$$

The second linear differential center in the region R_2 is

$$\begin{aligned} \dot{x} &= -\frac{x}{8} - \frac{y}{32} + \frac{66\sqrt{2}-62\sqrt{3}-18\sqrt{6}+525}{1128}, \\ \dot{y} &= x + \frac{y}{8} + \frac{-600\sqrt{6}+8\sqrt{6\left(232\sqrt{6}+2555\right)}-12627}{4512}, \end{aligned} \tag{46}$$

this system has the first integral

$$H_2(x,y) = \frac{1}{8(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)}32(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)x^2 + 2x(4(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)y + 1521\sqrt{2}+556\sqrt{3}-666\sqrt{6} - 2456) + y((-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)y - 8(-51\sqrt{2}-16\sqrt{3} + 24\sqrt{6}+76)).$$

In this case the unique real solution of the system of equations (38) for $i = 4$, is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (2, 2, 2, -2, 3, 3\sqrt{2}, 4, -4\sqrt{3})$.

We prove the statement for the first configuration of the class C_5. We consider the linear differential center in the region R_3

$$\begin{aligned}\dot{x} &= -\frac{x}{6} - \frac{13y}{144} + \frac{53\sqrt{2}+1134}{4896},\\ \dot{y} &= x + \frac{y}{6} + \frac{-114\sqrt{2}+91\sqrt{3}-24\sqrt{6}+843}{72(\sqrt{2}-6)},\end{aligned} \tag{47}$$

this system has the first integral

$$\begin{aligned}H_1(x,y) = \quad & \frac{1}{36(\sqrt{2}-6)}(144(\sqrt{2}-6)x^2 - 4x(-12(\sqrt{2}-6)y + 114\sqrt{2} - 91\sqrt{3}\\ & +24\sqrt{6} - 843) + y(13(\sqrt{2}-6)y - 48\sqrt{2} + 394)).\end{aligned}$$

The second linear differential center in the region R_1 is

$$\begin{aligned}\dot{x} &= \frac{x}{6} - \frac{5y}{18} + \frac{1}{612}\left(-89\sqrt{2} - 636\right),\\ \dot{y} &= x - \frac{y}{6} + \frac{1}{306}(432\sqrt{3} + 178\sqrt{3(\sqrt{3}+2)} - 795),\end{aligned} \tag{48}$$

this differential system has the first integral

$$\begin{aligned}H_2(x,y) = \quad & \frac{1}{9(\sqrt{2}-6)}(36(\sqrt{2}-6)x^2 - 4x(3(\sqrt{2}-6)y + 141\sqrt{2} + 142\sqrt{3}\\ & +6\sqrt{6} - 312) + 2y(5(\sqrt{2}-6)y + 6\sqrt{2} - 214)).\end{aligned}$$

For this piecewise linear differential centers the unique real solution of system (38) when $i = 5$ is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (1, \sqrt{2}, 2, -2\sqrt{3}, 2, 2\sqrt{3}, 3, -6)$. Hence, the discontinuous piecewise linear differential system (47)–(48) has a unique crossing limit cycle, see $(\mathcal{C}_5^1)$ of Figure 13.

Finally we prove the statement for the second configuration of the class C_5 .In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{x}{2} - \frac{5y}{4} + \frac{2}{5}, \quad \dot{y} = x - \frac{y}{2} + \frac{1}{640}(96\sqrt{2} + 88\sqrt{3} + 365), \tag{49}$$

this system has the first integral

$$H_1(x,y) = 4x^2 + \frac{1}{80}x\left(-320y + 96\sqrt{2} + 88\sqrt{3} + 365\right) + \frac{1}{5}y(25y - 16).$$

In the region R_2 we consider the linear center

$$\begin{aligned}\dot{x} &= -\frac{x}{4} - \frac{y}{8} + \frac{-22000\sqrt{2} - 3000\sqrt{3} + 34816\sqrt{5} + 24585}{640(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5}},\\ \dot{y} &= x + \frac{y}{4} + \frac{2400\sqrt{2} - 2200\sqrt{3} - \dfrac{4125(149\sqrt{11 - 4\sqrt{6}} + 200)}{275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5}} + 62593}{108800},\end{aligned} \tag{50}$$

this system has the first integral

$$\begin{aligned} H_2(x,y) = \; & \frac{1}{800(275\sqrt{2}+75\sqrt{3}-272\sqrt{5})}(3200(275\sqrt{2}+75\sqrt{3}-272\sqrt{5})x^2 \\ & +x(1600(275\sqrt{2}+75\sqrt{3}-272\sqrt{5})y+940225\sqrt{2}+312300\sqrt{3} \\ & -1001488\sqrt{5}-25000\sqrt{6}-38400\sqrt{10}+35200\sqrt{15})+10y(40(275\sqrt{2} \\ & +75\sqrt{3}-272\sqrt{5})y+22000\sqrt{2}+3000\sqrt{3}-34816\sqrt{5}-24585)). \end{aligned}$$

For this piecewise linear differential centers the unique real solution of system (38) when $i=5$ is $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(-1/4,\sqrt{3}/8,-4/5,4/(5\sqrt{5}),-1/2,$ $-1/(2\sqrt{2}),-4/5,-4/(5\sqrt{5}))$. Then the discontinuous piecewise linear differential system (49)–(50) has exactly one crossing limit cycle, see $(\mathcal{C}_5^2)$ of Figure 13. This completes the proof of statement (a) of Theorem 3. □

Proof of statement (*b*) *of Theorem* 3. First we prove the statement for class C_1. In the region R_1 we consider the linear differential center

(51)
$$\dot{x} = -0.0327708x - 0.167012y + 0.202324, \quad \dot{y} = x + 0.0327708y - 2.82026,$$

its first integral is

$$H_1(x,y) = 4(x+0.0327708y)^2 + 8(-2.82026x - 0.202324y) + 0.663752y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = \frac{x}{10} - \frac{13y}{50} - \frac{1}{5}, \quad \dot{y} = x - \frac{y}{10} - \frac{16}{5}, \tag{52}$$

its first integral is

$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{5} - \frac{16x}{5}\right) + y^2.$$

For the piecewise linear differential system (51)–(52) the real solutions of the system of equations

$$\begin{aligned} & H_1(\alpha_i,\beta_i) - H_1(\gamma_i,\delta_i) = 0, \\ & H_1(f_i,g_i) - H_1(h_i,k_i) = 0, \\ & H_2(\alpha_i,\beta_i) - H_1(f_i,g_i) = 0, \\ & H_2(\gamma_i,\delta_i) - H_1(h_i,k_i) = 0, \\ & c_s(\alpha_i,\beta_i) = c_s(\gamma_i,\delta_i) = 0, \quad i = 1,2, \\ & c_s(f_i,g_i) = c_s(h_i,k_i) = 0. \end{aligned} \tag{53}$$

when $s=1$, is $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.571172..,1.04103..,2.93153..,6.0596..,$ $0.449711..,-0.861917..,2.66907..,-5.36724..)$ and $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) =$ $(1.19066..,2.07275..,2.40283..,4.69567..,0.928007..,-1.60885..,2.25189..,-4.32923..)$. Hence these linear differential centers have exactly two crossing limit cycles, see $(\mathcal{C}_1)$ of Figure 14.

Now we prove the statement for the class C_2. In the region R_1 we consider the linear differential center

$$\dot{x} = -0.252669y, \quad \dot{y} = x - 0.498272, \tag{54}$$

its first integral is

$$H_1(x,y) = 4x^2 - 3.98617x + 1.01067y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = -\frac{y}{25}, \quad \dot{y} = x - \frac{1}{2}, \tag{55}$$

its first integral is

$$H_2(x,y) = 4x^2 - 4x + \frac{4y^2}{25}.$$

For these centers the real solutions of system (53) when $s = 2$ are $(0.898773.., 0.43723.., 0.169784.., 0.631617.., 0.898773.., -0.43723.., 0.169784.., -0.631617..)$ and $(0.758688.., 0.640578.., 0.368993.., 0.782685.., 0.758688.., -0.640578.., 0.368993.., -0.782685..)$. Hence the discontinuous piecewise linear differential system (54)–(55) has two crossing limit cycles, see $(\mathcal{C}_2^1)$ of Figure 15.

For the second configuration of the class C_2 we consider the linear center

$$\dot{x} = 0.057143x - 0.0550264y - 0.0521176, \quad \dot{y} = x - 0.057143y - 0.805757, \tag{56}$$

in the region R_1, its first integral is

$$H_1(x,y) = 4(x - 0.057143y)^2 + 8(0.0521176y - 0.805757x) + 0.207044y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = \frac{x}{10} - \frac{17y}{80} - \frac{1}{10}, \quad \dot{y} = x - \frac{y}{10} - \frac{11}{10}, \tag{57}$$

its first integral is

$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{10} - \frac{11x}{10}\right) + \frac{81y^2}{100}.$$

The real solutions of system (53) for these centers are $(0.253068.., 0.283285.., 1.26272.., 1.89942.., 0.204637.., -0.224601.., 1.16927.., -1.72215..)$ and $(0.145203.., 0.155388.., 1.36881.., 2.10673.., 0.119104.., -0.125997.., 1.24855.., -1.87223..)$. Hence the discontinuous piecewise linear differential system (56)–(57) has two crossing limit cycles, see $(\mathcal{C}_2^2)$ of Figure 15.

For the class C_3 and in the region R_1 we consider the center

$$\dot{x} = 0.0333015x - 0.132045y - 0.0410184, \quad \dot{y} = x - 0.0333015y - 1.97694, \tag{58}$$

its first integral is

$$H_1(x,y) = 4(x - 0.0333015y)^2 + 8(0.0410184y - 1.97694x) + 0.523744y^2.$$

The second linear differential center in the region R_2 is

$$\dot{x} = 0.1x - 0.26y - 0.2, \quad \dot{y} = x - 0.1y - 2.3, \tag{59}$$

its first integral is

$$H_2(x,y) = 4(x - 0.1y)^2 + 8(0.2y - 2.3x) + y^2.$$

The real solutions of system (53) when $s = 3$ are $(0.340737.., 0.198898.., 2.70162.., 4.44056.., 0.308678.., -0.171498.., 2.52616.., -4.01504..)$ and $(0.862714.., 0.801309.., 2.24826.., 3.37109.., 0.727481.., -0.620487.., 2.17185.., -3.2007..)$. Hence the discontinuous piecewise linear differential system (58)–(59) has two crossing limit cycles, see $(\mathcal{C}_3)$ of Figure 14.

We prove the statement for the class C_4. In the region R_1 we consider the center

$$\dot{x} = -0.142331x - 0.0908868y + 1.59716, \quad \dot{y} = x + 0.142331y - 9.92891, \tag{60}$$

its first integral is

$$H_1(x,y) = 4(x + 0.142331y)^2 + 8(-9.92891x - 1.59716y) + 0.282515y^2.$$

We consider the second linear differential center in the region R_2

$$\dot{x} = \frac{x}{10} - \frac{y}{10} - \frac{7}{10}, \quad \dot{y} = x - \frac{y}{10} - \frac{44}{5}, \tag{61}$$

its first integral is

$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{7y}{10} - \frac{44x}{5}\right) + \frac{9y^2}{25}.$$

The two real solutions of system (53) when $s = 4$ are $(2.34486.., 2.71929.., 9.31289.., 26.851.., 1.97922.., -1.95854.., 8.44735.., -23.0527..)$ and $(4.45798.., 8.2899.., 7.62439.., 19.6236.., 3.20672.., -4.7636.., 7.33329.., -18.455..)$. Hence the discontinuous piecewise linear differential system (60)–(61) has two crossing limit cycles, see $(\mathcal{C}_4)$ of Figure 14.

For the first configuration of the class C_5. We consider the linear differential center

$$\dot{x} = -0.453205y, \quad \dot{y} = x + 0.497252, \tag{62}$$

in the region R_3 with its first integral

$$H_1(x,y) = 4x^2 + 3.97802x + 1.81282y^2.$$

We consider the second linear differential center in the region R_2

$$\dot{x} = -\frac{y}{10}, \quad \dot{y} = x + \frac{1}{2}, \tag{63}$$

its first integral is

$$H_2(x,y) = 4x^2 + 4x + \frac{2y^2}{5}.$$

The two real solutions of system (53) when $s = 5$ are $(-0.0927079.., 0.088306.., -0.837019.., 0.337912.., -0.0927079.., -0.088306.., -0.837019.., -0.337912..)$ and $(-0.217826.., 0.192647.., -0.663868.., 0.38489.., -0.217826.., -0.192647.., -0.663868.., -0.38489..)$. Hence, the discontinuous piecewise linear differential system (62)–(63) has two crossing limit cycles, see $(\mathcal{C}_5)$ of Figure 16.

Finally we prove the statement for the second configuration of the class C_5. In the region R_3 we consider the linear differential center

$$\dot{x} = 0.057143x - 0.0550264y - 0.0521176, \quad \dot{y} = x - 0.057143y - 0.805757, \tag{64}$$

its first integral is

$$H_1(x,y) = 4(x - 0.057143y)^2 + 8(0.0521176y - 0.805757x) + 0.207044y^2.$$

We consider the second linear differential center in the region R_1

$$\dot{x} = \frac{x}{10} - \frac{17y}{80} - \frac{1}{10}, \quad \dot{y} = x - \frac{y}{10} - \frac{11}{10}, \tag{65}$$

its first integral is

$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{10} - \frac{11x}{10}\right) + \frac{81y^2}{100}.$$

The two real solutions of system (53) when $s = 5$ are $(0.253068.., 0.283285.., 1.26272.., 1.89942.., 0.204637.., -0.224601.., 1.16927.., -1.72215..)$ and $(0.145203.., 0.155388.., 1.36881.., 2.10673.., 0.119104.., -0.125997.., 1.24855.., -1.87223..)$. Hence the discontinuous piecewise linear differential system (64)–(65) has two crossing limit cycles, see $(\mathcal{C}_5^2)$ of Figure 16. □

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